MATH 227 (Vector Calculus) Notes

Morgan Arnold

This document was typeset on January 20, 2022

Contents

1 Vector-Valued Functions

Definition 1: Vector-Valued Functions

A vector valued function is a function $\vec{\boldsymbol{r}} : I \to \mathbb{R}^n$, where $I \subseteq \mathbb{R}$. A vector valued function in $\mathbb{R}^3$ may be written in the form

$$\vec{\boldsymbol{r}}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$$

Definition 2: Continuity

A vector valued function $\vec{\boldsymbol{r}}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$ is *continuous* on I if and only if $x(t)$, $y(t)$, and $z(t)$ are all continuous on I.

Definition 3: Velocity & Acceleration

Given some vector valued function $\vec{\boldsymbol{r}}(t)$, there exists a corresponding *velocity* $\vec{\boldsymbol{v}}(t)$ equal to $\vec{\boldsymbol{r}}'(t)$ and a corresponding *acceleration* $\vec{\boldsymbol{a}}(t)$ equal to $\vec{\boldsymbol{r}}''(t)$, or equivalently $\vec{\boldsymbol{v}}'(t)$. The magnitude of $\vec{\boldsymbol{v}}(t)$ is a scalar quantity sometimes called the *speed* of $\vec{\boldsymbol{r}}$.

A stright line is described by the parameterization

$$\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{r}}_0 + t\vec{\boldsymbol{v}}$$

but may also be equivalently parameterized like

$$\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{r}}_0 + 2t\vec{\boldsymbol{v}}$$

which describes the same curve, but has a different velocity. Generally, the parameterization of a curve is not unique.

A helix is descibed by the parameterization

$$\vec{\boldsymbol{r}}(t) = \cos t\hat{\boldsymbol{\imath}} + \sin t\hat{\boldsymbol{\jmath}} + t\hat{\boldsymbol{k}}$$

which corresponds to uniform circular motion in the x-y plane coupled with downward motion in the z direction with constant velocty, resulting in a curve with a helical shape. $z(t)$ may be modified to change the "compression" of the helix. For instance,

$$\vec{\boldsymbol{r}}(t) = \cos t\hat{\boldsymbol{\imath}} + \sin t\hat{\boldsymbol{\jmath}} + t^3\hat{\boldsymbol{k}}$$

is much less compressed at large values of z, and highly compressed near 0.

Example 1

Parameterize the curve of intersection of $x^2 + y^2 = 9$ with $z = x + y$.

In the x-y plane, the path is circular, so we have

$$x = 3\cos t$$
$$y = 3\sin t$$

and due to the fact that the curve lies on the surface $z = x + y$, it must be that

$$z = 3\cos t + 3\sin t$$

which is a parameterization on the interval $[0, 2\pi]$.

Example 2

Parameterize the intersection of $z = x^2 + y^2$ and $x = 2y$.

The intersection of these curves is a parabola, so we may simply set

$$y = t$$

which then yields equations for x and z

$$x = 2t$$
$$z = 5t^2$$

Definition 4: Differentiation Rules

The differentiation of vector valued functions proceeds by differentiating each of the components, so the normal rules of differentiation apply, namely

$$(\vec{\boldsymbol{u}}+\vec{\boldsymbol{v}})' = \vec{\boldsymbol{u}}' + \vec{\boldsymbol{v}}'$$

$$(\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{v}})' = \vec{\boldsymbol{u}}'\cdot\vec{\boldsymbol{v}} + \vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{v}}'$$

and similarly for the cross product

$$(\vec{\boldsymbol{u}}\times\vec{\boldsymbol{v}})' = \vec{\boldsymbol{u}}'\times\vec{\boldsymbol{v}} + \vec{\boldsymbol{u}}\times\vec{\boldsymbol{v}}'$$

2 Curves & Arc Length

Definition 5: Smoothness

A vector valued function $\vec{\boldsymbol{r}}(t)$ is *smooth* on an interval (a,b) if $\vec{\boldsymbol{r}}'(t)$ exists *and* $\vec{\boldsymbol{r}}'(t) \neq \vec{\mathbf{0}}$ on the interval.

In the above definition, the case where $\vec{\boldsymbol{r}}'(t) = \vec{\mathbf{0}}$ corresponds to coming to a full stop, at which point motion may resume in any direction while maintaining the existence of the derivative, so a stronger condition is necessary to eliminate these cases.

