Multiplication is multiple additions

Ways of thinking about multiplication:

Counting groups or serial addition of groups

On a number line

With algebra tiles

With vectors

As a function

Things, groups, variables, or sets that you are multiplying must be the same size

Use a dot ·, asterisk (star) *, parentheses 2(x), or no sign 2x

I never use a cross or x as that is used as a variable.

Multiple additions

3*2=2+2+2=6

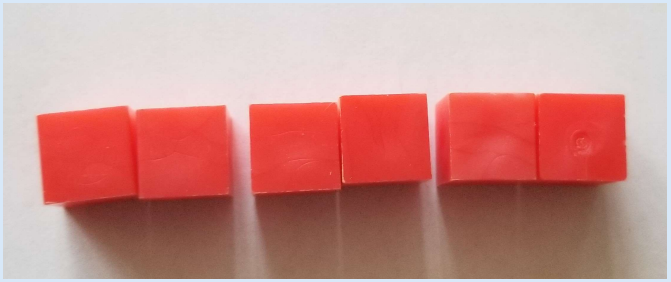
2*3=3+3=6

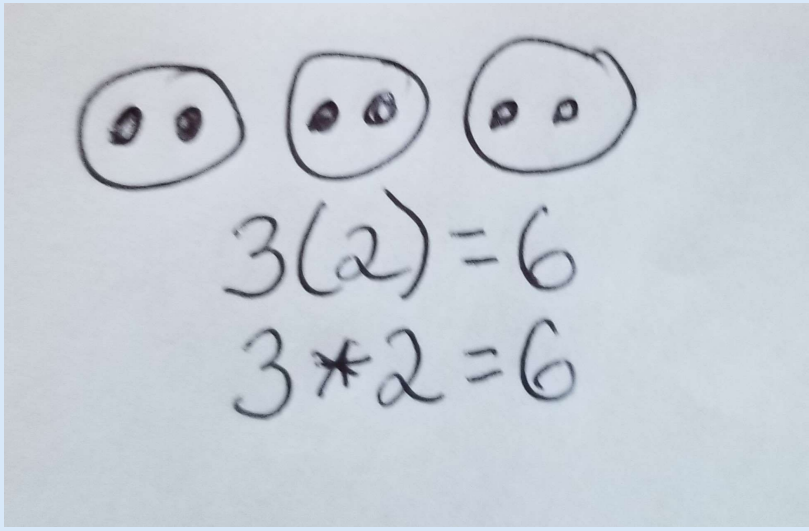
2x=x+x

x(2)=x+x

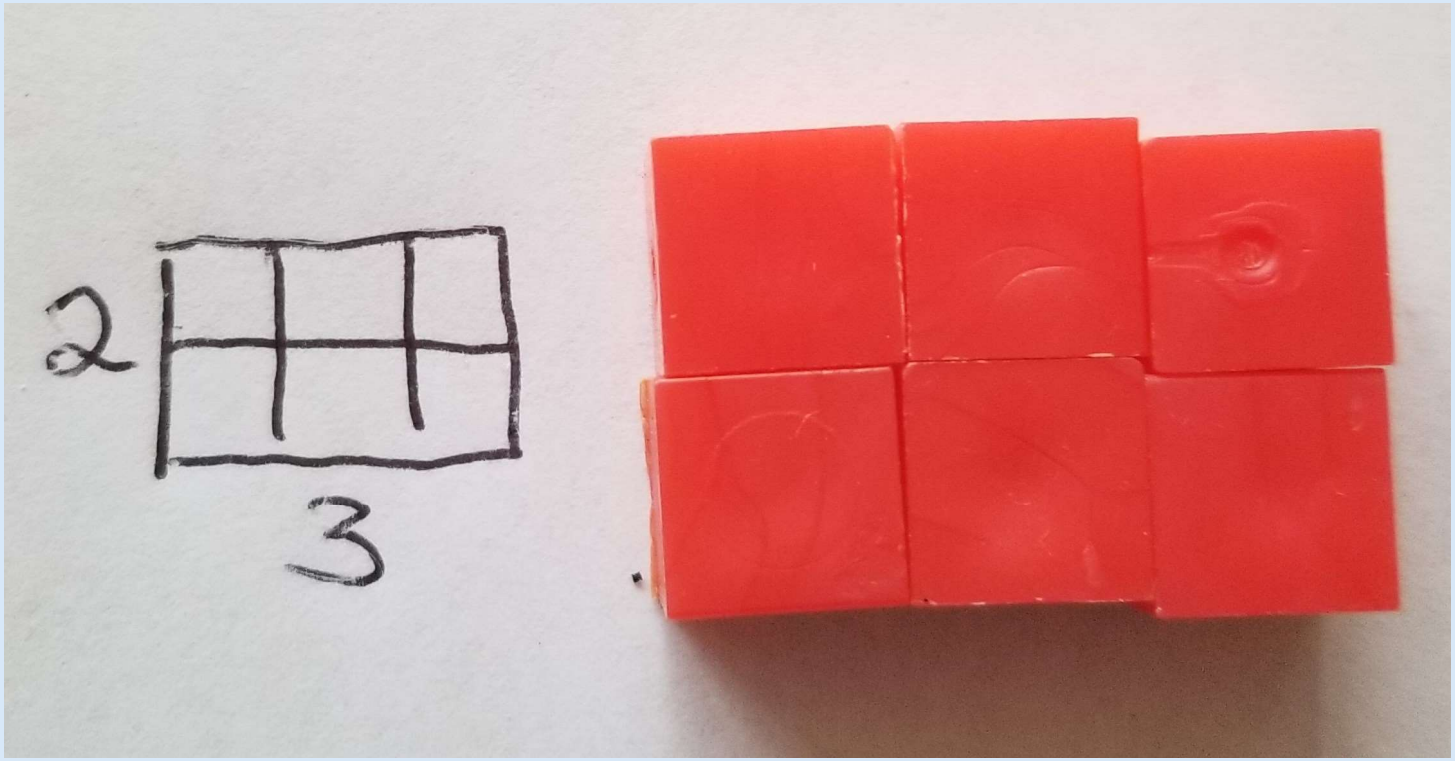
Multiplication is commutative and the order doesn’t matter.

