Problem 1

(a) $$p = \frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(-mc^2\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}\right)$$

$$= \frac{mc^2 \cdot \left(\frac{-1}{2}\right)}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} \cdot \left(-\frac{2\dot{x}}{c^2}\right) = \frac{m\dot{x}}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} = \gamma m \dot{x}$$

$$H = \dot{x}\frac{\partial L}{\partial \dot{x}} - L = \frac{m\dot{x}^2}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} + mc^2\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}$$

$$= \frac{mc^2}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} = \gamma mc^2 \quad \left(\gamma := \frac{1}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}}\right)$$

$\Rightarrow$ The same as special relativity result

(b) $\frac{\dot{x}}{c} \to 0$, $L \to -mc^2\left[1 - \frac{1}{2}\left(\frac{\dot{x}}{c}\right)^2\right]$

$$= -mc^2 + \frac{1}{2}m\dot{x}^2$$

$\Rightarrow$ Non-relativitistic Lagrangian

Also $H \to mc^2\left[1 + \frac{1}{2}\left(\frac{\dot{x}}{c}\right)^2\right]$

$$= \underbrace{mc^2}_{\text{Rest Mass energy}} + \underbrace{\frac{1}{2}m\dot{x}^2}_{\text{Non-rel Kinetic energy}}$$

$p \to m\dot{x}$ (Momentum of low vel)

(c)

$$L = T - U = -mc^2\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2} - \frac{1}{2}m\omega^2 x^2$$

$$\Rightarrow \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\Rightarrow -m\omega^2 x = \gamma m\ddot{x} + \underline{\gamma^3 m\dddot{x}\,\dot{x}/c}.$$

Negligible as $\frac{\dot{x}}{c} \to 0$

$$\Rightarrow \ddot{x} + \omega^2\sqrt{1-\left(\frac{\bar{V}}{c}\right)^2}\, x = 0$$

$\therefore$ New frequency

$$\omega' \approx \omega\left(1-\frac{\bar{V}^2}{c^2}\right)^{\frac{1}{4}} \approx \omega\left(1-\frac{\bar{V}^2}{4c^2}\right)$$

Note that the $\bar{V}$ here is some time-averaged velocity, not the maximum velocity V_{max}.

From numerical simulation, we found that $\bar{V} \approx \frac{1}{2}V_{max}$, this is understandable, as the velocity in one period varies with some sinusoidal function.