Definition 6: Closure

A vector valued function $\vec{\boldsymbol{r}}(t)$ is *closed* on an interval (a,b) if $\vec{\boldsymbol{r}}(a) = \vec{\boldsymbol{r}}(b)$.

Definition 7: Simplicity

A vector valued function $\vec{\boldsymbol{r}}(t)$ is *simple* on an interval (a,b) if it has no self-intersections on the interval, except possibly where $\vec{\boldsymbol{r}}(a) = \vec{\boldsymbol{r}}(b)$.

Given a curve described by some function $\vec{\boldsymbol{r}}(t)$ on some interval (a,b), we may wish to find its arc length. We assume that $\vec{\boldsymbol{r}}$ is continuous, smooth, and 1-to-1. This may be done by approximating the curve as a sequence of line segments, and taking the limit as their length goes to 0. We consider some set of points $\{t_0, \ldots, t_n\}$, where $a = t_0 < t_1 < \ldots < t_n = b$, which yield the corresponding line segments $\{\vec{\boldsymbol{r}}(t_1) - \vec{\boldsymbol{r}}(t_0), \ldots, \vec{\boldsymbol{r}}(t_n) - \vec{\boldsymbol{r}}(t_{n-1})\}$ whose lengths may be summed up to obtain an approximation.

Definition 8: Rectifiability

A curve is rectifiable if there exists some $k > 0$ such that the length of an approximation of the curve in terms of line segments is less than k for any number of line segments. In other words, if the length of the approximation approaches a limit.

If a curve is rectifiable with some k, then the smallest such k is the length of the curve. Formulaically,

$$L = \int_a^b \left|\vec{\boldsymbol{r}}'(t)\right| \, dt = \int_a^b v(t) \, dt$$

which may be proven by noting that the length of the line segment between $\vec{\boldsymbol{r}}(t_i)$ and $\vec{\boldsymbol{r}}(t_{i-1})$ is $|\vec{\boldsymbol{r}}(t_i) - \vec{\boldsymbol{r}}(t_{i-1})|$, which approaches $|\vec{\boldsymbol{v}}'(t_i)|$ as t_i and t_{i-1} become very close.

Example 3

Find the length of the helix described by $\vec{\boldsymbol{r}}(t) = a\cos t\hat{\boldsymbol{\imath}} + a\sin t\hat{\boldsymbol{\jmath}} + bt$ on the interval $0 \leq t \leq T$.

$\vec{\boldsymbol{r}}'(t) = -a\sin t\hat{\boldsymbol{\imath}} + a\cos t\hat{\boldsymbol{\jmath}} + b\hat{\boldsymbol{k}}$, so $v(t) = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2}$ and

$$L = \int_0^T \sqrt{a^2 + b^2} \, dt = \left[\sqrt{a^2 + b^2}t\right]_0^T = T\sqrt{a^2 + b^2}$$

Example 4

Find the length of the curve described by $\vec{\boldsymbol{r}}(t) = 2t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}} + \frac{1}{3}t^3\hat{\boldsymbol{k}}$ on the interval $1 \le t \le 2$.

$\vec{\boldsymbol{r}}'(t) = 2\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}} + t^2\hat{\boldsymbol{k}}$, so $v(t) = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2 + t^2)^2} = 2 + t^2$, and the length of the curve is

$$L = \int_1^2 2 + t^2 \, dt = \left[2t + \frac{t^3}{3} \right]_1^2 = \frac{13}{3}$$

Let's say that $\vec{\boldsymbol{r}} : [a, b] \to \mathbb{R}^d$ is a smooth parametric curve, and $s(t)$ is the length of the curve from $\vec{\boldsymbol{r}}(a)$ to $\vec{\boldsymbol{r}}(t)$. We may then use s to parametrize the curve instead of t, which yields a parameterization with a constant speed of 1.

Note that

$$s(t) = \int_a^t v(u) \, du$$

so

$$\frac{ds}{dt} = v(t)$$

by FTC.

Theorem 1

Prove that an arc length parametrized curve has constant speed 1.

