Problem 1

(a)
$$p = \frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(-mc^2\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}\right)$$

$$= \frac{mc^2 \cdot \left(\frac{-1}{2}\right)}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} \cdot \left(-\frac{2\dot{x}}{c^2}\right) = \frac{m\dot{x}}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} = \gamma m\dot{x}$$

$$H = \dot{x}\frac{\partial L}{\partial \dot{x}} - L = \frac{m\dot{x}^2}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} + mc^2\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}$$

$$= \frac{mc^2}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} = \gamma mc^2 \quad \left(\gamma := \frac{1}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}}\right)$$

$\Rightarrow$ The same as special relativity result

(b) $\frac{\dot{x}}{c} \to 0$, $L \to -mc^2\left[1 - \frac{1}{2}\left(\frac{\dot{x}}{c}\right)^2\right]$

$$= -mc^2 + \frac{1}{2}m\dot{x}^2$$

$\Rightarrow$ Non-relativitistic Lagrangian

Also $H \to mc^2\left[1 + \frac{1}{2}\left(\frac{\dot{x}}{c}\right)^2\right]$

$$= \underbrace{mc^2}_{\text{Rest Mass energy}} + \underbrace{\frac{1}{2}m\dot{x}^2}_{\text{Non-rel kinetic energy}}$$

$p \to m\dot{x}$ (Momentum of low vel)

(c)

$$L = T - U = -mc^2\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2} - \frac{1}{2}m\omega^2 x^2$$

$$\Rightarrow \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\Rightarrow -m\omega^2 x = \gamma m\ddot{x} + \underline{\gamma^3 m\dddot{x}\,\dot{x}/c}$$

Negligible as $\frac{\dot{x}}{c} \to 0$

$$\Rightarrow \ddot{x} + \omega^2\sqrt{1-\left(\frac{\bar{V}}{c}\right)^2}\, x = 0$$

$\therefore$ New frequency

$$\omega' \approx \omega\left(1-\frac{\bar{V}^2}{c^2}\right)^{\frac{1}{4}} \approx \omega\left(1-\frac{\bar{V}^2}{4c^2}\right)$$

Note that the $\bar{V}$ here is some time-averaged velocity, not the maximum velocity V_{max}.

From numerical simulation, we found that $\bar{V} \approx \frac{1}{2}V_{max}$, this is understandable, as the velocity in one period varies with some sinusoidal function.