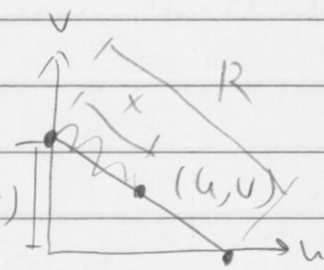
Problem 2.

(a) Since the two heavy beads have been already constrained by $h(t)$ and the rigid rod, the only geometric freedom happens at the third bead.

And because the third bead is constrained on the rod $\Rightarrow$ Only one degree of freedom x.

(b)
$$u = x\sin\theta$$
$$v = h - x\cos\theta$$

$$\begin{cases} \sin\theta = \dfrac{\sqrt{R^2-h^2}}{R} \\ \cos\theta = \dfrac{h}{R} \end{cases}$$



$$v = h - x\cdot\frac{h}{R} = h\left(1-\frac{x}{R}\right)$$

$$L = T - U$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\dot{\theta}^2 - mgv - \frac{1}{2}k(x-x_0^2)$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left(\frac{x}{R}\right)^2\cdot\frac{\dot{h}^2}{1-\left(\frac{h}{R}\right)^2} - mgh\left(1-\frac{x}{R}\right) - \frac{1}{2}k(x-x_0)^2$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\frac{\dot{h}^2}{R^2-h^2} - mgh\left(1-\frac{x}{R}\right) - \frac{1}{2}k(x-x_0)^2$$

(c) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$\Rightarrow \quad m\ddot{x} - m x \frac{\dot{h}^2}{R^2-h^2} - \frac{mgh}{R} + k(x-x_0) = 0$

$\Rightarrow \quad m\ddot{x} + \left(k - m\frac{\dot{h}^2}{R^2-h^2}\right) x = k x_0 + \frac{mgh}{R}$

We assume $h = h(t)$ has only some small oscillation

$\overline{h} = \overline{h(t)} = h_0 + \Delta h \cos w_h t$

$\overline{\dot{h}} \approx \overline{\Delta h \cdot w}$

$\Rightarrow \quad m\ddot{x} + \left(k - m\frac{\overline{\Delta h}^2 w_h^2}{R^2-h_0^2}\right) x = \text{Const.}$

$\Rightarrow \quad x = \underline{x_0 \cos(w't + \phi)} + x_0 + \frac{mgh_0}{Rk}$

$w' = \sqrt{\frac{k}{m} - \frac{\overline{\Delta h}^2 \cdot w_h^2}{R^2-h_0^2}}$

(d) $H = \dot{x}\frac{\partial L}{\partial \dot{x}} - L$

$$= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}mx^2\frac{\dot{h}^2}{R^2-h^2} + mgh\left(1-\frac{x}{R}\right) + \frac{1}{2}k(x-x_0)^2$$

Since $h=h(t)$ is some explicit function of t,
$H \neq E$

Here energy $E = T + U$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\frac{\dot{h}^2}{R^2-h^2} + mgh\left(1-\frac{x}{R}\right) + \frac{1}{2}k(x-x_0)^2$$

$$\neq H$$

And $\frac{dH}{dt} = -\frac{\partial L}{\partial t} \neq 0 \Rightarrow H$ is not conserved.

E is not conserved, we can give a simple example.
If all the oscillations and velocities are small
$\Rightarrow T \ll V$, $V \approx mgh\left(1-\frac{x}{R}\right)$
$E \approx V \approx mgh\left(1-\frac{x}{R}\right)$
As $h=h(t)$ changes with time t, E will change with $h=h(t)$

(e) For $h = R\cos(\omega t)$

$$m\ddot{x} + \left(k - m\frac{\dot{h}^2}{R^2-h^2}\right)x = kx_0 + \frac{mgh}{R}$$

$$\Rightarrow m\ddot{x} + (k - m\omega^2)x = kx_0 + mg\cos(\omega t)$$

$$\Rightarrow \ddot{x} + \omega'^2 x = f_0 + g\cos(\omega t)$$

$$\left(f_0 = \frac{kx_0}{m}\right) \quad \left(\omega' = \sqrt{\frac{k}{m} - \omega^2} \approx \sqrt{\frac{k}{m}}\right)$$

$$\Rightarrow x = X_1 \cos(\omega' t + \phi) + \frac{f_0}{\omega'^2} + \frac{g}{\omega'^2 - \omega^2}\cos\omega t$$

(f) As $\omega \to \omega' \approx \sqrt{\frac{k}{m}}$, amplitude of $x(t)$ $\to \infty$, which is the resonance case.

The amplitude of $x := x_s = \left|\frac{g}{\omega'^2 - \omega^2}\right|$ (as $t \to \infty$)

$$x_s = x_s(\omega) = \left|\frac{g}{\omega'^2 - \omega^2}\right|$$