Homework 5

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MATH 3003 - Theory of Numbers

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Exercise 3.4.1. Use the Euclidean algorithm to find each of the following greatest common divisors.[1]

Part (a) 45,75
Sol$^{\mathbf{n}}$:

$$
\begin{aligned}
gcd(45, 75) \implies& \\
75 &= 45()+ \\
75 &= 45(1) + 30 \\
45 &= 30()+ \\
45 &= 30(1) + 15 \\
30 &= 15()+ \\
30 &= \boxed{15}(2) + \mathbf{0}
\end{aligned}
$$

$\therefore gcd(45, 75) = 15. \quad \square$

Part (b) 102,222
Sol$^{\mathbf{n}}$:

$$
\begin{aligned}
gcd(102, 222) \implies& \\
222 &= 102(2) + 18 \\
102 &= 18(5) + 12 \\
18 &= 12(1) + 6 \\
12 &= \boxed{6}(2) + 0
\end{aligned}
$$

$\therefore gcd(102, 222) = 6. \quad \square$

[1] I'll keep this format for the rest of this document. Where the first part (a) I put the extra empty valued steps, and then the other parts (b) through {some letter in the alphabet} I'll just do quickly.

Part (c) 666,1414
Sol$^{\mathbf{n}}$:

$$
\begin{aligned}
gcd(666, 114) \implies& \\
1414 &= 666(2) + 82 \\
666 &= 82(8) + 10 \\
82 &= 10(8) + 2 \\
10 &= \boxed{2}(5) + 0 \\
\therefore gcd(666, 1414) = 2. \quad \square&
\end{aligned}
$$

Part (d) 20785,44350
Sol$^{\mathbf{n}}$:

$$
\begin{aligned}
gcd(20785, 44350) \implies& \\
44350 &= 20785(2) + 2780 \\
20785 &= 2780(7) + 1325 \\
2780 &= 1325(2) + 130 \\
1325 &= 130(10) + 25 \\
130 &= 25(5) + 5 \\
25 &= \boxed{5}(5) + 0 \\
\therefore gcd(20785, 44350) = 5. \quad \square&
\end{aligned}
$$

Exercise 3.4.3. For each pair of integers in Exercise 1, express the greatest common divisor of the integers as a linear combination of these integers.

Part (a) 45,75.
Sol$^{\mathbf{n}}$:

We know

$$15 = (1)45 + (-1)30$$

and since $30 = 75 - 45$,

$$15 = 45 - (75 - 45) = 2(45) - 75.$$

$\therefore 15 = 2(45) - 75. \quad \square$

Part (b) 102,222
Sol$^{\mathbf{n}}$:

We know

$$gcd(102, 222) = 6 = 18 - 12$$

and also $102 = 18(5) + 12 \implies 12 = 102 - 18(5)$. So,

$$6 = 18 - [102 - 18(5)] = 6(18) - 102$$

Now $222 = 102(2) + 18 \implies 18 = 222 - 102(2)$. So,

$$6 = 6[222 - 102(2)] - 102 = 6(222) + (-13)(102)$$

$\therefore 6 = 6(222) - 13(102)$. $\square$

Part (c) 666,1414
Sol$^{\mathbf{n}}$:

We have $82 = 10(8) + 2 \implies$

$$2 = 82 - 10(8)$$

And since $666 = 82(8) + 10 \implies 10 = 666 - 82(8)$, then

$$2 = 82 - [666 - 82(8)](8) = (-8)666 + 65(82).$$

And since $1414 = 666(2) + 82 \implies 82 = 1414 - 666(2)$, then

$$2 = (-8)666 + 65(1414 - 666(2)) = -138(666) + 65(1414).$$

$\therefore 2 = -138(666) + 65(1414)$. $\square$

Part (d) 20785,44350
Sol$^{\mathbf{n}}$:

So we have

$$5 = 130 - 25(5)$$

And since $25 = 1325 - 130(10)$, we know

$$\begin{aligned} 5 &= 130 - [1325 - 130(10)](5) \\ &= 51(130) - 5(1325) \end{aligned}$$

And since $130 = 2780 - 1325(2)$, we now know

$$\begin{aligned} 5 &= 51[2780 - 1325(2)] - 5(1325) \\ &= 51(2780) - 107(1325) \end{aligned}$$

And since $1325 = 20785 - 2780(7)$, we know

$$\begin{aligned} 5 &= 51(2780) - 107[20785 - 2780(7)] \\ &= -107(20785) + 800(2780) \end{aligned}$$

And since $2780 = 44350 - 20785(2)$, we know

$$\begin{aligned} 5 &= -107(20785) + 800[44350 - 20785(2)] \\ &= -1707(20785) + 800(44350) \end{aligned}$$

$\therefore 5 = -1707(20785) + 800(44350)$. $\square$

Exercise 3.4.5. Find the greatest common divisor of each of the following sets of integers.