Problem 1.c

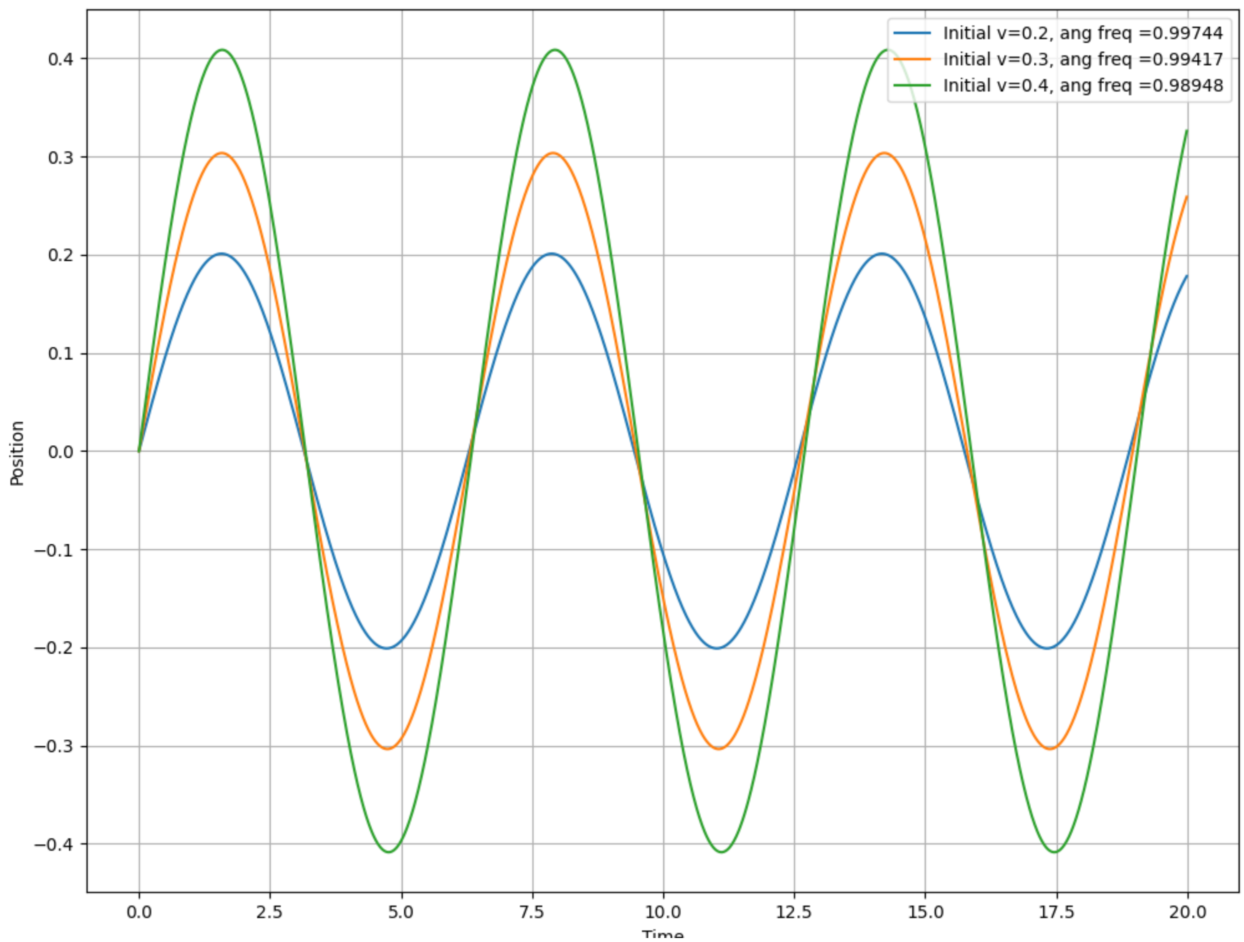
We set k = 1, m=1, and thus the low velocity limit angular frequency is , $\omega_0 = \sqrt{k/m} = 1$.

Now we set three cases of initial velocities to be $\frac{v_0}{c} = 0.2, 0.3, 0.4$.

We found that the effective angular velocities are equal to

$$\omega' = \omega_0 (1 - \frac{1}{4}\left(\frac{\bar{v}}{c}\right)^2),$$

And that the time averaged velocity $\bar{v} = 0.5 v_0$



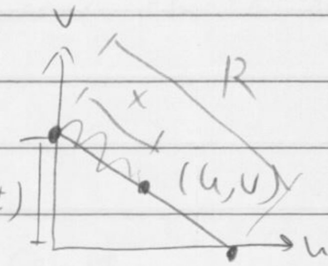
Problem 2.

(a) Since the two heavy beads have been already constrained by $h(t)$ and the rigid rod, the only geometric freedom happens at the third bead.

And because the third bead is constrained on the rod $\Rightarrow$ Only one degree of freedom x.

(b) $u = x\sin\theta$
$V = h - x\cos\theta$

$$\begin{cases} \sin\theta = \dfrac{\sqrt{R^2-h^2}}{R} \\ \cos\theta = \dfrac{h}{R} \end{cases}$$



$$V = h - x\cdot\frac{h}{R} = h\left(1-\frac{x}{R}\right)$$

$$L = T - U$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\dot{\theta}^2 - mgV - \frac{1}{2}k(x-x_0^2)$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left(\frac{x}{R}\right)^2\cdot\frac{\dot{h}^2}{1-\left(\frac{h}{R}\right)^2} - mgh\left(1-\frac{x}{R}\right) - \frac{1}{2}k(x-x_0)^2$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\frac{\dot{h}^2}{R^2-h^2} - mgh\left(1-\frac{x}{R}\right) - \frac{1}{2}k(x-x_0)^2$$

