Problem 1

20/20

(a) $\vec{F} = -\frac{mk^2}{r^3}\hat{e}_r$ O

m

force-center

$y \uparrow \quad \rightarrow x$

Potential: $V = -\frac{mk^2}{2r^2}$

$$E = T + V$$

$$= \frac{1}{2} m V_x^2 - \frac{mk^2}{2r^2}$$

$$= \frac{1}{2} m \cdot 0 - \frac{mk^2}{2d^2}$$

$$\Rightarrow \quad V_x = \sqrt{k^2\left(\frac{1}{r^2} - \frac{1}{d^2}\right)} \Rightarrow V_x = \infty$$ when it hits the center

$$= k\sqrt{\frac{1}{r^2} - \frac{1}{d^2}}$$

(b)

$$F_x = \frac{mk^2}{r^3} = m\frac{dV_x}{dt}$$

$$\Rightarrow \frac{k^2}{r^3} = \frac{k}{2}\frac{1}{\sqrt{\frac{1}{r^2} - \frac{1}{d^2}}} \cdot \left(\frac{-2}{r^3}\right) \cdot \frac{dr}{dt}$$

$$\Rightarrow k = \frac{1}{\sqrt{\frac{1}{r^2} - \frac{1}{d^2}}}\left(-\frac{dr}{dt}\right)$$

$$\Rightarrow -\int_0^{\Delta T} k\, d\tau = \int_d^0 \frac{dr}{\sqrt{\frac{1}{r^2} - \frac{1}{d^2}}} = \left(\sqrt{1 - \frac{r^2}{d^2}}\right)^{-1} r\, dr$$

$$\Rightarrow -k\Delta T = d^2\left(\sqrt{1 - \frac{r^2}{d^2}} - 1\right)\Big|_{r=d}^{0}$$

$$\Rightarrow \Delta T = \frac{d^2}{k}\left(1-\sqrt{1-\frac{r^2}{d^2}}\right)\Bigg|_{r=0}^{d}$$

$\therefore$ Total time from $r=d$ to $r=0$

$$\Delta T = \frac{d^2}{k}$$

(C) In relativity, $|v| < c$, thus it is impossible that $v_x \to \infty$

$$E = T + V = \gamma m c^2 - \frac{mk^2}{2r^2}$$

$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} mc^2 - \frac{mk^2}{2r^2} = mc^2 - \frac{mk^2}{2d^2}$$

$$\Rightarrow c^2\left(1-\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{k^2}{2}\left(\frac{1}{d^2}-\frac{1}{r^2}\right)$$

$$\Rightarrow 1-\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{k^2}{2c^2}\left(\frac{1}{d^2}-\frac{1}{r^2}\right)$$

$$\Rightarrow 1-\frac{k^2}{2c^2}\left(\frac{1}{d^2}-\frac{1}{r^2}\right) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \frac{1}{\left[1-\frac{k^2}{2c^2}\left(\frac{1}{d^2}-\frac{1}{r^2}\right)\right]^2} = 1-\left(\frac{v}{c}\right)^2$$

$$\Rightarrow v = c\cdot\sqrt{1-\frac{1}{\left[1-\frac{k^2}{2c^2}\left(\frac{1}{d^2}-\frac{1}{r^2}\right)\right]^2}}$$

$\therefore$ As $r \to 0$, $v \to c$

$$v = -\frac{dr}{dt} = c\sqrt{1 - \frac{1}{\left[1 - \frac{k^2}{2c^2}\left(\frac{1}{d^2} - \frac{1}{r^2}\right)\right]^2}}$$

$$\Rightarrow \frac{dr}{\sqrt{1 - \frac{1}{\left[1 - \frac{k^2}{2c^2}\left(\frac{1}{d^2} - \frac{1}{r^2}\right)\right]^2}}} = -c\,dt$$

$$\Rightarrow \Delta T = \frac{1}{c}\int_0^d \frac{dr}{\sqrt{1 - \frac{1}{\left[1 - \frac{k^2}{2c^2}\left(\frac{1}{d^2} - \frac{1}{r^2}\right)\right]^2}}}$$

When $\left(\frac{k}{cd}\right)^2 \ll 1$, we can make an approximation that

$$\sqrt{1 - \frac{1}{\left[1 - \left(\frac{k^2}{2c^2}\right)\left(\frac{1}{d^2} - \frac{1}{r^2}\right)\right]^2}} \approx \sqrt{1 - \left[1 + \frac{k^2}{c^2}\left(\frac{1}{d^2} - \frac{1}{r^2}\right)\right]}$$

$$= \sqrt{\left(\frac{k}{c}\right)^2\left(\frac{1}{d^2} - \frac{1}{r^2}\right)}$$

