# 02. Convex Sets – 凸集合

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## Contents



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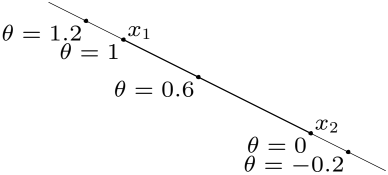
# 1 Affine and convex sets

**Line** Through $x_1$ and $x_2$, all points:

$$x = \theta x_1 + (1-\theta)x_2, \ (\theta \in \boldsymbol{R})$$




**Affine set (仿射集合)**: contains the line through any two distinct points in the set

- Example (look at analysis in next slide): solution set of linear equations $\{x | Ax = b\}$ (conversely, every affine set can be expressed as solution set of system of linear equations)

# 1 Affine and convex sets

**Alternative explanation of affine set**
Assume $C$ is affine, then

$$C = V + x_0 = \{v + x_0 | v \in V\}$$

for some $x_0$, where $V$ is a subspace

**Analysis**: For an affine set $C$ and $x_0 \in C$

$$V = C - x_0 = \{x - x_0 | x \in C\}$$

is a subspace (closed under sums and scalar multiplication)

proof ?

# 1 Affine and convex sets

*proof.*

Assume $x_0 \in C$ and $C$ is affine, there is:

(1) $0 \in C - x_0$ (do we need that?)

(2) Closed under scalar multiplication

$$x_1 \in C - x_0 \Leftrightarrow x_1 + x_0 \in C$$
$$ax_1 + x_0 = a(x_1 + x_0) + (1 - a)x_0 \in C \Rightarrow ax_1 \in C - x_0$$

(3) Closed under sums

$$x_1 \in C - x_0, x_2 \in C - x_0 \Rightarrow 2x_1 \in C - x_0, 2x_2 \in C - x_0$$
$$x_1 + x_2 + x_0 = \frac{1}{2}(2x_1 + x_0) + \frac{1}{2}(2x_2 + x_0) \in C$$

# 1 Affine and convex sets

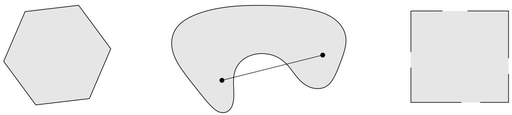
**Line segment** between $x_1$ and $x_2$: all points

$$x = \theta x_1 + (1-\theta)x_2, \ (\theta \in [0,1])$$

**Convex set**: contains line segment between any two points in the set

$$x_1, x_2 \in C, \ \theta \in [0,1] \Rightarrow \theta x_1 + (1-\theta)x_2 \in C$$

**Examples**: one convex, two non-convex



# 1 Affine and convex sets

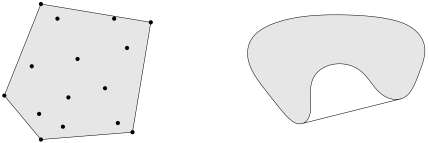
**Convex Combination (凸组合)** of $x_1, \ldots, x_k$: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k,$$

with $\theta_1 + \cdots + \theta_k = 1$, and $\theta_i \geq 0$

(Convex set: convex combination lies in the set)

**Convex hull (凸包)** $\mathbf{conv}\, S$: set of all convex combinations of points in $S$

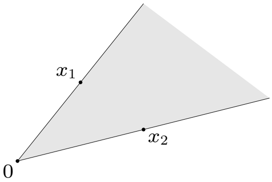


# 1 Affine and convex sets

**Conic (nonnegative) combination** of $x_1$ and $x_2$: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2,$$

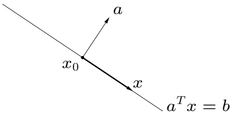
with $\theta_1 \geq 0$, and $\theta_2 \geq 0$



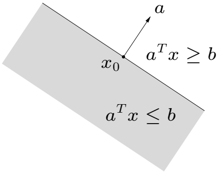

**Convex cone**: set that contains all conic combinations of points in the set

# 2 Important examples

**Hyperplane**: set of the form $\{x | a^T x = b\}$, $a \neq 0$




**Halfspace**: set of the form $\{x | a^T x \leq b\}$ $(a \neq 0)$




$a$ is the normal vector (确定了法线的方向)

hyperplanes are affine and convex; halfspaces are convex

# 2 Important examples

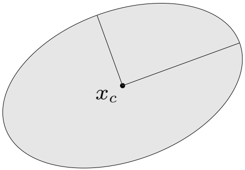
**(Euclidean) ball**: with center $x_c$ and radius $r$:

$$B(x_c, r) = \{x | \ \|x - x_c\|_2 \leq r\} = \{x_c + ru | \ \|u\|_2 \leq 1\}$$

**Ellipsoid（椭球）**：set of the form

$$\{x | \ \|(x - x_c)^T P^{-1}(x - x_c) \leq 1\}$$

with $P \in \boldsymbol{S}^n_{++}$ (*i.e.*, $P$ symmetric positive definite)




