Problem 1

(a)

Velocity in S_0 is:

$$\vec{v}' = \vec{v} + \vec{\Omega} \times \vec{r}$$

Kinetic energy T is:

$$T = \frac{1}{2} m |\vec{v}'|^2 = \frac{1}{2} m (\vec{v} + \vec{\Omega} \times \vec{r}) \cdot (\vec{v} + \vec{\Omega} \times \vec{r})$$

$$= \frac{1}{2} m |\vec{v}|^2 + m \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2$$

$$= \frac{1}{2} m |\vec{v}|^2 + m \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2} m [(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}] \cdot \vec{r}$$

$$= \frac{1}{2} m |\vec{v}|^2 + m \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2} m [\Omega^2 \vec{r} - (\vec{\Omega} \cdot \vec{r}) \vec{\Omega}] \cdot \vec{r}$$

$$= \frac{1}{2} m [|\vec{v}|^2 + r^2 \Omega^2 - (\vec{\Omega} \cdot \vec{r})^2] + m \vec{v} \cdot (\vec{\Omega} \times \vec{r})$$

$L = T - V$ (V is potential energy, if any)

$(\Omega^2 := |\vec{\Omega}|^2, \ r^2 := |\vec{r}|^2)$

(b) Euler-Lagrange eq. : (Assume $V = 0$, $L = T$)

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \vec{v}} \right) = \frac{\partial L}{\partial \vec{r}}$$

$$\Rightarrow \frac{d}{dt} [m \vec{v} + m (\vec{\Omega} \times \vec{r})] = m [\Omega^2 \vec{r} - (\vec{\Omega} \cdot \vec{r}) \vec{\Omega}] + m (\vec{v} \times \vec{\Omega})$$

$$\Rightarrow m \vec{a} + m (\vec{\Omega} \times \vec{v}) = m \, \vec{\Omega} \times (\vec{r} \times \vec{\Omega}) + m (\vec{v} \times \vec{\Omega})$$

$$\Rightarrow m \vec{a} = \underbrace{m \, \vec{\Omega} \times (\vec{r} \times \vec{\Omega})}_{\text{Centrifugal force}} + \underbrace{2m (\vec{v} \times \vec{\Omega})}_{\text{Coriolis force}}$$

Problem 2

(a)

The velocity moving on a sphere is:

$$\vec{V} = r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}$$

$\therefore$ Kinetic energy : $T = \frac{m}{2} r^2 (\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$

Potential : $V = mgr\cos\theta$

$\therefore$ Lagrangian :

$$L = T - V = \frac{m}{2} r^2 (\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) - mgr\cos\theta$$

Canonical Momenta:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}$$

$\therefore$ Hamiltonian:

$$H = \sum_{i=\theta,\phi} p_i v_i - L = \frac{m}{2} r^2 (\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + mgr\cos\theta$$

$$= \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + mgr\cos\theta$$

(b) Canonical Eqs.

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \quad \frac{dp_\theta}{dt} = \frac{-\partial H}{\partial \theta} = \frac{p_\phi^2}{mr^2}\frac{\cos\theta}{\sin^3\theta} + mgr\sin\theta$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta}, \quad \frac{dp_\phi}{dt} = \frac{-\partial H}{\partial \phi} = 0$$

(C) It seems $H = T + V$, thus $H = E$ (Energy)

But energy is not conserved because

$$\frac{dE}{dt} = \frac{dH}{dt} = \frac{-\partial L}{\partial t} \neq 0 \quad (\because r = r(t))$$

Problem 3   We use Einstein notation

(a) $$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial g}{\partial p_k}\frac{dp_k}{dt}$$

$$= \frac{\partial g}{\partial t} + \frac{\partial g}{\partial q_k}\left(\frac{\partial H}{\partial p_k}\right) + \frac{\partial g}{\partial p_k}\left(\frac{-\partial H}{\partial q_k}\right).$$

$$\left(\because \frac{\partial H}{\partial p_k} = \frac{dq_k}{dt},\ \frac{-\partial H}{\partial q_k} = \frac{dp_k}{dt}\right)$$

$$= \frac{\partial g}{\partial t} + [g, H]$$

(b) $$\frac{dq_j}{dt} = \frac{\partial q_j}{\partial t} + [q_j, H] = [q_j, H]$$

$(\because (a))$

$$\frac{dp_j}{dt} = \frac{\partial p_j}{\partial t} + [p_j, H] = [p_j, H]$$

(c) $$[p_k, p_j] = \frac{\partial p_k}{\partial q_i}\frac{\partial p_j}{\partial p_i} - \frac{\partial p_k}{\partial p_i}\frac{\partial p_j}{\partial q_i} = 0$$

