Homework for Week 8-9

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1 Quantum XY model

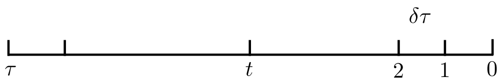
a) Denote the annihilation operator $a_i \equiv e^{i\Theta_i}$, creation operator $a_i^\dagger \equiv e^{-i\Theta_i}$, such that the J term in the Hamiltonian can be written as $\frac{J}{2}\left(a_i^\dagger a_{i'} + h.c.\right)$, so that

$$H = \sum_{vertex i} K N_i^2 - \sum_{link\langle ii'\rangle} \frac{J}{2}\left(a_i^\dagger a_{i'} + h.c.\right) \tag{1}$$

K describes onsite interaction between bosons, while J term describes hopping between different sites, which is the kinetic energy.

b) Assume the initial and final configurations are $\vec{\theta}_0$, so that the path integral can be written as

$$Z = \langle \vec{\theta}_0 | e^{-\tau H/\hbar} | \vec{\theta}_0 \rangle \tag{2}$$



Insert identity at integer time step, we have

$$Z = \left(\prod_{i,t} \int_{-\pi}^{\pi} \frac{d\theta_{i,t}}{2\pi} \right) \langle \vec{\theta}_t | e^{-\delta\tau H/\hbar} | \vec{\theta}_{t-1} \rangle \tag{3}$$

and then insert identity at half-integer time step,

$$Z = \left(\prod_{i,t} \int_{-\pi}^{\pi} \frac{d\theta_{i,t}}{2\pi} \sum_{n_{i,t}\in\mathbb{Z}} \right) \langle \vec{\theta}_t | \vec{n}_{t-1/2} \rangle \langle \vec{n}_{t-1/2} | \vec{\theta}_{t-1} \rangle \, e^{-\delta\tau H(\vec{n}_{t-1/2}, \vec{\theta}_{t-1})} \tag{4}$$

In the end, the Euclidean path integral is

$$\begin{aligned} Z = &\left(\prod_{i,t} \int_{-\pi}^{\pi} \frac{d\theta_{i,t}}{2\pi} \sum_{n_{i,t}\in\mathbb{Z}} \right) \\ &\times \exp\left[-i \sum_{i,t} n_{i,t-1/2}(\theta_{i,t} - \theta_{i,t-1}) - \frac{\delta\tau}{\hbar} \left(\sum_{i,t} K n_{i,t-1/2}^2 - \sum_{\langle ii'\rangle, t} J \cos(\theta_{i,t-1} - \theta_{i',t-1}) \right) \right] \end{aligned} \tag{5}$$

c) Map the Euclidean path integral to classical XY/Villain model.

- The $n_{i,t}$ related terms in the integrand can be extracted as

$$\left(\prod_{i,t}\sum_{n_{i,t}\in\mathbb{Z}}\right)\prod_{i,t}\exp\left[-\frac{\delta\tau}{\hbar}Kn_{i,t-1/2}^2 - in_{i,t-1/2}(\theta_{i,t}-\theta_{i,t-1})\right] \tag{6}$$

Defining $f(x)\equiv e^{\left[-\frac{\delta\tau}{\hbar}Kx^2-ix(\theta_{i,t}-\theta_{i,t-1})\right]}$, then we have $\sum_{n\in\mathbb{Z}}f(n)=\sum_{n\in\mathbb{Z}}\int dxf(x)\delta(x-n)$, using Poisson resummation $\sum_{n\in\mathbb{Z}}\delta(x-n)=\sum_{n\in\mathbb{Z}}e^{i2\pi nx}$, we get $\sum_{n\in\mathbb{Z}}f(n)=\sum_{n\in\mathbb{Z}}\int dxf(x)e^{i2\pi nx}$, the Gaussian integral of x can be carried out exactly

$$\sum_{n\in\mathbb{Z}}f(n)=\sum_{m\in\mathbb{Z}}\int dx\exp(\left[-\frac{\delta\tau}{\hbar}Kx^2-ix(\theta_{i,t}-\theta_{i,t-1}-2\pi m)\right]=\sum_{m\in\mathbb{Z}}C\exp(\frac{(\theta_{i,t}-\theta_{i,t-1}+2\pi m)^2}{4\delta\tau K/\hbar}) \tag{7}$$

with constant $C=\sqrt{\frac{\pi}{\delta\tau/\hbar}}$. The last equation above has changed $m\to -m$. Now the partition function for $D=d+1$ dimensional anisotropic classical XY model is

