Problem 1

$f(t) := |\sin(t)|$ is a periodic function with $T = \pi$

Fourier series of $f(t)$ is $(f(t+T) = f(t))$

$$f(t) = \sum_{n=-\infty}^{\infty} a_n \cos(2nt)$$

$\because$ $f(t)$ is an even function $\Rightarrow$ use cos to expand

Also, $\cos(2nt)$ has period of $T = \pi$

① $n = 0$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} |\sin(t)| \cdot \cos(0t)\, dt$$

$$= \frac{1}{\pi}\left(-\cos(t)\right)\Big|_{t=0}^{\pi} = \frac{2}{\pi}$$

② $n = k$

$$a_k = \frac{1}{\pi}\int_0^{\pi} \sin t \cdot \cos(2kt)\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}\sin[(1+2k)t] + \frac{1}{2}\sin[(1-2k)t]\, dt$$

$$= \frac{1}{2\pi}\left\{\frac{\cos[(1+2k)t]}{1+2k} + \frac{\cos[(1-2k)t]}{1-2k}\right\}\Big|_{\pi}^{0}$$

$$= \frac{1}{\pi}\left(\frac{1}{1+2k} + \frac{1}{1-2k}\right) = -\frac{2}{\pi}\,\frac{1}{4k^2-1}$$

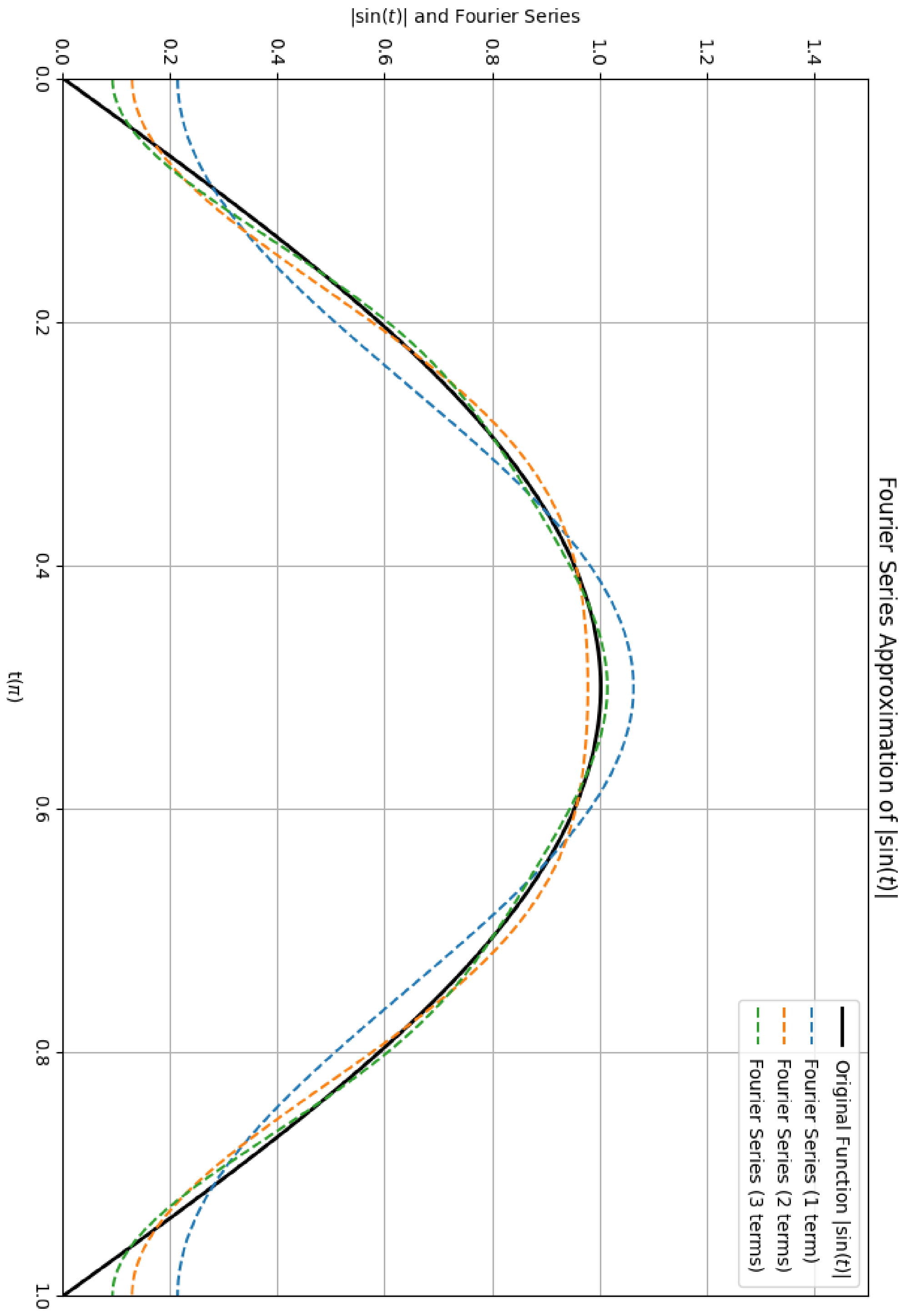
$$\therefore f(t) = \sum_{n=-\infty}^{\infty}\left(\frac{-2}{\pi}\right)\frac{1}{4k^2-1}\cos(2nt)$$