How many groups times how many in each group

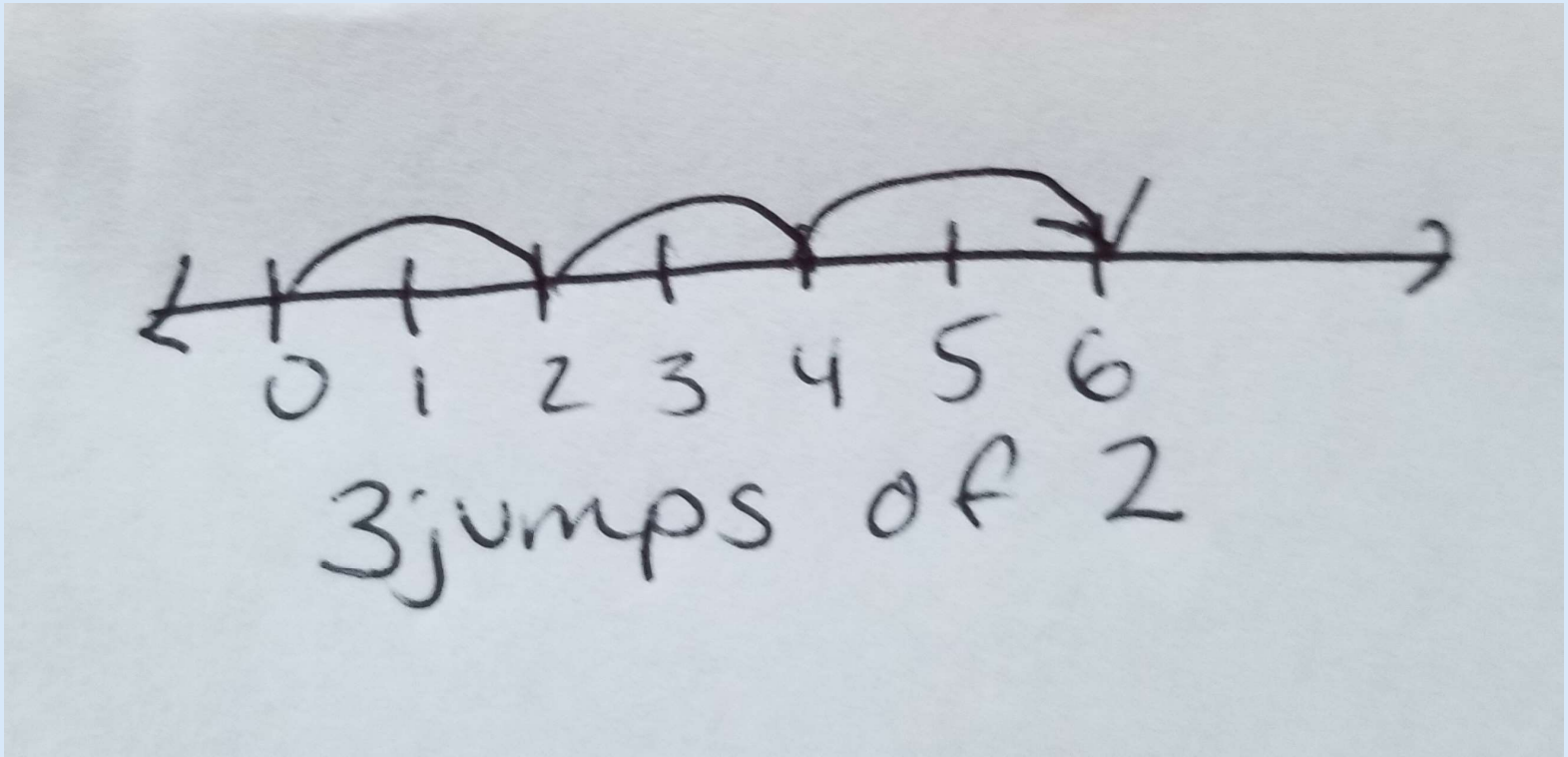




As area (or volume in 3 dimensions)

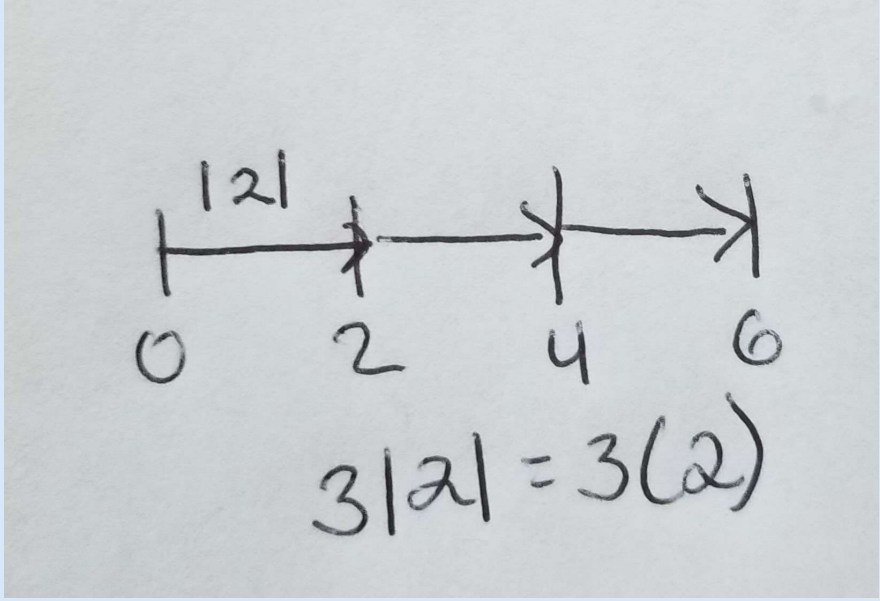


How many jumps times how big each jump is on a number line.