Other representation: $\{x_c + Au | \ \|u\| \leq 1\}$ with $A$ square and nonsingular （非奇异方阵）

# 2 Important examples

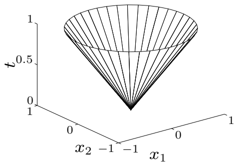
**Norm**: a function $\|\cdot\|$ that satisfies:

$$
\begin{aligned}
&(1) \quad \|x\| \geq 0;\ \ \|x\| = 0 \text{ if and only if } x = 0 \\
&(2) \quad \|tx\| = |t|\|x\|, t \in \boldsymbol{R} \\
&(3) \quad \|x + y\| \leq \|x\| + \|y\|
\end{aligned}
$$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\texttt{symb}}$ is particular norm

**Norm ball(范数球)**: with center $x_c$ and radius $r$: $\{x \mid \|x - x_c\| \leq r\}$

**Norm cone（范数锥）**：$\{(x,t) \mid \|x\| \leq t\}$ (Euclidean norm cone is called second-order cone)



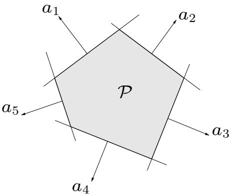

Norm balls and cones are convex

# 2 Important examples

**Polyhedra (多面体)**: solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \;\; Cx = d$$

($A \in R^{m\times n}$, $C \in R^{p\times n}$, $\preceq$ is componentwise inequality)




Polyhedron is intersection of finite number of halfspaces and hyperplanes

# 2 Important examples

**Notations**:

$\boldsymbol{S}^n$ is set of symmetric $n \times n$ matrices

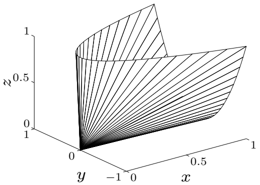
$\boldsymbol{S}^n_+ = \{X \in \boldsymbol{S}^n |\ X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \boldsymbol{S}^n_+ \Longleftrightarrow z^T X z \geq 0 \text{ for all } z$$

$\boldsymbol{S}^n_+$ is a convex cone

$\boldsymbol{S}^n_{++} = \{X \in \boldsymbol{S}^n |\ X \succ 0\}$: positive definite $n \times n$ matrices

**Example**: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \boldsymbol{S}^2_+$



# 1 Affine and convex sets

*proof.*

Assume $A \succeq 0$ and $B \succeq 0$

For any $\theta_1, \theta_2 \geq 0$

$$x^T(\theta_1 A + \theta_2 B)x = \theta_1 x^T A x + \theta_2 x^T B x \geq 0$$

That is $\theta_1 A + \theta_2 B \in \boldsymbol{S}_+^n$

# 3 Operations that preserve convexity

**Practical methods for establishing convexity of a set $C$:**

1. Apply definition

$$x_1, x_2 \in C, 0 \leq \theta \leq 1 \Rightarrow \theta x_1 + (1-\theta) x_2 \in C$$

2. Show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity

- Intersection
- Affine functions
- Perspective function – 透视函数
- Linear-fractional functions – 线性分式

# 3 Operations that preserve convexity

**Intersection**

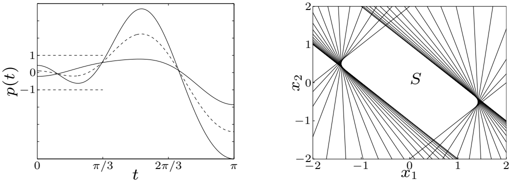
The intersection of (any number of) convex sets is convex

**Example**

$$S = \{x \in \boldsymbol{R}^m \mid |p(t)| \le 1 \text{ for } |t| \le \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$



# 3 Operations that preserve convexity

**Affine function**

Suppose $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \boldsymbol{R}^{m \times n}$, $b \in \boldsymbol{R}^m$), then:

1. The image of a convex set under $f$ is convex

$$S \subseteq \boldsymbol{R}^n \text{ convex} \Rightarrow f(S) = \{f(x) |\ x \in S\} \text{ convex}$$

2. The inverse image $f^{-1}(C)$ of a convex set under $f$ is convex

$$C \subseteq \boldsymbol{R}^m \text{ convex} \Rightarrow f^{-1}(C) = \{x \in \boldsymbol{R}^n |\ f(x) \in C\} \text{ convex}$$

**Examples**:

- Scaling, translation, projection
- Solution set of linear matrix inequality $\{x | x_1 A_1 + \cdots + x_m A_m \preceq B\}$ (with $A_i, B \in \boldsymbol{S}^p$)
- Hyperbolic cone $\{x |\ x^T P x \leq (c^T x)^2, c^T x \geq 0\}$ (with $P \in \boldsymbol{S}_+^n$)

# 1 Affine and convex sets

*proof.* of Example 2

Define $f(x) := B - A(x) = B - (x_1 A_1 + \cdots + x_m A_m)$

It is affine

For any $x$, the images $y = f(x)$ construct a subspace

With $f(x) \succeq 0$, $y = f(x)$ form a positive semidefinite cone, which is convex

$x$ satisfies $f(x) \succeq 0$ is also convex

*proof.* of Example 3

The above set is obtained from the inverse image of $\{(z,t) | z^T z \leq t^2, t \geq 0\}$ through affine function $f(x) = (P^{1/2} x, c^T x)$

# 3 Operations that preserve convexity

**Perspective function** $P : \boldsymbol{R}^{n+1} \to \boldsymbol{R}^n$:

$$P(x, t) = x/t, \ \mathbf{dom} P = \{(x, t) | \ t > 0\}$$

Suppose $f : \boldsymbol{R}^n \to \boldsymbol{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \boldsymbol{R}^{m \times n}, b \in \boldsymbol{R}^m$), then:

Images and inverse images of convex sets under perspective are convex

**Linear-fractional function**: $f : \boldsymbol{R}^n \to \boldsymbol{R}^m$:

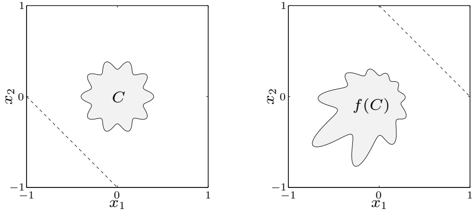
$$f(x) = \frac{Ax + b}{c^T x + d}, \ \mathbf{dom} f = \{x | \ c^T x + d > 0\}$$

Images and inverse images of convex sets under linear-fractional functions are convex

# 3 Operations that preserve convexity

**Example** of a linear-fractional function:

$$f(x) = \frac{1}{x_1 + x_2 + 1} x$$



# 3 Generalized inequalities – 广义不等式

A convex cone $K \subseteq \boldsymbol{R}^n$ is a proper cone (正常锥) if

- $K$ is closed (contains its boundary)
- $K$ is solid (has nonempty interior)
- $K$ is pointed (contains no line)

**Examples**:

1. Nonnegative orthant $K = \boldsymbol{R}^n_+ = \{x \in \boldsymbol{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$
2. Positive semidefinite cone $K = \boldsymbol{S}^n_+$
3. Nonnegative polynomials on $[0, 1]$:

$$K = \{x \in \boldsymbol{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

# 3 Generalized inequalities – 广义不等式

**Generalized inequality** defined by a proper cone $K$:

$$x \preceq_K y \Longleftrightarrow y - x \in K, \quad x \prec_K y \Longleftrightarrow y - x \in \textbf{int } K$$

**Examples**:

1. Componentwise inequality $(K = \boldsymbol{R}^n_+)$

$$x \preceq_{\boldsymbol{R}^n_+} y \Longleftrightarrow x_i \leq y_i, \quad i = 1, \dots, n$$

2. Matrix inequality $(K = \boldsymbol{S}^n_+)$

$$X \preceq_{\boldsymbol{S}^n_+} Y \Longleftrightarrow Y - X \text{ positive semidefinite}$$