Part (a) 6,10,15
Sol$^{\mathbf{n}}$:

Notice,

$$gcd(6, 10, 15) = gcd(6, gcd(10, 15))$$

Quickly,

$$\begin{aligned} & gcd(10, 15) \implies \\ & \qquad 15 = 10(1) + 5 \\ & \qquad 10 = 5(2) + 0 \\ \implies & gcd(10, 15) = 5 \end{aligned}$$

Now we have

$$\begin{aligned} gcd(6,(5)) &\implies \\ 6 &= 5(1)+1 \\ 5 &= 1(5)+0 \\ \implies gcd(6,5) &= 1. \end{aligned}$$

Therefore,

$$gcd(6,10,15) = 1. \quad \square$$

Part (b) 70,98,105
Sol$^{\mathbf{n}}$:

Notice,

$$gcd(70,98,105) = gcd(gcd(70,105),98)$$

Now,

$$\begin{aligned} gcd(70,105) &\implies \\ 105 &= 70(1)+35 \\ 70 &= 35(2)+0 \\ \implies gcd(70,105) &= 35 \end{aligned}$$

So then,

$$gcd(gcd(70,105),98) = gcd(35,98).$$

And

$$\begin{aligned} gcd(35,98) &\implies \\ 98 &= 35(2)+28 \\ 35 &= 28(1)+7 \\ 28 &= 7(4)+0 \end{aligned}$$

So

$$gcd(35,98) = 7$$

And

$$\therefore gcd(70,98,105) = 7. \quad \square$$

Part (c) 280,330,405,490
Sol$^{\mathbf{n}}$:

Note two calculations...

$$\begin{aligned} gcd(280, 330) &\implies \\ 330 &= 280(1) + 50 \\ 280 &= 50(5) + 30 \\ 50 &= 30(1) + 20 \\ 30 &= 20(1) + 10 \\ 20 &= 10(2) + 0 \\ \implies gcd(280, 330) &= 10. \end{aligned}$$

And also

$$\begin{aligned} gcd(405, 490) &\implies \\ 490 &= 405(1) + 85 \\ 405 &= 85(4) + 65 \\ 85 &= 65(1) + 20 \\ 65 &= 20(3) + 5 \\ 20 &= 5(4) + 0 \\ \implies gcd(405, 490) &= 5. \end{aligned}$$

Now lets consider,

$$\begin{aligned} gcd(280, 330, 405, 490) &= gcd(gcd(280, 330), gcd(405, 490)) \\ &= gcd(10, 5) \implies \\ &\qquad 10 = 5(2) + 0 \\ &\implies gcd(10, 5) = 5. \end{aligned}$$

Therefore

$$gcd(280, 330, 405, 490) = 5. \quad \square$$

Exercise 3.4.7 . Express the greatest common divisor of each set of numbers in Exercise 5 as a linear combination of the numbers in that set.

Part (a) 6,10,15
Sol$^{\mathbf{n}}$:

Note for $gcd(10, 15) = 5$,

$$15 = 10(1) + 5 \implies 5 = 15 - 10$$

Also $6 = 5(1) + 1 \implies 1 = 6 - 5$. So

$$1 = 6 - 5 = 6 - (15 - 10) = 6 - 15 + 10.$$

$\therefore 1 = 6 + 10 - 15. \quad \square$

Part (b) 70,98,105
Sol$^{\mathbf{n}}$:

Note that for

$$gcd(70, 98, 105) = 7 \implies$$
$$7 = 35 - 28$$

And since $28 = 98 - 35(2)$

$$7 = 35 - (98 - 35(2)) = 3(35) - 98$$

Now notice

$$gcd(105, 70) \implies$$
$$35 = 105 - 70$$

So

$$7 = 3(105 - 70) - 98$$
$$\therefore 7 = 3(105) - 3(70) - 98. \quad \square$$

Part (c) 280,330,405,490
Sol$^{\mathbf{n}}$:

Note that for $gcd(10, 5) = 5$,

$$5 = 10 - 5 \tag{*}$$

Now notice

$$\begin{aligned}
gcd(280, 330) = 10 \implies & \\
& 10 = 30 - \boxed{20} && \wedge\ 20 = 50 - 30 \\
\implies & 10 = 30 - (50 - 30) = 2(\boxed{30}) - 50 && \wedge\ 30 = 280 - 5(50) \\
\implies & 10 = 2(280 - 50(5)) - 50 = 2(280) - 11(\boxed{50}) && \wedge\ 50 = 330 - 280 \\
\implies & 10 = 2(280) - 11(330 - 280) = 13(280) - 11(330)
\end{aligned}$$

And also

$$\begin{aligned}
gcd(405, 490) = 5 \implies & \\
& 5 = 65 - \boxed{20}(3) && \wedge\ 20 = 85 - 65 \\
& 5 = 65 - (85 - 65)(3) = 4(\boxed{65}) - 3(85) && \wedge\ 65 = 405 - 85(4) \\
& 5 = 4(405 - 85(4)) - 3(85) = 4(405) - 19(\boxed{85}) && \wedge\ 85 = 490 - 405 \\
& 5 = 4(405) - 19(490 - 405) = 23(405) - 19(490)
\end{aligned}$$

Now finally,

$$\begin{aligned}
\therefore 5 &= 10 - 5 && (*) \\
&= [13(280) - 11(330)] - [23(405) - 19(490)] \\
&= 13(280) - 11(330) - 23(405) + 19(490) \quad \square
\end{aligned}$$