ECE 6320 Hwk 8

Cody Grogan
A02313514

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Hespanha Problems

11.3

a)

For LTI systems we know that $x \in \mathcal{C} \implies x \in Im(\Gamma)$. Now using the definition for the image of Γ,

$$\exists z \text{ s.t. } \Gamma z = x \tag{1}$$

Now decomposing this into the summation using the controllability matrix and z,

$$x = \sum_{i=0}^{n-1} A^i B z_i \tag{2}$$

Now expanding this out we see,

$$Ax = \sum_{i=1}^{n} A^i B z_{i-1} \tag{3}$$

$$Ax = ABz_0 + A^2Bz_1 + \ldots + A^nBz_{n-1} \tag{4}$$

Now we see that A^n can be expressed as a linear combination of lower powers due to Cayley-Hamilton,

$$Ax = ABz_0 + A^2Bz_1 + \ldots + (\sum_{k=0}^{n-1} A^k \alpha_k)Bz_{n-1} \tag{5}$$

$$Ax = ABz_0 + A^2Bz_1 + \ldots + \Gamma \alpha z_{n-1} \tag{6}$$

$$\tag{7}$$

Thus showing that the final term is simply some linear combination of the controllability matrix, meaning that this final term is in the $Im(\Gamma)$ thus $Ax \in Im(\Gamma)$.

b)

Again using that the image of the controllability matrix is the controllable subspace. We see the image of the controllablility matrix is,

$$\exists z \text{ s.t } \Gamma z = x \tag{8}$$

Now expanding this out we see,

$$x = Bz_0 + ABz_0 + \ldots + A^{n-1}Bz_{n-1} \tag{9}$$

This then shows that the controllable subspace contains the image of B because its columns are represented in the controllability matrix. This then means any x will have a linear combination of the B with the addition of the rest of the controllability matrix.

12.6

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \tag{10}$$

a)

Checking the controllability matrix of the system we see,

$$rank(\Gamma) = 4 \tag{11}$$

Thus the system is completely controllable.a

b)

For no radial thruster we see the rank of the controllability matrix,

$$rank(\Gamma) = 4 \tag{12}$$

Showing the system is completely controllable without the radial thruster. Now without the tangential thruster we see,

$$rank(\Gamma) = 3 \tag{13}$$

Checking the eigenvalues we see that an eigenvalue of 0 is not full rank and thus the system is not controllable or stabilizable.

12.7

$$\dot{x} = \begin{bmatrix} -x_1 + u_1 \\ -x_2 + u_2 \\ x_1u_1 - x_1u_2 \end{bmatrix} \tag{14}$$

$$y = x_1^2 + x_2^2 + x_3^2 \tag{15}$$

a)

Linearizing the system around zero-state we see $u_1 = u_2 = 0$,

$$\dot{\delta x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \delta u \tag{16}$$

Using the controllability matrix to evaluate the controllability we see,

$$rank(\Gamma) = 2 \tag{17}$$

So the system is not controllable.

b)

Now evaluating a $x_1 = x_2 = x_3 = 1$ our $u_1 = u_2 = 1$

$$\dot{\delta x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \delta u \tag{18}$$

Now evaluating the controllability we see,

$$rank(\Gamma) = 2 \tag{19}$$

Showing the system is not controllable

13.1

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{20}$$

$$B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{21}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{22}$$

$$D = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{23}$$

Evaluating the controllability of the system using the controllability matrix we see,

$$rank(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}) = 1 \tag{24}$$

Thus the system is not controllable. Now taking a controllable decomposition we see,

$$\hat{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{25}$$

$$\hat{B} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \tag{26}$$

$$\hat{C} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{27}$$

As we can see the system is stabilizable because we can place the top left pole.
Looking at the transfer function we get,

$$G(s) = C(sI - A)^{-1}B + D \tag{28}$$

$$G(s) = \begin{bmatrix} \frac{2s+1}{s+1} \\ \frac{s+2}{s+1} \end{bmatrix} \tag{29}$$

Which is the same as the previous transfer function.

