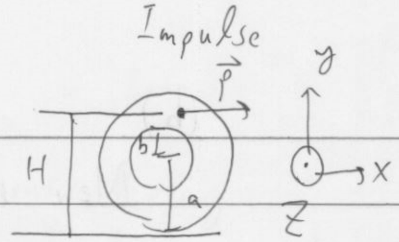
HW7

Shooting Pool



(a)

$$\text{Density}: \rho = \frac{M}{V} = \frac{M}{\pi(a^2-b^2)l}$$

Inertia tensor for a solid cylinder of mass m, length l and radius r is:

$$\overleftrightarrow{I} = \begin{bmatrix} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ 0 & 0 & I_\parallel \end{bmatrix} = \begin{bmatrix} \frac{1}{12}m(l^2+3r^2) & 0 & 0 \\ 0 & \frac{1}{12}m(l^2+3r^2) & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix}$$

$\therefore$ The hollow cylinder's inertia tensor is:

$$\begin{aligned} I_{\perp,tot} &= I_{\perp,a} - I_{\perp,b} \\ &= \frac{\pi}{12}(\rho a^2 l)(l^2+3a^2) - \frac{\pi}{12}(\rho b^2 l)(l^2+3b^2) \\ &= \frac{\pi}{12}\rho\left[l^3(a^2-b^2)+3l(a^4-b^4)\right] \\ &= \frac{\pi}{12}\rho l(a^2-b^2)\left[l^2+3(a^2+b^2)\right] = \frac{M}{12}\left[l^2+3(a^2+b^2)\right] \end{aligned}$$

$$\begin{aligned} I_{\parallel,tot} &= I_{\parallel,a} - I_{\parallel,b} \\ &= \frac{\pi}{2}(\rho a^2 l)a^2 - \frac{\pi}{2}(\rho b^2 l)b^2 \\ &= \frac{\pi}{2}\rho l(a^4-b^4) \\ &= \frac{M}{2}(a^2+b^2) \end{aligned}$$

$$\therefore \overleftrightarrow{I}_{tot} = \begin{bmatrix} \frac{1}{12}M\left[l^2+3(a^2+b^2)\right] & 0 & 0 \\ 0 & \frac{1}{12}M\left[l^2+3(a^2+b^2)\right] & 0 \\ 0 & 0 & \frac{M}{2}(a^2+b^2) \end{bmatrix}$$

(b)

New momentum: $\vec{p} = p_0 (1, 0, 0)$

New angular momentum:

$$\vec{L} = \vec{r}_p \times \vec{p} = (0, H-a, 0) \times (p_0, 0, 0)$$
$$= (0, 0, -p_0(H-a))$$
$$= \overleftrightarrow{I} \vec{\omega}$$
$$\Rightarrow \omega_z = \frac{-p_0(H-a)}{\frac{M}{2}(a^2+b^2)}$$

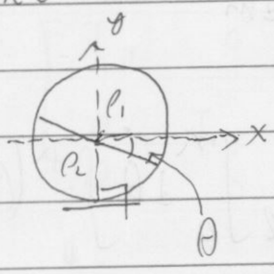
To let rolling no slip:

$$|\omega_z| \cdot a = |\vec{p}|/M$$
$$\Rightarrow \frac{p_0(H-a)a}{\frac{M}{2}(a^2+b^2)} = \frac{p_0}{M} \Rightarrow H = \frac{(a^2+b^2)}{2a} + a$$

(c) ① $b \to 0$, $H = \frac{a^2}{2a} + a = \frac{3}{2}a$

② $b \to a$, $H = \frac{a^2+a^2}{2a} + a = 2a$

Problem 1



(1) $\because$ Rolling without slipping

(2) Rotation only happens at $x-y$ plane.

$\therefore$ Only has one degree of freedom θ

$$T = \frac{1}{2} I_z \dot{\theta}^2 + \frac{1}{2} M (\dot{\theta} R)^2$$

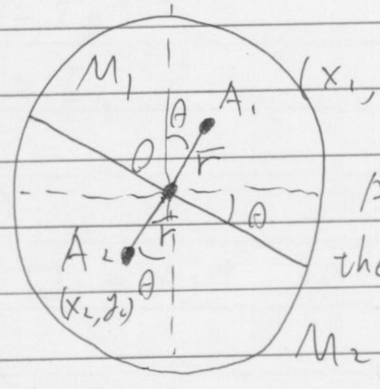
$$= \frac{1}{2}\left[\frac{1}{2}\left(\frac{\pi}{2}\rho_1 R^2 l\right) R^2 + \frac{1}{2}\left(\frac{\pi}{2}\rho_2 R^2 l\right) R^2\right]\dot{\theta}^2$$

$$+ \frac{1}{2}\left[\frac{\pi}{4}(\rho_1+\rho_2) R^2 l\right] \cdot R^2 \dot{\theta}^2$$

$$= \frac{\pi}{4}(\rho_1+\rho_2) R^4 l \dot{\theta}^2 \quad (l = \text{Disk length in } z\text{-direction})$$

$$V = M_2 g y_2 + M_1 g y_1$$

$$= \frac{\pi}{4}(\rho_1 - \rho_2) R^2 l \cdot \bar{r} \cos\theta$$



A_1, A_2 are the centers of mass of M_1 & M_2.

