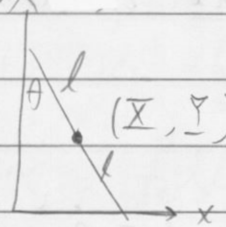
Problem 3.

(a)



$(\bar{X}, \bar{Y}) = (l\sin\theta, l\cos\theta)$

$$L = T - V = \frac{1}{2}Ml^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 - Mg\bar{Y}$$

$$= Ml^2\left(\frac{1}{2} + \frac{\alpha}{6}\right)\dot{\theta}^2 - Mgl\cos\theta$$

(b)

$$H = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$$

E.M $\downarrow$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow Ml^2\left(1 + \frac{\alpha}{3}\right)\ddot{\theta} - Mgl\sin\theta = 0.$$

$$= Ml^2\left(\frac{1}{2} + \frac{\alpha}{6}\right)\dot{\theta}^2 + Mgl\cos\theta = E$$

$$= Mgl$$

$$\Rightarrow \dot{\theta} = \sqrt{\frac{g(1-\cos\theta)}{l\left(\frac{1}{2} + \frac{\alpha}{6}\right)}} = \sqrt{\frac{2g}{l\left(\frac{1}{2} + \frac{\alpha}{6}\right)}}\sin\theta$$

(c) $\displaystyle\int \frac{d\theta}{\sin\theta} = \int a\,dt$ $\quad := a\sin\theta$

$$\Rightarrow \ln\left|\tan\left(\frac{\theta}{2}\right)\right| = at + C$$

$$\tan\left(\frac{\theta}{2}\right) = Ce^{at}, \quad \theta = 2\tan^{-1}\left(Ce^{At}\right)$$

(d) There are two geometrical constraints:

$$\begin{cases} \bar{x} = l\sin\theta \\ \bar{y} = l\cos\theta \end{cases}$$

$$\Rightarrow L = \frac{1}{2}M\left(\dot{\bar{x}}^2 + \dot{\bar{y}}^2\right) + \frac{1}{2}I\dot{\theta}^2$$

$$- Mgy + \underbrace{A}\left(y - l\cos\theta\right) + \underbrace{B}\left(x - l\sin\theta\right)$$

Lagrangian multipliers

$$\Rightarrow \begin{cases} I\ddot{\theta} = Al\sin\theta - Bl\cos\theta & (3.d.1) \\ M\ddot{x} = B & (3.d.2) \\ M\ddot{y} = -Mg + A & (3.d.3) \end{cases}$$

There are two constraint force equations (3.d.2) & (3.d.3), and from them, we can see that

$$\begin{cases} M\ddot{x} = B > 0 & (3.d.4) \\ M\ddot{y} + Mg = A > 0 & (3.d.5) \end{cases}$$

because the constraint forces of the walls can only be outward.

( ∵ Negative constraint forces mean there are adhesion forces, which are not real for a normal frictionless wall. )

(e)

Now, we use the solution $\dot{\theta} = a \sin\theta$ to see that whether inequalities (3.2.4-5) hold.

①

$$\ddot{x} = \frac{d^2}{dt^2}(l \sin\theta) = l \frac{d}{dt}(\cos\theta \cdot \dot{\theta})$$

$$= -l \sin\theta \cdot \dot{\theta}^2 + l \cos\theta \cdot \ddot{\theta}$$

$$= -a^2 l \sin\theta \cdot \sin^2\theta + l \cos\theta \, a \cos\theta \, \dot{\theta}$$

$$= -a^2 l \sin^3\theta + a^2 l \cos^2\theta \sin\theta$$

$$= a^2 l \sin\theta \cdot \cos 2\theta$$

$\ddot{x} > 0$ for $0 < \theta < \frac{\pi}{4}$ due to the $\cos 2\theta$ term but becomes negative when $\theta \geq \frac{\pi}{4} := \theta_{crit,x}$

② $$\ddot{y} + g = g + \frac{d^2}{dt^2}(l \cos\theta) = g - l \frac{d}{dt}(\sin\theta \cdot \dot{\theta})$$

$$= g - l \cos\theta \cdot \dot{\theta}^2 - l \sin\theta \cdot \ddot{\theta}$$

$$= g - l \cos\theta \cdot a^2 \sin^2\theta - l \sin\theta \cdot a \cos\theta \cdot \dot{\theta}$$

$$= g - a^2 l \sin^2\theta \cos\theta - a^2 l \sin^2\theta \cos\theta$$

$$= g - a^2 l \sin\theta \cdot \sin 2\theta$$

$\ddot{y} > 0$ for $0 < \theta < \theta_{crit,y}$ ; negative otherwise

$$\theta_{crit,y} = f^{-1}\left(\frac{g}{a^2 l}\right), \quad f(\theta) = \sin\theta \cdot \sin 2\theta$$

As long as $\theta_{crit,y} < \theta_{crit,x} = \frac{\pi}{4}$, the ladder will leave the ground first.

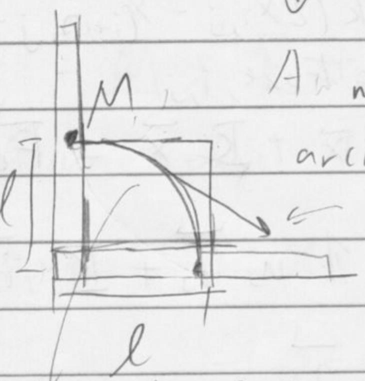
If $\theta_{crit,y} > \theta_{crit,x} = \frac{\pi}{4}$ the ladder will come off the wall first.

(f)

As $\alpha \to 0$

$$L = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2) - Mgy + A(x - l\sin\theta) + B(y - l\cos\theta)$$

$\updownarrow$ equivalent to




A mass point falling of an arch

← At some point, the mass point will fly away from the arch due to Newton 1st law.

Arch of frictionless of a circle of radius $l$