Control Design Review

A

Checking the eigenvalues of the system we see it has eigenvalues at $\pm 2i$ and checking the controllability.

$$rank([-2iI - A, B]) = 2 \tag{30}$$
$$rank([2iI - A, B]) = 2 \tag{31}$$
$$\tag{32}$$

We see that the system is then completely controllable. Now designing a control law we get $u = \begin{bmatrix} 1 & 3 \end{bmatrix} x$. This then places the eigenvalues at -1 and -2.

B

Checking the eigenvalues of the system we get $1, -1$ showing the system is unstable. However checking the controllability of the system we see,

$$rank([1I - A, B]) = 1 \tag{33}$$
$$rank([-1I - A, B]) = 2 \tag{34}$$
$$\tag{35}$$

This then shows that the system is not controllable or stabilizable. Performing a decomposition on the matrix we find,

$$\dot{\hat{x}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \hat{x} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \hat{u} \tag{36}$$
$$T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{37}$$

It is not possible to create a stabilizing control law because the system is unstabilizable.

C

Checking the eigenvalues of the matrix we see they are $-1, 0$. Now checking the controllability of the system we see,

$$rank([-1I - A, B]) = 1 \tag{38}$$
$$rank([1I - A, B]) = 2 \tag{39}$$
$$\tag{40}$$

From this we see the system is stabilizable. Now performing a decomposition we get,

$$\dot{\hat{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \hat{x} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \hat{u} \tag{41}$$
$$T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{42}$$

Designing a control law we get $\hat{u} = \begin{bmatrix} -3 & 0 \end{bmatrix} \hat{x}$

D

Looking at the system we see it has eigenvalues at -100 thus it is GAS. However we do not have a B matrix so the system is not controllable or stabilizable. In addition doing a decomposition is impossible because the controllability matrix will be all 0's. There is also no control law to stablilize the system because it is not stabilizable or controllable.

Segway-like Robot Control

1)

Substituting zero-state and input we see that the dynamics matrix evaluates to 0 (Symbolic form wont fit on document).

2)

Finding the eigenvalues of the matrix we see they are 0 and ± 8.518. Now evaluating the controllability of the system we see,

$$\begin{aligned} rank(0I - A) &= 6 && (43) \\ rank(8.518I - A) &= 7 && (44) \\ rank(-8.518I - A) &= 7 && (45) \\ & && (46) \end{aligned}$$

This then shows that the system is not completely controllable, and we do not know anything about the stabilizability of the system because it has eigenvalues of 0.

3)

Using the dynamics for the states in z we see the system has the form,

$$\dot{\delta z} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2.16615 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 72.5598 & 0 \end{bmatrix} \delta z + \begin{bmatrix} 0 & 0.029014 \\ -1.6687 & 0 \\ 0 & 0 \\ -24.1513 & 0 \end{bmatrix} \delta u \tag{47}$$

Now taking the eigenvalues we see they are 0 and ± 8.5182. Now evaluating the controlability,

$$\begin{aligned} rank(0I - A) &= 4 && (48) \\ rank(8.5182i - a) &= 4 && (49) \\ rank(-8.5182i - a) &= 4 && (50) \end{aligned}$$

Showing the system with z is completely controllable.

4)

Taking the controllability decomposition we see,

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 72.5 & -0.000056 & -2.17 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.000026 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.000026 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.015 & 0.00025 & 0 \\ 0 & 0 & 0 & 0 & -1 & -0.015 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} -24.2 & 0 \\ 0 & 0 \\ -0.94 & 0 \\ 0.0000246 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{51}$$

We then see that our states in z can be isolated from the system because they arent dependent on the uncontrollable states.

