

On Tensors

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1 Vector Spaces & Dual Spaces

We begin our exploration of tensors with a review of the theory of vector spaces. Once we have reviewed relevant definitions and theorems regarding vector spaces, we will move to examining linear mappings from vectors to scalars. This will allow us to define the dual space corresponding to any vector space.

Definition 1 (Vector Spaces). *A vector space is composed of a field F and a set V , along with two binary operations called vector addition and scalar multiplication, typically denoted $\mathbf{u} + \mathbf{v}$ and $a\mathbf{u}$ respectively, where $a \in F$ and $\mathbf{u}, \mathbf{v} \in V$. The following axioms must also be satisfied for all $a, b \in F$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$:*

- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $\exists \mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- $\exists (-\mathbf{u}) \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- $a(b\mathbf{u}) = (ab)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$, where 1 is the multiplicative identity of F
- $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

The elements of F are called scalars, and the elements of V are called vectors. V is called a vector space over F , or sometimes just a vector space if F is not relevant.

In the event that the reader is not familiar with the definition of a field, and is interested in tensors for their applications to physics, little is lost by thinking of F as being one of \mathbb{R} and \mathbb{C} . In particular, when doing general relativity it is typically the case that $F = \mathbb{R}$, and when doing quantum mechanics it is typically the case that $F = \mathbb{C}$.

Notation 1 (Einstein Summation Convention). *Consider some indexed sets $\{a^0, \dots, a^n\}$ and $\{b_0, \dots, b_n\}$. The Einstein summation convention states that*

$$\sum_{\mu=0}^n a^\mu b_\mu = a^\mu b_\mu.$$

In other words, if an unbound index variable occurs twice in a single term, once in superscript and once in subscript, summation is implied over all values of the index. In general relativity, the values of such indices are almost always $\{0, 1, 2, 4\}$, but in general it must be clear from context what the values taken on by the indices are. a^μ and b_μ are sometimes used to refer to the sets themselves.

Definition 2 (Linear Dependence & Independence). Let V be a vector space over F . Let \mathbf{v}_μ be a finite subset of V . \mathbf{v}_μ is called linearly dependent if there exists a subset c^μ of F , not all zero, such that $c^\mu \mathbf{v}_\mu = 0$. \mathbf{v}_μ is called linearly independent if it is not linearly dependent.

Definition 3 (Bases). Let V be a vector space over F . A linearly independent subset \mathbf{e}_μ of V is called a basis if for all $\mathbf{v} \in V$ there exists a subset c^μ of F such that $\mathbf{v} = c^\mu \mathbf{e}_\mu$.

Notation 2 (Cardinality). Let S be a finite set. The number of elements in S , also called the cardinality of S , is denoted $|S|$.

Theorem 1. Let V be a vector space. If \mathbf{u}_μ and \mathbf{v}_ν are bases of V , then $|\mathbf{u}_\mu| = |\mathbf{v}_\nu|$.

Proof. □

Definition 4 (Dimension). Let V be a vector space. Let \mathbf{e}_μ be a basis of V . $|\mathbf{e}_\mu|$ is called the dimension of V .

Notation 3 (Mappings). Given a mapping $f : A \rightarrow B$ and some $a \in A$, we denote the value of f at a by $\langle f, a \rangle$.

Definition 5 (Linear Mappings). Let V be a vector space over F . A mapping $f : V \rightarrow F$ is called linear if for all $\mathbf{u}, \mathbf{v} \in V$ and $a \in F$, the following hold:

- $\langle f, \mathbf{u} + \mathbf{v} \rangle = \langle f, \mathbf{u} \rangle + \langle f, \mathbf{v} \rangle$
- $\langle f, a\mathbf{u} \rangle = a\langle f, \mathbf{u} \rangle$

Definition 6 (Isomorphism). Let U and V be vector spaces. A bijective linear mapping $f : U \rightarrow V$ is called an isomorphism. If any isomorphisms exist, U and V are called isomorphic.

