ECE-6320 HW2

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Problem 2.3

Starting with the dynamic equation for the pendulum,

$$ml^2\ddot{\theta} = mgl\sin(\theta) - b\dot{\theta} + sat(u) \tag{1}$$
$$y = \theta \tag{2}$$

a)

Writing the dynamics in matrix form with $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

The equation then becomes,

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin(x_1) - \frac{b}{ml^2}x_2 + \frac{1}{ml^2}sat(u) \end{bmatrix} \tag{3}$$
$$y = x_1 \tag{4}$$

Now doing this for the general case of x^{eq} and u^{eq}. We start with the taylor expansion with respect to x and u,

$$\delta x = \frac{\partial f(x,u)}{\partial x}\delta x|_{x,u=x^{eq},u^{eq}} + \frac{\partial f(x,u)}{\partial u}\delta u|_{x,u=x^{eq}} \tag{5}$$

Now taking the Jacobian of the dynamics with respect to x and u,

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos(x_1) & -\frac{b}{ml^2} \end{bmatrix}\delta x + \begin{bmatrix} 0 \\ \frac{1}{ml^2}\frac{\partial sat(u^{eq})}{\partial u} \end{bmatrix}\delta u \tag{6}$$

And with respect to y,

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix}\delta x \tag{7}$$

Now solving around $\theta = 0$ and $u = 0$,

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \delta u \tag{8}$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x \tag{9}$$

The eigenvalues of this matrix are 0.24960267 -157.04960267

b)

Using the above general equation for the linearized system and a equillibrium point of $\theta = \pi$ and $u = 0$.

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \delta u \tag{10}$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x \tag{11}$$

The eigenvalues of this matrix are -0.25039987, -156.54960013

c)

Again using the general equation for the dynamics

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l\sqrt{2}} & -\frac{b}{ml^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \delta u \tag{12}$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x \tag{13}$$

We see this linearization about the equilibrium point is only valid when the following is true,

$$0 = mgl \sin(\frac{\pi}{4}) + sat(u) \tag{14}$$

$$-mgl\frac{\sqrt{2}}{2} = sat(u) \tag{15}$$

So this means the left term must be between -1 and 1 so the u term can cancel it out.

The eigen values for this are 0.17657784, -156.97657784

d)

For this section we know that the pendulum is moving at a constant velocity t so our trajectory has the form,

$$\theta = t, \quad \dot{\theta} = 1, \quad \ddot{\theta} = 0 \tag{16}$$

Substituting this into our dynamics equation we find the equation for the torque,

$$0 = mgl\sin(t) - b + T \tag{17}$$

$$T = -\frac{1}{4}\sin(t) + \frac{1}{2} \tag{18}$$

The linearized system is,

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos(t) & -\frac{b}{ml^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \delta u \tag{19}$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x \tag{20}$$

Problem 2.4

Starting with the dynamics equation,

$$\ddot{\theta} + k\sin(\theta) = \tau \tag{21}$$

a)

Defining $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ we get the dynamics of the system as,

$$\dot{x} = f(x, u) = \begin{bmatrix} x_2 \\ -ksin(x_1) + \tau \end{bmatrix} \tag{22}$$

$$y = g(x, u) = x_1 \tag{23}$$

b)

The equilibrium points for the system around $\tau = 0$ are where $\dot{\theta} = 0$ and $\theta = n\pi$. So the equilibrium points for the system are $x^{eq} = span\{\begin{bmatrix} \pi \\ 0 \end{bmatrix}\}$

c)

Taking the jacobian of the dynamics matrix with respect to x and u,

$$\dot{\delta x} \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u \tag{24}$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x \tag{25}$$

1 Problem 2.6

Starting with the dynamics equation,

$$I\ddot{\theta} = -b\dot{\theta} - gm\cos(\theta) + \tau \tag{26}$$

a)

Writing this in the state space model,

$$\dot{z} = f(z, u) = \begin{bmatrix} z_2 \\ -\frac{b}{I}z_2 - \frac{gm}{I}\cos(z_1) + \frac{\tau}{I} \end{bmatrix} \tag{27}$$

$$y = g(z, u) = l\sin(z_1) \tag{28}$$

b)

Solving for the equilibrium points for the system we find that z_2 must be 0 and τ must be 0 so this simplifies the problem to,

$$0 = -\frac{gm}{I}\cos(z_1) \tag{29}$$

$$0 = \cos(z_1) \tag{30}$$

We then find the system has equillibrium points $span\{\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}\}$

The linearized system has the state equation is,

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ \frac{gm}{I} & \frac{-b}{I} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} \delta u \tag{31}$$

$$\delta y = \begin{bmatrix} 0 & 0 \end{bmatrix} \delta x \tag{32}$$

Because both the matrices C and D are zeros. This shows that it will be extremely difficult to control the system based on y.