$$= \frac{2}{\pi} + \sum_{n=1}^{\infty}\left(\frac{-4}{\pi}\right)\frac{1}{4k^2-1}\cos(2nt)$$

We plot the first three terms in the following.

Problem 1

The figure is the first three fourier terms of the fourier series of the full rectifier



Problem 2

(a)

$$L\{x(t)\} = (\partial_t^2 + 2\beta\partial_t + \omega_0^2)x = F(t)/m$$

$$L\{G(t,t')\} = \delta(t-t')$$

$$\Rightarrow G(t,t') = \frac{\Theta(t-t')}{\omega_1} e^{-\beta(t-t')} \sin[\omega_1(t-t')]$$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$$ From the resule of HW05

$$\therefore x(t) = \int_0^{N(2\pi/\omega_1)} G(t,t')\cdot \frac{F_{ext}(t')}{m}\, dt'$$

$$= \frac{A}{\omega_1}\int_0^{NT_1} e^{-\beta(t-t')} \sin[\omega_1(t-t')]\cdot \sin(\omega_1 t')\, dt'$$

$$= \frac{A}{\omega_1}\int_0^{NT_1} e^{-\beta(t-t')} \left(\frac{-1}{4}\right)\left[e^{i\omega_1(t-t')} - e^{-i\omega_1(t-t')}\right] \cdot (e^{i\omega_1 t'} - e^{-i\omega_1 t'})\, dt'$$

$$\left(T_1 := \frac{2\pi}{\omega_1}\right)$$

$$= \frac{-A}{4\omega_1}\int_0^{NT_1} e^{-\beta(t-t')}\left(e^{i\omega_1 t} + e^{-i\omega_1 t} - e^{i\omega_1(t-2t')} - e^{-i\omega_1(t-2t')}\right) dt'$$

$$= \frac{-A}{2\omega_1}\int_0^{NT_1} e^{-\beta(t-t')}\left\{\cos(\omega_1 t) - \cos[\omega_1(t-2t')]\right\} dt'$$

For large N, the anti-rotating term $\cos[\omega_1(t-2t')]$ will have negligible contribute

$$\approx \frac{-A}{2\omega_1}\cos(\omega_1 t)\, e^{-\beta t}\left(e^{+\beta NT_1} - 1\right)/\beta$$

$$= \frac{-A}{2\beta\omega_1}\cos(\omega_1 t)\, e^{-\beta t}\left(e^{\beta NT_1} - 1\right)$$

(b) As $\beta \to 0$

$$\lim_{\beta \to 0} x(t) = \frac{-A}{2\omega_1} \cdot \lim_{\beta \to 0} e^{-\beta t} \cdot \frac{e^{\beta N T_1} - 1}{\beta} \cos(\omega_1 t)$$

$$= \frac{-A}{2\omega_1} \cos(\omega_0 t) \cdot (N T_1) \quad (\because \omega_1 \to \omega_0 \text{ as } \beta \to 0)$$

We can see that $x(t)$'s amplitude is proportional to N, which means, if the F_{ext} continues forever ($N \to \infty$), the amplitude will also go to infinity