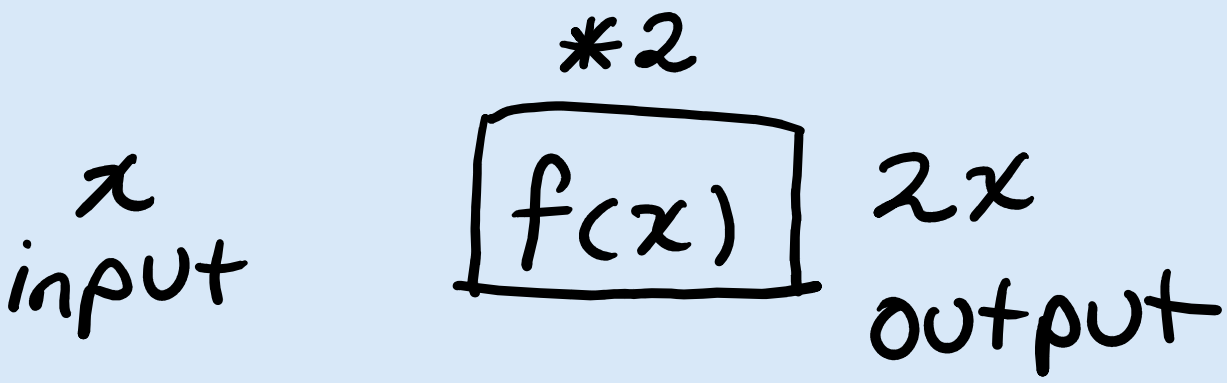


How many vectors get added together.

Number of vectors times the magnitude of each vector.



As a function



Warning! Here the f(x) is the name of the function with an input of x. The parentheses here don't represent multiplication.

One is the identity of multiplication

Any number times one is that number

1*2=2

1*n=n

n*1=n

x(1)=x

1x=x

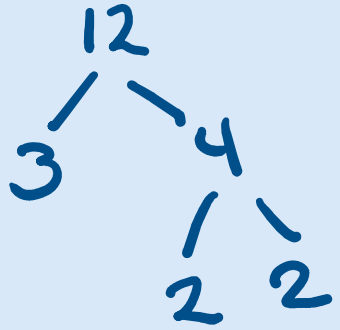
1n=n

Factoring undoes multiplication.

Factoring is when you break down something that you multiplied together and implies that there is no remainder.

6 factors into the prime factors of 2 and 3

Prime integers don’t factor into other integers.



Division is the inverse of multiplication

Division does the inverse of the multiplication process and can have a remainder.

Uses the symbol / or is written with a top and bottom where you multiply on top and divide on the bottom $\frac{\mathbf{2}}{\mathbf{3}}$

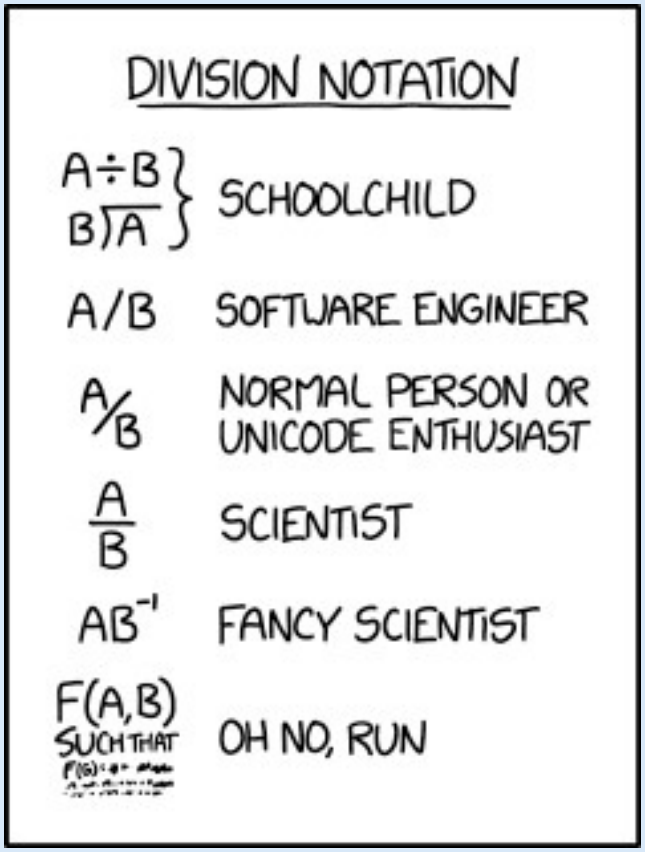
Some calculators use the division symbol, but I don't since it is too easy to misread as a +

On paper, I always use top and bottom since I am a very visual-spatial thinker.