Note that

$$\left| \frac{d\vec{\boldsymbol{r}}}{dt} \right| = \left| \frac{d\vec{\boldsymbol{r}}}{dt} \frac{dt}{ds} \right|$$

but by the definition of arc length parameterization we have $\frac{dt}{ds} = \frac{1}{v(t)}$, which is always defined because $\vec{\boldsymbol{r}}$ is smooth. In that case,

$$\frac{d\vec{\boldsymbol{r}}}{dt} = |v(t)| \frac{1}{|v(t)|} = 1$$

Example 5

Find the arc length parameterization of the helix described by $\vec{\boldsymbol{r}}(t) = a\cos t\hat{\boldsymbol{\imath}} + a\sin t\hat{\boldsymbol{\jmath}} + bt$ on the interval $0 \leq t \leq T$.

$\vec{\boldsymbol{r}}(t) = a\cos t\hat{\boldsymbol{\imath}} + a\sin t\hat{\boldsymbol{\jmath}} + b\hat{\boldsymbol{k}}$, so $v(t) = \sqrt{a^2+b^2}$ and

$$s(t) = \int_0^t \sqrt{a^2+b^2}\,du = t\sqrt{a^2+b^2}$$

which we may use this to reparameterize the curve in the form

$$\vec{\boldsymbol{r}} = a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\hat{\boldsymbol{\imath}} + a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)\hat{\boldsymbol{\jmath}} + b\frac{s}{\sqrt{a^2+b^2}}\hat{\boldsymbol{k}}$$

Example 6

Find the arc length parameterization of the curve described by $\vec{\boldsymbol{r}}(t) = 2t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}} + \frac{t^3}{3}\hat{\boldsymbol{k}}$ on the interval $0 \leq t \leq T$.

$\vec{\boldsymbol{r}}(t) = 2t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}} + \frac{t^3}{3}\hat{\boldsymbol{k}}$, so $v(t) = t^2 + 2$ and

$$s(t) = \int_0^t u^2 + 2\,du = \frac{t^3}{3} + 2t$$

which may be solved for t to find a parameterization. The algebra involved is tedious, and thus elided.

3 Curvature

We wish to construct a mathematical way to encode the "amount" which a given curve curves, so that we may distinguish straighter curves from more tightly winding curves.

The velocity vector $\vec{\boldsymbol{v}}$ of a vector valued function $\vec{\boldsymbol{r}}$ is tangent to the curve which $\vec{\boldsymbol{r}}$ describes, so this notion of curvature must be related to the rate of change of $\vec{\boldsymbol{v}}$, as the curvature is highest when the velocity vector changes rapidly. One issue is that parameterizations are generally not unique, but our definition of curvature should be independent of any particular parameterization, and thus an intrinsic property of the curve. To do this, we normalize $\vec{\boldsymbol{v}}$ to unit length, so that the speed with which we traverse the curve is irrelevant to the curvature.

Theorem 2

Consider a smooth curve parameterized by $\vec{\boldsymbol{r}}(t)$. Prove that $|\vec{\boldsymbol{r}}| = a$ is constant if and only if $\vec{\boldsymbol{v}} \perp \vec{\boldsymbol{r}}$ for all t.

$\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}} = |\vec{\boldsymbol{r}}|^2 = a^2$, which is constant if and only if $\frac{d}{dt}\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}} = 0$, but

$$\frac{d}{dt}\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}' \cdot \vec{\boldsymbol{r}} + \vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}' = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{r}} + \vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{v}} = 2\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{r}}$$

so $|\vec{\boldsymbol{r}}(t)| = a$ is constant if and only if $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{r}} = 0$, and thus $\vec{\boldsymbol{v}} \perp \vec{\boldsymbol{r}}$.

Definition 9: Unit Tangent Vector

Assuming that $\vec{\boldsymbol{r}}(t)$ is smooth, so that $\vec{\boldsymbol{r}}'(t) \neq \vec{\mathbf{0}}$, we define the *unit tangent vector*

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|}$$

which is always tangent to $\vec{\boldsymbol{r}}(t)$, and has unit length $|\vec{\boldsymbol{T}}| = 1$, so the unit tangent vector does not depend on parameterization, except for orientation.

The unit tangent vector may also be defined in terms of the arc length parameterization. By noting that because the arc length parameterization has constant unit speed, we have

$$\vec{\boldsymbol{T}}(s) = \frac{\frac{d\vec{\boldsymbol{r}}}{ds}}{\left|\frac{d\vec{\boldsymbol{r}}}{ds}\right|} = \frac{d\vec{\boldsymbol{r}}}{ds}$$

Definition 10: Curvature

We define the *curvature* κ of a curve in terms of the unit tangest vector as

$$\kappa = \left| \frac{d\vec{\boldsymbol{T}}}{ds} \right|$$

using the arc length parameterization so that the curvature is independent of any particular parameterization.

