Problem 1

(a) $L = \frac{1}{2} m |\dot{\vec{r}}|^2 + \frac{1}{4} m \Omega^2 \left[|\vec{r}|^2 - 3(\vec{r}\cdot\hat{e}_z)^2 \right] - \frac{1}{2} m \dot{\vec{r}} \cdot (\vec{\omega} \times \vec{r})$

$\vec{r} = \rho\,\hat{\rho}(\phi) + z\,\hat{z}$

$\dot{\vec{r}} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}$

$\therefore |\dot{\vec{r}}|^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2, \quad |\vec{r}|^2 = \rho^2 + z^2$

$\vec{r}\cdot\hat{e}_z = z$

$\dot{\vec{r}}\cdot(\vec{\omega}\times\vec{r}) = \vec{\omega}\cdot(\vec{r}\times\dot{\vec{r}}) = \frac{eB}{mc}\hat{z}\cdot(\rho^2\dot{\phi}\,\hat{z})$

$= \frac{eB}{mc}\rho^2\dot{\phi}$

$\therefore L = \frac{1}{2} m (\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \frac{1}{4} m\Omega^2 (\rho^2 - 2z^2)$

$- \frac{eB}{2c}\rho^2\dot{\phi}$

(b) $p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi} - \frac{eB}{2c}\rho^2, \quad p_z = m\dot{z}$

$H = \sum_i p_i v_i - L$

$= \frac{1}{2} m (\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - \frac{1}{4} m\Omega^2 (\rho^2 - 2z^2)$

$= \frac{p_\rho^2}{2m} + \frac{p_z^2}{2m} + \frac{\left(p_\phi + \frac{eB}{2c}\rho^2\right)^2}{2m\rho^2} - \frac{1}{4} m\Omega^2 (\rho^2 - 2z^2)$

$= \frac{p_\rho^2}{2m} + \frac{p_\phi^2}{2m\rho^2} + \frac{eB}{2mc} p_\phi + \left(\frac{e^2B^2}{8mc^2} - \frac{1}{4} m\Omega^2\right)\rho^2$

$+ \frac{p_z^2}{2m} + \frac{m\Omega^2}{2} z^2$

(c) $\frac{\partial H}{\partial P_\phi} = \frac{d\phi}{dt}$ $\quad -\frac{\partial H}{\partial \phi} = \frac{dP_\phi}{dt}$

$\Rightarrow \frac{d\phi}{dt} = \frac{P_\phi}{m\rho^2} + \frac{eB}{2mc}$ $\quad \Rightarrow \frac{dP_\phi}{dt} = 0$ — (c.2)

— (c.1)

$\frac{\partial H}{\partial P_\rho} = \frac{d\rho}{dt} \Rightarrow \frac{d\rho}{dt} = \frac{P_\rho}{m}$ — (c.3)

$\frac{\partial H}{\partial \rho} = -\frac{dP_\rho}{dt} \Rightarrow \frac{dP_\rho}{dt} = \frac{P_\phi^2}{m\rho^3} - \left(\frac{e^2B^2}{4mc^2} - \frac{1}{2}m\Omega^2\right)\rho$ — (c.4)

$$\begin{cases} \frac{\partial H}{\partial P_z} = \frac{dz}{dt} \Rightarrow \frac{dz}{dt} = \frac{P_z}{m} & \text{— (c.5)} \\ \frac{\partial H}{\partial z} = -\frac{dP_z}{dt} \Rightarrow \frac{dP_z}{dt} = -m\Omega^2 z & \text{— (c.6)} \end{cases}$$

$\Rightarrow$ z is a simple harmonic oscillator with Ω being the frequency.