Different symbols are used for division.

xkcd: Division Notation

Why, yes, I am a scientist.
Later we can all become
Fancy Scientists.



There is a false belief about teaching fractions, and it causes issues for many students.

Please let me know if this is an issue for you and we can do some work to let that go and unblock your brain around these issues.

Rational numbers or expressions

A rational number or rational expression is left as a ratio or fraction.
3/5 or 3:5 are ways of writing ratios.
If I give you a rational problem, you can give a rational solution!

$$x = \frac{3}{5} \qquad x = \frac{6}{10} = \frac{3}{5}$$

It is polite to reduce rational expressions to their simplest form.

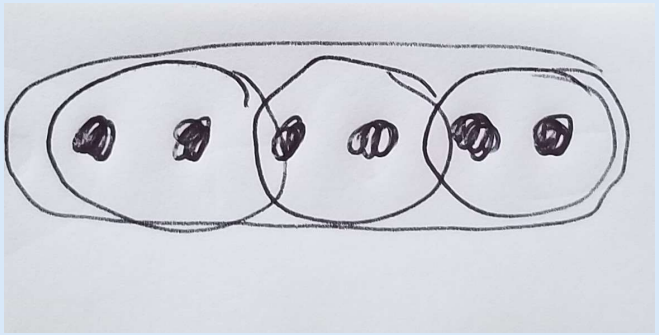
An exception to reducing is **percent** which is out of a hundred (cent is from Latin for hundred)

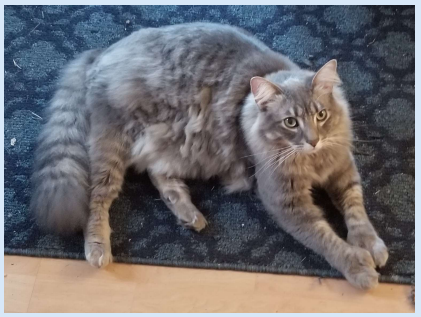
5/100 or 20/100 or 35/100 is allowed to stay over 100 when it is a percent.

$\frac{5}{100} = 5\%$ $\frac{20}{100} = 20\%$

Warning! In some programming languages, the % sign represents remainder and not percent.

How many groups or how many in each group





Assuming the cats each have four paws, if two cats have 8 paws, how many paws are on each cat?

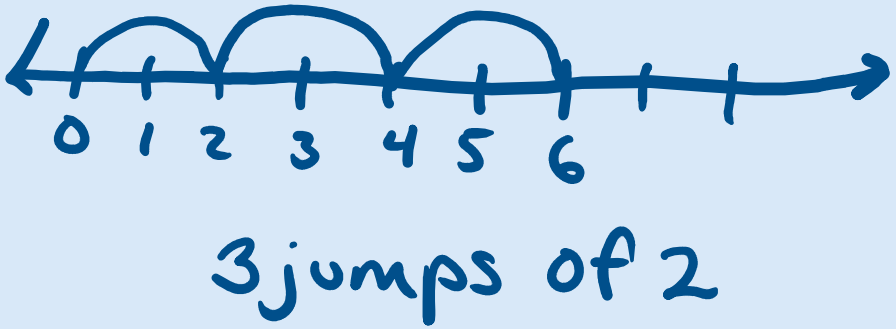
Each group must be the same size. May skip

Doc is not a good model for multiplication or division with his digits or claws because he has polydactyly. That is a genetic condition where he has 8 digits on each of his front paws, instead of the usual 5.

