# HW 10

## Problem 1

(a)

$$H = \frac{p^2}{2m} + \sigma|x|$$

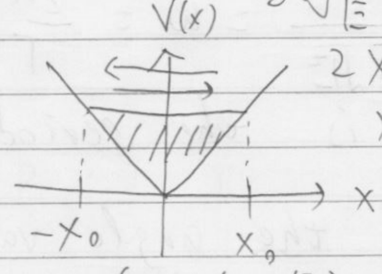
Since $\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$ and $H = E = T + V$

$\Rightarrow E$ is conserved

$$\Rightarrow p(x, E) = \pm\sqrt{2m}\sqrt{E - \sigma|x|}$$

Now we want to calculate for a constant energy $E$, the action variable $J(E)$

$$J(E) := \frac{1}{2\pi}\oint_E p\, dx$$




$2x_0$ is the range of the motion, $x_0(E) = \frac{E}{\sigma}$ (assume $E > 0$)

And we can determine the sign of $p(x, E)$:

$p(x, E) > 0$ for motion from $-x_0$ to $+x_0$

$< 0$ for motion from $+x_0$ to $-x_0$

And also the motion is symmetric

$$\therefore J = \frac{1}{2\pi}\oint_E p\, dx = \frac{2}{2\pi}\int_{-x_0}^{x_0} +\sqrt{2m}\sqrt{E - \sigma|x|}\, dx$$

$$= \frac{4}{2\pi}\int_0^{x_0}\sqrt{2m}\sqrt{E - \sigma x}\, dx$$

$$= \frac{2\sqrt{2m}\sqrt{E}}{\pi}\int_0^1 \sqrt{1-u}\, du \cdot \frac{E}{\sigma} \quad \left(u := \frac{\sigma x}{E}\right)$$

$$= \frac{2\sqrt{2m}\, E\sqrt{E}}{\pi\sigma}\cdot\frac{2}{3} = \frac{4\sqrt{2m}\, E\sqrt{E}}{3\pi\sigma}$$

Action variable is

$$\therefore J(E) = \frac{4\sqrt{2m}}{3\pi}\ \frac{E\sqrt{E}}{\sigma}$$

And the corresponding angle variable is

$$\dot{\theta} = \frac{\partial H}{\partial J}$$

$$H = E = E(J) = \left(\frac{3\pi\sigma J}{4\sqrt{2m}}\right)^{\frac{2}{3}}$$

$$\therefore \dot{\theta} = \frac{\partial H(J)}{\partial J} = \frac{2}{3J}\cdot\left(\frac{3\pi\sigma J}{4\sqrt{2m}}\right)^{\frac{2}{3}}$$

$$= \frac{2}{3}\frac{E}{J} = \frac{2E}{3}\cdot\frac{3\pi\sigma}{4\sqrt{2m}\,E\sqrt{E}}$$

$$= \frac{\pi\sigma}{2\sqrt{2m}\sqrt{E}} = \omega = \frac{2\pi}{T}$$

$$\Rightarrow T = \frac{4\sqrt{2m}\sqrt{E}}{\sigma} \text{ is the period}$$

$$\theta = \frac{\pi\sigma t}{2\sqrt{2m}\sqrt{E}} \text{ is the angle variable}$$

(b) In adiabatic process, $J(E)$ will not change

① Energy $E$

$\therefore J(E) = \frac{4\sqrt{2m}}{3\pi} \frac{E\sqrt{E}}{6}$ is conserved

$\therefore E \propto (6J)^{\frac{2}{3}} \propto 6^{\frac{2}{3}}$

$\therefore E = E_0 \cdot \left(\frac{6(t)}{6(t=0)}\right)^{\frac{2}{3}}$ will change with $6$ accordingly.

