ECE-6320 HWK 7

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October 20, 2023

Problem 1

2.3

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin(\theta) - \frac{b}{ml^2} + \frac{u}{ml^2} \end{bmatrix} \tag{1}$$

a)

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} 0 & 1 \\ 4g & -16g \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 16g \end{bmatrix} \tag{3}$$

We know the eigenvalues of A are $0.2496, -158.0496$. Checking for controllability with the unstable eigenvalue.

$$rank([0.2496I - A, B]) = 2 \tag{4}$$

$$rank([-158.0496I - A, B]) = 2 \tag{5}$$

Thus we know this linearized system is both controllable and stabilizable

b)

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -4g & -16g \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 16g \end{bmatrix} \tag{6}$$

For this matrix we know the eigenvalues are $-0.25, -156.55$ which is LAS by itself. However when we check the results of controllability we get,

$$rank([-0.25I - A, B]) = 2 \tag{7}$$
$$rank([-156.55I - A, B]) = 2 \tag{8}$$
$$\tag{9}$$

We see that the system is controllable and stabilizable.

c)

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ \frac{4g}{\sqrt{2}} & -16g \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 16g \end{bmatrix} \tag{10}$$

For this system we see the eigenvalues are $0.1766, -156.9766$. So checking the the eigenvalues we get,

$$rank([0.1766I - A, B]) = 2 \tag{11}$$
$$rank([-156.9766I - A, B]) = 2 \tag{12}$$
$$\tag{13}$$

Showing that the system is both controllable and stabilizable.

2.4

$$\dot{x} = \begin{bmatrix} x_2 \\ -\sin(x_1) + u \end{bmatrix} \tag{14}$$

c)

For the system we have,

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u \tag{15}$$

From this we know the eigenvalues are $\pm 1i$. So checking the controllability of the matrix we see,

$$rank([-iI - A, B]) = 2 \tag{16}$$
$$rank([iI - A, B]) = 2 \tag{17}$$
$$\tag{18}$$

From this we see that the system is both Controllable and stabilizable.

2.6)

$$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{gm}{I}\cos(x_1) - \frac{b}{I}x_2 + \frac{1}{I}u \end{bmatrix} \tag{19}$$

b)

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u \tag{20}$$

We know the system has eigenvalues $-2.4142, 0.4142$. Now checking if the system is controllable,

$$rank([-2.4142I - A, B]) = 2 \tag{21}$$
$$rank([0.4142I - A, B]) = 2 \tag{22}$$
$$\tag{23}$$

Showing the system is both controllable and stabilizable.

2.7

$$\dot{x} = \begin{bmatrix} x_2u_2 + u_1 \\ -u_2x_1 \\ u_2 \end{bmatrix} \tag{24}$$

b)

Finding the linearization around the equillibrium point,

$$\dot{\delta x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u \tag{25}$$

From this system we know the eigenvalues are all 0's. So checking if the system is controllable,

$$rank([0I - A, B]) = 2 \tag{26}$$

Thus the system is neither controllable or stabilizable.

d)

Finding the linearization around the trajectory,

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u \tag{27}$$

From this we see that the eigenvalues are $\pm i, 0$. So checking whether the system is controllable,

$$rank([-iI - A, B]) = 3 \tag{28}$$
$$rank([iI - A, B]) = 3 \tag{29}$$
$$rank([0I - A, B]) = 3 \tag{30}$$
$$\tag{31}$$

Thus showing the system is both controllable and stabilizable.

Controllability Proof

For this we need to prove,

$$\mathcal{C}[t_0, t_1] = Im(W_C(t_0, t_1))$$

Where the definition of Controllablity is

$$\mathcal{C}[t_0, t_1] = \{x_0 \in \mathcal{R} |\ \exists u(\cdot) \in \mathcal{U}, 0 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau\} \tag{32}$$

The controllability Grammian

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^T(\tau)\Phi^T(t_0, \tau)\ d\tau \tag{33}$$

1)

First proving $x_0 \in Im(W_C(t_0, t_1)) \implies x_0 \in C(t_0, t_1)$. Starting with the definition of the image space for the Controllability Grammian,

$$\exists\ \eta_1 \text{ s.t. } x_1 = W_C(t_0, t_1)\eta_1 \tag{34}$$

Now plugging in the our control $u = -B(\tau)\Phi^T(t_0,\tau)\eta_1$ into the controllability definition,

$$0 = \Phi(t_1,t_0)x_1 + \int_{t_0}^{t_1} -\Phi(t_1,\tau)B(\tau)B(\tau)\Phi^T(t_0,\tau)\eta_1\ d\tau \tag{35}$$