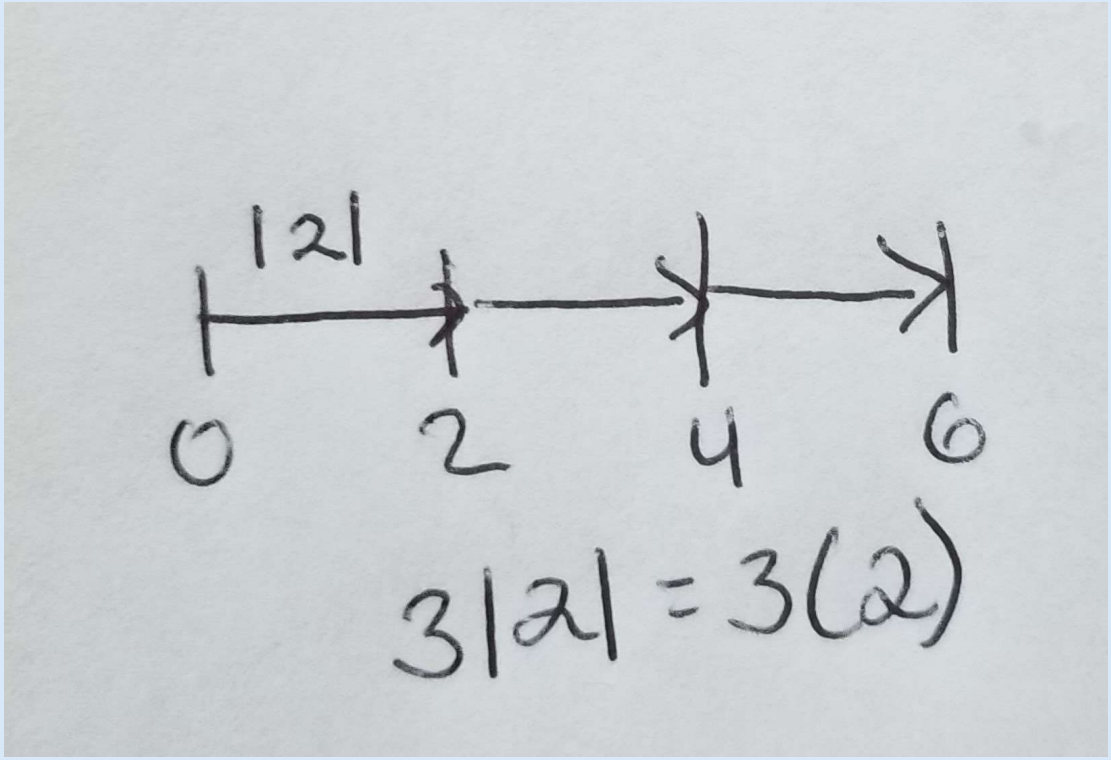
Most people are better models for multiplication and division with fingers and toes. May skip

Most people have 5 fingers or digits on each hand and 5 toes or digits on each foot, unlike Doc.

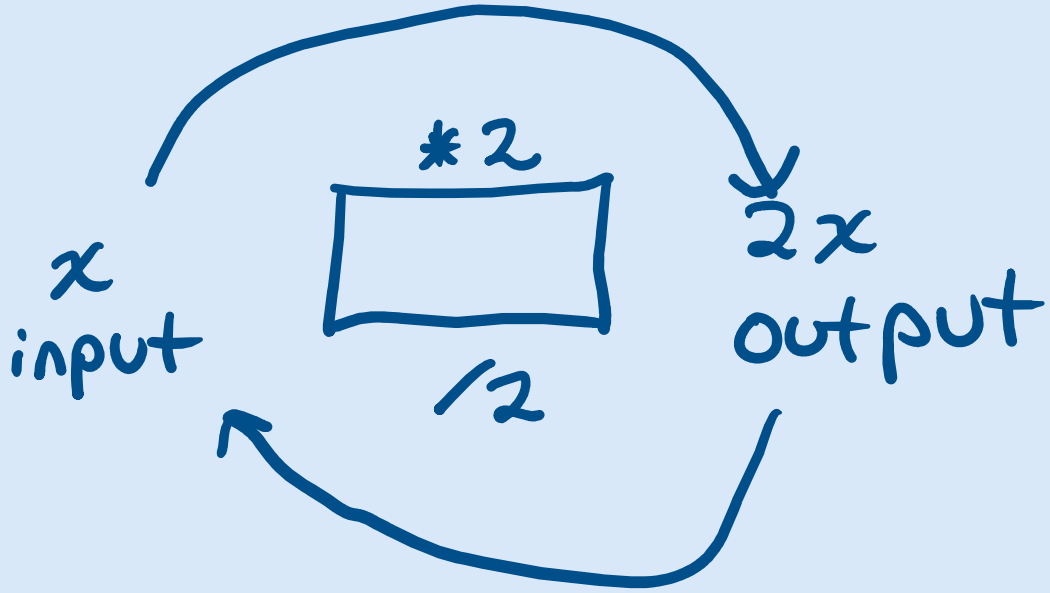
How many jumps or how many in each jump.