One difficulty with this definition is that it requires us to find the arc length parameterization to solve for the curvature, but finding the arc length parameterization of even simple curves may be difficult or impossible. Therefore, to compute κ, we use the chain rule, noting that $\frac{ds}{dt} = |\vec{\boldsymbol{r}}'(t)|$, so we may compute the curvature of a curve using

$$\kappa = \left| \frac{d\vec{\boldsymbol{T}}}{dt} \frac{dt}{ds} \right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|}$$

Example 7

Consider a straight line parameterized by $\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{r}}_0 + \vec{\boldsymbol{v}}t$. Show that this curve has $\kappa = 0$.

First, we find $\vec{\boldsymbol{r}}'(t) = \vec{\boldsymbol{v}}$, so $|\vec{\boldsymbol{r}}'(t)| = |\vec{\boldsymbol{v}}|$, which gives us

$$\vec{\boldsymbol{T}} = \frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}$$

but since $\vec{\boldsymbol{v}}$ is constant, we have

$$\vec{\boldsymbol{T}}' = 0$$

and consequently $\kappa = 0$.

Example 8

Consider a circle parameterized by $\vec{\boldsymbol{r}}(t) = a\cos(t)\hat{\boldsymbol{\imath}} + a\sin t\hat{\boldsymbol{\jmath}}$. Find its curvature.

First, we find

$$\vec{\boldsymbol{r}}'(t) = -a\sin t\hat{\boldsymbol{\imath}} + a\cos t\hat{\boldsymbol{\jmath}}$$

so

$$|\vec{\boldsymbol{r}}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$$

which gives us

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{a} = -\sin t\hat{\boldsymbol{\imath}} + \cos t\hat{\boldsymbol{\jmath}}$$

and

$$\vec{\boldsymbol{T}}'(t) = -\cos t\hat{\boldsymbol{\imath}} - \sin t\hat{\boldsymbol{\jmath}}$$

which means that $|\vec{\boldsymbol{T}}'(t)| = 1$, so

$$\kappa = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{1}{a}$$

Example 9

Consider a helix parameterized by $\vec{\boldsymbol{r}}(t) = a\cos t\hat{\boldsymbol{\imath}} + a\sin t\hat{\boldsymbol{\jmath}} + bt\hat{\boldsymbol{k}}$. Find its curvature.

First, we find

$$\vec{\boldsymbol{r}}'(t) = -a\sin t\hat{\boldsymbol{\imath}} + a\cos t\hat{\boldsymbol{\jmath}} + b\hat{\boldsymbol{k}}$$

so

$$|\vec{\boldsymbol{r}}'(t)| = \sqrt{a^2+b^2}$$

which gives us

$$\vec{\boldsymbol{T}}(t) = \frac{1}{\sqrt{a^2+b^2}}\vec{\boldsymbol{r}}'(t)$$

and

$$|\vec{\boldsymbol{T}}'(t)| = \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\cos^2 t + a^2\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}}$$

so the curvature is

$$\kappa = \frac{\frac{a}{\sqrt{a^2+b^2}}}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}$$

Note that in the limit as $b \to \infty$ the curvature approaches that of a straight line, and in the limit as $b \to 0$, the curvature approaches that of a circle.

This definition of curvature always works, but there is a formula for curvature which is often simpler to apply, namely

$$\kappa = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3}$$

the proof of which is elided.

Example 10

Consider the curve parameterized by $\vec{\boldsymbol{r}}(t) = 2t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}} + \frac{t^3}{3}\hat{\boldsymbol{\jmath}}$. Find its curvature.

