Problem 1

$$F(t) = F_0 \sum_{n=0}^{\infty} A^n \delta(t - \tfrac{n}{2}\tau)$$

To solve the full solution, we must first find the Green function of the equation of motion.

$$L\{x(t)\} = (\partial_t^2 + 2\beta\partial_t + \omega_0^2)\, x = F(t)/m$$
$$L\{G(t,t')\} = (\partial_t^2 + 2\beta\partial_t + \omega_0^2)\, G(t,t') = \delta(t-t')$$

We can solve it by Fourier transform:

$$\tilde{G}(\omega, t') = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, G(t,t')$$

$$\Rightarrow (-\omega^2 - 2i\beta\omega + \omega_0^2)\, \tilde{G}(\omega,t') = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \delta(t-t') = e^{i\omega t'}$$

$$\Rightarrow \tilde{G}(\omega,t') = \frac{e^{i\omega t'}}{-\omega^2 - 2i\beta\omega + \omega_0^2}$$

$$= \frac{-e^{i\omega t'}}{(\omega-\omega_+)(\omega-\omega_-)} \quad \left(\omega_\pm = -i\beta \pm \sqrt{\omega_0^2 - \beta^2}\right)$$

$$\Rightarrow G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-e^{i\omega(t'-t)}}{(\omega-\omega_+)(\omega-\omega_-)}\, d\omega$$

Since the integrand contains two poles $\omega_+$, $\omega_-$, which are both at the negative imaginary side of the complex frequency plane.

$\Rightarrow$ By Jordan's Lemma, if $\begin{cases} t > t', & G \text{ is finite} \\ t < t', & G = 0 \end{cases}$

Finite value of $G$ is

$$G(t, t') = 2\pi i \times (-1) \times \left[\frac{i}{2\pi}\left(\frac{-e^{-i\omega_+(t-t')}}{\omega_+ - \omega_-}\right) + \frac{i}{2\pi}\left(\frac{-e^{-i\omega_-(t-t')}}{\omega_- - \omega_+}\right)\right]$$

$$= i\left(\frac{e^{-i\omega_+(t-t')}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_-(t-t')}}{\omega_- - \omega_+}\right) \text{ if } t>t'$$

$\therefore$ In total,

$$G(t, t') = i\left(\frac{e^{-i\omega_+(t-t')}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_-(t-t')}}{\omega_- - \omega_+}\right)\Theta(t-t')$$

$$= i e^{-\beta(t-t')}\left(\frac{e^{i\omega(t-t')}}{2\omega} - \frac{e^{-i\omega(t-t')}}{2\omega}\right)\Theta(t-t')$$

$$= e^{-\beta(t-t')}\sin[\omega(t-t')]\,\Theta(t-t')$$

$(\omega := \sqrt{\omega_0^2 - \beta^2})$ ($\Theta(t-t')$ is Heaviside step function)

$\therefore$ For a delta response function $\delta(t-t_0)$

The displacement is

$$x(t) = \int_{-\infty}^{\infty} G(t, t')\,\delta(t'-t_0)\,dt'$$

$$= G(t, t_0)$$

$\therefore$ For $F(t) = \frac{F_0}{m}\sum_{n=0}^{\infty} A^n \delta(t - \frac{n}{2}\tau)$

$$\Rightarrow x(t) = \frac{F_0}{m}\sum_{n=0}^{\infty} A^n\, G(t, \tfrac{n}{2}\tau)$$

We shall use python to plot $x(t)$ for different $A$,

We tried to make some plots for different values of A.
Here we set the parameters to be

$$\beta = 1$$
$$\omega_0 = 20$$
$$\omega = \sqrt{\omega_0^2 - \beta^2} = 19.97$$
$$\tau = \frac{2\pi}{\omega} = 0.314$$

