Spring 2023

Practice Midterm #1 Exam

MATH 3031

Department of Mathematics
Temple University

Februrary 22, 2023

Name: ______________________________

Instructor/Section: ______________________________

This exam consists of 6 questions.
This exam will take
1hr + 10min to complete.
A 4-function calculator may be used.
Show all relevant work.
No work, no credit.
Good Luck!

Question	Max	Points
1	**8**	
2	**4**	
3	**8**	
4	**4**	
5	**4**	
6	**8**	
Total	**36**	

$\boxed{\text{8pts}}$ **1.** We have an urn with 3 red and 10 blue balls. We draw 5 balls, one by one, without replacement.

(a) State Ω. Find the probability that the colors we see in order are blue, blue, blue, red, blue.

Solution.

$$\begin{aligned}
P(b,b,b,r,b) &= \frac{\cancel{10}}{13} \cdot \frac{\cancel{9}}{\cancel{12}^{\,3}} \cdot \frac{\cancel{8}^{\,2}}{11} \cdot \frac{\boxed{3}}{\cancel{10}} \cdot \frac{7}{\cancel{9}} && \text{(since this one's red)} \\
&= \frac{2 \cdot \cancel{3} \cdot 7}{13 \cdot \cancel{3} \cdot 11} \\
&= \frac{2 \cdot 7}{13 \cdot 11} = \frac{14}{143} \approx 0.097902. \quad \square
\end{aligned}$$

(b) Find the probability that our sample of balls were all blue.

Solution.

$$\begin{aligned}
P(b,b,b,b,b) &= \frac{\binom{10}{5}}{\binom{13}{5}} = \frac{\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} \\
&= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} \left(\frac{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9} \right) \\
&= \frac{\cancel{10 \cdot 9} \cdot 8 \cdot 7 \cdot 6}{13 \cdot 12 \cdot 11 \cdot \cancel{10 \cdot 9}} = \frac{\cancel{8}^{\,2} \cdot 7 \cdot 6}{13 \cdot \cancel{4}^{\,3} \cdot 11} \\
&= \frac{2 \cdot 7 \cdot \cancel{6}^{\,2}}{13 \cdot \cancel{3}^{\,1} \cdot 11} = 2 \left(\boxed{\frac{14}{143}} \right) \approx 2(0.097902) = 0.195804. \quad \square && \text{(from part (a))}
\end{aligned}$$

(c) Consider the new sample space Ω', which is constructed after part (b), i.e. remove 5 blue balls. Suppose we changed our sample size to $k = 4$. State Ω'. Find the probability of the event $A = \{2 \text{ are red and } 2 \text{ are blue}\}$.

Solution.

Let $\Omega' = \Omega \backslash \{b,b,b,b,b\} = \{10b, 3r\} \backslash \{5b\} = \{3 \text{ red and } 5 \text{ blue}\}$. Now,

$$P(\{2r,2b\}) = P(A) = \frac{\binom{3}{2}\binom{5}{2}}{\binom{8}{4}} = \frac{\frac{3\cdot 2}{2\cdot 1}\left(\frac{5\cdot 4}{2\cdot 1}\right)}{\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1}}$$

$$= \frac{3\cdot 2\cdot 5\cdot \cancel{4}}{\cancel{4}}\left(\frac{\cancel{4}\cdot \cancel{3}^{1}\cdot \cancel{2}\cdot 1}{\cancel{8}\cdot 7\cdot \cancel{6}^{2}\cdot 5}\right)$$

$$= \frac{3\cdot \cancel{2\cdot 5}}{7\cdot \cancel{2\cdot 5}} = \frac{3}{7} \approx 0.4285714. \quad \square$$

(d) Find the probability that the new sample A has at least 2 blue balls. ($\omega_i' \in \Omega'$)

