pset 5

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2/19/26

https://yale.instructure.com/courses/113840/assignments/558953

1 Problem 1

Our goal is to find the smallest k so that can we can make k' cents postage for any integer $k' \geq k$ with 4-, 9- and 15-cent stamps. (k and k' are restricted to be integers throughout the prompts given below.)

1.1 solution

1. Prove that if you can make k cents postage with some combination of 4-, 9-, and 15-cent stamps that includes at least 2 4-cent stamps, then you can make k+1 cents postage

   (a) suppose there $\exists k \in \mathbb{Z}[4a + 9b + 15c = k]$ and that $a \geq 2$
   let a' = a-2 and b' = b+1 and c' = c and k' = 4a' + 9b' + 15c'; also that a',b',c',k' $\exists \in \mathbb{N}$ so

$$k' = 4(a-2) + 9(b+1) + 15c$$

$$= 4a - 8 + 9b + 9 + c$$

$$= k - 8 + 9 = k + 1$$

   conclusion: you can make k+1 cents postage given that you have at least 2 4-cent postages

*made in Overleaf

2. Prove that if you can make k cents postage with some combination of 4-, 9-, and 15-cent stamps that includes at least 2 9-cent stamps, then you can make k+1 cents postage

   (a) suppose there $\exists k \in \mathbb{Z}[4a + 9b + 15c = k]$ and that $b \geq 2$
   let a' = a+1 and b' = b-2 and c' = c+1 and k' = 4a' + 9b' + 15c'; also that a',b',c',k' $\exists \in \mathbb{N}$ so

$$k' = 4(a + 1) + 9(b - 2) + 15(c + 1)$$

$$= 4a + 4 + 9b - 18 + c + 15$$

$$= k - 18 + 19 = k + 1$$

   conclusion: you can make k+1 cents postage given that you have at least 2 9-cent postages

3. Prove that if you can make k cents postage with some combination of 4-, 9-, and 15-cent stamps that includes at least 1 15-cent stamps, then you can make k+1 cents postage

   (a) suppose there $\exists k \in \mathbb{Z}[4a + 9b + 15c = k]$ and that $b \geq 2$
   let a' = a+4 and b' = b and c' = c-1 and k' = 4a' + 9b' + 15c'; also that a',b',c',k' $\exists \in \mathbb{N}$ so

$$k' = 4(a + 4) + 9(b) + 15(c - 1)$$

$$= 4a + 16 + 9b + c - 15$$

$$= k - 15 + 16 = k + 1$$

   conclusion: you can make k+1 cents postage given that you have at least 1 15-cent postages

4. Prove that if you have k cents postage for some $k \geq 14$ then you must have either 1) at least 2 4-cent stamps; 2) at least 2 9-cent stamps; or 3) at least 1 15-cent stamp.

   (a) $\therefore$ where $\exists a, b, c \in \mathbb{N}[k = 4a + 9b + 15c \geq 14]$
   base case: a,b,c = 0; then k ¡ 14 so cannot be true

i. lemma: if $c \geq 1 \rightarrow k \geq 14$ in this case, $k \geq 15 \cdot c$

$$= 15 \geq 14$$

ii. cases where 2 of a,b,c are 0 but one is not.

A. a $\neq$ 0; note that if $a < 4, k < 14$ by algebra (in this case, k = 4a); by contrapositive (modus ponens), $a \geq 4 > 2$ in this case

B. b $\neq$ 0; note that if $b < 2, k < 14$ by algebra (in this case, k = 9b); by contrapositive (modus ponens), $b \geq 2$ in this case

C. c $\neq$ 0; see lemma

iii. cases where 1 of a,b,c are 0 but two are not.

A. a,b $\neq$ 0;$k = 4a + 9b$ if $a \leq 1 \wedge b \leq 1$, then k < 14. By contrapositive (modus ponens), a,b must be $\geq$ 2 each

B. b,c $\neq$ 0; see lemma

C. c,a $\neq$ 0; see lemma

iv. case where none are 0

A. see lemma

5. Find the smallest k so that you can make k' cents postage for any $k' \geq k$ with 4-, 9-, and 15-cent stamps.

(a) suppose that $\exists a, b, c \in \mathbb{N}[k = 4a + 9b + 15c]$

this is a case of $p \vee q \vee r$ where

i. $p \equiv a \geq 2$

ii. $q \equiv b \geq 2$

iii. $r \equiv c \geq 1$ (if any of the above conditions are met, k+1 can be performed.)