$\Rightarrow$ Same results as frictionless resonance for equation of motion

$$\frac{d^2}{dt^2} x + \omega_0 x = Am \cos(\omega t)$$

$\Rightarrow$ $Amp(x) = \infty$ as $\omega \to \omega_0$

For the phase delay ϕ,

$$\tan\phi = \left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right), \text{ as } \omega \to \omega_0,$$

$$\tan\phi \to +\infty, \quad \phi \to \frac{\pi}{2}$$

And $x(t) \propto \cos(\omega_0 t)$ as $\beta \to 0$ has a phase delay of $\frac{\pi}{2}$ compared with the $F_{ext} \propto \sin(\omega_0 t)$ as $\beta \to 0$

Problem 3

(a) $T_1 = \frac{1}{2} m_1 r^2 \cdot \omega^2 + \frac{1}{2} m_1 \dot{r}^2$

$= \frac{1}{2} m_1 (r^2 \omega^2 + \dot{r}^2)$

$T_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$

$= \frac{1}{2} m_2 \left[\left| \frac{d}{dt} [r \cos(\omega t) + l \cos(\omega t + \theta)] \right|^2 + \left| \frac{d}{dt} [r \sin(\omega t) + l \sin(\omega t + \theta)] \right|^2 \right]$

$= \frac{1}{2} m_2 (r^2 \omega^2 + \dot{r}^2)$

$+ \frac{1}{2} m_2 l^2 (\omega + \dot{\theta})^2$

$+ m_2 [- \dot{r} \cos(\omega t) \, l \sin(\omega t + \theta) (\omega + \dot{\theta})$

$+ \dot{r} \sin(\omega t) \, l \cos(\omega t + \theta) (\omega + \dot{\theta})$

$+ \omega r \sin(\omega t) \, l \sin(\omega t + \theta) (\omega + \dot{\theta})$

$+ \omega r \cos(\omega t) \, l \cos(\omega t + \theta) (\omega + \dot{\theta})]$

$= m_2 l (\omega + \dot{\theta}) [\dot{r} \sin(\omega t - \omega t - \theta)$

$+ \omega r \cos(\omega t - \omega t - \theta)]$

$+ \frac{1}{2} m_2 [r^2 \omega^2 + \dot{r}^2 + l^2 (\omega + \dot{\theta})^2]$

$= \frac{1}{2} m_2 [r^2 \omega^2 + \dot{r}^2 + l^2 (\omega + \dot{\theta})^2]$

$+ m_2 l (\omega + \dot{\theta}) [-\dot{r} \sin\theta + \omega r \cos\theta]$

$V = \frac{1}{2} k (r - r_0)^2$

$L = T_1 + T_2 - V$

(b) $p_r = \frac{\partial L}{\partial \dot{r}}$

$= m_1 \dot{r} + m_2 \dot{r} - m_2 l(\omega + \dot{\theta}) \sin\theta$

$p_\theta = \frac{\partial L}{\partial \dot{\theta}}$

$= m_2 l^2 (\omega + \dot{\theta}) - m_2 l [\dot{r} \sin\theta + \omega r \cos\theta]$

(c) $H = \dot{r} p_r + \dot{\theta} p_\theta - L$

$= \frac{1}{2}(m_1 + m_2)(\dot{r}^2 - \omega^2 r^2)$

$+ \frac{1}{2} m_2 l^2 (-\omega^2 + \dot{\theta}^2)$

$- m_2 l \omega [\dot{r} \sin\theta + \omega r \cos\theta]$

$+ \frac{1}{2} k (r - r_0)^2$

(d) $E = T_1 + T_2 + V$

H is clearly not E ; rather, H is some very weird form.

$$\frac{dH}{dt} = \frac{-\partial L}{\partial t} = 0$$

($\because$ L is no explicit function of t, unless $\omega = \omega(t)$ is not const.)

E is not conserved because we must exert some external torque to maintain the rotational speed of m_1 as ω ($\because$ m_2 and the rigid rod will apply torque on m_1).

$$\therefore \frac{dE}{dt} = \tau_{ext} \cdot \omega \neq 0 \Rightarrow E \text{ is not conserved}$$

(e) For $r = r_0 = \text{const.}$,

$$L = \frac{1}{2} m_2 \left[r^2 \omega^2 + l^2 (\omega + \dot{\theta})^2 \right]$$
$$+ m_2 \omega l r (\omega + \dot{\theta}) \cos \theta$$

$$\to L = \frac{1}{2} m_2 l^2 (\omega + \dot{\theta})^2$$
$$+ m_2 \omega l r_0 (\omega + \dot{\theta}) \cos \theta$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right)$$

$$\Rightarrow - m_2 \omega (\omega + \dot{\theta}) l r_0 \sin \theta$$
$$= \frac{d}{dt} \left[m_2 l^2 (\omega + \dot{\theta}) + m_2 \omega l r_0 \cos \theta \right]$$

$$\Rightarrow - m_2 \omega (\omega + \dot{\theta}) l r_0 \sin \theta$$
$$= m_2 l^2 \ddot{\theta} - m_2 \omega \dot{\theta} l r_0 \sin \theta$$