First, we find $\vec{\boldsymbol{r}}'(t) = 2\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}} + t^2\hat{\boldsymbol{k}}$, so

$$|\vec{\boldsymbol{r}}'(t)| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2t + t^2)^2} = 2t + t^2$$

We may then calculate $\vec{\boldsymbol{r}}''(t) = 2\hat{\boldsymbol{\jmath}} + 2t\hat{\boldsymbol{k}}$, and $\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}'' = 2(t^2\hat{\boldsymbol{\imath}} - 2t\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}})$, which gives us

$$|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''| = 2\sqrt{t^4 + 4t^2 + 4} = 2(2 + t^2)$$

Finally, we may calculate

$$\kappa = \frac{2(2 + t^2)}{(2 + t^2)^3} = \frac{2}{(2 + t^2)^2}$$

4 Practice Problems

Example 11

Consider the curve described by $\vec{\boldsymbol{r}} = \left(\frac{t^3}{3} - 2t\right)\hat{\boldsymbol{\imath}} + \left(\frac{t^3}{3} + 2t\right)\hat{\boldsymbol{\jmath}} + \sqrt{2}t^2\hat{\boldsymbol{k}}$ for $t \in \mathbb{R}$. Find the arc length from $t = 0$ to $t = 1$.

First, we find that

$$\vec{\boldsymbol{r}}'(t) = \left(t^2 + 2\right)\hat{\boldsymbol{\imath}} + \left(t^2 + 2\right)\hat{\boldsymbol{\jmath}} + 2\sqrt{2}t\hat{\boldsymbol{\jmath}}$$

so

$$|\vec{\boldsymbol{r}}'(t)|^2 = \left(t^2 + 2\right)^2 + \left(t^2 + 2\right)^2 + (2\sqrt{2}t)^2 = 2\left(t^2 + 2\right)^2$$

which gives us $|\vec{\boldsymbol{r}}'(t)| = \sqrt{2}(t^2 + 2)$, and thus the arc length is given by

$$\int_0^1 \sqrt{2}(t^2 + 2)\, dt = \frac{7\sqrt{2}}{3}$$

Example 12

For the above curve, find the curvature at $t = 0$.

We previously found

$$\vec{\boldsymbol{r}}'(t) = \left(t^2 + 2\right) \hat{\boldsymbol{\imath}} + \left(t^2 + 2\right) \hat{\boldsymbol{\jmath}} + 2\sqrt{2}t\hat{\boldsymbol{\jmath}}$$

so we may calculate

$$\vec{\boldsymbol{r}}''(t) = 2t\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}} + 2\sqrt{2}\hat{\boldsymbol{k}}$$

The curvature at $t = 0$ is then given by

$$\kappa = \frac{|\vec{\boldsymbol{r}}'(0) \times \vec{\boldsymbol{r}}''(0)|}{|\vec{\boldsymbol{r}}'(0)|^3}$$

so we may begin by calculating

$$\vec{\boldsymbol{r}}'(0) \times \vec{\boldsymbol{r}}''(0) = 4\sqrt{2}\hat{\boldsymbol{\imath}} + 4\sqrt{2}\hat{\boldsymbol{\jmath}}$$

which has magnitude $|\vec{\boldsymbol{r}}'(0) \times \vec{\boldsymbol{r}}''(0)| = 8$. Finally, we calculate $|\vec{\boldsymbol{r}}'(0)| = 2\sqrt{2}$, so we obtain the final solution

$$\kappa = \frac{8}{\sqrt{8}^3} = \frac{1}{\sqrt{8}}$$

Example 13

Verify that $\frac{d}{dt}\vec{\boldsymbol{u}}'(t) \times \vec{\boldsymbol{u}}''(t) = \vec{\boldsymbol{u}}'(t) \times \vec{\boldsymbol{u}}'''(t)$.

By the derivative rule for the cross product, we have

$$\frac{d}{dt}\vec{\boldsymbol{u}}'(t) \times \vec{\boldsymbol{u}}''(t) = \vec{\boldsymbol{u}}''(t) \times \vec{\boldsymbol{u}}''(t) + \vec{\boldsymbol{u}}'(t) \times \vec{\boldsymbol{u}}'''(t)$$

but for any vector we always have $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{v}} = 0$, so this simplifies to

$$\frac{d}{dt}\vec{\boldsymbol{u}}'(t) \times \vec{\boldsymbol{u}}''(t) = \vec{\boldsymbol{u}}'(t) \times \vec{\boldsymbol{u}}(t)'''$$

Example 14

Expand and simplify $\frac{d}{dt}\left((\vec{\boldsymbol{u}} \times \vec{\boldsymbol{u}}') \cdot (\vec{\boldsymbol{u}}' \times \vec{\boldsymbol{u}}'')\right)$.

