ECE 6320 HWK 13

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Hespanha 24.4

Starting with the equations,

$$Ax_{eq} + Bu_{eq} = 0 \tag{1}$$
$$Gx_{eq} + Hu_{eq} = r \tag{2}$$

Making these equations into a linear system,

$$\begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} -A & B \\ -G & H \end{bmatrix} \begin{bmatrix} -x_{eq} \\ u_{eq} \end{bmatrix} \tag{3}$$

We also know that the system matrix is simply the Rosenbrock matrix with s = 0,

$$\begin{bmatrix} 0 \\ r \end{bmatrix} = P(0) \begin{bmatrix} -x_{eq} \\ u_{eq} \end{bmatrix} \tag{4}$$

Now using substitution of the proposed solution,

$$\begin{bmatrix} -x_{eq} \\ u_{eq} \end{bmatrix} = P(0)^T (P(0)P(0)^T)^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix} \tag{5}$$

We get,

$$\begin{bmatrix} 0 \\ r \end{bmatrix} = P(0)P(0)^T (P(0)P(0)^T)^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix} \tag{6}$$
$$\begin{bmatrix} 0 \\ r \end{bmatrix} = I \begin{bmatrix} 0 \\ r \end{bmatrix} \tag{7}$$

Confirming it is a solution to the system.

Lavretsky and Wise

Modified 2.1

$$\dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -2x_1 - 3x_2 + u \end{bmatrix} \tag{8}$$
$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{9}$$
$$t_0 = 0,\ T = 10 \tag{10}$$
$$J = \int_0^T (x_1^2 + u^2) d\tau + x_1^2(T) + x_2^2(T) \tag{11}$$

System Matrices

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \tag{12}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{13}$$

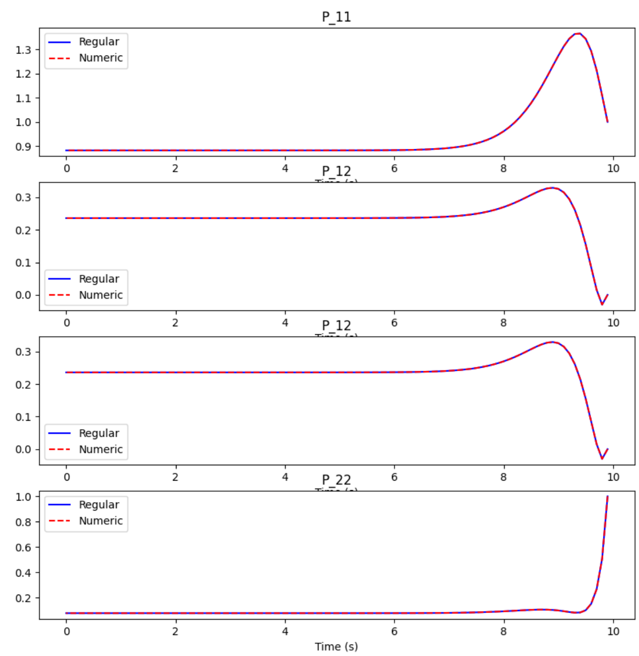
LQR Matrices

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{14}$$

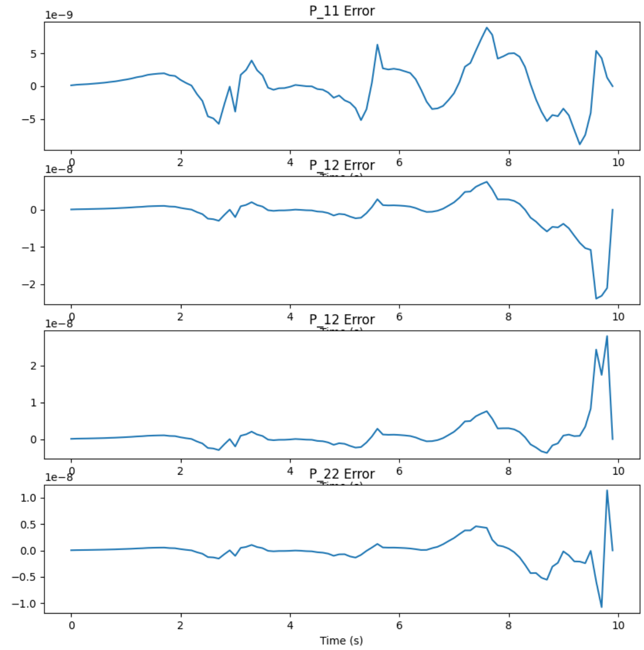
$$R = \begin{bmatrix} 1 \end{bmatrix} \tag{15}$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{16}$$