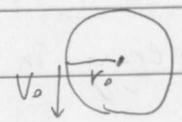
$$\Rightarrow l \ddot{\theta} + \omega^2 r_0 \sin \theta = 0$$

$$\Rightarrow g = \omega^2 r_0 \Rightarrow \text{Centrifugal acceleration}$$

Problem 4.

(a)

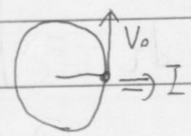
In the old orbit,



$$m\frac{V_0^2}{r_0} = \frac{GMm}{r_0^2} \Rightarrow V_0^2 = \frac{GM}{r_0} = \frac{k}{mr_0}$$

Angular Mom. $L = m r_0^2 \omega$

$$= m V_0 r_0 = m r_0 \sqrt{\frac{k}{mr_0}} = \sqrt{mkr_0}$$



$$E_{new} = \frac{1}{2} m V_0^2 - \frac{k}{r} + \frac{1}{2}\frac{I^2}{m}$$

$$= E_{old} + \frac{1}{2}\frac{I^2}{m}$$

In the new orbit, the angular momentum is not changed by the impulse $\vec{I}$ because $\vec{I}$ is along the radial direction.

$$\therefore \alpha_{new} = \frac{L^2}{mk} = \alpha_{old} = \alpha = r_0$$

$$\varepsilon_{new} = \sqrt{1 + \frac{2E_{new}L^2}{mk^2}} = \sqrt{1 + \frac{2\left(E_{old} + \frac{I^2}{2m}\right)L^2}{mk^2}}$$

$$= \sqrt{1 + \frac{2E_{old}L^2}{mk^2} + \frac{I^2L^2}{m^2k^2}}$$

$$= \sqrt{0 + \frac{I^2L^2}{m^2k^2}} \quad \left(\because \varepsilon_{old} = \sqrt{1 + \frac{2E_{old}L^2}{mk^2}} = 0\right)$$

$$= \frac{IL}{mk} = \frac{I}{mk}\sqrt{mkr_0}$$

$$= I\sqrt{\frac{r_0}{mk}}$$

Problem 4(b)

We plotted the old and new orbits. For the new orbits, we plotted the case for $\epsilon = 0.8$, for both cases of applying impulse outward and inward.

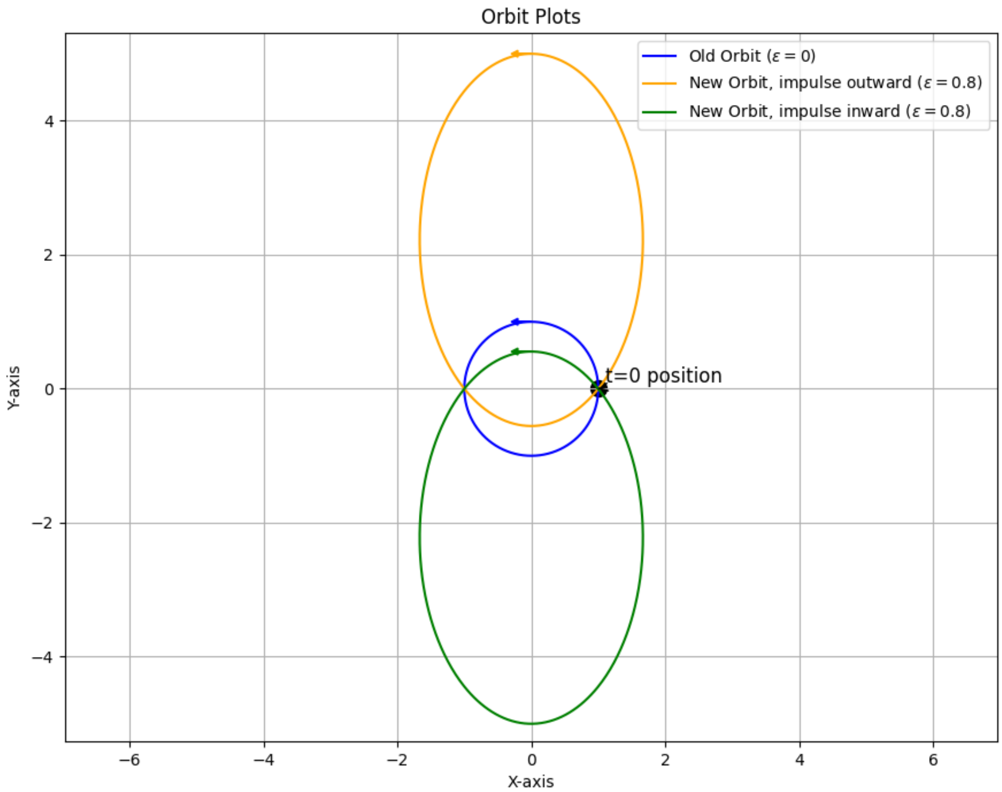
The orbit equation for the new orbit is:

$$r(\theta) = \frac{\alpha}{1 + \epsilon\cos(\theta + \theta_0)}$$

Because at t=0, $\theta = 0$, and $r(\theta = 0) = \alpha = r_0$, θ_0 must be $\pm\pi/2$.

If $\theta_0 = \pi/2$, we will have the case that impulse is applied outward.

If $\theta_0 = -\pi/2$, we will have the case that impulse is applied inward.



(C)

$$P_{star} := \frac{P_0 r_0^2}{r^2}$$

$E_0 = P_0 \cdot T_0$ is total energy in one period T_0 (Period of the old orbit)

In the new orbit,

$$E_{new} = \int_0^{T_{new}} P(r)\,dt = \int_0^{T_{new}} P_0 r_0^2 \frac{1}{r^2}\,dt$$

$$= \int_0^{T_{new}} P_0 r_0^2 \frac{m}{mr^2 \frac{d\theta}{dt}} \frac{d\theta}{dt}\,dt$$

$$= \frac{P_0 r_0^2 m}{L_{new}} \int_0^{2\pi} d\theta \quad (L_{new} = mr^2 \frac{d\theta}{dt})$$

$$= \frac{2\pi P_0 r_0^2 m}{L_{new}} = P_0 \left(\frac{2\pi}{T_0} \cdot m r_0^2\right) \cdot \frac{1}{L_{new}} \cdot T_0$$

$$= P_0 \frac{L_{old} T_0}{L_{new}} = P_0 T_0 = E_0 \quad (\because L_{new} = L_{old}) \quad (C.1)$$

$\therefore$ The total energy for one period is the same for both old and old orbits.

$\overline{P}_{new}$ is the average power in one period in the new orbit

$$\overline{P}_{new} = \frac{E_0}{T_{new}} < \frac{E_0}{T_0} = P_0 = \overline{P}_{old}$$

$\because T_{new} > T_0$

For old orbit

$$\frac{k}{r_0^2} = \frac{m}{r_0}\left(\frac{2\pi r_0}{T_0}\right)^2 \Rightarrow T_0 = 2\pi r_0\sqrt{\frac{m r_0}{k}} = 2\pi\sqrt{\frac{m r_0^3}{k}}$$

For new orbit, due to Kepler's 3rd law, we know

$$T_{new} = 2\pi\sqrt{\frac{m a^3}{k}}, \text{ where } \frac{\alpha}{1-\varepsilon} + \frac{\alpha}{1+\varepsilon} = 2a$$

$$\Rightarrow a_{new} = \frac{\alpha_{new}}{1-\varepsilon^2_{new}} = \frac{r_0}{1-\varepsilon^2_{new}} > r_0$$

$$\Rightarrow T_{new} > T_0$$

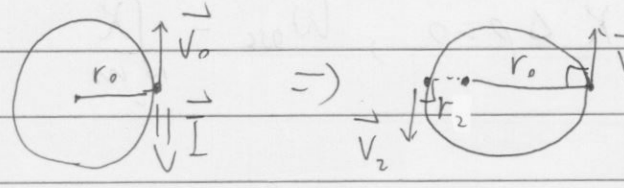
$\because$ (C.1)

$\therefore$ To increase the average power, one should

① increase $E_{new} = E_{old} \cdot \frac{L_{old}}{L_{new}} \to$ decrease L_{new}

② decrease $T_{new} \to$ decrease $a_{new} = \frac{\alpha_{new}}{1-\varepsilon^2_{new}}$

We can achieve both ① & ② by reducing L_{new} by giving $\vec{I}$ opposite to the current velocity $\vec{V_0}$.