Now left multiplying by $\Phi(t_0,t_1)$ and using the properties of the state transition matrix,

$$0 = \Phi(t_0,t_0)x_1 + \int_{t_0}^{t_1} -\Phi(t_0,\tau)B(\tau)B(\tau)\Phi^T(t_0,\tau)\eta_1\ d\tau \tag{36}$$

$$0 = Ix_1 + \int_{t_0}^{t_1} -\Phi(t_0,\tau)B(\tau)B(\tau)\Phi^T(t_0,\tau)\eta_1\ d\tau \tag{37}$$

$$x_1 = \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B(\tau)\Phi^T(t_0,\tau)\eta_1\ d\tau \tag{38}$$

$$x_1 = W_C(t_0,t_1)\eta_1 \tag{39}$$

Which is the definition for the image space of the Controllability Grammian and thus showing that $x_0 \in \mathcal{C}(t_0,t_1)$

2)

Second proving $x_0 \in \mathcal{C}(t_0,t_1) \implies x_0 \in W_C(t_0,t_1)$. Starting with the null space of the controllability grammian.

$$\exists\eta_0 \text{ s.t. } W_C(t_0,t_1)\eta_0 = 0 \tag{40}$$

Now looking at $\eta_0^T W_C \eta_0$,

$$0 = \int_{t_0}^{t_1} \eta_0^T \Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)\eta_0\ d\tau \tag{41}$$

Now defining $B^T(\tau)\Phi^T(t_0,\tau)\eta_0$ as V we get,

$$0 = \int_{t_0}^{t_1} V^T V\ d\tau \tag{42}$$

$$0 = \int_{t_0}^{t_1} ||V||^2\ d\tau \tag{43}$$

Then seeing that the norm cannot be zero unless V is zero we get,

$$B^T(\tau)\Phi^T(t_0,\tau)\eta_0 = 0 \tag{44}$$

Now using the definition of the orthogonal compliment,

$$x_0^T \eta_0 = 0 \implies x_0 \in (N(W_C))^{\perp} \implies x_0 \in Im(W_C) \tag{45}$$

To make this equivalence we start with our definition of controllability and left multiply by $\Phi(t_0, t_1)$

$$0 = \Phi(t_0, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\ d\tau \tag{46}$$

$$-x_0 = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\ d\tau \tag{47}$$

Now using our definition for orthogonal compliment,

$$-x_0^T \eta_0 = \int_{t_0}^{t_1} u^T(\tau)B^T(\tau)\Phi^T(t_0, \tau)\eta_0\ d\tau \tag{48}$$

Now seeing we proved $B^T(\tau)\Phi^T(t_0, \tau)\eta_0 = 0$ earlier we know that $x_0^T \eta_0 = 0$. Showing x_0 is in the image space of the controllability grammian.

Control Design

Simple System Design

S1

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{49}$$

Checking the controllability of the system we find,

$$rank([-.3722I - A, B]) = 2 \tag{50}$$

$$rank([5.3722I - A, B]) = 2 \tag{51}$$

So the system is controllable. Now defining u as $-kx$ we get,

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 - k_1 & 4 - k_2 \end{bmatrix} x \tag{52}$$

Now finding the eigenvalues of the system,

$$0 = det\left(\begin{bmatrix} \lambda - 1 & -2 \\ -(3-k_1) & \lambda - (4-k_2) \end{bmatrix}\right) \tag{53}$$

$$0 = (\lambda - 1)(\lambda - 4 + k_2) - 2(3 - k_1) \tag{54}$$

$$0 = \lambda^2 + (-5 + k_2)\lambda + 4 - k_2 - 6 + 2k_1 \tag{55}$$

$$0 = \lambda^2 + (-5 + k_2)\lambda + (-k_2 - 2 + 2k_1) \tag{56}$$

Now we know a GAS system with poles at -1, -2 has the form, $\lambda^2 + 3\lambda + 2$. So solving for the k's, we get $k_1 = 6, k_2 = 8$ so the control law is $u = -\begin{bmatrix} 6 & 8 \end{bmatrix} x$

S2

$$\dot{x} = \begin{bmatrix} -1 & -2 \\ 6 & 7 \end{bmatrix} x + \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} u \tag{57}$$

Checking the controllability of the system we see,

$$rank([5I - A, B]) = 2 \tag{58}$$

$$rank([I - A, B]) = 2 \tag{59}$$

So we see the system is controllable
Now adding Bk

$$\dot{x} = \begin{bmatrix} -1 + .5k_1 & -2 + .5k_2 \\ 6 - k_1 & 7 - k_2 \end{bmatrix} x \tag{60}$$

Now finding the eigenvalues of the matrix,

$$0 = det\left(\begin{bmatrix} \lambda + 1 - .5k_1 & 2 - .5k_2 \\ k_1 - 6 & \lambda + k_2 - 7 \end{bmatrix}\right) \tag{61}$$

$$0 = \lambda^2 + (-.5k_1 + k_2 - 6)\lambda + (1.5k_1 - 2k_2 + 5) \tag{62}$$

Now using the coefficients for eigenvalues of -1, -2 we get $k_1 = 30$ and $k_2 = 24$. Where the control law is $u = -\begin{bmatrix} 30 & 24 \end{bmatrix} x$

