ECE-6320 HW1

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1 Matrix Multiplication

Problem 1

$$\begin{bmatrix} -3 & 5 \\ 7 & -10 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} (-1)(-3) + (3)(5) \\ (-1)(7) + (3)(-10) \end{bmatrix} = \begin{bmatrix} 18 \\ -37 \end{bmatrix} \tag{1}$$

Problem 2

$$\begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 10 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ 47 \\ 4 \end{bmatrix} \tag{2}$$

2 Orthogonality

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}, \quad z = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{3}$$

Problem 3

$$x \cdot z = (1)(1) + (-1)(1) = 0 \tag{4}$$

$\boxed{\text{x and z are orthogonal vectors}}$

Problem 4

Let o_1 equal $\begin{bmatrix} o_{11} \\ o_{12} \end{bmatrix}$

$$x \cdot o_1 = 0 \tag{5}$$
$$(1)(o_{11}) + (1)(o_{12}) = 0 \tag{6}$$
$$o_{11} = -o_{12} \tag{7}$$

So an orthogonal vector to x would be anything satisfying the following conditions $\begin{bmatrix} -o_{12} \\ o_{12} \end{bmatrix}$ which has a solution: $\boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

Let o_2 equal $\begin{bmatrix} o_{21} \\ o_{22} \\ o_{23} \end{bmatrix}$

$$y \cdot o_2 = 0 \tag{8}$$
$$-3o_{21} + 2o_{22} + 4o_{23} = 0 \tag{9}$$

$\boxed{\text{This has an infinite number of solutions but one of them is: } \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}$

Let o_3 equal $\begin{bmatrix} o_{31} \\ o_{32} \end{bmatrix}$

$$z \cdot o_3 = 0 \tag{10}$$
$$o_{31} - o_{32} = 0 \tag{11}$$
$$o_{31} = o_{32} \tag{12}$$

So the orthogonal vector must satisfy $\begin{bmatrix} o_{31} \\ o_{31} \end{bmatrix}$

$\boxed{\text{which has a solution: } \begin{bmatrix} 2 \\ 2 \end{bmatrix}}$

3 Span and basis set

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix} \tag{13}$$

Problem 5

A basis set of the $span\{x_1, x_2, x_3\}$ is: $\boxed{S = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \}}$

4 Null and Image sub-spaces

$$A_1 = \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{14}$$

Problem 6

Let n_1 equal $\begin{bmatrix} n_{11} \\ n_{12} \\ n_{13} \end{bmatrix}$

$$2n_{11} + 3n_{12} = -5n_{13} \tag{15}$$
$$-4n_{11} + 2n_{12} = -3n_{13} \tag{16}$$

Simplifying and eliminating n_{13}

$$-\frac{2}{5}n_{11} - \frac{3}{5}n_{12} = \frac{4}{3}n_{11} - \frac{2}{3}n_{12} \tag{17}$$
$$-6n_{11} - 9n_{12} = 20n_{11} - 10n_{12} \tag{18}$$
$$-26n_{11} = -n_{12} \tag{19}$$
$$n_{11} = \frac{1}{26}n_{12} \tag{20}$$

Now using this we find n_{13},

$$\frac{2}{26}n_{12} + 3n_{12} = -5n_{13} \tag{21}$$
$$\frac{80}{26}n_{12} = -5n_{13} \tag{22}$$
$$-\frac{8}{13}n_{12} = n_{13} \tag{23}$$

$\boxed{\text{So we know the Null space of } A_1 \text{ is } span\{\begin{bmatrix} 1 \\ 26 \\ -16 \end{bmatrix}\}}$

Let n_2 equal $\begin{bmatrix} n_{21} \\ n_{22} \\ n_{23} \end{bmatrix}$

$$n_{21} + n_{23} = 0 \tag{24}$$
$$5n_{21} + 2n_{22} + n_{23} = 0 \tag{25}$$
$$n_{21} + 2n_{22} + 2n_{23} = 0 \tag{26}$$