We found that for three different values, x(t) behaves differently

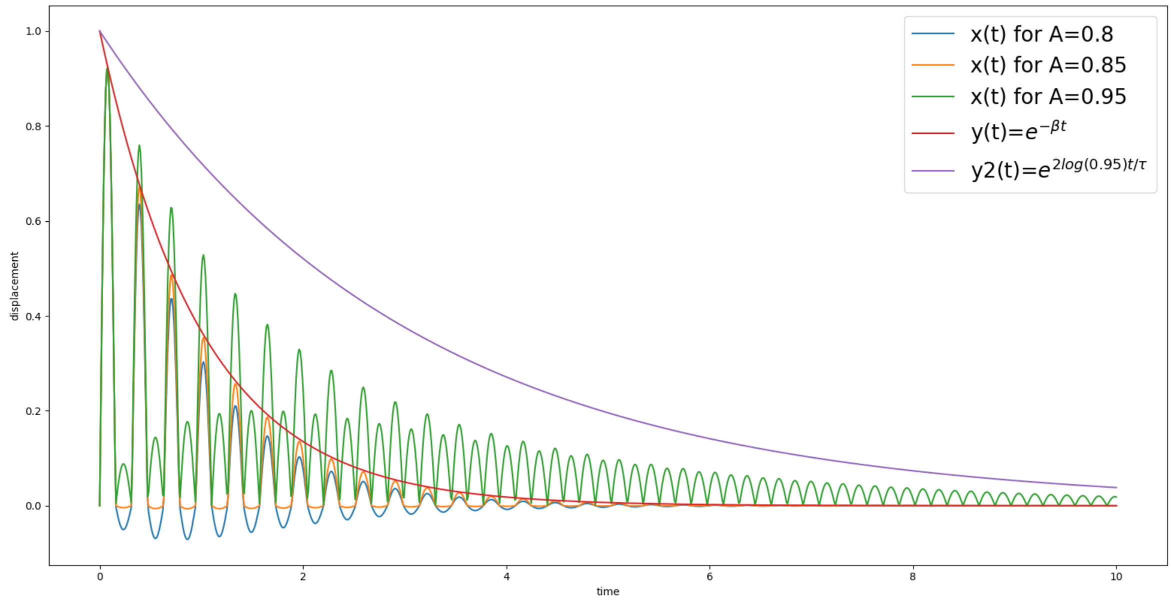


Fig. 2.1 Blue, orange, green lines are the x(t) for different A. Red line is a reference line for the damping rate $y(t) = e^{-\beta t}$, and the purple line is the reference line for $A^N(A = 0.95)$, or equivalently $y2(t) = e^{2\log(A=0.95)t/\tau}$.

We found that for A=0.95>0.85, $x(t) \geq 0$ for all t, and every half period $\tau/2$, the peak height changes a lot, from big to small, and from small to big interchangeably and repetitively, but all the peaks are above 0.

For A=0.85, there is half a period that x(t) almost equals to 0 and then at the next half period, peaks appear again.
Thus the critical value is at around 0.85.

The reason for the critical value of A, $A_{crit}$, is because of the interference between the new response and the old response.

For each n,

$$x_n(t) = A^n G\left(t, \frac{n}{2}\tau\right) = A^n e^{-\beta(t-\frac{n}{2}\tau)} \sin\left(\omega\left(t - \frac{n}{2}\tau\right)\right) \theta\left(t - \frac{n}{2}\tau\right)$$

And we can see easily that

$$x_n(t) + x_{n+1}(t) = A^n e^{-\beta(t-\frac{(n+1)}{2}\tau)} \sin\left(\omega\left(t - \frac{(n+1)}{2}\tau\right)\right)\left(-e^{-\beta\frac{1}{2}\tau} + A\right) =$$

$$\left(-e^{-\beta\frac{1}{2}\tau} + A\right) x_{n+1}(t) = (-b + A) x_{n+1}(t) \text{ at } t > \frac{n+1}{2}\tau$$

$$b = e^{-\beta\frac{1}{2}\tau}$$

Thus, if $b = A$, the interference between the n-th and (n+1)-th response function will cause half a period to have 0 value, and at next half period, the new response function will again give a positive peak.
Thus the critical value of A is

$$A_{crit} = b = e^{-\frac{\beta}{2}\tau} = 0.8544678930067565 \sim 0.85$$

in this case.
Moreover, it is also easy to see whether x(t) will be an exponential decrease or growth for $A > A_{crit} = b$.

We particularly look at the times where the peak may happen, i.e. when $t = \frac{N}{2}\tau + \frac{\tau}{4}$ .

$$x\left(t = \frac{N}{2}\tau + \frac{\tau}{4}\right) = \sum_{n=0}^{N} A^n G\left(t, \frac{n}{2}\tau\right) = \sum_{n=0}^{N} A^n e^{-\beta\left(t-\frac{n}{2}\tau\right)} \sin\left(\omega\left(t - \frac{n}{2}\tau\right)\right)$$

$$= \sum_{n=0}^{N} A^n b^{N-n} \sin\left[(N-n)\pi + \frac{\pi}{2}\right] \delta = \sum_{n=0}^{N} A^n (-b)^{N-n} \delta = a_N \delta$$