Solution.
Let $P(A) = P(kb)$ such that $k \in \mathbb{N}$. Now, since at least 2 blue (and cannot exceed 4 since $\#\omega = 4$),

then $\implies \displaystyle\sum_{k=2}^{4} P(kb) = P(2b) + P(3b) + P(4b)$

$$= \frac{\binom{5}{2}\binom{3}{2}}{\binom{8}{3}} + \frac{\binom{5}{3}\binom{3}{1}}{\binom{8}{3}} + \frac{\binom{5}{4}\cancel{\binom{3}{0}}}{\binom{8}{3}}$$

$$= \frac{\frac{5\cdot 4}{2\cdot 1}\left(\frac{3\cdot \cancel{2}}{\cancel{2}\cdot 1}\right) + \frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1}(3) + \frac{5\cdot 4\cdot 3\cdot 2}{4\cdot 3\cdot 2\cdot 1}}{\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1}}$$

$$= \left[\frac{5\cdot 4\cdot 3}{2} + \frac{5\cdot 4\cdot 3\cdot 3}{3\cdot 2} + \frac{5\cdot \cancel{4\cdot 3\cdot 2}}{\cancel{4\cdot 3\cdot 2}\cdot 1}\right]\left(\frac{4\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5}\right)$$

$$= \frac{5\cdot 2\cdot 3 + 5\cdot 2\cdot 3 + 5}{2\cdot 7\cdot 5} = \frac{65}{70} = \frac{13}{14} \approx 0.9285714. \quad \square$$

$\boxed{\text{4pts}}$ **2.** If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \cap B) = \frac{1}{4}$,

(a) Find $P(A^C \cap B^C)$

Solution.
Notice $P(A^C \cap B^C) = P((A \cup B)^C)$ by a de Morgan Law.
Now note using the complement definition, we have
$P((A \cup B)^C) = 1 - P(A \cup B)$.
And of course, by the inclusion exclusion principle, we get

$$\begin{aligned} 1 - P(A \cup B) &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) \\ &= 1 - \left(\frac{12 + 8 - 6}{24}\right) \\ &= 1 - \left(\frac{14}{24}\right) = \frac{12}{12} - \left(\frac{7}{12}\right) = \frac{5}{12} \approx 0.41666. \quad \square \end{aligned}$$

(b) Find $P(A \cup B^C)$

Solution.
By the inclusion exclusion principle, we have
$P(A \cup B^C) = P(A) + P(B^C) - P(AB^C)$.

Now we know by definition, $P(A) = P(AB) + P(AB^C)$ (which is a property of complements).
But of course this implies $P(AB^C) = P(A) - P(AB)$.

So now,

$$\begin{aligned} P(A \cup B^C) &= P(A) + P(B^C) - P(AB^C) \\ &= P(A) + P(B^C) - P(A) - P(AB) \\ &= \cancel{P(A)} + (1 - P(B)) - \cancel{P(A)} - P(AB) && \text{(by } def^n \text{ of complement)} \\ &= 1 - P(B) - P(AB) \\ &= \frac{12}{12} - \frac{4}{12} - \frac{3}{12} = \frac{5}{12} \approx 0.41666. \quad \square \end{aligned}$$

$\boxed{\text{8pts}}$ **3.** An urn contains 3 balls labeled 2, 3, and 4. We draw 2 balls one by one at random with replacement. Let X be the sum of the two numbers of the sample.

(a) Find the possible values of X.

Solution. $X = \{4,5,6,7,8\}$. $\square$

(b) Find the probability mass function of X.

Solution.