$$\left(\because \frac{\partial p_a}{\partial q_b} = 0,\ \forall a, b\right)$$

$$[q_k, q_j] = \frac{\partial q_k}{\partial q_i}\frac{\partial q_j}{\partial p_i} - \frac{\partial q_k}{\partial p_i}\frac{\partial q_j}{\partial q_i} = 0$$

$$\left(\because \frac{\partial q_a}{\partial p_b} = 0,\ \forall a, b\right)$$

(d) $$[q_k, p_j] = \frac{\partial q_k}{\partial q_i}\frac{\partial p_j}{\partial p_i} - \frac{\partial q_k}{\partial p_i}\frac{\partial p_j}{\partial q_i}$$

$$= \delta_{ik}\delta_{ij} - 0 = \delta_{kj}$$

(e)

If a quantity Q is

(i) Not explicitly dep. on t,

$$\frac{\partial Q}{\partial t} = 0$$

(ii) Commute with H,

$$[Q, H] = 0$$

Then

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + [Q, H] = 0$$

$\Rightarrow$ Const. of motion

Problem 4.

(a)

In cylindrical coordinates,

$$\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}, \quad \vec{\mu} = \mu\hat{z}$$

$$\vec{r} = \rho\,\hat{\rho}(\phi) + z\hat{z}$$

$$\therefore L = \frac{1}{2}mv^2 + \frac{q\vec{V}\cdot(\vec{\mu}\times\vec{r})}{r^3}$$

$$= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + \frac{q(\dot{\rho}, \rho\dot{\phi}, \dot{z})\cdot\mu\rho(0,1,0)}{(\rho^2+z^2)^{\frac{3}{2}}}$$

$$= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + q\mu\frac{\rho^2\dot{\phi}}{\left(\sqrt{\rho^2+z^2}\right)^3}$$

(b) $$p_\rho = \frac{\partial L}{\partial\dot{\rho}} = m\dot{\rho}$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = m\rho^2\dot{\phi} + q\mu\frac{\rho^2}{r^3} \quad \left(r := \sqrt{\rho^2+z^2}\right)$$

$$p_z = \frac{\partial L}{\partial\dot{z}} = m\dot{z}$$

$$\therefore H = \sum_i p_i v_i - L = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) = T$$

$$= \frac{p_\rho^2}{2m} + \frac{p_z^2}{2m} + \frac{\left(p_\phi - q\mu\frac{\rho^2}{r^3}\right)^2}{2m\rho^2}$$

(c) From (b), we can see that

$H = T$ (Kinetic Energy)

Since the magnetic field does no work to a moving charged particle, there is no explicit potential energy V.

$\therefore$ E (energy) $= T = H$

And since $\frac{dH}{dt} = \frac{-\partial L}{\partial t} = 0$

$\therefore$ $\frac{dE}{dt} = \frac{dH}{dt} = 0$, $E = H$ are conserved.

(d) In L, we see that

① $L = \frac{m}{2} \cdot (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) + q\mu \frac{\rho^2 \dot{\phi}}{r^3}$

$\Rightarrow \frac{\partial L}{\partial \phi} = 0$

② And $H = \frac{P_\rho^2}{2m} + \frac{P_z^2}{2m} + \frac{\left(P_\phi - q\mu \frac{\rho^2}{r^3}\right)^2}{2m\rho^2}$

$\frac{\partial H}{\partial \phi} = 0$

①, ② $\Rightarrow$ ϕ is a cyclic function ($\because$ L, H do not depend on ϕ explicitly)

$\therefore$ Conserved quantity $l = \frac{\partial L}{\partial \dot{\phi}} = P_\phi$ $\left(\because \frac{\partial L}{\partial \phi} = \frac{d}{dt} P_\phi = 0\right)$

$= m\rho^2 \dot{\phi} + q\mu \frac{\rho^2}{r^3}$

(e)

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

$$\frac{dp_z}{dt} = \frac{-\partial H}{\partial z} = \frac{\left(p_\phi - q_\mu \frac{\rho^2}{r^3}\right)}{m\rho^2} \cdot \frac{q_\mu \rho^2 \cdot (-3)}{r^4} \quad \frac{z}{r}$$

$$= \left(p_\phi - q_\mu \frac{\rho^2}{r^3}\right)\left(\frac{-3q_\mu}{m}\right) \cdot \frac{z}{r^5}$$

$$= \left(l - q_\mu \frac{\rho^2}{r^3}\right)\left(\frac{-3q_\mu}{m}\right) \frac{z}{r^5}$$

$$\frac{d\rho}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m}$$

$$\frac{dp_\rho}{dt} = \frac{-\partial H}{\partial \rho} = \frac{\left(l - q_\mu \frac{\rho^2}{r^3}\right)^2}{m\rho^3}$$