(c) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$$\Rightarrow \quad m\ddot{x} - mx\frac{\dot{h}^2}{R^2-h^2} - \frac{mgh}{R} + k(x-x_0) = 0$$

$$\Rightarrow \quad m\ddot{x} + \left(k - m\frac{\dot{h}^2}{R^2-h^2}\right)x = kx_0 + \frac{mgh}{R}$$

We assume $h = h(t)$ has only some small oscillation

$$\overline{h = h(t)} = h_0 + \Delta h \cos w_h t$$

$$\overline{\dot{h}} \approx \overline{\Delta h} \cdot w$$

$$\Rightarrow \quad m\ddot{x} + \left(k - m\frac{\overline{\Delta h}^2 w_h^2}{R^2-h_0^2}\right)x = \text{Const.}$$

$$\Rightarrow \quad x = \underline{x_0 \cos(w't+\phi)} + x_0 + \frac{mgh_0}{Rk}$$

$$w' = \sqrt{\frac{k}{m} - \frac{\overline{\Delta h}^2 \cdot w_h^2}{R^2-h_0^2}}$$

(d) $H = \dot{x}\dfrac{\partial L}{\partial \dot{x}} - L$

$$= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}mx^2\frac{h'^2}{R^2-h^2} + mgh\left(1-\frac{x}{R}\right) + \frac{1}{2}k(x-x_0)^2$$

Since $h = h(t)$ is some explicit function of t,
$H \neq E$

Here energy $E = T + U$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\frac{h'^2}{R^2-h^2} + mgh\left(1-\frac{x}{R}\right) + \frac{1}{2}k(x-x_0)^2$$

$$\neq H$$

And $\dfrac{dH}{dt} = -\dfrac{\partial L}{\partial t} \neq 0 \Rightarrow H$ is not conserved.

E is not conserved, we can give a simple example.
If all the oscillations and velocities are small
$\Rightarrow T \ll V, \quad V \approx mgh\left(1-\frac{x}{R}\right)$
$E \approx V \approx mgh\left(1-\frac{x}{R}\right)$
As $h = h(t)$ changes with time t, E will change with $h = h(t)$

(e) For $h = R\cos(\omega t)$

$$m\ddot{x} + \left(k - m\frac{\dot{h}^2}{R^2 - h^2}\right)x = kx_0 + \frac{mgh}{R}$$

$$\Rightarrow m\ddot{x} + (k - m\omega^2)x = kx_0 + mg\cos(\omega t)$$

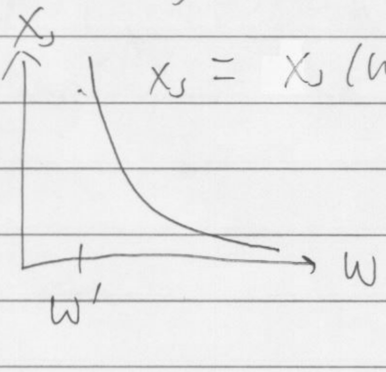
$$\Rightarrow \ddot{x} + \omega'^2 x = f_0 + g\cos(\omega t)$$

$$\left(f_0 = \frac{kx_0}{m}\right) \quad \left(\omega' = \sqrt{\frac{k}{m} - \omega^2} \approx \sqrt{\frac{k}{m}}\right)$$

$$\Rightarrow x = X_1 \cos(\omega' t + \phi) + \frac{f_0}{\omega'^2} + \frac{g}{\omega'^2 - \omega^2}\cos\omega t$$

(f) As $\omega \to \omega' \simeq \sqrt{\frac{k}{m}}$, amplitude of $x(t)$ $\to \infty$, which is the resonance case.