Then

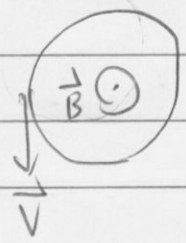
$$\Delta T \approx \frac{1}{c}\int_0^d \frac{dr}{\sqrt{\left(\frac{k}{c}\right)^2\left(\frac{1}{d^2} - \frac{1}{r^2}\right)}} = \frac{d^2}{k}$$

The same as the non-relativity result

Problem 2

20/20

(a)



$$\vec{F} = e\,\vec{V}\times\vec{B}$$

$$= eB_0(V_x, V_y, 0)\times(0,0,1)$$

$$= eB_0\,(V_y, -V_x, 0)$$

$$= m\,(a_x, a_y, 0)$$

$$\Rightarrow \begin{cases} \dfrac{dV_x}{dt} = \omega_c V_y \\ \dfrac{dV_y}{dt} = -\omega_c V_x \end{cases} \Rightarrow \frac{d}{dt}(V_x + iV_y) = -i\omega_c(V_x + iV_y)$$

$$\Rightarrow \frac{d}{dt}V_+ = -i\omega_c V_+$$

$$V_+ = V_x + iV_y = V_0 e^{-i\omega_c t}, \quad r_+ = x + iy = r_0 e^{i\theta} = r_0 e^{-i\omega_c t} \quad (r_0 = V_0/\omega_c)$$

$\Rightarrow$ Circle radius,

(angular) frequency $= \omega_c$

(b) Force caused by $B = e\,\vec{V}\times\vec{B}$

$\rightarrow$ has no component in z-direction

$\rightarrow$ Only consider effect of $\vec{E}$ on z-dir.

$$\therefore F_z = m a_z = e E_z \Rightarrow a_z = \frac{e}{m}E_z = \frac{d^2 z}{dt^2}$$

$$\Rightarrow z = \frac{eE_z}{2m}t^2 + z_0' t + z_0$$

(C) $\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$

$$= eE_0(0, 1, 0) + eB_0(v_x, v_y, 0) \times (0, 0, 1)$$

$$= e(B_0 v_y, -B_0 v_x + E_0, 0) = m(a_x, a_y, a_z)$$

$$\Rightarrow \begin{cases} \dfrac{dv_x}{dt} = \omega_c v_y \\ \dfrac{dv_y}{dt} = -\omega_c v_x + \dfrac{eE_0}{m} \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dv_x'}{dt} = \omega_c v_y \\ \dfrac{dv_y}{dt} = -\omega_c v_x' \end{cases} \qquad v_x' = v_x - \frac{eE_0}{m\omega_c}$$

$$\Rightarrow v_+' = v_0 e^{-i\omega_c t} = v_x' + iv_y$$

$$\Rightarrow \begin{cases} v_x = R_e\left\{v_0 e^{-i\omega_c t} + \dfrac{eE_0}{m\omega_c}\right\} \\ v_y = R_e\left\{v_0 e^{-i(\omega_c t + \frac{\pi}{2})}\right\} \end{cases}$$

$$\therefore At = 0, \quad v_x = v_0 + \frac{eE_0}{m\omega_c}$$

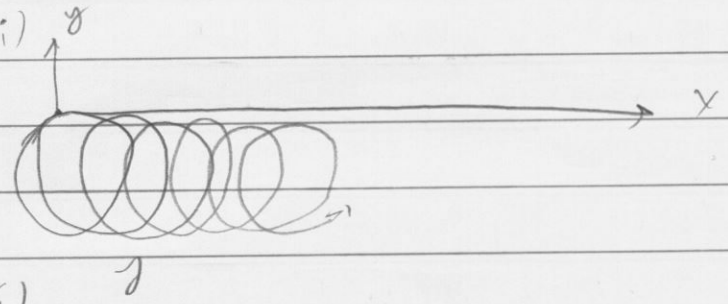
$$= \underbrace{v_0}_{=A} + \underbrace{\frac{E_0}{B}}_{=U}$$

(d)