How many vectors or what is the magnitude of each vector.



Inverse function of multiplication. It undoes the multiplication and brings you back to the input.



You can't divide by zero

Dividing by zero has no meaning, and we will later see that it causes an asymptote on a graph. You can't divide by zero! We will later see that if you are dividing by a variable, you have to put a restriction on the variable so that you aren't dividing by zero.

$\frac{0}{x} = 0$ $\frac{x}{0}$

You can’t divide by zero

If you have zero in a group, then you can divide it by any number and each group still has zero.

It makes no sense to talk about taking and group and not dividing by anything. That isn’t dividing.

You can’t divide by zero

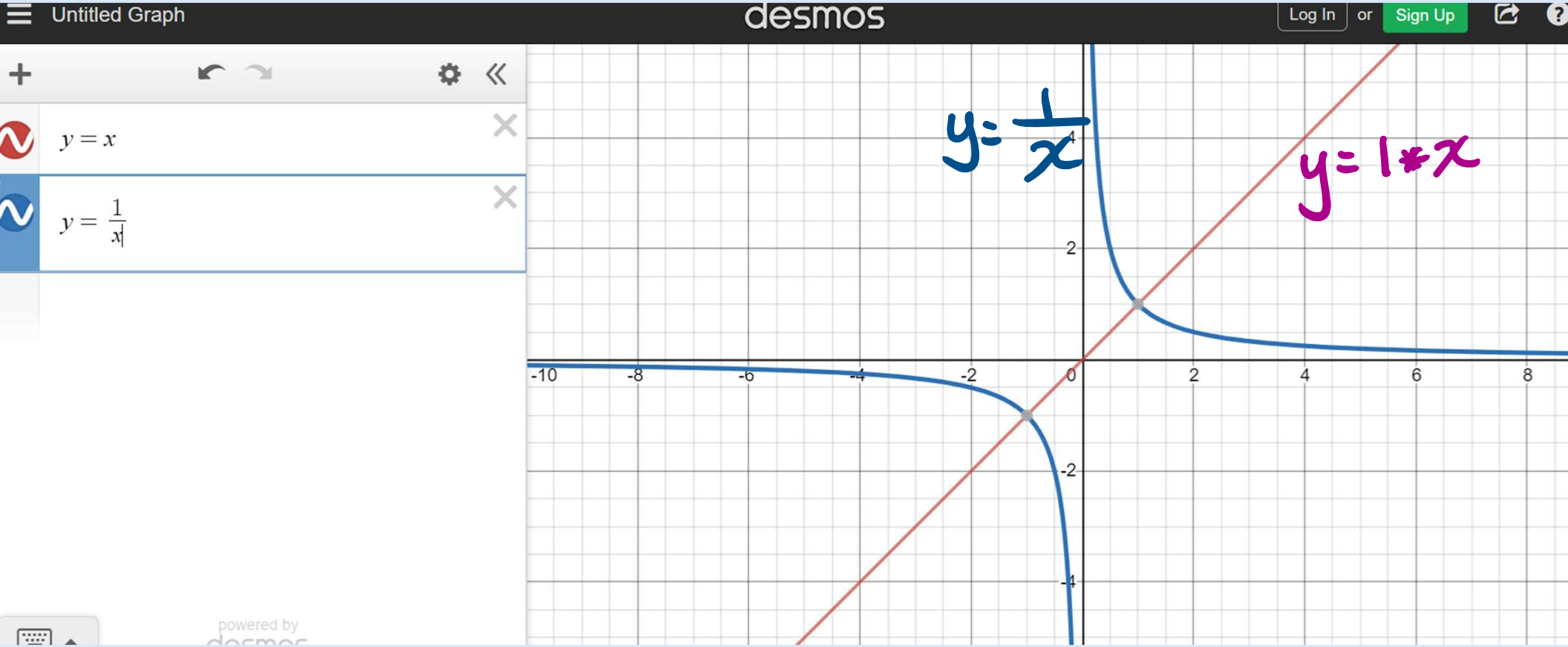
$\frac{0}{2}$ is 0

nothing divided by something is still nothing

$\frac{2}{0}$ is not defined

it doesn’t make sense to divide by nothing

Linear vs rational functions (Graph done in Desmos.com)



Any number divided by itself or over itself is one.