5)

Designing a control law for the system that will make it LAS we use the place function with poles at -1, -2, -3, and -4. This the yields a control law of

$$\delta u = \begin{bmatrix} -.00001877 & 0.163 & -3.584 & -0.2979 \\ 0.01034 & -0.425 & 1.543 & 0.41303 \end{bmatrix} \delta x \tag{52}$$

Controllability Proofs

We need to prove,

$$Im(\Gamma) = Im(W_C) \tag{53}$$

i)

Showing $x \in Im(W_C) \implies x \in Im(\Gamma)$. Now using the matrix exponential as the state transition function, and that fact that the controllable subspace is equal to the image of the controllability grammian

$$0 = e^{A(t_1 - t_0)} x_1 + \int_{t_0}^{t_1} e^{A(t_1 - \tau)} B u(\tau) d\tau \tag{54}$$

Now left multiplying by $e^{A(t_0 - t_1)}$.

$$-x_1 = \int_{t_0}^{t_1} e^{A(t_0 - \tau)} B u(\tau) d\tau \tag{55}$$

Now using Cayley Hamilton we get,

$$-x_1 = \int_{t_0}^{t_1} (\sum_{i=0}^{n-1} \alpha_i (t_0 - \tau) A^i) B u(\tau) d\tau \tag{56}$$

$$-x_1 = \int_{t_0}^{t_1} \sum_{i=0}^{n-1} A^i B u(\tau) \alpha_i (t_0 - \tau) d\tau \tag{57}$$

$$-x_1 = \sum_{i=0}^{n-1} A^i B \int_{t_0}^{t_1} u(\tau) \alpha_i (t_0 - \tau) d\tau \tag{58}$$

This can then be expressed in matrix multiplication form,

$$-x_1 = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \begin{bmatrix} \int_{t_0}^{t_1} u(\tau)\alpha_0(t_0 - \tau) d\tau \\ \int_{t_0}^{t_1} u(\tau)\alpha_1(t_0 - \tau) d\tau \\ \vdots \\ \int_{t_0}^{t_1} u(\tau)\alpha_{n-1}(t_0 - \tau) d\tau \end{bmatrix} \tag{59}$$

$$-x_1 = \Gamma \eta_1 \tag{60}$$

Thus we know that $-x_1$ is in the image of the controllability matrix then we know x_1 is in the image of the controllability matrix.

ii)

Showing $x_2 \in Im(\Gamma) \implies x_2 \in Im(W_C)$. Using the definition of being in the image of a matrix,

$$\exists v \text{ s.t. } x_2 = \Gamma v \tag{61}$$

From the proof that the image of controllability grammian is equal to the controllability subspace we know,

$$\eta_2 \in N(W_C) \implies 0 = B^T(\tau)\Phi^T(t_0,\tau)\eta_2 n \tag{62}$$

$$\tag{63}$$

Now using the LTI assumption,

$$0 = B^T(e^{A(t_0-\tau)})^T\eta_2 = \eta_2^T e^{A(t_0-\tau)}B \tag{64}$$

Now taking the time derivative of the above function,

$$0 = \frac{d}{d\tau}\eta_2^T e^{A(t_0-\tau)}B = -\eta_2^T A e^{A(t_0-\tau)}B \tag{65}$$

$$0 = \frac{d^2}{d\tau^2}\eta_2^T e^{A(t_0-\tau)}B = \eta_2^T A^2 e^{A(t_0-\tau)}B \tag{66}$$

$$0 = \frac{d^k}{d\tau^k}\eta_2^T e^{A(t_0-\tau)}B = (-1)^k\eta_2^T A^k e^{A(t_0-\tau)}B \tag{67}$$

Now at $\tau = t_0$ we get,

$$0 = \frac{d^k}{d\tau^k}\eta_2^T B = (-1)^k\eta_2^T A^k B \tag{68}$$

Now expanding this out we see,

$$\eta_2^T \begin{bmatrix} A & AB & \dots & A^{n-1}B \end{bmatrix} = 0 \tag{69}$$

$$\eta_2^T\Gamma = 0 \tag{70}$$

Now using the definition of the orthogonal compliment and that x_2 is in the image of Γ.