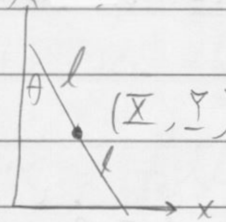
The amplitude of $x := X_s = \left|\frac{g}{\omega'^2 - \omega^2}\right|$ (as $t \to \infty$)



$$X_s = X_s(\omega) = \frac{g}{|\omega'^2 - \omega^2|}$$

Problem 3.

(a)



$$(\bar{X}, \bar{Y}) = (l\sin\theta, l\cos\theta)$$

$$L = T - V = \frac{1}{2}Ml^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 - Mg\bar{Y}$$

$$= Ml^2\left(\frac{1}{2} + \frac{\alpha}{6}\right)\dot{\theta}^2 - Mgl\cos\theta$$

E.M ↓

(b) $$H = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$$ $$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow Ml^2\left(1+\frac{\alpha}{3}\right)\ddot{\theta} - Mgl\sin\theta = 0.$$

$$= Ml^2\left(\frac{1}{2} + \frac{\alpha}{6}\right)\dot{\theta}^2 + Mgl\cos\theta = E$$

$$= Mgl$$

$$\Rightarrow \dot{\theta} = \sqrt{\frac{g(1-\cos\theta)}{l\left(\frac{1}{2}+\frac{\alpha}{6}\right)}} = \sqrt{\frac{2g}{l\left(\frac{1}{2}+\frac{\alpha}{6}\right)}}\sin\theta$$

(c) $$\int \frac{d\theta}{\sin\theta} = \int a\,dt \qquad := a\sin\theta$$

$$\Rightarrow \ln\left|\tan\left(\frac{\theta}{2}\right)\right| = at + C$$

$$\tan\left(\frac{\theta}{2}\right) = Ce^{at}, \quad \theta = 2\tan^{-1}\left(Ce^{At}\right)$$

(d) There are two geometrical constraints:

$$\begin{cases} \bar{x} = l\sin\theta \\ \bar{y} = l\cos\theta \end{cases}$$

$$\Rightarrow L = \frac{1}{2}M(\dot{\bar{x}}^2 + \dot{\bar{y}}^2) + \frac{1}{2}I\dot{\theta}^2$$

$$- Mgy + \underbrace{A}_{}(y - l\cos\theta) + \underbrace{B}_{d}(x - l\sin\theta)$$

Lagrangian multipliers

$$\Rightarrow \begin{cases} I\ddot{\theta} = A l \sin\theta - B l \cos\theta & (3.d.1) \\ M\ddot{x} = B & (3.d.2) \\ M\ddot{y} = -Mg + A & (3.d.3) \end{cases}$$

There are two constraint force equations (3.d.2) & (3.d.3), and from them, we can see that

$$\begin{cases} M\ddot{x} = B > 0 & (3.d.4) \\ M\ddot{y} + Mg = A > 0 & (3.d.5) \end{cases}$$ because the constraint forces of the walls can only be outward.

(∵ Negative constraint forces mean there are adhesion forces, which are not real for a normal frictionless wall.)

(e)

Now, we use the solution $\dot{\theta}^2 = a \sin\theta$ to see that whether inequalities (3.2.4-5) hold.

(1)

$$\ddot{x} = \frac{d^2}{dt^2}(l \sin\theta) = l\frac{d}{dt}(\cos\theta \cdot \dot{\theta})$$

$$= -l \sin\theta \cdot \dot{\theta}^2 + l \cos\theta \cdot \ddot{\theta}$$

$$= -a^2 l \sin\theta \cdot \sin^2\theta + l \cos\theta \, a \cos\theta \, \dot{\theta}$$

$$= -a^2 l \sin^3\theta + a^2 l \cos^2\theta \sin\theta$$

$$= a^2 l \sin\theta \cdot \cos 2\theta$$

$\ddot{x} > 0$ for $0 < \theta < \frac{\pi}{4}$ due to the $\cos 2\theta$ term

but becomes negative when $\theta \geq \frac{\pi}{4} := \theta_{crit,x}$

(2) $$\ddot{y} + g = g + \frac{d^2}{dt^2}(l \cos\theta) = g - l \frac{d}{dt}(\sin\theta \cdot \dot{\theta})$$