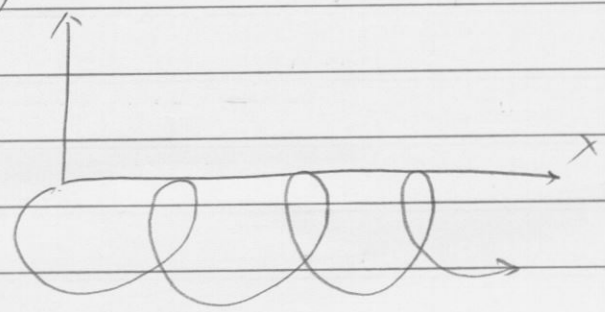
$$\begin{cases} V_x = V_0 \cos(\omega_c t) + U = \frac{dx}{dt} \\ V_y = -V_0 \sin(\omega_c t) = \frac{dy}{dt} \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{V_0}{\omega_c} \sin(\omega_c t) + Ut \\ y = \frac{V_0}{\omega_c} \cos(\omega_c t) - \frac{V_0}{\omega_c} \end{cases}$$

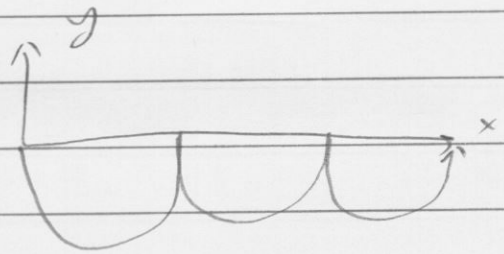
(e) (i)



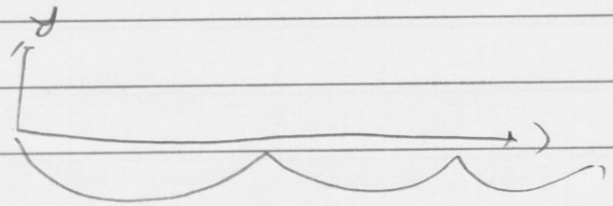
(ii)



(iii)



(iv)



Problem 3

(a)

$$\vec{F} = m\vec{a} = q(\vec{E} + \vec{V}\times B) - \gamma\vec{V}$$

Where γ is the resistivity strength.

We consider the case of orthogonal

$\vec{E} = (0, E_0, 0)$, $\vec{B} = (0, 0, B_0)$

$$\Rightarrow \begin{cases} \frac{d}{dt}V_x = \omega_c V_y - \gamma V_x \\ \frac{d}{dt}V_y = -\omega_c V_x - \gamma V_y + \frac{qE_0}{m} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d}{dt}V_x' = \omega_c V_y' - \gamma V_x' \\ \frac{d}{dt}V_y' = -\omega_c V_x' - \gamma V_y' \end{cases}$$

$$(V_x', V_y' = V_x - \underbrace{U_x}_{\text{Const.}}, V_y - \underbrace{U_y}_{\text{Const.}})$$

$$\Rightarrow V_+' = V_0 e^{-i\omega_c t - \gamma t}$$

$$\begin{cases} V_x' = V_0 \cos(\omega_c t) e^{-\gamma t} \\ V_y' = -V_0 \sin(\omega_c t) e^{-\gamma t} \end{cases}$$

$$\begin{cases} V_x = V_0 \cos(\omega_c t) e^{-\gamma t} + U_x \\ V_y = -V_0 \sin(\omega_c t) e^{-\gamma t} + U_y \end{cases}$$

We will calculate U_x, U_y, as they are related to the velocities at large times

(b) $$\begin{cases} 0 = \omega_c U_y - \gamma U_x \\ 0 = -\omega_c U_x - \gamma U_y + \frac{qE_0}{m} \end{cases}$$

$$\Rightarrow \begin{cases} \gamma U_x - \omega_c U_y = 0 \\ \omega_c U_x + \gamma U_y = \frac{qE_0}{m} \end{cases}$$