These two types are so common that we drop the subscript in $\preceq_K$

Many properties of $\preceq_K$ are similar to $\leq$ on $\boldsymbol{R}$, *e.g.*,

$$x \preceq_K y, \; u \preceq_K v \Rightarrow x + u \preceq_K y + v$$

# 3 Generalized inequalities – 广义不等式

**Minimum and minimal elements**

$\preceq_K$ is not in general a linear ordering: we can have $x \not\preceq_K y$ and $y \not\preceq_K x$

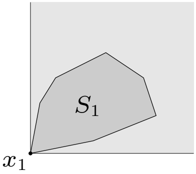
$x \in S$ is the minimum element of $S$ with respect to $\preceq_K$ if

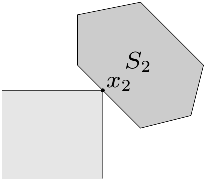
$$y \in S \Rightarrow x \preceq_K y$$

$x \in S$ is the minimal element of $S$ with respect to $\preceq_K$ if

$$y \in S, \ x \preceq_K y \Rightarrow y = x$$

**Example**:



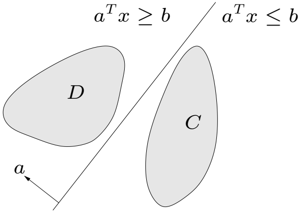



# 5 Separating and supporting hyperplanes

**Separating hyperplane theorem**

If $C$ and $D$ are disjoint convex sets, then there exists a $a \neq 0, b$ such that

$$a^T x \leq b \text{ for } x \in C, \; a^T x \geq b \text{ for } x \in D$$




The hyperplane $\{x | a^T x = b\}$ separates $C$ and $D$

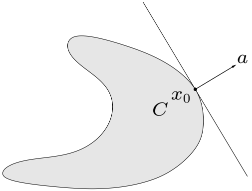
Strict separation requires additional assumptions($e.g.$, $C$ is closed, $D$ is a singleton)

# 5 Separating and supporting hyperplanes

Supporting hyperplane to set $C$ at boundary point $x_0$:

$$\{x | a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$




**supporting hyperplane theorem**: if $C$ is convex, then there exists a supporting hyperplane at every boundary point of $C$

# 5 Dual cones and generalized inequalities

**Dual cone** of a cone $K$:

$$K^* = \{y | y^T x \geq 0 \text{ for all } x \in K\}$$

**Examples**:

- $K = \boldsymbol{R}_+^n$: $K^* = \boldsymbol{R}_+^n$
- $K = \boldsymbol{S}_+^n$: $K^* = \boldsymbol{S}_+^n$
- $K = \{(x,t) | \|x\|_2 \leq t\}$: $K^* = \{(x,t) | \|x\|_2 \leq t\}$
- $K = \{(x,t) | \|x\|_1 \leq t\}$: $K^* = \{(x,t) | \|x\|_\infty \leq t\}$

First three examples are **self-dual cones**

Dual cones of proper cones are proper, hence define generalized inequalities:

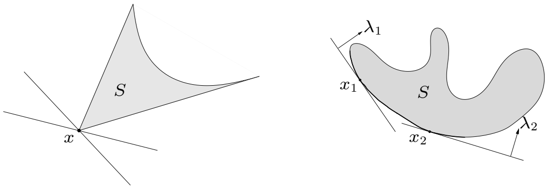
$$y \succeq_{K^*} \Longleftrightarrow y^T x \geq 0 \text{ for all } x \succeq_K 0$$

# 5 Dual cones and generalized inequalities

**Minimum and minimal elements via dual inequalities**

**Minimum element with respect to $\preceq_K$:** $x$ is minimum element of $S$ iff for all $\lambda \succ_{K^*} 0$, $x$ is the unique minimizer of $\lambda^T z$ over S

$$K^* = \{y | y^T x \geq 0 \text{ for all } x \in K\}$$




**Minimal element with respect to $\preceq_K$:** if $x$ minimizes $\lambda^T z$ over $S$ for some $\lambda \succ_{K^*} 0$, then $x$ is minimal

If $x$ is a minimal element of a convex set $S$, then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that $x$ minimizes $\lambda^T z$ over $S$

# 5 Dual cones and generalized inequalities

**Optimal production frontier**

- Different production methods use different amounts of resources $x \in \boldsymbol{R}^n$
- Production set $P$: resource vectors $x$ for all possible production methods
- Efficient (Pareto optimal) methods correspond to resource vectors $x$ that are minimal w.r.t. $\boldsymbol{R}^n_+$

**example (n =2)**: $x_1, x_2, x_3$ are efficient; $x_4, x_5$ are not