$$\begin{aligned}\frac{d}{dt}\left((\vec{\boldsymbol{u}} \times \vec{\boldsymbol{u}}') \cdot (\vec{\boldsymbol{u}}' \times \vec{\boldsymbol{u}}'')\right) &= (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{u}}')' \cdot (\vec{\boldsymbol{u}}' \times \vec{\boldsymbol{u}}'') + (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{u}}') \cdot (\vec{\boldsymbol{u}}' \times \vec{\boldsymbol{u}}'')' \\ &= (\vec{\boldsymbol{u}}' \times \vec{\boldsymbol{u}}' + \vec{\boldsymbol{u}} \times \vec{\boldsymbol{u}}'') \cdot (\vec{\boldsymbol{u}}' \times \vec{\boldsymbol{u}}'') \\ &\quad + (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{u}}') \cdot (\vec{\boldsymbol{u}}'' \times \vec{\boldsymbol{u}}'' + \vec{\boldsymbol{u}}' \times \vec{\boldsymbol{u}}''') \\ &= (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{u}}'') \cdot (\vec{\boldsymbol{u}}' \times \vec{\boldsymbol{u}}'') + (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{u}}') \cdot (\vec{\boldsymbol{u}}' \times \vec{\boldsymbol{u}}''')\end{aligned}$$

Example 15

Consider a smooth vector valued function $\vec{\boldsymbol{r}}$ for which there exists a point P which lies on every normal plane to $\vec{\boldsymbol{r}}$. Prove that the curve lies on a sphere.

First, note that we may take $P = 0$ without loss of generality, as any curve with such a point my be translated such that that $P = 0$ without changing the geometric properties of the curve.
The normal plane to the curve at $\vec{\boldsymbol{r}}(t_0)$ must be perpendicular to $\vec{\boldsymbol{r}}'(t_0)$, so we have

$$\vec{\boldsymbol{r}}'(t) \cdot ((x - x_0)\hat{\boldsymbol{\imath}} + (y - y_0)\hat{\boldsymbol{\jmath}} + (z - z_0)\hat{\boldsymbol{k}}) = 0$$

but since $P = 0$ must always satisfy the above, it must be that

$$\vec{\boldsymbol{r}}'(t) \cdot (-x_0\hat{\boldsymbol{\imath}} + -y_0\hat{\boldsymbol{\jmath}} + -z_0\hat{\boldsymbol{k}}) = \vec{\boldsymbol{r}}'(t) \cdot (-\vec{\boldsymbol{r}}(t)) = 0$$

which we have previously shown requires that $\vec{\boldsymbol{r}}$ lies on the surface of a sphere.

Example 16

Consider the parabola described by $\vec{\boldsymbol{r}}(t) = t\hat{\boldsymbol{\imath}} + \frac{t^2}{2]\hat{\boldsymbol{\jmath}}}$, with unit tangent vector $\vec{\boldsymbol{T}}(t)$. Find a parameterization $\vec{\boldsymbol{r}}(u)$ such that $|\frac{d\vec{\boldsymbol{T}}}{du}| = 1$.

First, we find that $\vec{\boldsymbol{r}}'(t) = \hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}}$, so $|\vec{\boldsymbol{r}}'(t)| = \sqrt{1+t^2}$ and

$$\vec{\boldsymbol{T}}(t) = \frac{1}{\sqrt{1+t^2}}(\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}})$$

The solution may then be completed by noting that since we require that

$$\left|\frac{d\vec{\boldsymbol{T}}}{du}\right| = 1$$

it must be that

$$\frac{du}{dt} = |\vec{\boldsymbol{T}}'(t)|$$

which may be integrated to find an expression for u. This process is elided for brevity.

5 Geometric Features of Curves

We have previously examined curvature, but curves have other geometric properties which may be of interest, so we wish to look at those and define them mathematically. One of these is the principal normal, which is always perpendicual to the tangent vector. The principal normal describes the rate of change of the tangent vector. Another is osculating planes and circles, where we use planes and circles to approximate curves, rather than linear line segments.