(d) In the formula of H, we can see that

H is not explicitly dependent on ϕ

$\rightarrow$ A cyclic variable

$\therefore$ By (c.2), we know P_ϕ is a conserved quantity

Make $P_\phi = l$

$\therefore \frac{d\phi}{dt} = \frac{l}{m\rho^2} + \frac{eB}{2mc}$

(e) From (C.3) & (C.4), we have

$$\frac{d^2\rho}{dt^2} = \frac{1}{m}\frac{dP_\rho}{dt} = \frac{P_\phi^2}{m^2\rho^3} - \left[\left(\frac{eB}{2mc}\right)^2 - \frac{1}{2}\Omega^2\right]\rho$$

$$= \frac{l^2}{m^2}\frac{1}{\rho^3} + \left[\frac{1}{2}\Omega^2 - \left(\frac{eB}{2mc}\right)^2\right]\rho$$

$\Rightarrow$ Similar to E.M in an effective potential of ρ $V_{eff} = V_{eff}(\rho)$, and being applied with some conservative force $F_{eff} = \frac{l^2}{m}\frac{1}{\rho^3} + \left[\frac{m}{2}\Omega^2 - \left(\frac{eB}{2mc}\right)^2 m\right]\rho$

so that $F_{eff} = m\frac{d^2\rho}{dt^2}$

We can look at the Hamiltonian to see more clearly:

$$H_{\rho\phi} = \frac{P_\rho^2}{2m} + \frac{P_\phi^2}{2m\rho^2} + \frac{eB}{2mc}P_\phi + \left(\frac{e^2B^2}{8mc^2} - \frac{1}{4}m\Omega^2\right)\rho^2$$

(Here $H_{\rho\phi}$ is the Hamiltonian containing only the ρ, ϕ parts because the z contribution is merely an independent S.H.O. Hamiltonian, and thus can be removed)

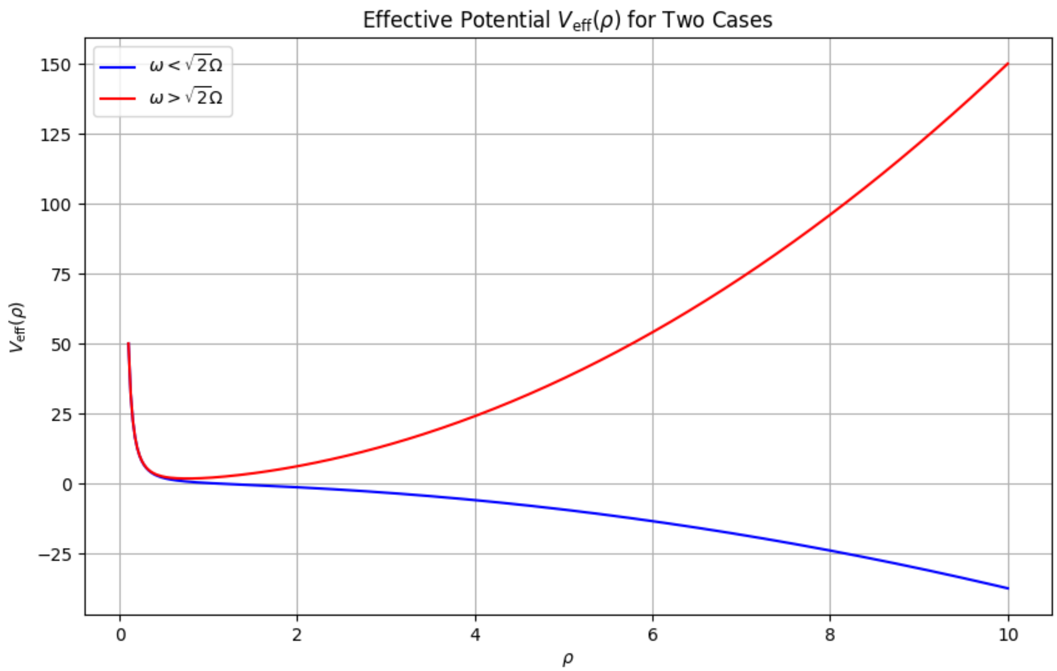
$$= \frac{P_\rho^2}{2m} + \underbrace{\frac{l^2}{2m\rho^2} + \left(\frac{e^2B^2}{8mc^2} - \frac{1}{4}m\Omega^2\right)\rho^2}_{= V_{eff}(\rho)} + \underbrace{\frac{eBl}{2mc}}_{\text{Const. can be neglected}}$$

$$= \frac{l^2}{2m\rho^2} + \frac{m}{4}\left[W^2 - 2\Omega^2\right]\rho^2$$

We can see clearly, if $W < \sqrt{2}\,\Omega$, then the ρ^2 term will become negative coefficient.