$$= g - l \cos\theta \cdot \dot{\theta}^2 - l \sin\theta \cdot \ddot{\theta}$$

$$= g - l \cos\theta \cdot a^2 \sin^2\theta - l \sin\theta \cdot a \cos\theta \cdot \dot{\theta}$$

$$= g - a^2 l \sin^2\theta \cos\theta - a^2 l \sin^2\theta \cos\theta$$

$$= g - a^2 l \sin\theta \cdot \sin 2\theta$$

$\ddot{y} > 0$ for $0 < \theta < \theta_{crit,y}$; negative otherwise

$$\theta_{crit,y} = f^{-1}\left(\frac{g}{a^2 l}\right), \quad f(\theta) = \sin\theta \cdot \sin 2\theta$$

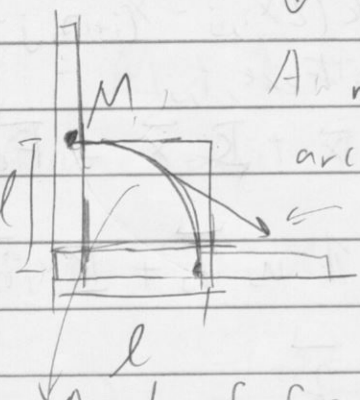
As long as $\theta_{crit,y} < \theta_{crit,x} = \frac{\pi}{4}$,
the ladder will leave the ground first.
If $\theta_{crit,y} > \theta_{crit,x} = \frac{\pi}{4}$
the ladder will come off the wall first.

(f)

As $\alpha \to 0$

$$L = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2) - Mgy + A(x - l\sin\theta) + B(y - l\cos\theta)$$

$\updownarrow$ equivalent to



A mass point falling of an arch

← At some point, the mass point will fly away from the arch due to Newton 1st law.

Arch of frictionless of a circle of radius l

Problem 4

(a)

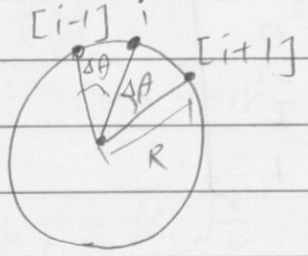
The total number of modes of motion will be as many as the number of atoms.

$N_{tot} = n \times \underbrace{1}_{\text{rotation on the 1-dim space of the ring}}$

$N_{vibration} = N_{tot} - \underbrace{1}_{\text{rotation of all atoms}}$

$= n-1$

(b)



$[i+1] := \begin{cases} 1 & , \text{if } i=n \\ i+1 & , \text{otherwise} \end{cases}$

$[i-1] := \begin{cases} n & , \text{if } i=1 \\ i-1 & , \text{otherwise} \end{cases}$

$L = T - V$

$= \sum_{i=1}^{n} \frac{m}{2} R^2 \dot{\theta}_i^2 - \frac{k}{2}\left(\Delta S_{i,[i+1]}^2 + \Delta S_{i,[i-1]}\right) \times \frac{1}{2}$

Reduced by half due to repetition of indices

$= \sum_{i=1}^{n} \frac{mR^2}{2} \dot{\theta}_i^2 - \frac{kR^2}{4} \cdot [(\theta_i - \theta_{[i+1]})^2 + (\theta_i - \theta_{[i-1]})^2]$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0$

$\Rightarrow \quad mR \ddot{\theta}_i + \frac{kR}{2}(2\theta_i - \theta_{[i+1]} - \theta_{[i-1]}) = 0$

$\vec{\theta} := [\theta_i]$ column vector of all i

Assume $\vec{\theta} = \vec{q}\, e^{i\omega t} + c.c.$

$\Rightarrow -mR\,\omega^2\, \mathbb{1}\cdot\vec{q} + \underline{\underline{K}}\cdot\vec{q} = 0$