$$\eta_2^T x_2 = \eta_2^T\Gamma v \tag{71}$$

$$\eta_2^T x_2 = 0 \tag{72}$$

This then says that $x_2 \in (N(W_C))^\perp$ then showing that $x_2 \in Im(W_C)$

Exact Control Design

a)

Using the equation for our state transition for an LTI system,

$$x_2 = e^{A(t_1-t_0)}x_1 + \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau \tag{73}$$

We then substitute in a control law of $u(\tau) = B^T(e^{A(t_1-t)})^T\eta$. This then give us the following,

$$x_2 = e^{A(t_1-t_0)}x_1 + \int_{t_0}^{t_1} e^{A(t_1-\tau)}BB^T(e^{A(t_1-t)})^T\eta d\tau \tag{74}$$

We then see that this is simply the reachability gramian. And from this we can solve for eta so we can find out control law.

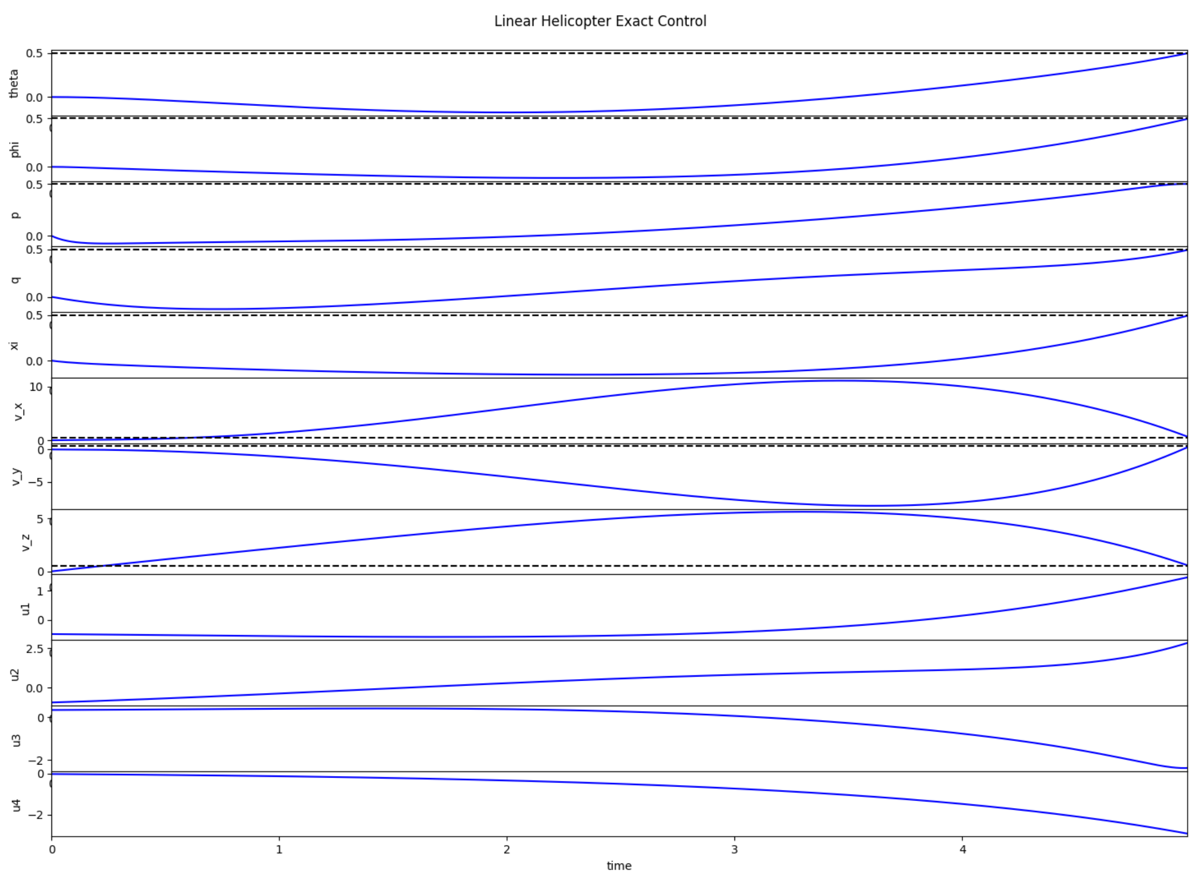
$$(x_2 - e^{A(t_1-t_0)}x_1) = W_R(t_0,t_1)\eta \tag{75}$$
$$\eta = W_R(t_0,t_1)^{-1}(x_2 - e^{A(t_1-t_0)}x_1) \tag{76}$$