$\rightarrow$ Trap will not work.

Problem 1 Plotting



We can see clearly that in the case of $\omega > \sqrt{2}\,\Omega$, there is a local minimum in the effective potential, so it can form an efficient trap.

On the other hand, for $\omega < \sqrt{2}\,\Omega$, there is not a local minimum in the effective potential, so it cannot form an efficient trap.

$w := (Q_1, Q_2, P_1, P_2)$

$Q := (x, y, p_x, p_y)$

Problem 2

(a)

The transform matrix $\frac{\partial \Omega_b}{\partial w_a} = \mathcal{J}$ is

$$\mathcal{J} = \begin{bmatrix} \frac{1}{\alpha}\sqrt{2P_1}\cos Q_1 & 0 & \frac{1}{\alpha}\frac{1}{\sqrt{2P_1}}\sin Q_1 & \frac{1}{\alpha} \\ \frac{-1}{\alpha}\sqrt{2P_1}\sin Q_1 & \frac{1}{\alpha} & \frac{1}{\alpha}\frac{1}{\sqrt{2P_1}}\cos Q_1 & 0 \\ \frac{-\alpha}{2}\sqrt{2P_1}\sin Q_1 & \frac{-\alpha}{2} & \frac{\alpha}{2}\frac{1}{\sqrt{2P_1}}\cos Q_1 & 0 \\ \frac{-\alpha}{2}\sqrt{2P_1}\cos Q_1 & 0 & -\frac{\alpha}{2}\frac{1}{\sqrt{2P_1}}\sin Q_1 & +\frac{\alpha}{2} \end{bmatrix}$$

$$\mathcal{J} J = \begin{bmatrix} \frac{-1}{\alpha}\frac{1}{\sqrt{2P_1}}\sin Q_1 & \frac{-1}{\alpha} & \frac{1}{\alpha}\sqrt{2P_1}\cos Q_1 & 0 \\ \frac{-1}{\alpha}\frac{1}{\sqrt{2P_1}}\cos Q_1 & 0 & \frac{-1}{\alpha}\sqrt{2P_1}\sin Q_1 & \frac{1}{\alpha} \\ \frac{-\alpha}{2}\frac{1}{\sqrt{2P_1}}\cos Q_1 & 0 & \frac{-\alpha}{2}\sqrt{2P_1}\sin Q_1 & -\frac{\alpha}{2} \\ \frac{\alpha}{2}\frac{1}{\sqrt{2P_1}}\sin Q_1 & \frac{-\alpha}{2} & \frac{-\alpha}{2}\sqrt{2P_1}\cos Q_1 & 0 \end{bmatrix}$$

$$\mathcal{J} J \mathcal{J}^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = J$$

$\Rightarrow$ Symplectic Transform

$\rightarrow$ is a canonical transform

(b) $\vec{B} = 0$

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2$$

$$\Rightarrow H = \frac{\alpha^2}{8m}\left(2P_1 + Q_2^2 + P_2^2 - 2\sqrt{2P_1}\cos Q_1 Q_2 - 2\sqrt{2P_1}\sin Q_1 P_2\right)$$
$$+ \frac{m\omega^2}{2\alpha^2}\left(2P_1 + Q_2^2 + P_2^2 + 2\sqrt{2P_1}\cos Q_1 Q_2 + 2\sqrt{2P_1}\sin Q_1 P_2\right)$$