Problem 2.7

Starting with the state,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} p_x\cos(\theta) + (p_y - 1)\sin(\theta) \\ -p_x\sin(\theta) + (p_y - 1)\cos(\theta) \\ \theta \end{bmatrix} \tag{33}$$

Now taking the partial derivatives with respect to variable p_x, p_y, and θ,

$$\dot{x_1} = \cos(\theta)\dot{p_x} + \sin(theta)\dot{p_y} + (-p_x\sin(\theta) + (p_y - 1)\cos(\theta))\dot{\theta} \quad (34)$$
$$= v\cos^2(theta) + v\sin^2(theta) + x_2\dot{\theta} \quad (35)$$
$$= v + x_2\omega \quad (36)$$
$$\dot{x_2} = -\sin(\theta)\dot{p_x} + \cos(\theta)\dot{p_y} + (-p_x\cos(\theta) - (p_y - 1)\sin(\theta))\dot{\theta} \quad (37)$$
$$= -v\sin(\theta)\cos(\theta) + v\sin(\theta)\cos(\theta) - (x_1)\dot{\theta} \quad (38)$$
$$= -x_1\omega \quad (39)$$
$$\dot{x_3} = \omega \quad (40)$$

b)

Taking the jacobian of the dynamics equations and evaluated at the equilibrium points we get,

$$\dot{\delta x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u \quad (41)$$
$$\delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x \quad (42)$$

c)

In cartesian coordinates the solution gives,

$$x = \begin{bmatrix} \sin(t)\cos(t) - \cos(t)\sin(t) \\ -\sin^2(t) - \cos^2(t) \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix} \quad (43)$$

So the derivative of x we get,

$$\dot{x} = \begin{bmatrix} v + \omega x_2 \\ -x_1\omega \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (44)$$

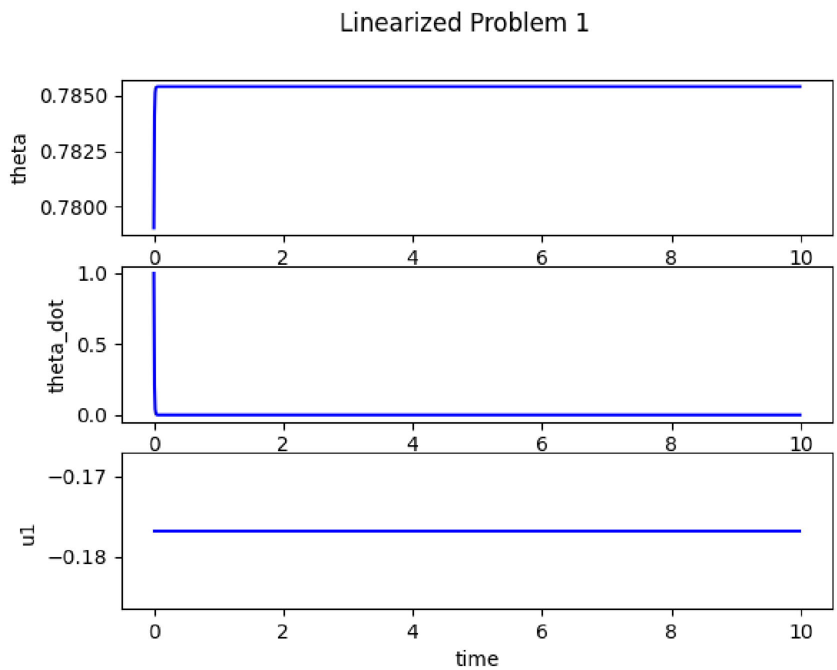
d)