$$\begin{aligned}Z_{XY}=&\left(\prod_{i,t}\int_{-\pi}^{\pi}\frac{d\theta_{i,t}}{2\pi}\sum_{\tilde{m}_{i,t}\in\mathbb{Z}}\sqrt{\frac{\pi}{\delta\tau/\hbar}}\right)\\&\times\exp\left[\frac{1}{4\delta\tau K/\hbar}\sum_{i,t}(\theta_{i,t}-\theta_{i,t-1}+2\pi\tilde{m}_{i,t})^2+\frac{\delta\tau}{\hbar}J\sum_{\langle ii'\rangle,t}\cos(\theta_{i,t-1}-\theta_{i',t-1})\right]\end{aligned} \tag{8}$$

and the partition function for Villain model is

$$\begin{aligned}Z_{V}=&\left(\prod_{i,t}\int_{-\pi}^{\pi}\frac{d\theta_{i,t}}{2\pi}\sum_{\tilde{m}_{i,t}\in\mathbb{Z}}\sum_{m_{l,t}\in\mathbb{Z}}\sqrt{\frac{\pi}{\delta\tau/\hbar}}\right)\\&\times\exp\left[\frac{1}{4\delta\tau K/\hbar}\sum_{i,t}(d\tilde{\theta}_{i,t}+2\pi\tilde{m}_{i,t})^2+\frac{\delta\tau}{2\hbar}J\sum_{l,t}(d\theta_{l,t-1}+2\pi m_{l,t})^2\right]\end{aligned} \tag{9}$$

where $l=\langle ii'\rangle$ and $d\tilde{\theta}_{i,t}=\theta_{i,t}-\theta_{i,t-1}$.

Mapping coefficients to classical temperature,

$$\begin{aligned}\tilde{T}_\tau&=2\delta\tau K/\hbar\\\tilde{T}_s&=\frac{\hbar}{\delta\tau J}\end{aligned} \tag{10}$$

- According to Fourier series expansion,

$$\exp\left[\frac{\delta\tau J}{\hbar}\cos(d\theta_{l,t-1})\right]=\sum_{j\in\mathbb{Z}}I_j(\frac{\delta\tau J}{\hbar})e^{-ij(d\theta)_{l,t-1}} \tag{11}$$

so that

$$\begin{aligned}Z=&\left(\prod_{i,t}\int_{-\pi}^{\pi}\frac{d\theta_{i,t}}{2\pi}\sum_{n_{i,t}\in\mathbb{Z}}\sum_{j_{l,t}\in\mathbb{Z}}I_{j_{l,t}}(\frac{\delta\tau J}{\hbar})\right)\\&\times\exp\left[-i\sum_{i,t}n_{i,t-1/2}(\theta_{i,t}-\theta_{i,t-1})-\frac{\delta\tau K}{\hbar}\sum_{i,t}n_{i,t-1/2}^2-\sum_{l,t}ij_{l,t-1}(d\theta)_{l,t-1}\right]\end{aligned} \tag{12}$$

use $\sum_l j_l(d\theta)_l = \sum_i (\nabla \cdot j)_i \theta_i$, the coefficient of $\theta_{i,t}$ is $-i[(\nabla \cdot j)_i + (n_{i,t+1/2} - n_{i,t-1/2})]$, after integrating out $\theta_{i,t}$, we get

$$\begin{aligned} Z =& \left(\prod_{i,t} \sum_{n_{i,t}\in\mathbb{Z}} \sum_{j_{l,t}\in\mathbb{Z}} I_{j_{l,t}}(\frac{\delta\tau J}{\hbar}) \right) \\ &\times \exp\left[-\frac{\delta\tau K}{\hbar} \sum_{i,t} n_{i,t-1/2}^2 \right] \prod_{i,t} \delta\left((\nabla \cdot j_t)_i + \frac{\partial n_i}{\partial t} \right) \end{aligned} \tag{13}$$

The delta function indicated the current conservation

$$(\nabla \cdot j_t)_i + \frac{\partial n_i}{\partial t} = 0 \tag{14}$$

n is the particle number, while j is the current.

d) According to the equation (**??**),
• "high T" in the classical model correspond to large K and small J, we can neglect J term and get the Hamiltonian constructed by boson number operator. Each term in the H is commute with each other. So the Hamiltonian becomes classical.
In "high T" limit, focus on current representation (**??**), and change imaginary time back to real time, the integrand of will oscillate strikingly as K is large, so only the classical $n_{i,t}$ contributes. And in current representation, n is regarded as boson particle number. So "high T" corresponds to boson perspective classical.
• "low T" correspond to small K and large J, we can neglect K term and get the Hamiltonian classical.
In the same way, according to (**??**), only classical $\theta_{i,t}$ contributes most, while θ is interpreted as rotor angular coordinate.