$$\Rightarrow U_x = \frac{\omega_c}{\gamma^2 + \omega_c^2} \frac{qE_0}{m}$$

$$U_y = \frac{\gamma}{\gamma^2 + \omega_c^2} \frac{qE_0}{m}$$

As $t \to \infty$, $e^{-\gamma t} \to 0$, thus

$$v_x \to U_x = \frac{\omega_c}{\gamma^2 + \omega_c^2} \frac{qE_0}{m}$$

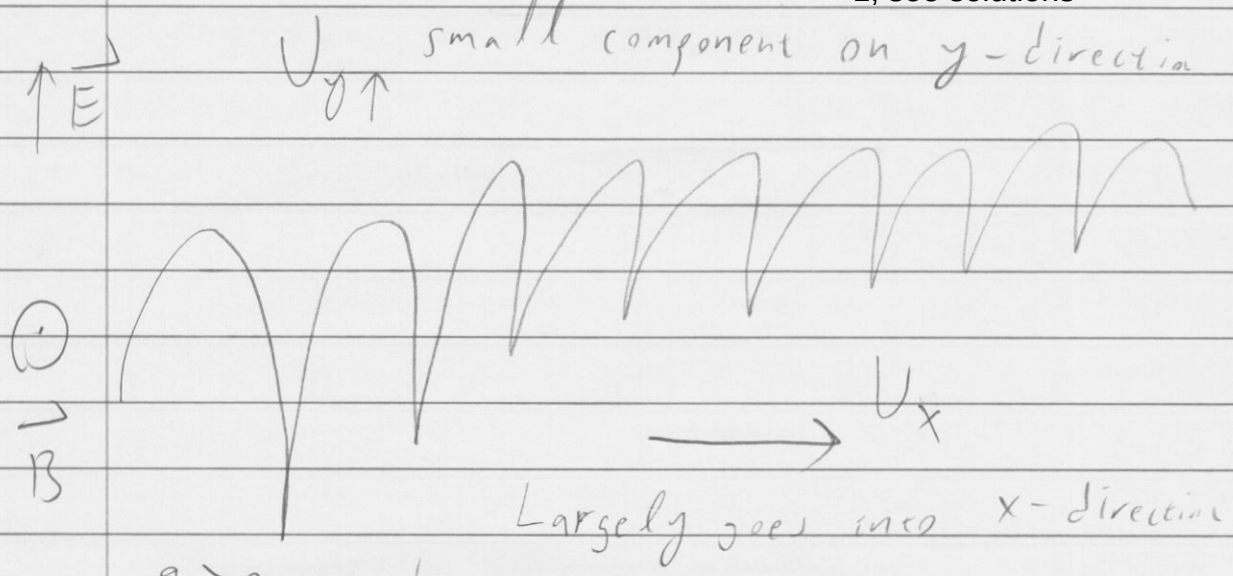
$$v_y \to U_y = \frac{\gamma}{\gamma^2 + \omega_c^2} \frac{qE_0}{m}$$

We can see that as γ becomes stronger, the direction of terminal velocity becomes more parallel in the y-direction; otherwise, parallel with x-direction.

1(c) When γ is relatively weak compared with W_c, we can see

$U_y \ll U_x$, and the orbit is like

-1, see solutions



$q > 0$ in this case

This can be intuitively understand as, when γ is small, the field by $\vec{B}$ and $\vec{E}$ must balance, more specifically,

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

$$\Rightarrow \because \vec{E} \parallel +\hat{y}$$

$$\vec{B} \parallel +\hat{z}$$

$$\therefore \vec{v} \parallel +\hat{x}$$

Problem 4 15/20

$F = ma$

$\Rightarrow m\ddot{x} = -m\omega^2 x - 2b\dot{x} + F_0 \cos(\omega t) + \lambda x^3$

$\Rightarrow m\ddot{x} + 2b\dot{x} + m\omega^2 x - \lambda x^3 = F_0 \cos(\omega t)$

At $\lambda = 0$, we have the unperturbed solution

$\bar{x}(t) = x_1 e^{+i\omega t} + x_2 e^{-i\omega t}$

$$\begin{cases} (-m\omega^2 + 2ib\omega + m\omega^2)\, x_1 = F_0 \\ (-m\omega^2 - 2ib\omega + m\omega^2)\, x_2 = F_0 \end{cases}$$

$\Rightarrow x_1 = \dfrac{+F_0}{4b\omega i}, \quad x_2 = x_1^* = \dfrac{-F_0}{4b\omega i}$

$\therefore \bar{x} = \dfrac{F_0}{2b\omega} \sin(\omega t)$

Now we add some small perturbations to the frequency and amplitudes to the solution $x(t)$ for small λ.

$x(t) = (x_1 + \Delta x_1) e^{+i(\omega+\Delta\omega)t} + (x_2 + \Delta x_2) e^{-i(\omega+\Delta\omega)t}$

$$\Rightarrow \begin{cases} \{[-m(\omega+\Delta\omega)^2 + 2ib(\omega+\Delta\omega) + m\omega^2](x_1+\Delta x_1) - 3\lambda |x_1|^2 x_1\} e^{+i\omega' t} \\ = F_0 e^{+i\omega t} \\ \{[-m(\omega+\Delta\omega)^2 - 2ib(\omega+\Delta\omega) + m\omega^2](x_2+\Delta x_2) - 3\lambda |x_2|^2 x_2\} e^{-i\omega' t} \\ = F_0 e^{-i\omega t} \end{cases}$$

(High-freq terms like $e^{\pm 3 i\omega' t}$ were neglected)