We then substitute this into our control law and input our time interval,

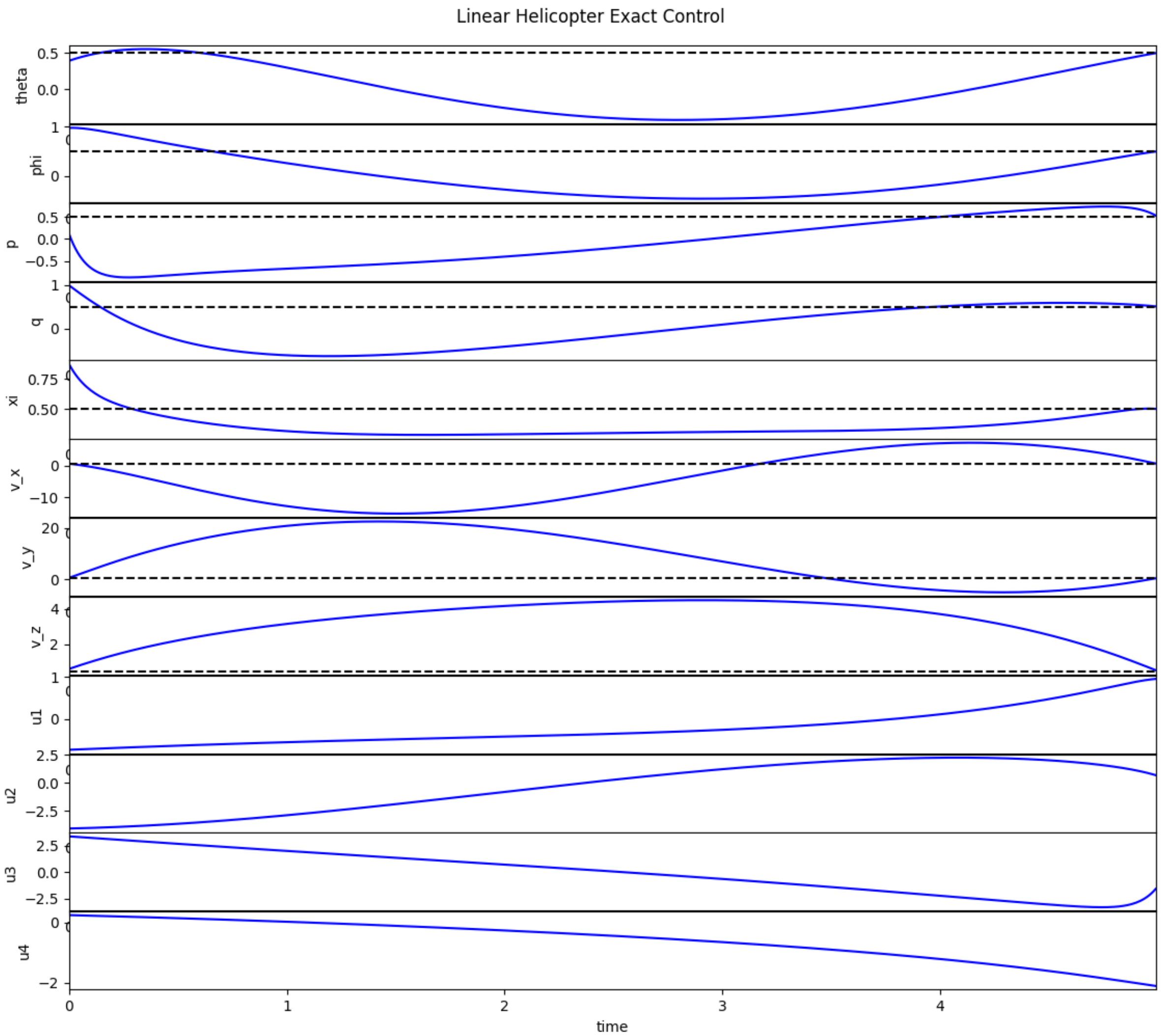
$$u(t) = B^T e^{A^T(t_1-t)}W_R(t_0,t_1)^{-1}(x_2 - e^{A(t_1-t_0)}x_1) \tag{77}$$
$$u(t) = B^T e^{A^T(5-t)}W_R(0,5)^{-1}(x_2 - e^{5A}x_1) \tag{78}$$