Where

$$\delta = e^{-\frac{\beta\tau}{4}}$$

$$N = t//(\frac{\tau}{2})$$

$$a_N = \sum_{n=0}^{N} A^n(-b)^{N-n}$$

And it has an iteration behavior

$$a_{N+1} = -ba_N + A^{N+1}$$

All we need to see is that whether $a_{N+1}$ converges or diverges as $N \to \infty$.
We can use mathematical induction rule to prove that $a_N$ will converge.
We assume that $0 < a_N \leq A^N$
For N=0,

$$a_0 = 1$$

And for N=1,

$$0 < a_1 = -b + A \leq A^1 \ (A > A_{crit} = b)$$

And if $0 < a_k \leq A^k$,
Then

$$a_{k+1} = -ba_N + A^{N+1} = \left(A - \frac{ba_N}{A^N}\right)A^N$$

Since $a_n > 0$, $a_{k+1} \leq A^{N+1}$, and since $\frac{a_n}{A^N} \leq 1$, $a_{k+1} = \left(A - \frac{ba_N}{A^N}\right)A^N \geq (A-b)A^N > 0$.
Thus if $0 < a_k \leq A^k$, $0 < a_{k+1} = -ba_N + A^{N+1} \leq A^{N+1}$
Thus we know by mathematical induction, $0 < a_N \leq A^N$ for all N.
Thus $a_N$ must converge as $N \to \infty$ and converge faster than $A^N$
(see Fig. 2.1's purple line).
$x\left(t = \frac{N}{2}\tau + \frac{\tau}{4}\right) = a_N\delta$ (the peak heights) will decrease exponentially.
Thus, Kay is right.

Problem 2

(a)

For transformation $\delta q_i = F_i \delta\eta$

$$(\delta L)_q = \sum_i \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i$$

$$= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i$$

$$= \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}F_i\right)\delta\eta$$

$$= \frac{d}{dt}\left(\sum_i p_i F_i\right)\delta\eta \qquad p_i := \frac{\partial L}{\partial \dot{q}_i}$$

For transformation $\delta t = G\delta\eta$

$$(\delta L)_t = \frac{\partial L}{\partial t}\delta t = -\frac{dH}{dt}G\delta\eta \quad \left(\because -\frac{\partial L}{\partial t} = \frac{dH}{dt}\right)$$

$\Rightarrow$ For total transformation

$$\begin{cases} \delta \vec{q} = \vec{F}\,\delta\eta \\ \delta t = G\,\delta\eta \end{cases} \Rightarrow (\delta L)_{total} = \frac{d}{dt}\left(\vec{p}\cdot\vec{F} - GH\right)\delta\eta$$

$\vec{p}\cdot\vec{F} - GH$ is the conserved quantity

Here we assume $G$ is a constant of time; if otherwise, the conserved quantity will have a more complex form.

(b) For $L = L(r, \theta - \omega t, \dot{r}, \dot{\theta})$

The transformation has symmetry

$\delta\theta = \delta\eta$

$\delta t = \frac{1}{\omega}\delta\eta$

Since $\delta(\theta - \omega t) = (1 - \frac{1}{\omega}\cdot\omega)\delta\eta = 0$

$\Rightarrow \delta L = 0$

$\therefore$ The conserved quantity is

$$Q := 1\cdot\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{1}{\omega}H = \frac{1}{\omega}(\omega L_z - H)$$

$Q$ is conserved.

To understand more specifically, we assume the Lagrangian is

$$L(\theta - \omega t, \dot{\theta}) = \frac{1}{2}I(\dot{\theta} - \omega)^2 - \frac{1}{2}k(\theta - \omega t)^2$$

Which is a rotational harmonic oscillator in the constant rotational frame.

$\Rightarrow \quad L_z = I(\dot{\theta} - \omega)$

$\quad H = \dot{\theta}L_z - L = \frac{1}{2}I(\dot{\theta}^2 - \omega^2) + \frac{1}{2}k(\theta - \omega t)^2$

Which looks quite unnatural and probably not the expression for energy $E$.

Then

$\omega L_z - H = I\omega(\dot{\theta} - \omega) - \frac{1}{2}I(\dot{\theta}^2 - \omega^2) - \frac{1}{2}k(\theta - \omega t)^2$

$= -\frac{1}{2}I(\dot{\theta} - \omega)^2 - \frac{1}{2}k(\theta - \omega t)^2$

$= -\left[\frac{1}{2}I(\dot{\theta} - \omega)^2 + \frac{1}{2}k(\theta - \omega t)^2\right]$

We can see that

$$E := \frac{1}{2} I(\dot{\theta} - \omega)^2 + \frac{1}{2} k(\theta - \omega t)^2$$

is a conserved charge, and is the total energy before the rotational transformation $\begin{cases} \theta \to \theta - \omega t \\ \dot{\theta} \to \dot{\theta} - \omega \end{cases}$

$\therefore$ Noether's theorem tells us the conserved quantity before and after coordinate transformation, also including the effect of Galileen boost (for both translational and rotational)