② Amplitude $x_0(E)$

Since $x_0 = x_0(E) = \frac{E(6)}{6}$, and $E \propto 6^{\frac{2}{3}}$

$\therefore x_0 \propto 6^{\frac{-1}{3}} \Rightarrow x_0 = x_0(6)$
$= x_0(\underset{\underset{6(t=0)}{\parallel}}{6_0}) \cdot \left(\frac{6(t)}{6(t=0)}\right)^{\frac{-1}{3}}$

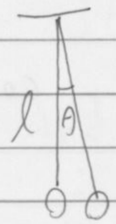
③ Period $T$,

$T = T(E) = 4\sqrt{2m}\frac{\sqrt{E}}{6} \propto \frac{\sqrt{E}}{6} \propto 6^{\frac{-2}{3}}$

$\therefore T(6) = T(6(t=0)) \cdot \left(\frac{6(t)}{6(t=0)}\right)^{\frac{-2}{3}}$

Problem 2

$$I = ml^2$$



$\downarrow g$

$$H = \frac{p_\theta^2}{2I} + mgl(1-\cos\theta)$$

$$\simeq \frac{p_\theta^2}{2I} + \frac{mgl}{2}\theta^2 \quad (\because \theta \text{ is small})$$

$H = E$ is conserved $(\because \frac{\partial H}{\partial t} = 0)$

$$\therefore \quad p_\theta = \pm\sqrt{2I\left(E - \frac{mgl}{2}\theta^2\right)}$$

$$= \pm\sqrt{2EI - mglI\theta^2}$$

Action is

$$J(E) = \frac{1}{2\pi}\oint_E p_\theta\, d\theta = \frac{4}{2\pi}\int_0^{\theta_{max}} \sqrt{2EI - mglI\theta^2}\, d\theta$$

$$= \frac{2}{\pi}\sqrt{2EI}\int_0^{b_{max}} \sqrt{1-b^2}\, db \sqrt{\frac{2E}{mgl}}$$

$$\left(b := \sqrt{\frac{mgl}{2E}}\,\theta,\ b_{max} = \sqrt{\frac{mgl}{2E}}\,\theta_{max} = \sqrt{\frac{mgl}{2E}}\cdot\sqrt{\frac{2E}{mgl}} = 1\right)$$

$$= \frac{4E\sqrt{I}}{\pi\sqrt{mgl}}\int_0^1 \sqrt{1-b^2}\, db$$

$$= \frac{4}{\pi}\sqrt{\frac{l}{g}}\,E\cdot\frac{\pi}{4} = \sqrt{\frac{l}{g}}\,E$$

$$\Rightarrow H = E = \sqrt{\frac{g}{l}}\,J(E)$$

$\therefore$ Corresponding angle variable $\phi$ is

$$\dot{\phi} = \frac{\partial H}{\partial J} = \sqrt{\frac{g}{l}} = \omega = \frac{2\pi}{T}$$

$\Rightarrow T = 2\pi\sqrt{\frac{l}{g}}$ is the period which is the same as the period from our understanding of small angle pendulum.

Amplitud $\theta_{max}(E) = \sqrt{\frac{2E}{mgl}} \propto \sqrt{\frac{E}{l}}$

Now let's look at adiabatic process.

$J(E) = \sqrt{\frac{l}{g}}\, E$ will be conserved

$\Rightarrow E = \sqrt{\frac{g}{l}}\, J \propto \sqrt{\frac{1}{l}}$ as $l$ changes slowly

① Amplitude $\theta_{max}$

$\theta_{max}(E) \propto \sqrt{\frac{E}{l}} \propto \frac{1}{l}$ as $l$ changes

② Energy $E$

$E \propto \sqrt{\frac{1}{l}}$

③ frequency $\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{l}} \propto \frac{1}{\sqrt{l}}$

Problem 3

(a)

$$L = T - V$$

$$= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\Rightarrow H = p_r \cdot \dot{r} + p_\theta \dot{\theta} - L$$

$$= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r} = T + V$$

Since

$$H = T + V = E, \text{ and}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \Rightarrow \frac{dE}{dt} = 0, \ E \text{ is conserved}$$