e) Using current representation, and include background A and change iA on the time link to μ,

$$\begin{aligned} Z =& \left(\prod_{i,t} \int_{-\pi}^{\pi} \frac{d\theta_{i,t}}{2\pi} \sum_{n_{i,t}\in\mathbb{Z}} \sum_{j_{l,t}\in\mathbb{Z}} I_{j_{l,t}}(\frac{\delta\tau J}{\hbar}) \right) \\ &\times \exp\left[\sum_{i,t} n_{i,t-1/2}(\theta_{i,t-1} - \theta_{i,t} - q\mu) - \frac{\delta\tau K}{\hbar} \sum_{i,t} n_{i,t-1/2}^2 - \sum_{l,t} i j_{l,t-1}(d\theta - qA)_{l,t-1} \right] \end{aligned} \tag{15}$$

chemical potential couple to particle number in the form of $e^{-\mu n}$. So when $\mu \uparrow$, $n \downarrow$, when $\mu \downarrow$, $n \uparrow$. Agree with intuition.

2 Ising Model from Breaking XY Model

a) Residual global symmetry: $\theta_v \to \theta_v + \alpha_v$, keep $d\alpha_l = 0$ and $\alpha_v = \frac{2\pi m}{n}, m = 0, 1, \ldots, n-1$. So now global symmetry becomes discrete.
When $n = 2$, α_v can only be $0, \pi$. Also, when $V \gg 1$, $e^{V[\cos(n\theta_v)-1]}$ peaks at $n\theta_v = 2\pi m, m \in \mathbb{Z}$, i.e. $\theta_v = 0, \pi$ contributes most. This setup behaves like Ising model very much.

b) As we see in a), when $V \to +\infty$, $e^{V[\cos(n\theta_v)-1]}$ can be approximated by $\sum_{m\in\mathbb{Z}} \delta(\frac{n\theta_v}{2\pi} - m)$. Using Poisson resummation, we have

$$\sum_{m\in\mathbb{Z}} \delta(\frac{n\theta_v}{2\pi} - m) = \sum_{k_v\in\mathbb{Z}} e^{ink_v\theta_v} \tag{16}$$

such that two partition function becomes equivalent.
When $n = 2$, sum over k_v, we get

$$Z = \left[\prod_v \left(\sum_{\theta_v = 0, \pi} \right) \right] \prod_l \left(\delta_{d\theta_l = 0} + e^{-2/T} \delta_{d\theta_l = \pi} \right) \tag{17}$$

This is exactly the same as Ising model. When n is other value, $\theta_v \in \{\frac{2\pi m}{n}, m = 0, 1, \ldots, n-1\}$, and $d\theta_l \in \{\frac{2\pi m}{n}, m = 0, 1, \ldots, n-1\}$.

c) Using the current representation

$$Z = \left(\prod_v \int_{-\pi}^{\pi} \frac{d\theta_v}{2\pi} \sum_{j_l \in \mathbb{Z}} I_{j_l}(\frac{1}{T}) \right) \prod_l e^{-ij_l (d\theta)_l} \prod_v e^{V[\cos(n\theta_v) - 1]} \tag{18}$$

use Fourier series again, also with $\sum_l j_l (d\theta)_l = \sum_v (\nabla \cdot j)_v \theta_v$, we have

$$Z = \left(\prod_v \int_{-\pi}^{\pi} \frac{d\theta_v}{2\pi} \sum_{j_l \in \mathbb{Z}} I_{j_l}(\frac{1}{T}) \sum_{k_v \in \mathbb{Z}} I_{k_v}(V) \right) \prod_v e^{-i(\nabla \cdot j)_v \theta_v - ik_v n\theta_v} e^{-V} \tag{19}$$

after integrating out θ_v, we get $\nabla \cdot j + nk_v = 0$, which means current only conserved mod n.
When $n = 2$, focus on equation (**??**), when j_l is even, sum over $d\theta_l = 0, \pi$, we get 2, but when j_l is odd, alter summing over $d\theta_l$ we get zero. So only even j contributes. This is the same picture of the high expansion of Ising model.

d) $\theta_v \to \theta_v + \alpha_v$, $A_l \to A_l + d\alpha_l$, as $\alpha_v = \frac{2\pi m}{n}, m = 0, 1, \ldots, n-1$, $d\alpha_l = \frac{2\pi m}{n}, m = 0, 1, \ldots, n-1$, so

$$e^{iA_l} \in \{e^{i2\pi m/n}, m = 0, 1, \ldots, n-1\} \tag{20}$$

for $n = 2$, $e^{iA_l} = 1, -1$, corresponds to $e^{[\pm\cos(d\theta)_l - 1]/T}$, i.e. flipping the sign of the Ising couplings. So gauge transformation in terms of Ising model is just flip the sign of the couplings locally.