$\vec{V_1}$, $V_1 < V_0$, lower velocity

$\Rightarrow$ Weaker centrifugal force, the ship is pulled inside by the gravitational force more

$\Rightarrow r_2 < r_0$

$$a_{new} = \frac{r_0 + r_2}{2} < r_0$$

$\Rightarrow T_{new}$ is decreased

L_{new} decreased $\Rightarrow E_{new} = E_{old} \cdot \frac{L_{old}}{L_{new}} > E_{old}$ increased

Problem 5

(a)

$$m\frac{d^2r}{dt^2} = F(r) + F_{centrifugal}$$

$$= -\frac{k}{r^2} e^{\left(\frac{-r}{a}\right)} + \frac{l^2}{\mu r^3}$$

For stable orbit, $r = \rho = \text{const.}$

$$m\frac{d^2r}{dt^2} = 0 = \frac{l^2}{\mu\rho^3} - \frac{k}{\rho^2} e^{\left(-\frac{\rho}{a}\right)} \Rightarrow \frac{l^2}{\mu\rho^3} = \frac{k}{\rho^2} e^{\left(-\frac{\rho}{a}\right)}$$

For small perturbation $\rho' = \rho + \Delta\rho$

$(|\Delta\rho| << \rho)$

$$\Rightarrow \mu\frac{d^2\Delta\rho}{dt^2} = \frac{-3l^2}{\mu\rho^4}\Delta\rho + \left(\frac{2k}{\rho^3} + \frac{k}{a\rho^2}\right)\Delta\rho\, e^{-\left(\frac{\rho}{a}\right)}$$

$$= \left[\frac{k}{\rho^2}\left(\frac{2}{\rho} + \frac{1}{a}\right)e^{\left(-\frac{\rho}{a}\right)} - \frac{3l^2}{\mu\rho^4}\right]\Delta\rho$$

$$\Rightarrow K := \frac{3l^2}{\mu\rho^4} - \frac{k}{\rho^2}\left(\frac{2}{\rho} + \frac{1}{a}\right)e^{\left(-\frac{\rho}{a}\right)} = \frac{k}{\rho^2}\left(\frac{1}{\rho} - \frac{1}{a}\right)$$

$$\therefore \mu\frac{d^2\Delta\rho}{dt^2} + K\Delta\rho = 0, \quad \omega_{osc} = \sqrt{\frac{K}{\mu}}$$

$K > 0$ for

$$\frac{3k}{\rho^3} e^{\left(-\frac{\rho}{a}\right)} \; \frac{3l^2}{\mu\rho^4} > \frac{k}{\rho^2}\left(\frac{2}{\rho} + \frac{1}{a}\right)e^{-\frac{\rho}{a}} \Rightarrow 3 > 2 + \frac{\rho}{a}$$

$$\Rightarrow \rho < \rho_{stab} = a$$

$\therefore$,

(b) $\frac{l^2}{\mu\rho^3} = \frac{k}{\rho^2} e^{-\frac{\rho}{a}}$

$\Rightarrow l^2 = \mu k \rho e^{-\frac{\rho}{a}} = \mu^2 \rho^4 w^2$

$\Rightarrow w = \sqrt{\frac{k}{\mu\rho^3} e^{-\frac{\rho}{a}}} = \sqrt{\frac{k}{\mu\rho^3}}\, e^{-\frac{\rho}{2a}}$

$$\theta_{Aps} = \frac{2\pi}{w_{osc}} \times w$$

$$= \frac{2\pi}{\sqrt{\frac{k}{\mu\rho^2}\left(\frac{1}{\rho} - \frac{1}{a}\right)}} \times \sqrt{\frac{k}{\mu\rho^3}}\, e^{-\frac{\rho}{2a}}$$

$$= 2\pi e^{-\frac{\rho}{2a}} \cdot \sqrt{\frac{1}{\left(1 - \frac{\rho}{a}\right)}}$$

Problem 5(c)

Here we made the plot for different values of initial radius:

$$r_0 = \rho = 0.75, 0.95, and\ 1.2$$

Here we set $a = 1$.

We can see that when r_0 is smaller than 1, the orbit is very stable in a circle (like the 0.75 and 0.95 case), but when $r_0 = 1.2 > 1$, it becomes quite unstable after several turns.