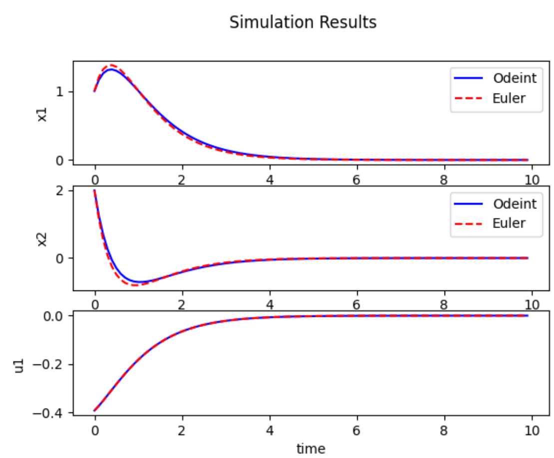
Solution Plots



Error Plot



System Simulation



LQR using Bryson's method

Expressing System Matrices

$$A = \begin{bmatrix} 3 & 6 & 4 \\ 9 & 6 & 10 \\ -7 & -7 & -9 \end{bmatrix} \tag{17}$$

$$B = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \tag{18}$$

Controllability

Looking at the controlability matrix we see it is not full rank. Checking the controllable decomposition we find,

$$\hat{A_{11}} = \begin{bmatrix} -3.182 & 1.597 & -7.109 \\ -0.134 & -1.817 & 19.116 \\ 0 & 0 & 5 \end{bmatrix} \tag{19}$$

$$\hat{B_1} = \begin{bmatrix} .656 & -.748 \\ .485 & .325 \end{bmatrix} \tag{20}$$

As can be seen, the uncontrollable part of the system has a positive eigenvalue of 5 so the system is uncontrollable.

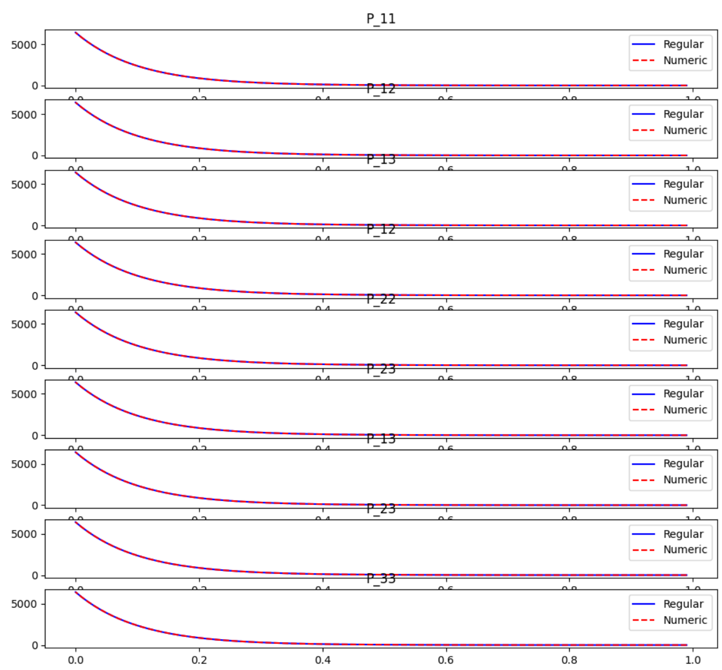
LQR Matrices

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{100^2} & 0 \\ 0 & 0 & \frac{1}{100^2} \end{bmatrix} \tag{21}$$