And $\frac{dp_\theta}{dt} = \frac{-\partial H}{\partial \theta} = 0 \Rightarrow p_\theta$ is also conserved

let's make $p_\theta = l$ = angular momentum afterwards

(b) Now let's look at the Hamiltonian

$$H = \frac{p_r^2}{2m} + \left(\frac{l^2}{2mr^2} - \frac{GMm}{r}\right)$$

$$:= \frac{p_r^2}{2m} + V_{eff}(r) \quad \left(V_{eff} := \frac{l^2}{2mr^2} - \frac{GMm}{r}\right)$$

$$\therefore \; p_r(r, E) = \pm\sqrt{2m\left(E - V_{eff}(r)\right)}$$
$(H = E)$

$\therefore$ Action variable

$$I_r(E, p_\theta, M) = \oint \frac{p_r}{2\pi} dr$$

$$= \frac{2}{2\pi}\int_{r_{min}}^{r_{max}} dr \sqrt{2m\left(E - \frac{l^2}{2mr^2} + \frac{GMm}{r}\right)}$$

$$= \frac{l}{\pi}\int_{r_{min}}^{r_{max}} \sqrt{\left(\frac{1}{r_{min}} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_{max}}\right)}\, dr$$

$\because$ $E - \frac{l^2}{2mr^2} + \frac{GMm}{r} = 0$ the two roots are the solutions of $\frac{1}{r_{min}}, \frac{1}{r_{max}}$, at which the $p_r = 0$ ($\because$ At min or max point of $r$)

Make $r_{max} = \alpha$, $r_{min} = \beta$

$$\therefore \quad I = \frac{l}{\pi}\int_{\beta}^{\alpha}\sqrt{\left(\frac{1}{\beta}-\frac{1}{r}\right)\left(\frac{1}{r}-\frac{1}{\alpha}\right)}\,dr$$

$$= \frac{l}{\pi}\int_{\beta}^{\alpha}\sqrt{\left(\frac{r}{\beta}-1\right)\left(1-\frac{r}{\alpha}\right)}\left(\frac{dr}{r}\right)$$

$$= \frac{l}{\pi\sqrt{\alpha\beta}}\int_{\beta}^{\alpha}\sqrt{(r-\beta)(\alpha-r)}\left(\frac{dr}{r}\right)$$

We can use contour integral to solve this integral:

① Pole at origin $r=0$

$$\Rightarrow \quad I_0 = \frac{l}{\pi\sqrt{\alpha\beta}}\cdot 2\pi i\cdot\sqrt{-\alpha\beta}$$

$$= -2l$$

② Pole at infinity

$$I_\infty = \frac{l}{\pi\sqrt{\alpha\beta}}\oint\left(1-\frac{\alpha}{2z}\right)\left(1-\frac{\beta}{2z}\right) i\,dz$$

$$= 2\pi i\cdot\frac{l}{\pi\sqrt{\alpha\beta}}\left(-\frac{i}{2}\right)\cdot(\alpha+\beta)$$

$$= \frac{l(\alpha+\beta)}{\sqrt{\alpha\beta}}$$

$$\therefore \quad I_r(E) = I_0 + I_\infty$$

$$= -2l + \frac{l(\alpha+\beta)}{\sqrt{\alpha\beta}}$$

From our previous lecture notes, we know that

$$r_{max/min} = \frac{a}{1 \pm \varepsilon} = \frac{a}{1-\varepsilon} \ (max) \ \text{or} \ \frac{a}{1+\varepsilon} \ (min)$$

$$a = \frac{\ell}{\sqrt{2m|E|}}, \ \varepsilon = \sqrt{1 - \frac{2|E| \ell^2}{(G^2 M^2 m^3)}}$$

$$\therefore \ \alpha + \beta = \frac{2a}{1-\varepsilon^2} = \frac{2\ell}{\sqrt{2m|E|}} \cdot \frac{(GMm)^2 m}{2|E| \ell^2}$$

$$= \frac{(GMm)^2 \sqrt{m}}{\ell \sqrt{2}\sqrt{|E|}^3}$$

$$\sqrt{\alpha\beta} = a\sqrt{\frac{1}{1-\varepsilon^2}} = \frac{\ell}{\sqrt{2m|E|}} \cdot \frac{\sqrt{(GMm)^2 m}}{\sqrt{2|E| \ell^2}}$$

$$= \frac{1}{2} \frac{(GMm)}{|E|}$$

$$\therefore I_r(E) = -2\ell + \ell \frac{\sqrt{2} GMm\sqrt{m}}{\ell\sqrt{|E|}}$$

$$= -2\ell + \frac{\sqrt{2} GMm\sqrt{m}}{\sqrt{|E|}}$$

(c)