S3

$$\dot{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u \tag{63}$$

Checking the controllability of the system we find,

$$rank([-1.117I - A, B]) = 3 \quad (64)$$
$$rank([1.117I - A, B]) = 3 \quad (65)$$
$$rank([-A, B]) = 3 \quad (66)$$

So the system is controllable. Now defining our k as $\begin{bmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \end{bmatrix}$ we get,

$$\dot{x} = \begin{bmatrix} 1 - k_1 & 2 - k_2 & 3 - k_3 \\ 4 - k_4 & 5 - k_5 & 6 - k_6 \\ -k_1 - k_4 + 7 & -k_2 - k_5 + 8 & -k_3 - k_6 + 9 \end{bmatrix} x \quad (67)$$
$$0 = det\left(\begin{bmatrix} \lambda + k_1 - 1 & k_2 - 2 & k_3 - 3 \\ k_4 - 4 & \lambda + k_5 - 5 & k_6 - 6 \\ (k_1 + k_4 - 7 & k_2 + k_5 - 8 & \lambda + k_3 + k_6 - 9 \end{bmatrix}\right) \quad (68)$$
$$(69)$$

Using the place function for eigenvalues of -1, -2, and -3 we get a control law of, $u = -\begin{bmatrix} 3.50 & 1.66 & 4.16 \\ 3.99 & 6.66 & 6.65 \end{bmatrix} x$.

Orbit-plane Motion

1)

Starting with our dynamics,

$$\dot{x} = \begin{bmatrix} x_3 \\ x_4 \\ x_4^2 x_1 - \frac{\mu}{x_1^2} + u_1 \\ -2\frac{x_3 x_4}{x_1} + \frac{u_2}{x_1} \end{bmatrix} \quad (70)$$

Solving for a trajectory around R we know that $x_1 = R$, $x_3 = 0$, and $\dot{x_3} = 0$. We also assume inputs are 0

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ 0 \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 \\ x_4 \\ x_4^2 R - \frac{\mu}{R^2} \\ 0 \end{bmatrix} \quad (71)$$
$$x_4^2 R = \frac{\mu}{R^2} \quad (72)$$
$$x_4 = \sqrt{\frac{\mu}{R^3}} \quad (73)$$

Now we know we have a trajectory around $x_1 = R,\ x_2 = \sqrt{\frac{\mu}{R^3}}t,\ x_3 = 0,\ x_4 = \sqrt{\frac{\mu}{R^3}}$ and $a_i = a_r = 0$

Finding the jacobians we get,

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x_4^2 + 2\frac{\mu}{x_1^3} & 0 & 0 & 2x_4x_1 \\ 2\frac{x_3x_4}{x_1^2} - \frac{a_i}{x_1^2} & 0 & -2\frac{x_4}{x_1} & -2\frac{x_3}{x_1} \end{bmatrix} \tag{74}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{x_1} \end{bmatrix} \tag{75}$$

Now evaluating about the trajectory,

$$\dot{\delta x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\frac{\mu}{R^3} & 0 & 0 & 2\sqrt{\frac{\mu}{R}} \\ 0 & 0 & -2\sqrt{\frac{\mu}{R^5}} & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{R} \end{bmatrix} \delta u \tag{76}$$

Where $\delta x = \begin{bmatrix} x_1 - R \\ x_2 - \sqrt{\frac{\mu}{R^3}}t \\ x_3 \\ x_4 - \sqrt{\frac{\mu}{R^3}} \end{bmatrix}$ and $\delta u = \begin{bmatrix} a_r \\ a_i \end{bmatrix}$

2)

After substituting in μ and R we get eigenvalues of $0, 0, \pm 2.3189 \cdot 10^{-5} i$. So since we have all eigenvalues at Re(0) we know the system is unstable.

3)

Using the definition for controllability we check the rank of the following matrices,

$$Rank([0I - A, B]) = 4 \tag{77}$$

$$Rank([\pm 2.3189 \cdot 10^{-5} iI - A, B]) = 4 \tag{78}$$

This makes sense because the the last 2 columns contain the remaining directions in the space that the B matrix is missing.

4)

For this question we define our K as $\begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix}$. Using a control law $\delta u = -BK$ we get,

$$\dot{\delta x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\frac{\mu}{R^3} - k_1 & -k_2 & k_3 & 2\sqrt{\frac{\mu}{R}} - k_4 \\ -k_5 & -k_6 & -2\sqrt{\frac{\mu}{R^5}} - k_7 & -k_8 \end{bmatrix} \tag{79}$$

Using the place function with eigenvalues of -1, -1, -2, and -2 we find the control law to be

$$u = -\begin{bmatrix} 2 & 0 & 3 & .00927 \\ 4.44e-14 & 400 & -4.6378e-05 & 600 \end{bmatrix} \delta x$$

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5)