A Mathematician's Guide to Measurement

A Brief Introduction to Measure Theory

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Goals for Presentation

- Motivation for Measure Theory (Improvements from Real Analysis)
- Algebras, σ-algebras, measures, integration, and differentiation.

Prerequisites

- Logic and Proof Writing
- (Naive) Set Theory
- Single and Multi-variable Calculus
- Linear Algebra
- (Undergraduate) Real Analysis
- Basic Topology

Isn't Real Analysis Hard Enough? Well yes, but...

- Extending the usability of "basic" Analysis
- Reworking fundamental definitions
- Where to start?

Set Theory, of course! So what are we looking for?

Ideal Function

Want a function, λ, such that

1: $\lambda : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+ \cup \ \infty$

2: $\lambda(a, b] = b - a$

3: $\lambda(A + x] = \lambda(A)$

4: $\lambda\left(\bigcup_{j\geq 1} A_j\right) = \sum_{j\geq 1} \lambda(A_j)$

Can we do it? As it turns out, ***no***. A function with the above properties does not exist! (Proof relies on the Axiom of Choice.) Therefore, there exist ***non-measureable subsets of*** $\mathbb{R}$.

Alegbras, σ–algebras, and measures - Part 1

Let X be a (universal) set, S be a collection of subsets of X, T be an arbitrary set in X.

Definition of an algebra

S is an algebra if

1: $X \in S$

2: $\mathcal{T} \in S \implies \mathcal{T}^c \in S$

3: *if* $T_1, T_2, ... \in S \implies \bigcup_{i=1}^{n} T_i \in S$

Definition of a σ–algebra

S is an algebra if

1: $X \in S$

2: $\mathcal{T} \in S \implies \mathcal{T}^c \in S$

3: *if* $T_1, T_2, ... \in S \implies \bigcup_{i=1}^{\infty} T_i \in S$

Want the full context, any partial context and its completion, and the collection of contexts to be a context itself. (X, S) is called a ***measurable space***.

Quick Proof

Let S be an algebra.

S is also $\sigma-$algebra if it is closed under countable, *disjoint* unions.

Proof

Suppose $\{E_j\}_{j=1}^{\infty} \subset S$

Define $F_k = E_k - \bigcup_{j=1}^{k-1} E_j = E_k \cap \left\{ \bigcup_{j=1}^{k-1} E_j \right\}^c$

By construction, $F_k \in S$ and all F_k's are disjoint. Lastly, $\bigcup_{j=1}^{\infty} E_j = \bigcup_{k=1}^{\infty} F_k$. Therefore, S is a $\sigma-$algebra $\square$

Let X is a metric/topological space. The collection of open or closed sets in X are called the ***Borel sets***, and the $\sigma-$algebra generated by the Borel sets is called the ***Borel$\sigma-$algebra***.

Where do we go from algebras and σ–algebra?

The atomic building blocks.
Combine with measuring function.
Let X be a (universal) set, S be a σ–algebra, and μ be a function : $S \to [0, \infty]$

Definition of a measure

μ is a ***measure***
1: $\mu(\emptyset) = 0$
2: If $\{E_j\}_{j=1}^{\infty} \subset S$ is a sequence of disjoint sets in S, then $\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$

Note:

- There can be ***multiple measures*** that can be applied to X or S! (Counting Measure, Dirac Measure, Probability Measure, Haar Measure, Hausdorff Measure, etc.)
- $\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j) \implies \mu(\bigcup_{j=1}^{n} E_j) = \sum_{j=1}^{n} \mu(E_j)$
- Loosening definitions allows for other types of measures

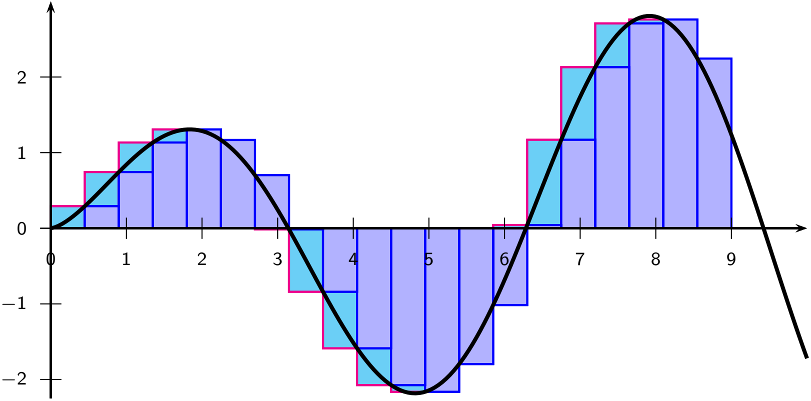
There's more where that came from...

Other measure-related topics

Almost everywhere
Null set
μ–null
Completion
Outer measure
Premeasure
Caratheodory's Theorem

Measures that have the Borel σ–algebra as the domain are called ***Borel measures***.
$(X, \mathcal{B}_{\mathbb{R}}), (X, \mathcal{B}_{\mathbb{C}})$

Let's refresh ourselves on Riemann Integration:



$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\,dx$$

$f : [a, b] \to \mathbb{R}$, $P = \{[a_1, b_1], [a_2, b_2], ..., [a_n, b_n]\}$, $\Delta x_i = x_i - x_{i-1}$, $x_i^* \in [x_{i-1}, x_i]$

(Riemann, Riemann-Stieltjes, Darboux, etc.)

Recognizing problems and expanding horizons

Problems with Riemann Integration:

- Restricted to Real Numbers.
- Difficulty with interchanging integrals, derivatives, summations, and limits.
- Dependence on continuity.

Indicator Function Definition

$1_A : X \to \{0,1\}, A \subset X$

$1_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & otherwise \end{cases}$

f: X $\to \mathbb{C}$

$f = \sum_{i=1}^{n} z_j \chi_{Aj}$, where $A_j = f^{-1}(\{z_j\})$ ***The Standard Representation of*** f

Recognizing problems and expanding horizons

Let (X, S, μ) be a measure space. Let L^+ be the space of all measureable functions from X to $[0, \infty]$. Then define the integral of a function from L^+:

Integral of Measureable Functions (using specific μ)

$$\int f \, d\mu = \sum_{j=1}^{n} z_j \mu(A_j)$$

Note:

- Sum is finite!

References

- ***Real Analysis, 2nd Edition*** by Gerald B. Folland
- ***Real and Complex Analysis, 3rd Edition*** by Walter Rudin
- ***Functional Analysis, 2nd Edition*** by Walter Rudin
- ***Measure and Integral*** by Martin Brokate and Götz Kersting
- ***Measure Theory YouTube Playlist*** by The Bright Side of Mathematics Link to Lecture Playlist
- ***Masters Program Measure Thoery*** by Instituto de Matemática Pura e Aplicada Link to Lecture Playlist
- ***Why Use Measure Theory for Probability*** by Chris Evans Link to 1st Video
- ***(Copious Amounts of) Wikipedia and Wolfram Mathworld***

Conclusion

Thank you! Questions?