The adiabatic invariant for angle motion is

$$I_\theta(E) = \frac{1}{2\pi}\oint p_\theta \, d\theta = l$$

$\Rightarrow$ $l$ (angular momentum) will be conserved for adiabatic process

And since $I_r(E, p_\theta = l, M)$ will also be conserved

$$\Rightarrow I_r = 2l + \frac{\sqrt{2}\, G M m \sqrt{m}}{\sqrt{|E|}}$$

will not change when $M$ changes

$\therefore$ The ratio $\frac{M}{\sqrt{|E|}}$ will not change

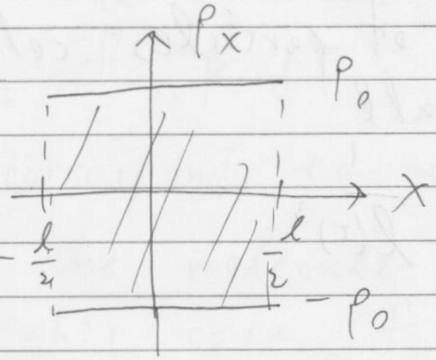
$\Rightarrow \sqrt{|E|} \propto M$

$|E| \propto M^2$ over long times

Problem 4

(a)

In the motion of a single particle over one period T:




The adiabatic invariant

$$I(E) = \frac{1}{2\pi}\oint_E p_x\, dx = \frac{1}{2\pi} p_0 \cdot 2l = \frac{p_0 l}{\pi}$$

In an adiabatic process $l = l(t)$,

$I(E)$ will be conserved

$$\Rightarrow \quad p_0 l = \text{const.} \Rightarrow p_0 \propto \frac{1}{l(t)}$$

Moreover, the period to complete one cycle also changes with $l(t)$,

$$T = \frac{2l}{(p_0/m)} \propto l^2(t)$$

We know that pressure P is the average rate of exchange of momentum at the boundary, i.e.

$$P \propto \underbrace{\frac{N}{T}}_{\text{\# rate of particles colliding with the wall}} \cdot p_0 \quad \to \text{average momentum}$$

$$\therefore P \propto \frac{p_0}{T} \propto \frac{1}{l(t)^3}$$

(b) We assume that temperature Temp will be proportional to the average kinetic energy

$$\text{Temp}(\bar{E}) = \frac{p_0^2}{2m} \cdot C \propto \frac{1}{l(t)^2}$$

(c) From (a), we know

$P \propto \frac{1}{l^3}$, and since $V \propto l$

$\therefore PV^3$ is a constant in this case

And from our understanding of classical thermodynamics, the adiabatic expansion / compression has the following formula:

$PV^{\gamma} = \text{const.}$, $\gamma := \frac{f+2}{f}$

and the internal energy will be

$dU = \frac{f}{2} nR\,dT$

$f$ corresponds to the degree of freedom of the molecule motion.

In this case, $f = 1 \Rightarrow \gamma = \frac{f+2}{f} = 3$

$\therefore \gamma = 3$ is consistent.

(d) Since the walls expand too fast, it has no time to do work on the particles.

$\Rightarrow$ Kinetic energy of gas particles are approximately conserved

$\Rightarrow T = \text{Temp}$ is conserved

$\therefore \text{Temp} = \text{Temp}$ (E)

$V$ definitely increases.

$P \propto \dfrac{p_0}{T(\text{Period})} = \dfrac{p_0}{\frac{2l}{(p_0/m)}} \propto \dfrac{1}{l}$ ($\because p_0$ conserved)

reduces