Then,

$$n_{21} = -n_{23} \tag{27}$$

Then,

$$n_{22} = 2n_{23} \tag{28}$$

Then,

$$5n_{23} = 0 \tag{29}$$
$$n_{23} = 0 \tag{30}$$

$$\boxed{\text{The Null space of } A_2 \text{ is } \{\begin{bmatrix}0\\0\\0\end{bmatrix}\}}$$

Let n_3 equal $\begin{bmatrix}n_{31}\\n_{32}\\n_{33}\end{bmatrix}$

$$2n_{31} + n_{32} + n_{33} = 0 \tag{31}$$
$$n_{31} + n_{32} = 0 \tag{32}$$
$$n_{31} + n_{33} = 0 \tag{33}$$
$$\tag{34}$$

We know,

$$n_{31} = -n_{32} \tag{35}$$
$$n_{31} = -n_{33} \tag{36}$$

We then substitute and find,

$$2n_{31} - n_{31} - n_{31} = 0 \tag{37}$$
$$0 = 0 \tag{38}$$

So, n_{31} can be any number.

$$\boxed{\text{So the null space of } A_3 \text{ is } \textit{span}\{\begin{bmatrix}1\\-1\\-1\end{bmatrix}\}}$$

Problem 7

Reducing A_1

$$\begin{bmatrix}1 & \frac{3}{2} & \frac{5}{2}\\ -4 & 2 & 3\end{bmatrix} \tag{39}$$
$$\begin{bmatrix}1 & \frac{3}{2} & \frac{5}{2}\\ 0 & 8 & 13\end{bmatrix} \tag{40}$$
$$\begin{bmatrix}1 & \frac{3}{2} & \frac{5}{2}\\ 0 & 1 & \frac{13}{8}\end{bmatrix} \tag{41}$$
$$\begin{bmatrix}1 & 0 & \frac{1}{16}\\ 0 & 1 & \frac{13}{8}\end{bmatrix} \tag{42}$$

$$\boxed{\text{So the Image space of } A_1 \text{ is } span\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}}$$

Reducing A_2

$$\begin{bmatrix}1 & 0 & 1\\0 & 2 & -4\\0 & 2 & 1\end{bmatrix} \tag{43}$$

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 5\end{bmatrix} \tag{44}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \tag{45}$$

$$\boxed{\text{So the Image space of } A_2 \text{ is } span\{\begin{bmatrix}1\\5\\1\end{bmatrix}, \begin{bmatrix}0\\2\\2\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}\}}$$

Reducing A_3,

$$\begin{bmatrix}2 & 1 & 1\\1 & 1 & 0\\1 & 0 & 1\end{bmatrix} \tag{46}$$

$$\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{2}\\0 & \frac{1}{2} & -\frac{1}{2}\\0 & -\frac{1}{2} & \frac{1}{2}\end{bmatrix} \tag{47}$$

$$\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{2}\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix} \tag{48}$$

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix} \tag{49}$$

$$\boxed{\text{So the Image space of } A_3 \text{ is } span \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}}$$

Problem 8

First we set up the problem

$$z^T N(A_2) = 0 \tag{50}$$

$$\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \tag{51}$$

$\boxed{\text{So } z \text{ is } \{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} |a_1, a_2, a_3 \in R\} \text{ which is the image space of } A_2.}$

Setting up the problem for A_3,

$$z^T N(A_3) = 0 \tag{52}$$

$$\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 0 \tag{53}$$

Multiplying, and adding on identities of the z variables

$$\begin{bmatrix} z_1 - z_2 - z_3 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ z_2 \\ z_3 \end{bmatrix} \tag{54}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_2 + z_3 \\ z_2 \\ z_3 \end{bmatrix} \tag{55}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = z_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{56}$$

$\boxed{\text{So the orthogonal compliment is } \mathit{span}\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}\text{which is the image space of } A_3}$