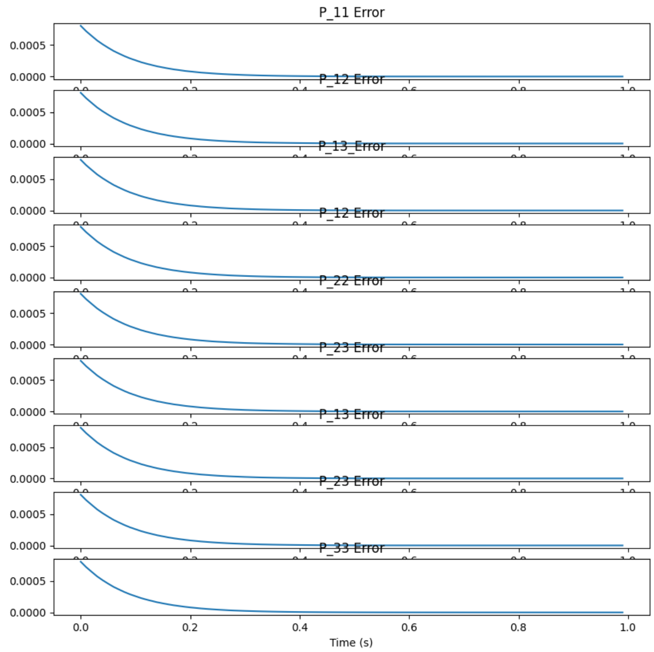
$$R = \begin{bmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{10^2} \end{bmatrix} \tag{22}$$

$$S = \begin{bmatrix} \frac{1}{10^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \tag{23}$$