Make $\frac{\alpha^2}{8m} = \frac{m\omega^2}{2\alpha^2}$

$$\Leftrightarrow \alpha^4 = 4m^2\omega^2 \Leftrightarrow \alpha = \sqrt{2m\omega}$$

$$\therefore H = \frac{\omega}{2}\left[P_1 + \left(\frac{P_2^2 + Q_2^2}{2}\right)\right] \cdot 2$$
$$= \omega\left(P_1 + \frac{P_2^2 + Q_2^2}{2}\right)$$

For Q_1, P_1,

$$\frac{\partial H}{\partial P_1} = \frac{dQ_1}{dt} = \omega, \quad \frac{\partial H}{\partial Q_1} = -\frac{dP_1}{dt} = 0$$

$\Rightarrow Q_1 = \omega t + \phi_0$, $P_1 = P_{1_0}$ $(= P_1(t=0))$

The Q_1 and P_1 are just describing a constant frequency $(=\omega)$ circular motion. (radius $= \sqrt{2P_1}$)

For Q_2, P_2, we see a S.H.O. Hamiltonian form, independent of Q_1 & P_1

$$\begin{cases} \frac{\partial H}{\partial P_2} = \frac{dQ_2}{dt} = \omega P_2 \\ \frac{\partial H}{\partial Q_2} = -\frac{dP_2}{dt} = \omega Q_2 \end{cases} \Rightarrow \begin{cases} Q_2 = A \cdot \sin(\omega t + \psi_0) \\ P_2 = A \cos(\omega t + \psi_0) \end{cases}$$

$\rightarrow$ S.H.O motion with freq $= \omega$

(c) $\omega = 0$

$$\therefore H = \frac{(\vec{p} - q\vec{A})^2}{2m}$$

$$= \left[\vec{p} - \frac{q}{2}(\vec{B}\times\vec{r})\right]\cdot\left[\vec{p} - \frac{q}{2}(\vec{B}\times\vec{r})\right]\frac{1}{2m}$$

$$= \frac{p^2}{2m} - \frac{q\vec{B}}{2m}\cdot(\vec{r}\times\vec{p}) + \frac{q^2}{8m}\left[B^2r^2 - (\vec{B}\cdot\vec{r})^2\right]$$

Since motion of particle confined to $z=0$, we can only consider B_z, $\vec{B}\cdot\vec{r} = 0$, $\vec{B} = B_z\hat{z}$

$$H = \frac{p_x^2 + p_y^2}{2m} - \frac{qB_z}{2m}(xp_y - yp_x) + \frac{q^2B_z^2}{8m}(x^2+y^2)$$

$$= \frac{1}{2m}\left(p_x + \frac{qB_z}{2}y\right)^2 + \left(p_y - \frac{qB_z}{2}x\right)^2\cdot\frac{1}{2m}$$

$$= \frac{1}{2m}\left[\frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right) + \frac{qB_z}{2\alpha}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right)\right]^2$$

$$+ \frac{1}{2m}\left[\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right) + \frac{qB_z}{2\alpha}\left(\sqrt{2P_1}\sin Q_1 + P_2\right)\right]^2$$

Make $\frac{\alpha}{2} = \frac{qB_z}{2\alpha} \Leftrightarrow \alpha = \sqrt{qB_z}$

$$H = \frac{1}{2m}\left[\frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right) + \frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right)\right]^2$$

$$+ \frac{1}{2m}\left[\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right) + \frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 + P_2\right)\right]^2$$

$$= \frac{\alpha^2}{m}P_1$$

So as a result, P_2, Q_2 and Q_1 are cyclic variables for H.

$$\frac{dQ_2}{dt} = [Q_2, H] + \frac{\partial Q_2}{\partial t} = 0 \Rightarrow P_2, Q_2 \text{ are just some constants of motion}$$

$$\frac{dP_2}{dt} = [P_2, H] + \frac{\partial P_2}{\partial t} = 0 \rightarrow \text{Displacement from the origin on } x, y\text{-plane}$$