furthermore, k+1 must also contain these conditions ($p \vee q \vee r$ must also apply to k+1)

method: find smallest of the base cases

(b) strong induction: for all natural numbers p(i) ¿ 19, if p(n) is true for n = 18, ... , k - 1 then p(k) is true

i. base case: is p(n) true for 18? yes, because 18 = $9 \cdot 2$ = 2b

ii. base case: is p(n) true for 19? yes, because 19 = 15 + 4 = c + a

iii. base case: is p(n) true for 20? yes, because 20 = 4 ·5= 5a

6. For your chosen k , prove that you can make k' cents postage for any $k' \geq k$ with some combination of 4-, 9-, and 15-cent stamps.

   (continued from proof above)

   (a) strong induction: for all natural numbers p(i) ¿ 19, if p(n) is true for n = 18, . . . , k - 1 then p(k) is true

      i. suppose p(i) is true for all i where $18 \leq i \leq k - 1$ p(k) = 4a + 9b + 15c
      since $a \geq 2 \vee b \geq 2 \vee c \geq 1$ for P(i), p(k) is true

2 problem 2

Problem 2 (10 points): Prove by induction that $\sum_{i=1}^{n}(2i+3) = n \cdot (n+4)$ for all integers $n \geq 0$ (An empty sum $\sum_{i=1}^{0}$ is defined to be 0. Do not use the result shown in class that $\sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$).

Figure 1: problem 2

2.1 solution

base case: $n = 0$

$$\sum_{i=1}^{n} 2i + 3 = n(n+4)$$

$$= \sum_{i=1}^{0} 2i + 3 = n(n+4)$$

$$0 = 0$$

(empty set rule used above)

inductive step: suppose $n \geq 0$, and the rule applies to n

$$\sum_{i=1}^{n+1} 2i + 3 = \sum_{i=1}^{n} 2i + 3 + (2(n+1) + 3)$$
$$= n(n+4) + (2(n+1) + 3)$$
$$= n^2 + 4n + (2(n+1) + 3)$$
$$= n^2 + 6n + 5$$
$$= (n+1)(n+5)$$
$$= (n+1)((n+1) + 4)$$

3 problem 3

Problem 3 (10 points): Define a sequence by $a_1 = 8$ and $a_i = a_{i-1} + 4i - 2$ for $i \geq 2$. Prove by induction that for any positive odd integer n, a_n is a multiple of 8. (There are many approaches to this, but there must be a proof by induction in your answer. That may be weak induction with a strengthened induction hypothesis, induction on odd integers, or, if you figure out some closed-form formula for a_n, then you would need to prove that your formula is correct by induction.)

Figure 2: problem 3

4 solution

base case: n = 1

$$a_1 = 8$$
$$8 \mid 8$$

induction step: suppose for all $n \geq 1$, $odd(n) \rightarrow 8 \mid a_n$

$$a_{n+2} = [a_n + 4(n+2) - 2] + 4((n+1) + 1) - 2$$
$$= [a_n + 4n + 2] + 4(n+2) - 2$$
$$= a_n + 4n + 2 + 4n + 6$$
$$= a_n + 8n + 8$$

since $8 \mid a_n \wedge 8 \mid 8n \wedge 8 \mid 8$ and since $n+2$ is odd $8 \mid a_{n+2}$ given n is an odd natural number

5 problem 4

Problem 4 (10 points): Define a sequence by $a_0 = 4$, $a_1 = 16$, and $a_n = 2 \cdot a_{n-1} + a_{n-2} - 2$ for $a \geq 2$. Prove by strong induction that $a_n \equiv 4 \pmod{6}$ for all integers $n \geq 0$.

Figure 3: problem 4

5.1 solution

base cases:

$a_0 = 4$ (mod 6)

$a_1 = 16 = 4$ (mod 6)

induction step (stronk):

for some index i where $\exists i \in \mathbb{N}[0 \leq i \leq n-1]$ and that p(i) is true (that $a_i = 4$ (mod 6)

$a_n = 2 \cdot a_{n-1} + a_{n-2} - 2$

a_n (mod 6) = 2$\cdot$ 4 (mod 6) + 4 (mod 6) - 2 (mod 6)

= 8 (mod 6) + 4 (mod 6) - 2

= 2 (mod 6) - 2 (mod 6) + 4 (mod 6)

= 4 (mod 6)

$a_n = 4$ (mod 6) for all integers $n \geq 0$