$$L = T - V$$

$$= \frac{\pi}{4}(\rho_1 + \rho_2) R^4 l \dot{\theta}^2$$

$$+ \frac{\pi}{4}(\rho_2 - \rho_1) R^2 l \bar{r} \cos\theta$$

$$= A\dot{\theta}^2 + B\cos\theta \quad (A, B > 0)$$

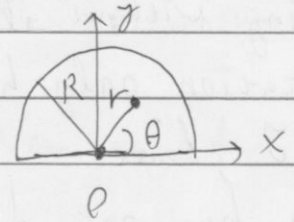
$y_1 = \bar{r}\cos\theta$

$y_2 = -\bar{r}\cos\theta$

$\bar{r} = \frac{4R}{3\pi}$ (by Appendix 1.)

$$\Rightarrow \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \Rightarrow 2A\ddot{\theta} = -B\sin\theta \Rightarrow \text{Harmonic Oscillator (small } \theta)$$

Appendix 1.



$y = r\sin\theta$

$$\bar{y} = \frac{\int y\,dm}{M}$$

$$= \frac{1}{\frac{\pi}{2}\rho R^2 l}\int_0^\pi d\theta \int_0^R \overbrace{(\rho l r\,dr)}^{dm}\, r\sin\theta$$

$$= \frac{\rho l}{\frac{\pi}{2}\rho R^2 l}\int_0^\pi \sin\theta\,d\theta \int_0^R r^2\,dr$$

$$= \frac{1}{\frac{\pi}{2}R^2}\,\frac{R^3}{3}\cdot 2$$

$$= \frac{4R}{3\pi}$$

Problem 2

$$\overleftrightarrow{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \qquad \begin{aligned} I_x = I_y &:= I_1 \\ I_z &:= I_2 \end{aligned}$$

$$\left(\frac{d\vec{L}}{dt}\right)_{rot} + \vec{\omega} \times \vec{L} = \vec{\tau}$$

(in body frame)

$$\Rightarrow \overleftrightarrow{I} \cdot \dot{\vec{\omega}} + \vec{\omega} \times (\overleftrightarrow{I} \cdot \vec{\omega}) = \vec{\tau}$$

$$\Rightarrow (I_x \dot{\omega}_x, I_y \dot{\omega}_y, I_z \omega_z) + (\omega_x, \omega_y, \omega_z) \times (I_x \omega_x, I_y \omega_y, I_z \omega_z) = \vec{\tau}$$

$$\Rightarrow \begin{cases} I_x \dot{\omega}_x + (I_z - I_y) \omega_y \omega_z = \tau_x \\ I_y \dot{\omega}_y + (I_x - I_z) \omega_z \omega_x = \tau_y \\ I_z \dot{\omega}_z + (I_y - I_x) \omega_x \omega_y = \tau_z \end{cases}$$

$$\Rightarrow I_z \dot{\omega}_z = \tau_z \quad (\because I_y = I_x)$$

$$\Rightarrow \omega_z = \omega_3 + \frac{\tau_z}{I_z} t := \omega_3 + \alpha_3 t \quad \left(\alpha_3 := \frac{\tau_z}{I_z}\right)$$

$$\begin{cases} I_1 \dot{\omega}_x + (I_2 - I_1) \omega_y (\omega_3 + \alpha_3 t) = \tau_x = 0 \\ I_1 \dot{\omega}_y + (I_1 - I_2) \omega_x (\omega_3 + \alpha_3 t) = \tau_y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} I_1 (\dot{\omega}_x + i\dot{\omega}_y) + i(I_1 - I_2)(\omega_x + i\omega_y)(\omega_3 + \alpha_3 t) = 0 \\ I_1 (\dot{\omega}_x - i\dot{\omega}_y) - i(I_1 - I_2)(\omega_x - i\omega_y)(\omega_3 + \alpha_3 t) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} I_1 \dot{u}(t) + i(I_1 - I_2)(\omega_3 + \alpha_3 t) u(t) = 0 \\ I_1 \dot{v}(t) - i(I_1 - I_2)(\omega_3 + \alpha_3 t) v(t) = 0 \end{cases}$$

$$u := \omega_x + i\omega_y, \quad v := \omega_x - i\omega_y$$

$$\int \frac{du}{u} = \int i\left(\frac{I_1 - I_2}{I_1}\right)(\omega_3 + \alpha_3 t)\, dt$$

$$\int \frac{dv}{v} = -\int i\left(\frac{I_1 - I_2}{I_1}\right)(\omega_3 + \alpha_3 t)\, dt$$

$$\Rightarrow \quad u = u(t=0)\, e^{i\left(\frac{I_1 - I_2}{I_1}\right)\left[\omega_3 t + \frac{\alpha_3}{2} t^2\right]}$$

$$v = v(t=0)\, e^{-i\left(\frac{I_1 - I_2}{I_1}\right)\left(\omega_3 t + \frac{\alpha_3}{2} t^2\right]}$$

$$u(t=0) = v(t=0) = \omega_1$$

$$\therefore \quad \omega_x = \frac{u+v}{2} = \omega_1 \cos\left[\left(\frac{I_1 - I_2}{I_1}\right)\left(\omega_3 t + \frac{\alpha_3}{2} t^2\right)\right]$$

$$\omega_y = \frac{u-v}{2i} = \omega_1 \sin\left[\left(\frac{I_1 - I_2}{I_1}\right)\left(\omega_3 t + \frac{\alpha_3}{2} t^2\right)\right]$$