For Q_1, P_1

$$\frac{dQ_1}{dt} = \frac{\partial H}{\partial P_1} = \frac{\alpha^2}{m} = \frac{qB_z}{m} := W_c \text{ (Cyclotron Freq.)}$$

$$\therefore Q_1 = W_c t + \phi_0$$

$$\frac{dP_1}{dt} = \frac{\partial H}{\partial Q_1} = 0, \quad P_1 \text{ is a constant}$$

$\therefore$ Q_1, P_1 are describing a circular motion

(radius $= \sqrt{2P_1}$, freq $= W_c$)

Problem 2

(d) From (b), (c) we know the full Hamiltonian is

$$H = \frac{1}{2m}\left[\left(p_x + \frac{qB_z}{2} y\right)^2 + \left(p_y - \frac{qB_z}{2} x\right)^2\right] + \frac{1}{2} m\omega^2 (x^2+y^2)$$

$$= \frac{1}{2m}\left[\frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right) + \frac{qB_z}{2\alpha}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right)\right]^2$$

$$+ \frac{1}{2m}\left[\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right) + \frac{qB_z}{2\alpha}\left(\sqrt{2P_1}\sin Q_1 + P_2\right)\right]^2$$

$$+ \frac{m\omega^2}{2\alpha^2}\left[\left(2P_1 + Q_2^2 + P_2^2\right) + 2\sqrt{2P_1}\left(Q_2\cos Q_1 + P_2 \sin Q_1\right)\right]$$

$$= \frac{1}{2m}\left[\left(\frac{\alpha^2}{4} + \frac{q^2B_z^2}{4\alpha^2}\right)\left(2P_1 + Q_2^2 + P_2^2\right) + \left(\frac{q^2B_z^2}{4\alpha^2} - \frac{\alpha^2}{4}\right)\left(2\sqrt{2P_1}\right)\left(Q_2\cos Q_1 + P_2\sin Q_1\right) + \left(\frac{qB_z}{2}\right)\left(2P_1 - Q_2^2 - P_2^2\right)\right]$$

$$+ \left(\frac{m\omega^2}{2\alpha^2}\right)\left[\left(2P_1 + Q_2^2 + P_2^2\right) + \left(2\sqrt{2P_1}\right)\left(Q_2\cos Q_1 + P_2\sin Q_1\right)\right]$$

$$= \left[\frac{1}{2m}\left(\frac{\alpha^2}{4} + \frac{q^2B_z^2}{4\alpha^2}\right) + \left(\frac{m\omega^2}{2\alpha^2}\right)\right]\underline{X}$$

$$+ \frac{1}{2m}\left(\frac{qB_z}{2}\right)\underline{Y}$$

$$+ \left[\frac{1}{2m}\left(\frac{q^2B_z^2}{4\alpha^2} - \frac{\alpha^2}{4}\right) + \left(\frac{m\omega^2}{2\alpha^2}\right)\right]\underline{U}$$

$$\left(\begin{aligned} \underline{X} &:= 2P_1 + Q_2^2 + P_2^2 \\ \underline{Y} &:= 2P_1 - Q_2^2 - P_2^2 \\ \underline{U} &:= \left(2\sqrt{2P_1}\right)\cdot\left(Q_2\cos Q_1 + P_2\sin Q_1\right) \end{aligned}\right)$$

So in the formula of $H = H(\underline{X}, \underline{Y}, \underline{U})$,

the most "evil" term we want to eliminate is $\underline{U}$. Hence, we choose α to make the coefficient before $\underline{U}$ to be 0, that is

$$0 = \frac{1}{2m}\left(\frac{q^2 B_z^2}{4\alpha^2} - \frac{\alpha^2}{4}\right) + \frac{m\omega^2}{2\alpha^2} = \left(\frac{q^2 B_z^2}{8m} + \frac{m\omega^2}{2}\right)\frac{1}{\alpha^2} - \frac{\alpha^2}{8m}$$

$$\Rightarrow \alpha^2 = \sqrt{q^2 B_z^2 + 4m^2\omega^2}, \quad \left(\frac{q^2 B_z^2}{8m} + \frac{m\omega^2}{2}\right)\frac{1}{\alpha^2} = \frac{\alpha^2}{8m}$$