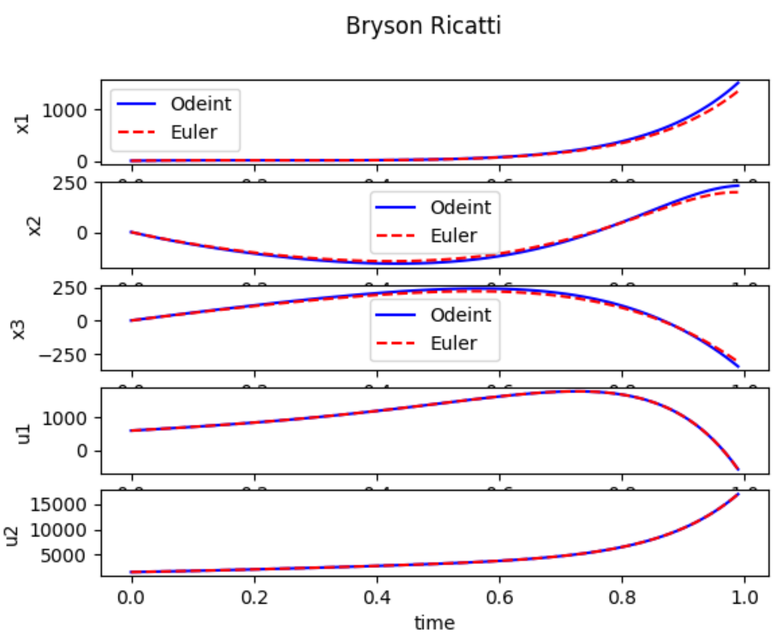
Solution Plot

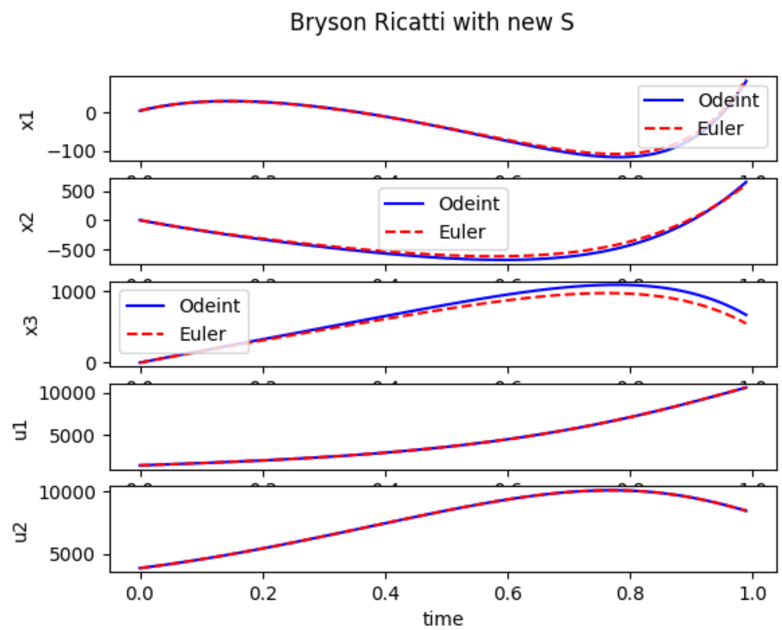


Error Plot



Simulation Results





Problem 2.3

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{24}$$

$$J = \int_0^\infty (x_1^2 + u^2) dt \tag{25}$$

First checking the controllability matrix we see the system is completely controllable. Expressing the Q and R matrices,

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{26}$$

$$R = 1 \tag{27}$$

Now using the ARE,

$$0 = A^T P + PA + Q - PBR^{-1}B^T P \tag{28}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p_{12} & p_{22} \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} p_{12} & p_{11} \\ p_{22} & p_{12} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & p_{12} \\ 0 & p_{22} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \tag{30}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2p_{12} & p_{11} + p_{22} \\ p_{11} + p_{22} & 2p_{12} \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{22}p_{12} \\ p_{22}p_{12} & p_{22}^2 \end{bmatrix} \tag{31}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p_{12}^2 - 2p_{12} - 1 & -p_{22}p_{12} + p_{11} + p_{22} \\ -p_{22}p_{12} + p_{11} + p_{22} & -p_{22}^2 + 2p_{12} \end{bmatrix} \tag{32}$$

From the top left equation we know p_{12} has solutions of $1 \pm \sqrt{2}$. The bottom right equation give solutions for p_{22} of $\pm\sqrt{2p_{12}}$. The top right equation gives,

$$p_{11} = p_{22}p_{12} - p_{22} \tag{33}$$

$$p_{11} = p_{22}(p_{12} - 1) \tag{34}$$

$$p_{11} = 2\sqrt{p_{12}} \tag{35}$$

Giving a total solution to the P matrix as,

$$p_1 1 = 2\sqrt{p_{12}} \tag{36}$$
$$p_1 2 = 1 \pm \sqrt{2} \tag{37}$$
$$p_2 2 = \pm\sqrt{2p_{12}} \tag{38}$$

Checking the eigenvalues of all the solutions I find the solution which yields a PD P is,

$$P = \begin{bmatrix} 2\sqrt{1+\sqrt{2}} & 1+\sqrt{2} \\ 1+\sqrt{2} & \sqrt{2+2\sqrt{2}} \end{bmatrix} \tag{39}$$

This then yields an optimal control law of,

$$u = -R^{-1}B^T P x \tag{40}$$
$$u = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{1+\sqrt{2}} & 1+\sqrt{2} \\ 1+\sqrt{2} & \sqrt{2+2\sqrt{2}} \end{bmatrix} x \tag{41}$$
$$u = -\begin{bmatrix} 1+\sqrt{2} & \sqrt{2+2\sqrt{2}} \end{bmatrix} x \tag{42}$$

2.4

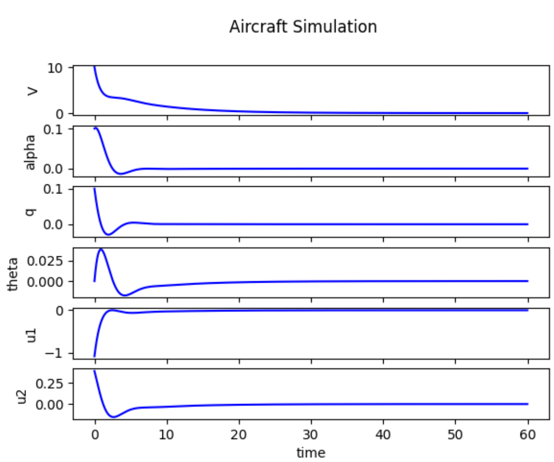
$$\begin{bmatrix} \dot{V} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.038 & 18.984 & 0 & -32.174 \\ -0.001 & -0.632 & 1 & 0 \\ 0 & -0.759 & -0.518 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 10.1 & 0 \\ 0 & -0.0086 \\ 0.025 & -0.011 \\ 0 & 0 \end{bmatrix} u \tag{43}$$