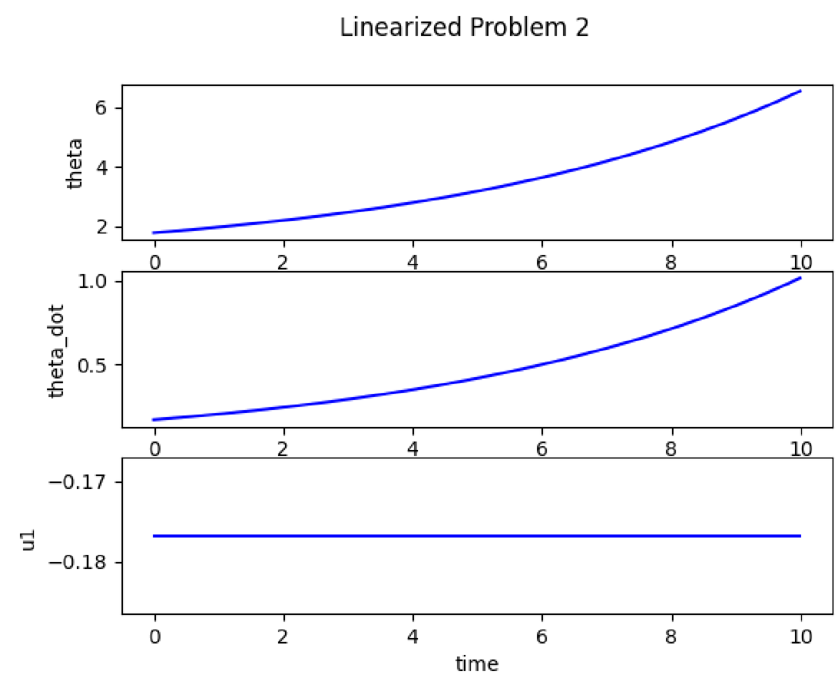
$$\dot{\delta x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u \quad (45)$$
$$\delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x \quad (46)$$

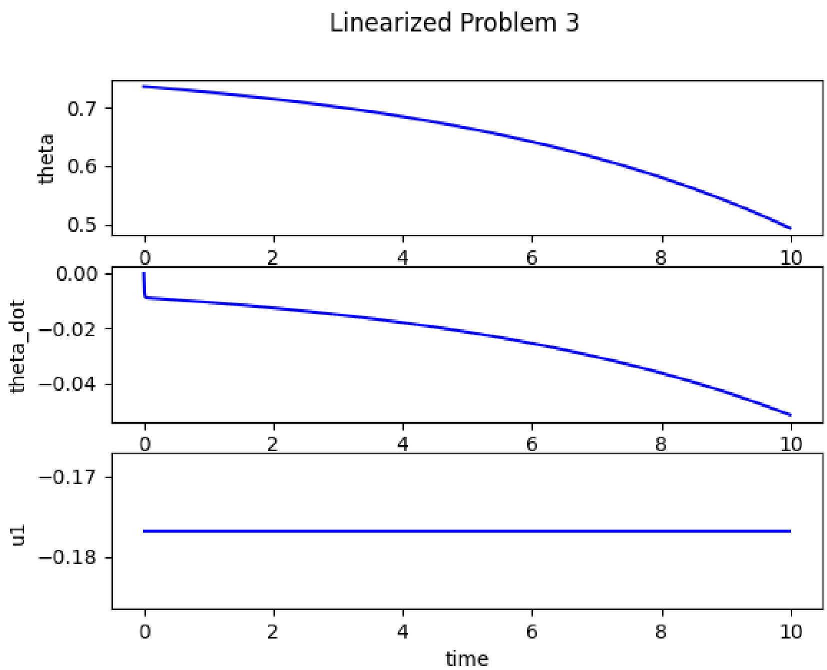
Since none of the A, B, C or D matrices depend on time this system is LTI.