I often say over for “divided by”.

$\frac{2}{2} = 1 \quad \frac{365}{365} = 1 \quad \frac{n}{n}\ n \neq 0 = 1$

One is the identity for multiplication and division

$1 * x = x$

$\frac{x}{1} = x$

$\frac{x}{x} = 1$

$x \neq 0$

Any number divided by one is that number. Any integer can be written as that number over one.

$$\frac{x}{1} = x \qquad \frac{2}{1} = 2$$

$$x = \frac{x}{1} \qquad 2 = \frac{2}{1}$$

We use improper fractions and don't use mixed numbers because that gets very complicated with addition and division in the same number.
Mixed numbers are not rational numbers because they are not the ratio of two integers.

$\frac{x}{2}$

If x is 5, we write $\frac{5}{2}$

$2\frac{1}{2}$ is $2+\frac{1}{2}$ is not a ratio of two integers but is a whole number plus a fraction.
This is the last mixed number you will see in Crainix maths.

Dividing by a number is the same as multiplying by the reciprocal.

$$\frac{1}{2} = 1 \cdot \frac{1}{2} \qquad \frac{x}{2} = x \cdot \frac{1}{2} = \frac{1}{2}x$$

$$\frac{2}{3}x = 2 \cdot \frac{1}{3} \cdot x$$

$$\frac{a}{b} = a \cdot \frac{1}{b}$$

$$b \neq 0$$

When I see fractions, I think of multiplying by the top and dividing by the bottom.

$\frac{2}{3}x$

Take x, multiply by 2 and divide by 3 or divide by 3 and multiply by 2

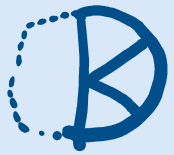
$\frac{\text{mult}}{\text{div}}$

$\frac{a}{b}x$

Take x, multiply by a and divide by b or divide by b and multiply by a.

$b \neq 0$

Dividing by a number is the same as multiplying by the reciprocal.
Dividing a fraction by a whole number is the same as multiplying the fraction by one over that number.



$\frac{\frac{1}{2}}{3} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ $\frac{1}{2}$ divided into 3 pieces is $\frac{1}{6}$ of the original whole.

$$\frac{\frac{1}{a}}{b} = \frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$$

$a, b \neq 0$

$$\frac{\frac{\text{mult}}{\text{div}}}{\text{div}}$$

If you are dividing by a fraction, then you are multiplying by the reciprocal of the bottom.

If I have a whole and I am dividing it into half size pieces, how many pieces do I have?

$$\frac{1}{\frac{1}{2}} = 1 \cdot \frac{2}{1} = 2$$

$$\text{whole} \cdot \frac{2 \text{ pieces}}{1 \text{ whole}} = 2 \text{ pieces}$$

2 pieces

$$\frac{a}{\frac{b}{c}} = a \cdot \frac{c}{b} = \frac{ac}{b}$$

$b, c \neq 0$

Dividing by a number is the same as multiplying by the reciprocal. This can get confusing so I think of the bottom doing the inverse of the top.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \qquad b,c,d \neq 0$$

$$\frac{\text{mult}}{\text{divide}} \Big/ \frac{\text{divide}}{\text{mult}}$$

$$\frac{\frac{1}{2}}{\frac{3}{5}} = \frac{1}{2} \cdot \frac{5}{3} = \frac{5}{6}$$

Remainder or Modulo operation

Sometimes when we do division, we just care about the remainder.

This happens with modular arithmetic or clock arithmetic. On a twelve-hour clock, 13:00 is the same as 1 pm.

Looking at the remainder after division is called the modulo operation.

You have a group of kids, and everyone gets the same number of apples. You may want to know how many apples you have left over.

I think of this as the leftover operation to use when I only care about what the leftovers are. % sign is often used for modulo