X	4	5	6	7	8
$P_X(k)$	P(2,2)	$P(2,3)+P(3,2)$	$P(2,4)+P(4,2)+P(3,3)$	$P(3,4)+P(4,3)$	$P(4,4)$
	$\Vert$	$\Vert$	$\Vert$	$\Vert$	$\Vert$
	$\left(\frac{1}{3}\cdot\frac{1}{3}\right)$	$\left(\frac{1}{3}\cdot\frac{1}{3}\right)+\left(\frac{1}{3}\cdot\frac{1}{3}\right)$	$\left(\frac{1}{3}\cdot\frac{1}{3}\right)+\left(\frac{1}{3}\cdot\frac{1}{3}\right)+\left(\frac{1}{3}\cdot\frac{1}{3}\right)$	$\left(\frac{1}{3}\cdot\frac{1}{3}\right)+\left(\frac{1}{3}\cdot\frac{1}{3}\right)$	$\left(\frac{1}{3}\cdot\frac{1}{3}\right)$
	$\Vert$	$\Vert$	$\Vert$	$\Vert$	$\Vert$
	$\left(\frac{1}{9}\right)$	$2\left(\frac{1}{9}\right)$	$3\left(\frac{1}{9}\right)$	$2\ \left(\frac{1}{9}\right)$	$\left(\frac{1}{9}\right)$
	$\Vert$	$\Vert$	$\Vert$	$\Vert$	$\Vert$
	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

(c) Let $Y = 3X-21$. Find the probability mass function of Y.

Solution.

X	-9	6	3	0	3
$P_Y(k)$	P(2,2)	$P(2,3)+P(3,2)$	$P(2,4)+P(4,2)+P(3,3)$	$P(3,4)+P(4,3)$	$P(4,4)$
	$\Vert$	$\Vert$	$\Vert$	$\Vert$	$\Vert$
	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

A.k.a. the same as part (b), $P_X(k)$, but with $A \in X \to A \in Y$ by the function $f : X \to Y$, where $x \mapsto 3x-21$.

(d) Prove that X and Y are dependent for some $(i,j) \in \Omega^2$.

Proof.
By way of contradiction (BWOC), assume that X and Y are independent event spaces of Ω. Then by definition, we have $P(X=i,Y=j) = P(X=i)P(Y=j)$. Where $i \in X$ amd $j \in Y$.

Set $i = 6, j = 0$. Notice that $P(X=6,Y=0) = P(X=6)P(Y=0)$
$\implies P(\{(2,4),(4,2),(3,3)\} \cap \{(3,4),(4,3)\}) = P(\{(2,4),(4,2),(3,3)\})P(\{(3,4),(4,3)\})$.

But then the LHS implies $P(\{(2,4),(4,2),(3,3)\} \cap \{(3,4),(4,3)\}) = P(\emptyset) = 0$.
However, the RHS implies $P(\{(2,4),(4,2),(3,3)\})P(\{(3,4),(4,3)\}) = \dfrac{3}{9}\cdot\dfrac{2}{9} = \dfrac{6}{81}$.
Contradiction! $0 \neq \dfrac{6}{81}$.

Hence, X and Y are not independent. Therefore X and Y are dependent. *q.e.d.*

$\boxed{\text{4pts}}$ **4.** We have 3 urns. Urn I has 1 green and 2 red balls, urn II has 2 green and 3 red balls, and urn III has 3 green and 4 red balls.

(a) We first choose an urn at random and then choose a ball randomly from the chosen urn. Find the probability that the ball is red.

Solution. This is a conditional probability problem with sum of multiplication rule.

$$\begin{aligned}
P(R) &= P(I)P(R|I) + P(II)P(R|II) + P(III)P(R|III) \\
&= \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{3}{5} + \frac{1}{3}\cdot\frac{4}{7} = \frac{2}{9} + \frac{3}{15} + \frac{4}{21} \\
&= \frac{1}{3}\left(\frac{2}{3} + \frac{3}{5} + \frac{4}{7}\right) = \frac{1}{3}\left(\frac{70+63+60}{105}\right) \\
&= \frac{1}{3}\left(\frac{193}{105}\right) = \frac{193}{315} \approx 0.6126984. \quad \square
\end{aligned}$$

(b) Suppose we draw a random ball from urn I and transfer it to urn III. Then we choose a ball randomly from urn III. What is the probability that we draw a green ball from urn III.