Simulation Results

Linearization Results

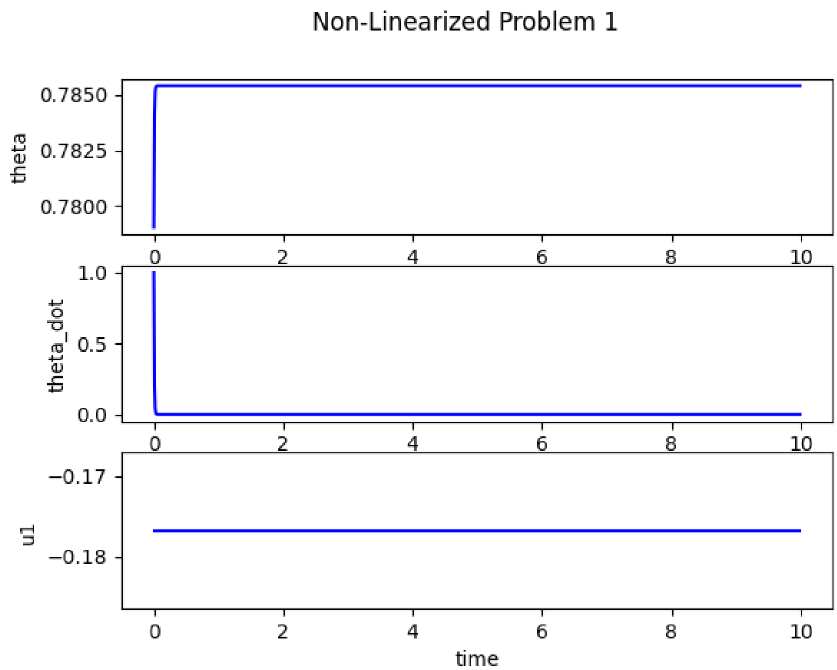


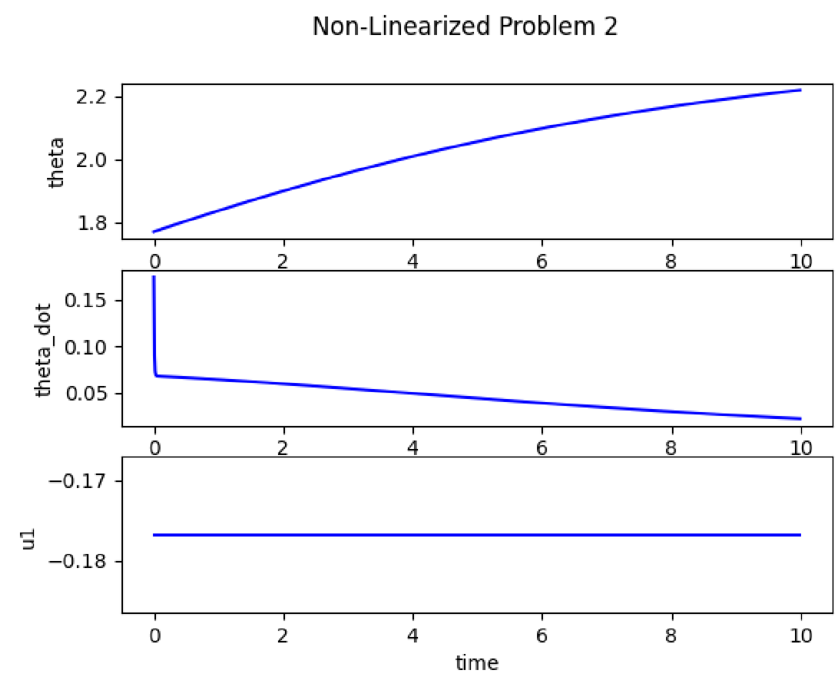


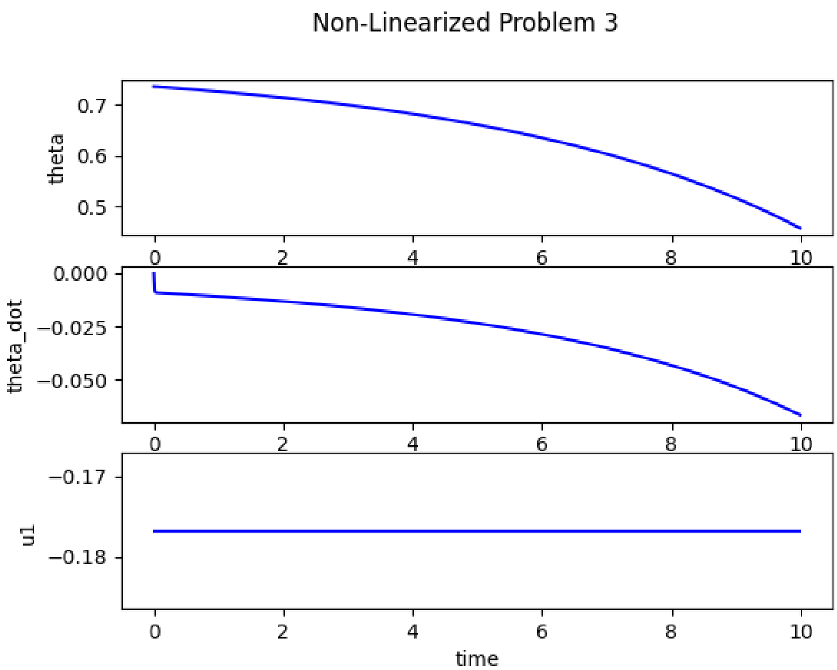


Linearized Problem 3
theta
theta_dot
u1
time
0.7
0.6
0.5
0.00
−0.02
−0.04
−0.17
−0.18
0
2
4
6
8
10

Non-Linearized Results







The comparison shows that the difference between the linearized and non-linearized models are relatively close when the operating point is around $\frac{\pi}{4}$. As seen in plots for Problem 1 and 3 there are only slight differences in the plots. However, our linearization is not accurate outside a small region around $\frac{\pi}{4}$. As seen in the problem 2 plots, the graphs are extremely different. Most notably that the linearized plot is accelerating $\dot{\theta}$ and the nonlinearized plot decelerating. This then shows that linearizations are only good around the linearization point.