5 Eigen vectors and values

Problem 9

For A_2,

$$\begin{vmatrix} \lambda - 1 & 0 & -1 \\ -5 & \lambda - 2 & -1 \\ -1 & -2 & \lambda - 2 \end{vmatrix} \tag{57}$$

$$(\lambda - 1)((\lambda - 2)^2 - 2) - (10 + (\lambda - 2)) \tag{58}$$

$$\lambda^3 - 5\lambda^2 + 5\lambda - 10 \tag{59}$$

Eigen values of $\lambda_1 = 4.3797$, $\lambda_2 = .3101 + 1.479$i, $\lambda_3 = .3101$ - 1.479i
Eigen vectors of v_1, v_2, v_3 = $\begin{bmatrix} 0.296 \\ 1.042 \\ 1 \end{bmatrix}, \begin{bmatrix} -0.259 - 0.556i \\ -0.7154 + 1.017i \\ 1 \end{bmatrix} \begin{bmatrix} -0.259 + 0.556i \\ -0.7154 - 1.017i \\ 1 \end{bmatrix}$

For A_3,

Eigen values of $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = 0$
Eigen vectors of v_1, v_2, v_3 = $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

For A_4,

Eigen values of $\lambda_1 = 10.2942$, $\lambda_2 = 0.852911 + 2.17083$i, $\lambda_3 = 0.852911$ - 2.17083i
Eigen vectors of v_1, v_2, v_3 = $\begin{bmatrix} 9.29418 \\ 11.4998 \\ 1 \end{bmatrix}, \begin{bmatrix} -0.147089 + 2.17083i \\ -1.04992 - 1.43022i \\ 1 \end{bmatrix}, \begin{bmatrix} -0.147089 - 2.17083i \\ -1.04992 + 1.43022i \\ 1 \end{bmatrix}$

For A_5,

Eigen values of $\lambda_1 = 7$, $\lambda_2 = 4$, $\lambda_3 = 1$
Eigen vectors of v_1, v_2, v_3 = $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For A_6,

Eigen values of $\lambda_1 = 8$, $\lambda_2 = 5$, $\lambda_3 = 2$
Eigen vectors of v_1, v_2, v_3 = $\begin{bmatrix} 13 \\ 60 \\ 18 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Problem 10

A_2, A_4, A_5, and A_6 are full rank

Problem 11

Both are upper diagonal and the eigen values are the numbers on the diagonal

6 Fundamental Theorem of linear equations

Prove $Im(V) = N(V^T)^{\perp}$

Stating the definition of the Image space of matrix V,

$$x \in Im(V) \implies \exists \eta : x = V\eta \tag{60}$$
$$\tag{61}$$

Stating the definition of the Null space of matrix V^T

$$z \in N(V^T) \implies V^T z = 0 \tag{62}$$

Then we know that $Im(V) = N(V^T)^{\perp}$ if x is the orthogonal compliment of null space z. So left multiplying x by z^T.

$$z^T x = 0 \tag{63}$$
$$z^T V\eta = 0(V^T z)^T \eta = 0 \tag{64}$$
$$\tag{65}$$

We then see the definition for the null space which we know to be zero. So this means that x is the orthogonal compliment of z and thus proving the Lemma.

7 State Representation

Problem 1

Starting with the variable mappings,

$$x_1 = y_1 \tag{66}$$
$$x_2 = \dot{y_1} = \dot{x_1} \tag{67}$$
$$x_3 = \ddot{y_1} = \dot{x_2} \tag{68}$$
$$x_4 = y_2 \tag{69}$$
$$x_5 = \dot{y_2} = \dot{x_4} \tag{70}$$
$$x_6 = y_3 \tag{71}$$
$$x_7 = \dot{y_3} = \dot{x_6} \tag{72}$$
$$\tag{73}$$

Now keeping the highest terms on the left and substituting variable mappings,

$$\dot{x_1} = x_2 \tag{74}$$
$$\dot{x_2} = x_3 \tag{75}$$
$$\dot{x_3} = -2x_3 - 3x_2 + 4x_7 - 5x_5 + 4u_1 - u_3 \tag{76}$$
$$\dot{x_4} = x_5 \tag{77}$$
$$\dot{x_5} = -4x_1 + 3x_7 + u_2 \tag{78}$$
$$\dot{x_6} = x_7 \tag{79}$$
$$\dot{x_7} = -x_3 + 2u_1 + 4u_3 \tag{80}$$