$\Rightarrow -mR\,(\omega^2 - \omega^2)\,\vec{q} = 0$

Where $\vec{q}$ is an eigenvector of $\mathbb{K}$ and with eigenfrequency ω

The force constant matrix $\mathbb{K}$ in the special case of $n=4$ is

$$\mathbb{K} = kR \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$\Rightarrow$ Eigenvecs & freqs are

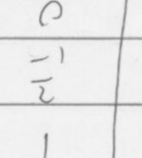
① $\omega_1 = 0$ ② $\omega_2 = \sqrt{\frac{k}{m}}$ ③ $\omega_3 = \sqrt{\frac{k}{m}}$

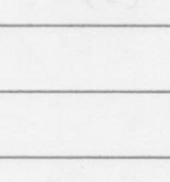
$$\vec{q}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$\Rightarrow$ Rotation, not considered here

Vibrational modes:

(1) $\omega_2 = \sqrt{\frac{k}{m}}$ (2) $\omega_3 = \sqrt{\frac{k}{m}}$

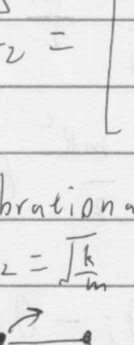




④ $\omega_4 = \sqrt{\frac{2k}{m}}$

$$\vec{q}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

(3)



$\omega_4 = \sqrt{\frac{2k}{m}}$

$\Rightarrow \vec{\theta} = \sum_s \vec{q}_s\, e^{i\omega_s t} + c.c.$

(c) $n=6$

$$\overline{K} = kR \begin{bmatrix} 1 & \frac{-1}{2} & 0 & 0 & 0 & \frac{-1}{2} \\ \frac{-1}{2} & 1 & \frac{-1}{2} & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 1 & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & \frac{-1}{2} & 1 & \frac{-1}{2} \\ \frac{-1}{2} & 0 & 0 & 0 & \frac{-1}{2} & 1 \end{bmatrix}$$

① $w_1 = 0$

$$\vec{q}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Rotation, no consider

② $w_2 = \frac{1}{\sqrt{2}}\sqrt{\frac{k}{m}}$

$$\vec{q}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

③ $w_3 = \frac{1}{\sqrt{2}}\sqrt{\frac{k}{m}}$

$$\vec{q}_3 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

④ $w_4 = \sqrt{\frac{3}{2}}\sqrt{\frac{k}{m}}$

$$\vec{q}_4 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

⑤ $w_5 = \sqrt{\frac{3}{2}}\sqrt{\frac{k}{m}}$

$$\vec{q}_5 = \begin{bmatrix} 1 \\ -2.8 \\ 1.8 \\ 1 \\ -2.8 \\ 1.8 \end{bmatrix}$$

⑥ $w_6 = \sqrt{2}\sqrt{\frac{k}{m}}$

$$\vec{q}_6 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Assume $\vec{\theta} = \sum_s u_s \vec{q}_s e^{iwt} + \text{c.c.}$, $\overline{K}\vec{q}_s = mw_s^2 \vec{q}_s$ (R)

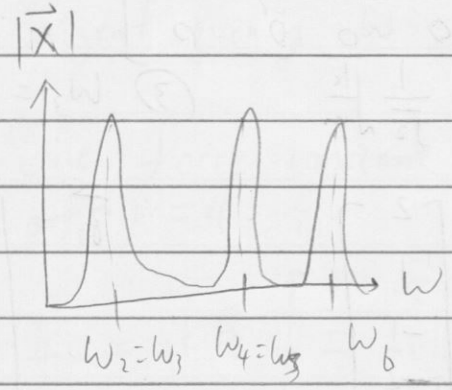
Adding damping factor β to force equation

$$mR D_t^2 \vec{\theta} + \beta R D_t \vec{\theta} + \overline{K} \cdot \vec{\theta} = \vec{F}_{ext}$$