Solution.
We will use multiplication rule for the conditional probability (transfering the ball), then add intersection of getting a green from III if I say transfered a red ball (which you calculate by multiplying the probabilities since they are "independent"[1]).

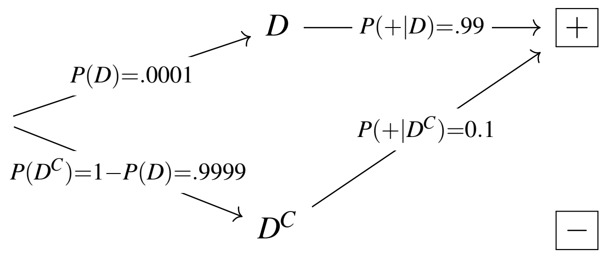
$$\begin{aligned}
P(G\&I \cap G\&III) &= P(G\&I)P(G\&III|G\&I) + P(G\&III \cap R\&I) \\
&= \frac{1}{3}\left(\frac{2+1}{6}\right) + P(G\&III)P(R\&I) \\
&= \frac{1}{6} + \frac{3}{7}\cdot\frac{2}{3} = \frac{1}{6} + \frac{2}{7} = \frac{7+12}{42} = \frac{19}{42} \approx 0.4523809. \quad \square
\end{aligned}$$

[1]keep in mind, the sample size chaged for $P(G\&III)$, so its different

$\boxed{\text{4pts}}$ **5.** There is a new test for a disease that occurs in about 0.01% of the population. It detects the disease 99% of the time. However it has a false positive rate of 10%.

(a) What is the probability that the person's result is positive.

Solution.



So of course (by my method of adding branches),

$$\begin{aligned} P(+) &= P(D)P(+|D) + P(D^C)P(+|D^C) \\ &= (.0001 * .99) + (.9999 * .1) \\ &= .000099 + .09999 = .100089. \quad \square \end{aligned}$$

(b) What is the probability that the person actually has the disease if they test positive.

Solution. Using bayes formula, (my method = branch you want/sum of branches)

$$\begin{aligned} P(D|+) &= \frac{P(D)P(+|D)}{P(+)} \\ &= \frac{.000099}{.100089} = .0009891. \quad \square \end{aligned}$$

$\boxed{\text{8pts}}$ **6.** Suppose $A, B, C \in \mathcal{F},\ s.t.\ A, B, C$ are mutually independent.

Set $P(A) = .1,\ P(B) = .2,\ P(C) = .3.$

(a) $P(A \cap B \cap C^C)$

Solution.

$$\begin{aligned} P(A \cap B \cap C^C) &= P(A)P(B)P(C^C) && \text{(by independence)} \\ &= P(A)P(B)(1 - P(C)) \\ &= .1 * .2 * (1 - .3) = .02 * .7 = .014. \quad \square \end{aligned}$$

(b) $P(A \cup C)$

Solution. By the inclusion-exclusion principle.

$$\begin{aligned} P(A \cup C) &= P(A) + P(C) - P(A \cap C) \\ &= P(A) + P(C) - (P(A)P(C)) && \text{(by independence)} \\ &= .1 + .3 - (.1 * .3) = .4 - (.03) = .37 \quad \square \end{aligned}$$

(c) $P((A \cup C) \cap B)$

Solution. By independence,

$$P((A \cup C) \cap B) = P(A \cup C)P(B)$$

and by part (b), we already know $P(A \cup C)$. So of course,

$$P(A \cup C)P(B) = .37(.2) = .074. \quad \square$$

(d) Prove that $A \cup C$ is independent of B.

Proof. Let $A, B, C \in \mathcal{F},\ s.t.\ A, B, C$ are mutually independent.

Notice, since A and C are independent, and mutually independent from B, then a proper subset of B, say $\alpha \subset B$, is not part of $A \cup C$. i.e. $\alpha \not\subset A \cup C$.

But then $P(B \cap (A \cup C)) = P(B)P(A \cup C)$. So B is independent of $A \cup C$. $q.e.d.$