Now, the state space representation is,

$$\boxed{\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & 0 & -5 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} u && (81) \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x && (82)\end{aligned}}$$

7.1 Problem 2

Starting with the variable mappings,

$$x_1 = y_1 \tag{83}$$
$$x_2 = \dot{y_1} = \dot{x_1} \tag{84}$$
$$x_3 = y_2 \tag{85}$$
$$x_4 = \dot{y_2} = \dot{x_3} \tag{86}$$
$$x_5 = y_3 \tag{87}$$
$$x_6 = \dot{y_3} = \dot{x_5} \tag{88}$$

Now keeping the highest order terms and variable mappings,

$$\dot{x_1} = x_2 \tag{89}$$
$$\dot{x_2} = -.4x_2 + .5x_6 + .4u_1 \tag{90}$$
$$\dot{x_3} = x_4 \tag{91}$$
$$\dot{x_4} = -9x_1 + 3x_6 + u_1 + 7u_2 \tag{92}$$
$$\dot{x_5} = x_6 \tag{93}$$
$$\dot{x_6} = -8x_2 - 5x_4 + u_1 + 3u_2 \tag{94}$$

So the state space representation is,

$$\boxed{\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -.4 & 0 & 0 & 0 & .5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -9 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -8 & 0 & -5 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ .4 & 0 \\ 0 & 0 \\ 1 & 7 \\ 0 & 0 \\ 1 & 3 \end{bmatrix} u & (95) \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x & (96)\end{aligned}}$$

8 Simulation of Unstable System

8.1 Plots

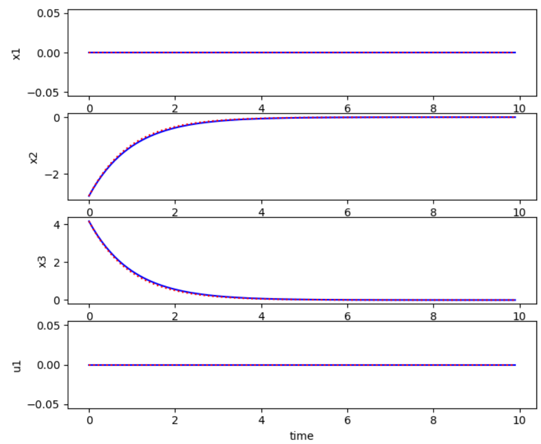

Figure 1: Initial state $x_0 = \begin{bmatrix} 0 \\ -2.77 \\ 4.16 \end{bmatrix}$ and u = 0

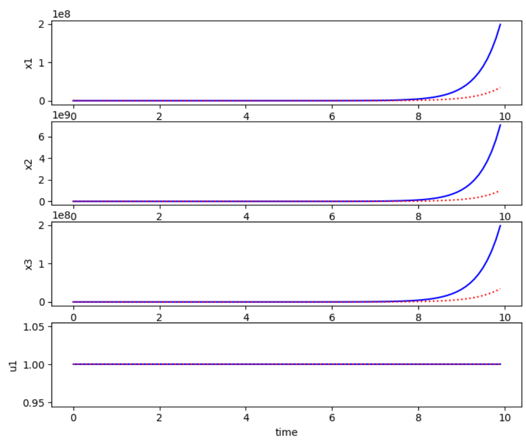

Figure 2: Initial state $x_0 = \begin{bmatrix} 0 \\ -2.77 \\ 4.16 \end{bmatrix}$ and u = 1

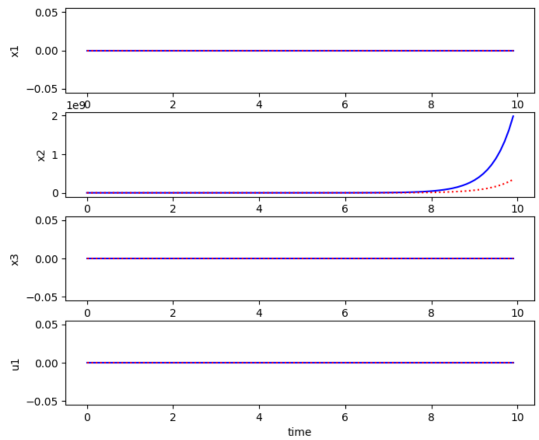

Figure 3: Initial state $x_0 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$ and u = 0