Leave out high-order terms, we have

$$\Rightarrow \begin{cases} 0 = (-2m\omega\Delta\omega + 2ib\Delta\omega)x_1 + 2ib\omega\Delta x_1 - 3\lambda|x_1|^2 x_1 \\ 0 = (2m\omega\Delta\omega - 2ib\Delta\omega)x_2 - 2ib\omega\Delta x_2 - 3\lambda|x_2|^2 x_2 \end{cases}$$

$$\Rightarrow \begin{cases} 0 = (-2m\omega\Delta\omega - 3\lambda|x_1|^2)x_1 + 2ib(\omega\Delta x_1 + \Delta\omega x_1) \\ 0 = (-2m\omega\Delta\omega - 3\lambda|x_2|^2)x_2 - 2ib(\omega\Delta x_2 + \Delta\omega x_2) \end{cases}$$

$\Rightarrow$ we have a simple solution that

$$\Delta\omega = \frac{-3\lambda|x_1|^2}{2m\omega} = \frac{-3\lambda|x_2|^2}{2m\omega} = \frac{-3\lambda F_0^2}{32m\omega^3 b^2}$$

$$\Delta x_1 = -x_1 \cdot \frac{\Delta\omega}{\omega}$$

$$\Delta x_2 = -x_2 \cdot \frac{\Delta\omega}{\omega} = \Delta x_1^* \quad (\because x_2 = x_1^*)$$

$$\Rightarrow x(t) \approx (x_1 + \Delta x_1)e^{+i(\omega+\Delta\omega)t} + c.c.$$

$$= x_1\left(1 - \frac{\Delta\omega}{\omega}\right)e^{+i(\omega+\Delta\omega)t} + c.c.$$

$$= \frac{F_0}{2b\omega}\left(1 - \frac{\Delta\omega}{\omega}\right)\sin(\omega+\Delta\omega)t$$

$$\Delta\omega = \frac{-3\lambda F_0^2}{32m\omega^3 b^2}$$

Problem 5

(a) I just made a simple guess that it will not affect the $\vec{E}$, $\vec{B}$ fields, as the absolute value of gravitational potential will not affect the gravitational force.

Now let's prove my thinking.

$$\vec{A} \to \vec{A}' = \vec{A} + \nabla\psi$$

$$\phi \to \phi' = \phi - \frac{1}{c}\frac{\partial\psi}{\partial t}$$

$$\vec{B} = \nabla\times\vec{A} \to \vec{B}' = \nabla\times\vec{A}' = \nabla\times(\vec{A}+\nabla\psi)$$

$$= \nabla\times\vec{A} = \vec{B} \quad (\because \nabla\times\nabla\psi = 0)$$

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$$

$$\to \vec{E}' = -\nabla\phi' - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}'$$

$$= -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A} + \frac{1}{c}\nabla\left(\frac{\partial\psi}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\psi)$$

$$= -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A} = \vec{E}$$

$\therefore$ Gauge transformation will not affect the fields.

(b)

$$L = \frac{1}{2} m \sum_j \dot{r}_j^2 - q\phi(\vec{r},t) + q\vec{A}(\vec{r},t)\cdot\dot{\vec{r}}$$

$$\rightarrow L' = \frac{1}{2} m \sum_j \dot{r}_j^2 - q\phi' + q\vec{A}'\cdot\dot{\vec{r}}$$

$$= L + q\left(\frac{1}{c}\frac{\partial\psi}{\partial t} + \vec{\nabla}\psi\cdot\dot{\vec{r}}\right)$$

$$= L + \frac{q}{c}\frac{d\psi}{dt}$$

(-1) No 1/c

We will see in below whether the total time-derivative term will affect the equations of motion, which I guess will not

(C) $L' = L + \Delta L$, $\Delta L = \frac{q}{c}\frac{d\psi}{dt}$

$$= \frac{q}{c}\frac{\partial\psi}{\partial t} + q\nabla\psi$$

$$\frac{d}{dt}\left(\frac{\partial \Delta L}{\partial \dot{\vec{r}}}\right) - \frac{\partial \Delta L}{\partial \vec{r}}$$

$$= \frac{d}{dt}(q\nabla\vec{\psi}) - \nabla\left(\frac{q}{c}\frac{d\psi}{dt}\right) = 0$$

$\Rightarrow$ No affect original Euler-Lagrange eq.

From the point of least action principle,

$$S'[q] = \int_a^b L'\,dt = \int_a^b L\,dt + \frac{q}{c}\int_a^b \frac{d\psi}{dt}\,dt$$

$$= S[q] + \frac{q}{c}\underline{\left[\psi(b) - \psi(a)\right]}$$

Not affected by the path, only determined by start-end.