b)

For the initial state of 0 we get,



For the random initial state we get,



I would not advise to use the open loop control law I made because it assumes the dynamics are modelled perfectly. In addition it uses extremely computationally expensive operations such as integration, matrix exponentials and inverting 8x8 matrices. Finally, it can only be done on completely controllable matrices because the gramian must be invertible.

Hespanha 14.2

$$T = \Gamma \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix} \tag{79}$$

a)

Starting with the equation and simply multiplying it out,

$$B = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{80}$$

$$B = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{81}$$

$$B = B \tag{82}$$

b)

Computing the right side of the matrix,

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{83}$$

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} 0 & 0 & -\alpha_3 \\ 1 & \alpha_1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{84}$$

$$\begin{bmatrix} AB \\ \alpha_1 AB + A^2B \\ -\alpha_3 B \end{bmatrix} \tag{85}$$

Now doing the left side,

$$A \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix} \tag{86}$$

$$A \begin{bmatrix} B \\ \alpha_1 B + AB \\ \alpha_2 B + \alpha_1 AB + A^2B \end{bmatrix} \tag{87}$$

$$\begin{bmatrix} AB \\ \alpha_1 AB + A^2B \\ \alpha_2 AB + \alpha_1 A^2B + A^3B \end{bmatrix} \tag{88}$$

$$\tag{89}$$

Now using the characteristic equation where,

$$A^3 = -\alpha_1 A^2 - \alpha_2 A - \alpha_3 \tag{90}$$

Now substituting this into our matrix,

$$\begin{bmatrix} AB \\ \alpha_1 AB + A^2B \\ (\alpha_2 A + \alpha_1 A^2 + A^3)B \end{bmatrix} \tag{91}$$

$$\begin{bmatrix} AB \\ \alpha_1 AB + A^2B \\ (\alpha_2 A + \alpha_1 A^2 + -\alpha_1 A^2 - \alpha_2 A - \alpha_3)B \end{bmatrix} \tag{92}$$

$$\begin{bmatrix} AB \\ \alpha_1 AB + A^2B \\ -\alpha_3 B \end{bmatrix} \tag{93}$$

$$\tag{94}$$

Showing both sides are equivalent.

0.1 c)

Starting with the equation,

$$T = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix} \tag{95}$$

Then if the system is controllable we know that Γ is full rank. Then we also know the matrix on the right is full rank so a full rank matrix times a full rank matrix is also full rank and thus non-singular.

d)

Finding the characterisitic equation of A we get $s^3 - s^2 - 2s - 3$. Now setting up the tranformation T we get,

$$T = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{96}$$

$$T = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} \tag{97}$$

Now using the transformation to find the controllable canonical form,

$$\hat{A} = T^{-1}AT = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{98}$$

$$\hat{B} = T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{99}$$

We then see this is the controllable canonical form,