$\therefore$ Then H is:

$$H = \frac{\alpha^2}{4m}\underline{X} + \frac{qB_z}{4m}\underline{Y}$$

$$= \frac{\alpha^2}{4m}(2P_1 + Q_2^2 + P_2^2) + \frac{qB_z}{4m}(2P_1 - Q_2^2 - P_2^2)$$

$$= \left(\frac{\alpha^2 + qB_z}{2m}\right)P_1 + \left(\frac{\alpha^2 - qB_z}{4m}\right)(Q_2^2 + P_2^2)$$

$$:= A_+ P_1 + \frac{A_-}{2}(Q_2^2 + P_2^2), \quad A_\pm := \frac{\alpha^2 \pm qB_z}{2m}$$

From discussion of (b), we know that

① Q_1, P_1 are describing a circular motion

radius $= \sqrt{2P_1}$

freq. $= A_+$ $\quad (B_z \to 0,\ A_+ \to \omega$

$\qquad \omega \to 0,\ A_+ \to \omega_c = \frac{qB_z}{m})$

② Q_2, P_2 are describing an S.H.O. motion

freq. $= A_-$ $\quad (B_z \to 0,\ A_- \to \omega$

$\qquad \omega \to 0,\ A_- \to 0)$

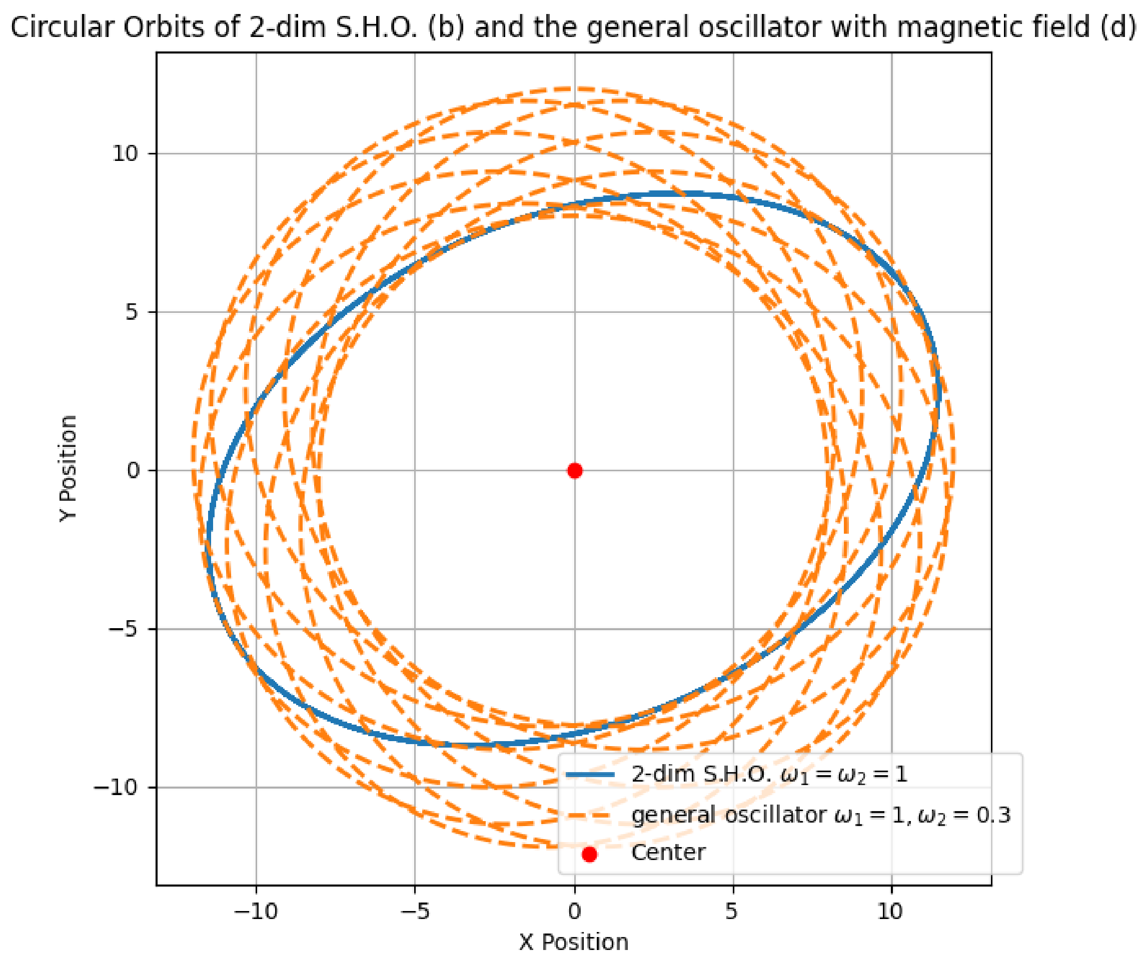
Thus, we can see that in general, motion in (d) is also the addition of a circular motion and an S.H.O. motion, but their frequencies are not necessarily the same ($\because A_+ \neq A_-$ in general) unlike in (b) they are all ω.