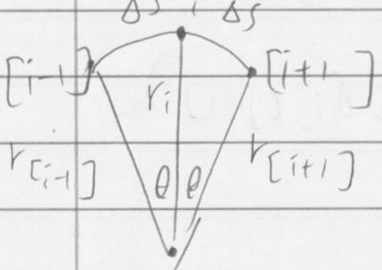
$$\Rightarrow \sum_s \left(-mR\omega^2 \mathbb{1} + i\beta R\omega \mathbb{1} + mR\omega_s^2 \mathbb{1}\right) u_s \vec{q_s}$$

$$= \sum_s \left(\vec{F_{ext}} \cdot \vec{q_s}\right) \vec{q_s}$$

$$\Rightarrow \quad u_s = \frac{F_{0,s}/(mR)}{(\omega_s^2 - \omega^2) + \frac{i\beta\omega}{m}} \Rightarrow |u_s|^2 \propto \frac{1}{(\omega_s^2-\omega^2)^2 + \frac{\beta^2\omega^2}{m^2}}$$



(d) We can investigate the radial motion modes behaving



$$\Delta S^2_{[i-1],i} = \left(\frac{r_{[i-1]}+r_i}{2} - R\right)^2\theta^2 + \left(r_i - r_{[i-1]}\right)^2$$

$$\Delta S^2_{i,[i+1]} = \left(\frac{r_{[i+1]}+r_i}{2} - R\right)^2\theta^2 + \left(r_i - r_{[i+1]}\right)^2$$

Here we only consider the radial motion

$$L = T - V$$

$$= \sum_{i=1}^{n} \frac{m}{2}\dot{r}_i^2 - \frac{k}{4}\theta^2\left[\left(\frac{r_i + r_{[i-1]}}{2} - R\right)^2 + \left(\frac{r_i + r_{[i+1]}}{2} - R\right)^2\right]$$

$$- \frac{k}{4}\left[\left(r_i - r_{[i-1]}\right)^2 + \left(r_i - r_{[i+1]}\right)^2\right]$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \frac{\partial L}{\partial r_i} = 0$$

$$\Rightarrow m\ddot{r}_i + \frac{k}{4}\theta^2\left(r_i + \frac{1}{2}r_{[i-1]} + \frac{1}{2}r_{[i+1]} - 2R\right)$$

$$+ \frac{k}{2}\left(2r_i - r_{[i-1]} - r_{[i+1]}\right) = 0$$

$$\Rightarrow m\ddot{\tilde{r}}_i + \frac{k}{4}\theta^2\left(\tilde{r}_i + \frac{1}{2}\tilde{r}_{[i-1]} + \frac{1}{2}\tilde{r}_{[i+1]}\right)$$

$$+ \frac{k}{2}\left(2\tilde{r}_i - \tilde{r}_{[i-1]} - \tilde{r}_{[i+1]}\right) = 0$$

$$\tilde{r}_i := r_i - R$$

$$\Rightarrow m D_t^2 \vec{\tilde{r}} + \underline{K} \cdot \vec{\tilde{r}} = 0$$

$$-m\omega^2 \underline{1}\, \vec{q} + \underline{K} \cdot \vec{q} = 0$$

$\underline{K}$ is now

$$\underline{K} = k\left(\mathbb{1} - \frac{1}{2}\delta_{i,[j+1]} - \frac{1}{2}\delta_{i,[j-1]}\right)$$
$$+ \frac{k}{4}\theta^2\left(\mathbb{1} + \frac{1}{2}\delta_{i,[j+1]} + \frac{1}{2}\delta_{i,[j-1]}\right)$$

For $n=4$

$$\underline{K} = k\begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{bmatrix} + \frac{k\theta^2}{4}\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

Note that this matrix has a finite lowest frequency due to the additional $\frac{k\theta^2}{4}$ term.

For lowest frequency ω_0,

$$\vec{q}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{K}\,\vec{q}_0 = m\omega_0^2\,\vec{q}_0$$

$$\Rightarrow \omega_0 = \sqrt{\frac{k\theta^2}{2m}} = \theta \cdot \sqrt{\frac{k}{2m}}$$
$$= \frac{2\pi}{N}\sqrt{\frac{k}{2m}}$$

$\Rightarrow$ Finite lowest freq. Unlike angular motion $\omega_0 = 0$