Control Matrices

Choosing Q and R because the system is completely controllable,

$$Q = \begin{bmatrix} \frac{1}{10^2} & 0 & 0 & 0 \\ 0 & \frac{1}{.1^2} & 0 & 0 \\ 0 & 0 & \frac{1}{.1^2} & 0 \\ 0 & 0 & 0 & \frac{1}{.05^2} \end{bmatrix} \tag{44}$$
$$R = \begin{bmatrix} \frac{1}{(\pi/2)^2} & 0 \\ 0 & \frac{1}{(\pi/2)^2} \end{bmatrix} \tag{45}$$

Simulation Results



2.5

From the system we see the control is $u = -kx$ and the system and cost matrices are,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{46}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{47}$$

$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \tag{48}$$

$$R = 1 \tag{49}$$

Looking at the system we know it is completely controllable. Finding the k matrix using lqr we find the full control is,

$$u = -\begin{bmatrix} 2 & 2 \end{bmatrix} x \tag{50}$$

This yields eigenvalues of $-1 \pm i$

Output Feedback Control

Control

Using the system we find u_{ff} using the equation $0 = Bu_{ff} + Ax_d$. This yields a feed forward term of,

$$u_{ff} = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} \tag{51}$$

Now checking the controllability of the system using a decomposition,

$$\hat{A} = \begin{bmatrix} 4.205 & -0.629 & 2.318 & 0.999 & 0.821 \\ 0.382 & 2.825 & 4.339 & 1.883 & 1.549 \\ -0.030 & -0.017 & -0.950 & -0.838 & 2.270 \\ 0.003 & 0.010 & -0.186 & 0.919 & -4.507 \\ 0.000 & 0.000 & 0.000 & 0.000 & -2.000 \end{bmatrix} \tag{52}$$

This then shows the system is stabilizable because the uncontrollable pole is stable. Finding the feedback control term using LQR we find,

$$\hat{u} = -\begin{bmatrix} 3.201 & -2.889 & 9.061 & 13.443 & -2.057 \\ 9.851 & -0.262 & 1.162 & 11.581 & -0.299 \\ 0.069 & 3.248 & -3.074 & -2.197 & -0.058 \end{bmatrix} (\hat{x} - x_d) \tag{53}$$

Giving a full control input of $u = \hat{u} + u_{ff}$ which yields a eigenvalues of $-3.68 \pm i$, -3.72, -1.21, and -2.

Observability

Checking the observability of the system we see,

$$\hat{A} = \begin{bmatrix} 4.0604 & -0.1501 & -0.0019 & 0.0000 & 0.0000 \\ 0.4221 & 2.9192 & 0.0754 & -0.0012 & 0.0000 \\ 0.3274 & 1.2992 & -1.9495 & 0.0180 & 0.0000 \\ -0.3866 & -1.7829 & 4.8983 & 0.9698 & 0.0000 \\ -1.5248 & -5.1151 & -1.1454 & -0.3499 & -1.0000 \end{bmatrix} \tag{54}$$

Thus showing the system is detectable because the unobservable eigenvalue is negative. Now finding the observer matrix we find,

$$L = \begin{bmatrix} 5.161 & -1.955 & -0.177 \\ -0.357 & 0.737 & 2.152 \\ -1.807 & 3.779 & -0.157 \\ 3.000 & 2.551 & 0.173 \\ 0.703 & -1.449 & -0.196 \end{bmatrix} \tag{55}$$

This then yields eigenvalues of A- LC of -1, -4.28, -3.21, and -2.55 ± 1.05