Problem 2 Simulation

Here, we drew the orbits for (b) and (d) for comparison.

We can see that in (b), since the frequency for the circular motion and the S.H.O. motion are the same, they will form a clos orbit, and the orbit shape is in general an oval.

However, in (d), the frequency for the circular motion (ω_1) and the frequency for the S.H.O motion (ω_2) are in general different, it will not form a close orbit, and will form a complicated pattern as follows.



Problem 3

$$H = \frac{p_x^2}{2m} + \frac{k}{x^2}$$

Since $\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \Rightarrow H = E$ is conserved

$$H = H(t=0) \Rightarrow \frac{p_x^2}{2m} + \frac{k}{x^2} = \frac{k}{x_0^2} \quad (\because p_x = m\frac{dx}{dt}, \text{ at } t=0, v=0)$$

$$\Rightarrow p_x = \sqrt{2mk\left(\frac{1}{x_0^2} - \frac{1}{x^2}\right)}$$

Then $y \equiv x^2$,

$$\frac{dy}{dt} = [y, H] + \frac{\partial y}{\partial t} = [y, H] = [y, \frac{p_x^2}{2m} + \frac{k}{x^2}]$$

$$= [y, \frac{p_x^2}{2m}] \quad (\because [y, \frac{k}{x^2}] = [x^2, \frac{k}{x^2}] = 0)$$

$$= \frac{1}{2m}[x^2, p_x^2] = \frac{1}{2m}[(2x)\cdot(2p_x) - 0] = \frac{2xp_x}{m}$$

$$= \frac{2\sqrt{2mk}}{m}\sqrt{\left(\frac{x}{x_0}\right)^2 - 1} = \sqrt{\frac{8k}{m}}\sqrt{\left(\frac{y}{y_0}\right) - 1}$$

$$\Rightarrow \int \frac{dy}{\sqrt{\left(\frac{y}{y_0}\right) - 1}} = \sqrt{\frac{8k}{m}}\int dt$$, and apply initial condition, $t=0, y = y_0 := x_0^2$

$$\Rightarrow 2y_0\sqrt{\left(\frac{y}{y_0}\right) - 1} = \sqrt{\frac{8k}{m}}\, t$$

$$\Rightarrow y = y_0\left[1 + \left(\frac{2k}{my_0^2}t^2\right)\right]$$

$$\Rightarrow x = \sqrt{y} = x_0\sqrt{1 + \left(\frac{2k}{mx_0^4}t^2\right)}$$

Problem 4

(a)

$$Q = p + iaq$$

$$P = \frac{p - iaq}{2ia} = \frac{1}{2ia}p - \frac{1}{2}q$$

The transform matrix $\mathcal{J}$ is:

$$\mathcal{J} = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix} = \begin{bmatrix} ia & 1 \\ \frac{-1}{2} & \frac{1}{2ia} \end{bmatrix}$$

$$\mathcal{J} J \mathcal{J}^T = \begin{bmatrix} ia & 1 \\ \frac{-1}{2} & \frac{1}{2ia} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} ia & \frac{-1}{2} \\ 1 & \frac{1}{2ia} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & ia \\ \frac{1}{2a} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} ia & \frac{-1}{2} \\ 1 & \frac{1}{2ia} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = J$$