$$+ \frac{\left(l - q_\mu \frac{\rho^2}{r^3}\right)}{m\rho^2} q_\mu \cdot \left(\frac{2r^3\rho - 3r^2 \cdot \frac{\rho}{r} \rho^2}{r^6}\right)$$

$$= \frac{\left(l - q_\mu \frac{\rho^2}{r^3}\right)^2}{m\rho^3} + \left(l - q_\mu \frac{\rho^2}{r^3}\right) \frac{q_\mu}{m\rho^2} \cdot \left(\frac{2\rho}{r^3} - \frac{3\rho^3}{r^5}\right)$$

$$= \frac{1}{m\rho^2}\left(l - q_\mu \frac{\rho^2}{r^3}\right)\left[\frac{l}{\rho} - q_\mu \frac{\rho}{r^3} + q_\mu \left(\frac{2\rho}{r^3} - \frac{3\rho^3}{r^5}\right)\right]$$

$$= \frac{1}{m\rho^2}\left(l - q_\mu \frac{\rho^2}{r^3}\right)\left[\frac{l}{\rho} + \frac{q_\mu \rho}{r^3}\left(1 - \frac{3\rho^2}{r^2}\right)\right]$$

(f)

If $z = 0 \ \forall t$

Then

$$H = \frac{p_\rho^2}{2m} + \frac{p_z^2}{2m} + \frac{\left(p_\phi - q\mu \frac{\rho^2}{r^3}\right)^2}{2m\rho^2}$$

$$= \frac{p_\rho^2}{2m} + 0 + \frac{1}{2m\rho^2}\left(l - \frac{q\mu}{\rho}\right)^2$$

$(\because p_z \propto z)$

$$= \frac{p_\rho^2}{2m} + V_{eff}(\rho)$$

Where

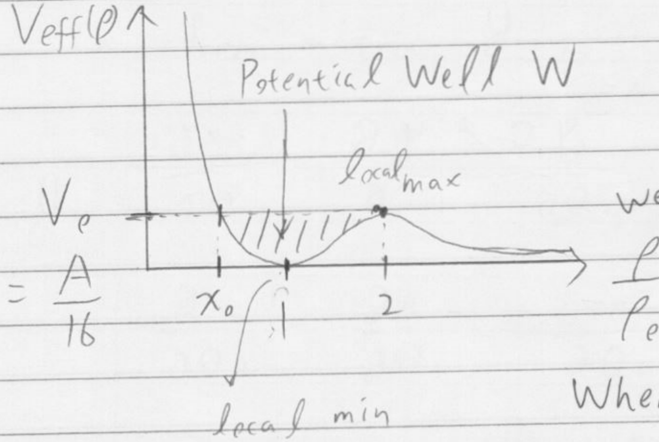
$$V_{eff}(\rho) = \frac{\left(l - \frac{q\mu}{\rho}\right)^2}{2m\rho^2}$$

Meanwhile, since $\frac{p_\rho}{m} = \frac{d\rho}{dt}$

$\Rightarrow$ The above Hamiltonian is describing equivalently a one-dim motion on ρ, with

$V = V_{eff}(\rho)$

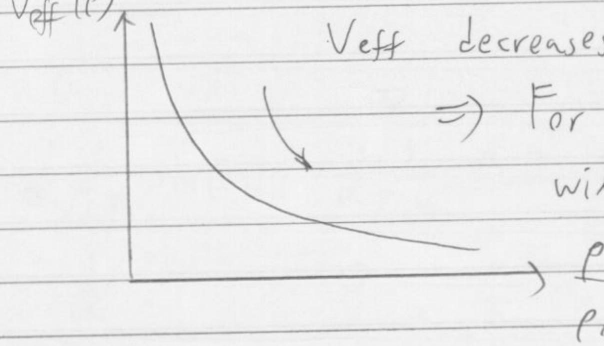
(g) ① $l > 0$ $V_{eff}(\rho) = A \cdot \left(\frac{1}{\rho} - \frac{\rho_0}{\rho^2}\right)^2$



When $V_0 > E > 0$, Orbit is within potential well $x_0 < \frac{\rho}{\rho_0} < 2$.

When $E > V_0$, orbit will go outside the potential well W, but will never touch the center $\rho = 0$ ($\because$ Strong repulsion)

② $l < 0$, $V_{eff}(\rho) = A \cdot \left(\frac{1}{\rho} + \frac{\rho_0}{\rho^2}\right)^2$



Veff decreases monotonically with ρ

$\Rightarrow$ For negative l, the charged point will be repulsed.

The result of ② can be explained as if the rotation direction of the charged particle is against the magnetic dipole direction, and then the rotation around the magnetic dipole will become repulsive and will be repulsed away. $\Rightarrow$ Pauli exclusion principle.