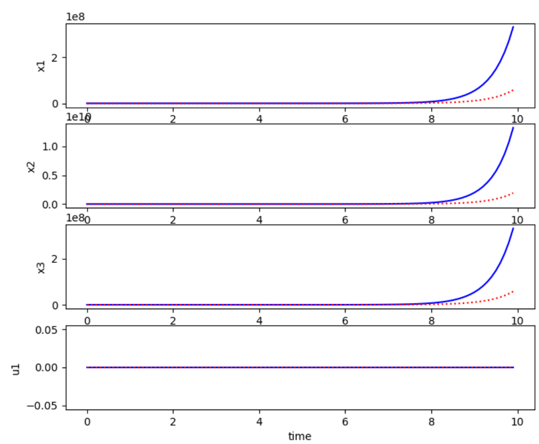

Figure 4: Initial state $x_0 = \begin{bmatrix} 3.39 \\ 3.78 \\ 3.71 \end{bmatrix}$ and u = 0

8.2 Questions

What is the behavior of the system for each of the described initial conditions?

In the first figure, the eigenvector corresponding to the negative eigenvalue only excites x_2 and x_3. In addition, because the initial state is an eigenvector of a stable eigenvalue of A the system converges back to stability.

In the second figure, we see the system becomes unstable with the addition of an input value. This is likely because the system stability relies on the state being a linear multiple of the eigenvector. Thus if the input is not is not also a linear multiple of the eigenvector it will become unstable.

In the third figure, we see complete instability of x_2 because the eigenvector has a positive eigenvalue. In addition, looking at A shows that the 2nd row is the only one dependent on x_2 causing only x_2 to become unstable.

In the fourth graph, we see the system is entirely unstable with a random initial state. This is because the random initial state is not a linear multiple of any of the eigenvectors.

How does Matlab's ode45 solution compare with the Euler integration for each of the described initial conditions?

Matlab's ode45 is only significantly better than the Euler integration methods at sufficiently high derivatives of x. This is because ode45 has a adaptive time step which scales with the derivatives of x. Where as Euler has a constant timestep which can accrue a significant amount of error as the derivatives of x become sufficiently high.

Which solution do you trust more (ode45 or Euler) and why?

I trust the ode45 integrator more because it has an adaptive time step which allows it to have significantly less error. This is obvious on the unstable cases because the Euler approximation converges to the ode45 solution when the timestep is decreased.

9 Extra Credit

Here is the Plot for the extra credit, it can be reproduced by running the NonLinSimulationScript.py.

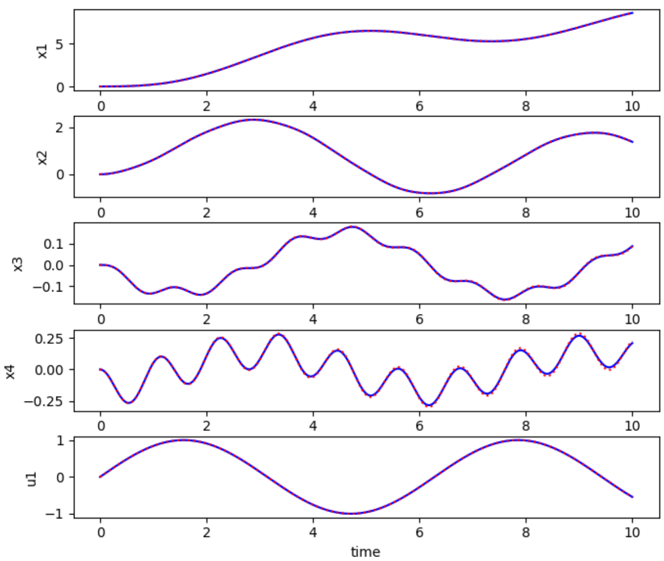

Figure 5: Initial state $x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and u = sin(t)