$\therefore$ $(p, q) \to (P, Q)$ is a canonical transform

(b)

If we look at the product of P and Q, we can find that

$$PQ = (p + iaq)\left(\frac{p}{2ia} - \frac{1}{2}q\right)$$
$$= \frac{1}{2ia}p^2 - \frac{ia}{2}q^2$$
$$= -i\left(\frac{1}{2a}p^2 + \frac{a}{2}q^2\right)$$
$$\therefore iPQ = \frac{1}{2a}(p^2 + a^2q^2)$$

Recall that the Hamiltonian for S.H.O. is

$$H_{SHO} = \frac{1}{2m}p^2 + \frac{m}{2}\omega^2 q^2 = \frac{1}{2m}\left[p^2 + (m\omega)^2 q^2\right]$$

$\therefore$ It's natural to make $a = m\omega$

$$\therefore iPQ = \frac{1}{2m\omega}\left(p^2 + (m\omega)^2 q^2\right) = \frac{H_{SHO}}{\omega}$$

$\therefore H_{SHO} = i\omega PQ$, $a \equiv m\omega$

$\therefore$ The Hamiltonian E.Ms are:

$$\frac{\partial H}{\partial Q} = -\frac{dP}{dt}, \quad \frac{\partial H}{\partial P} = \frac{dQ}{dt}$$

$$\Rightarrow \begin{cases} \frac{dP}{dt} = -i\omega P & \Rightarrow P(t) = P_0 e^{-i\omega t} \\ \frac{dQ}{dt} = i\omega Q & \quad\; Q(t) = Q_0 e^{+i\omega t} \end{cases}$$

(c) Continuing (b), we can solve $q(t)$ by P, Q

$$q(t) = \frac{1}{2ia}\left(P - 2ia\, Q\right)$$

$$= \frac{p(t)}{2ia} - Q(t)$$

$$= \frac{P_0}{2ia} e^{-i\omega t} - Q_0 e^{+i\omega t}$$

(d) Clearly, P, Q correspond to the annihilation and creation operators in the QM S.H.O.

In QM,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2$$

$$= \frac{1}{2m}\left(\hat{p}^2 + \alpha \hat{q}^2\right) \quad (\alpha := m\omega)$$

$$= \frac{1}{2m}\left[(\hat{p} + i\alpha\hat{q})(\hat{p} - i\alpha\hat{q}) - i\alpha[\hat{q}, \hat{p}]\right]$$

$$= \hbar\omega\left[\left(\frac{\hat{p}}{\sqrt{2m\omega\hbar}} + i\frac{\alpha}{\sqrt{2m\omega\hbar}}\hat{q}\right)\left(\frac{\hat{p}}{\sqrt{2m\omega\hbar}} - \frac{i\alpha}{\sqrt{2m\omega\hbar}}\hat{q}\right) - \frac{i\alpha}{2m\hbar\omega}[\hat{q}, \hat{p}]\right]$$

$$= \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$$

$$\hat{a}^\dagger := (\hat{p} + i\alpha\hat{q})\frac{1}{\sqrt{2m\omega\hbar}}, \quad \hat{a} := (\hat{p} - i\alpha\hat{q})\frac{1}{\sqrt{2m\omega\hbar}}$$

$\Downarrow$ similar to $\Downarrow$

P $\qquad\qquad$ Q

The only difference is the $\frac{1}{2}$ term, corresponding to the vacuum energy

$\Rightarrow$ A pure quantum phenomenon.