Definition 7 (Dual Space). Let V be a vector space over F . We call the set of all linear mappings $\mathbf{v}^* : V \rightarrow F$ the dual space of V , and denote it by V^* . V^* is a vector space over F once vector addition and scalar multiplication are defined as follows:

- $\langle \mathbf{u}^* + \mathbf{v}^*, \mathbf{v} \rangle = \langle \mathbf{u}^*, \mathbf{v} \rangle + \langle \mathbf{v}^*, \mathbf{v} \rangle$
- $\langle a\mathbf{v}^*, \mathbf{v} \rangle = a\langle \mathbf{v}^*, \mathbf{v} \rangle$

The elements of V^* are called covectors. Covectors are canonically indexed with a superscript, rather than the subscript which is used for vectors.

Definition 8 (Dual Basis). Let V be a vector space over F , and let V^* be its dual space. Given a basis \mathbf{e}_μ of V , we define the dual basis to be the set of $\mathbf{e}^\nu \in V^*$ such that $\langle \mathbf{e}^\nu, \mathbf{e}_\mu \rangle = \delta_\mu^\nu$.

Theorem 2. Let V be a vector space. Given a basis \mathbf{e}_μ of V , there exists one and only one dual basis.

Proof.

□

Observe that the definition of the dual basis allows for the operation of a covector on a vector to be evaluated especially easily. Let V be a vector space over F , and let V^* be its dual space. Let \mathbf{e}_μ be a basis of V , then if $\mathbf{v} \in V$ we may write $\mathbf{v} = a^\mu \mathbf{e}_\mu$ for some a^μ . Similarly, let \mathbf{e}^ν be the dual basis of V^* corresponding to \mathbf{e}_μ , then if $\mathbf{v}^* \in V^*$ we may write $\mathbf{v}^* = b_\nu \mathbf{e}^\nu$ for some b_ν . Therefore, in all generality we have

$$\begin{aligned} \langle \mathbf{v}^*, \mathbf{v} \rangle &= \langle b_\nu \mathbf{e}^\nu, a^\mu \mathbf{e}_\mu \rangle \\ &= a^\mu b_\nu \langle \mathbf{e}^\nu, \mathbf{e}_\mu \rangle \\ &= a^\mu b_\nu \delta_\mu^\nu \\ &= a^\mu b_\mu, \end{aligned}$$

where the critical simplification $\langle \mathbf{e}^\nu, \mathbf{e}_\mu \rangle = \delta_\mu^\nu$ would not have occurred if \mathbf{e}^ν was not chosen to be the dual basis corresponding to \mathbf{e}_μ .

Having shown that every vector space has a corresponding dual space, it is natural to ask about the dual space of the dual space. At first glance, it seems as if our creation of a single vector space has lead to an infinite family of vector spaces simultaneously springing into existence. Fortunately (or unfortunately, depending on the reader's affinity for vector spaces), it turns out that the only vector spaces which are of any interest to us are the one which we explicitly constructed and its dual space. This is due to a particularly strong relationship between vector spaces and the dual spaces of their dual spaces.

Theorem 3. *Let V be a vector space over F . For every linear mapping $\mathbf{v}^{**} : V^* \rightarrow F$, there is one and only one vector $\mathbf{v} \in V$ such that $\langle \mathbf{v}^{**}, \mathbf{v}^* \rangle = \langle \mathbf{v}^*, \mathbf{v} \rangle$ for all $\mathbf{v}^* \in V^*$. This defines an isomorphism.*

Proof.

□

This correspondence is called the natural correspondence, and it is so strong that it is sensible to refer to elements of $(V^*)^*$ by their corresponding elements of V . In this sense, we can state that a vector $\mathbf{v} \in V$ is a mapping $\mathbf{v} : V^* \rightarrow F$, analogously to how a covector $\mathbf{v}^* \in V^*$ is a mapping $\mathbf{v}^* : V \rightarrow F$. One particularly nice way of thinking about this correspondence is to observe that just as $\langle \mathbf{v}^*, \circ \rangle$ can be thought of as taking a vector and returning a scalar, $\langle \circ, \mathbf{v} \rangle$ can be thought of as taking a covector and returning a scalar.