Definition 11: Principal Normal Vector

Consider a curve parameterized by $\vec{\boldsymbol{r}}(t)$ on some interval. There exists a vector $\vec{\boldsymbol{N}}$ of unit length called the *principal normal vector* which is always perpendicular to $\vec{\boldsymbol{T}}$, and describes the direction in which $\vec{\boldsymbol{T}}$ is moving.
In the arc length parameterization, we have

$$\vec{\boldsymbol{T}} = \frac{d\vec{\boldsymbol{r}}}{ds}$$

then if $\kappa \neq 0$, we define

$$\vec{\boldsymbol{N}}(s) = \frac{\vec{\boldsymbol{T}}'(s)}{|\vec{\boldsymbol{T}}'(s)|}$$

which is of unit length and points in the direction in which $\vec{\boldsymbol{T}}$ is curving.
There is also an expression for $\vec{\boldsymbol{N}}$ in terms of an arbitrary parameterization, using that

$$\frac{d\vec{\boldsymbol{T}}}{ds} = \frac{d\vec{\boldsymbol{T}}}{dt}\frac{dt}{ds}$$

we have

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|}$$

so the formula is independent of parameterization.

Theorem 3

Prove that $\vec{\boldsymbol{N}} \perp \vec{\boldsymbol{T}}$.

Since $|\vec{\boldsymbol{T}}| = 1$, we have $\vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{T}}' = 0$, but since $\vec{\boldsymbol{N}}$ is a scalar multiple of $\vec{\boldsymbol{T}}'$, it must be that $\vec{\boldsymbol{N}} \cdot \vec{\boldsymbol{T}} = 0$ and $\vec{\boldsymbol{N}} \perp \vec{\boldsymbol{T}}$.

Definition 12: The Osculating Plane

Consider a curve described by $\vec{\boldsymbol{r}}(t)$ on some interval. We define the *osculating plane* at a point P on the curve as the plane through P which is spanned by $\vec{\boldsymbol{N}}$ and $\vec{\boldsymbol{T}}$.

Definition 13: The Osculating Circle

Consider a curve described by $\vec{r}(t)$ on some interval. We define the *osculating circle* at a point P on the curve to be a circle which lies in the osculating plane, having a radius of $\rho = \frac{1}{\kappa}$, and a centre at $P + \rho\vec{N}$.

The osculating circle represents a best circular approximation of the curve at a given point. Note that a circle of radius a has curvature $\kappa = \frac{1}{a}$, so the osculating circle has the same curvature as the curve it approximates.

Example 17

Consider the circle parameterized by $\vec{r}(t) = a\cos t\hat{\imath} + a\sin t\hat{\jmath}$ in $\mathbb{R}^3$. Find an expression for the osculating circle.

It has previously been found that

$$\vec{T}(t) = -\sin t\vec{i} + \cos t\hat{\jmath}$$

and therefore

$$\vec{T}'(t) = -\cos t\vec{i} - \sin t\hat{\jmath}$$

but since $|\vec{T}'| = 1$, we have

$$\vec{N}(t) = \vec{T}'(t)$$

The curvature of this circle is $\kappa = \frac{1}{a}$, and therefore the radius of the osculating circle is a, identical to that o f the original circle. The centre of the osculating circle is at $\vec{r} - a\vec{N} = \vec{0}$, so the osculating circle is precisely equal to the original circle.

Example 18

Consider the helix parameterized by $\vec{\boldsymbol{r}} = a\cos t\hat{\boldsymbol{\imath}} + a\sin t\hat{\boldsymbol{\jmath}} + bt\hat{\boldsymbol{k}}$. Find an expression for the osculating circle.

We have previously found that

$$\vec{\boldsymbol{T}}'(t) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\cos t\hat{\boldsymbol{\imath}} - a\sin t\hat{\boldsymbol{\jmath}}\right)$$

and

$$|\vec{\boldsymbol{T}}'(t)| = \frac{a}{\sqrt{a^2+b^2}}$$

but this yields

$$\vec{\boldsymbol{N}} = -\cos t\hat{\boldsymbol{\imath}} - \sin t\hat{\boldsymbol{\jmath}}$$

which gives the same osculating circle as that of the circle in the previous example, appropriately offset in the z-axis.

Example 19

Find an expression for the osculating plane of the helix from the previous example.

We require our osculating plane be spanned by $\vec{\boldsymbol{N}}$ and $\vec{\boldsymbol{T}}$, so the plane's normal vector must be

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{T}} \times \vec{\boldsymbol{N}}$$

where $\vec{\boldsymbol{B}}$ is known as the *binormal vector*. An expression describing the plane may then be obtained using the usual method.