Problem 1

(a) $\psi(x,t) = A(x,t)\, e^{i\varphi(x,t)}$, and we insert into Schrödinger eq.:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi$$

$$\Rightarrow i\hbar\left(\frac{\partial A}{\partial t} + iA\frac{\partial \varphi}{\partial t}\right)e^{i\varphi} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2 A}{\partial x^2} - A\left(\frac{\partial \phi}{\partial x}\right)^2 + 2i\left(\frac{\partial A}{\partial x}\right)\left(\frac{\partial \phi}{\partial x}\right) + iA\left(\frac{\partial^2 \varphi}{\partial x^2}\right)\right]e^{i\phi} + V(x)Ae^{i\phi}$$

$$\Rightarrow \text{Real Part: } -A\left(\frac{\partial \phi}{\partial t}\right)\hbar = \frac{\hbar^2}{2m}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 A - \left(\frac{\partial^2 A}{\partial x^2}\right)\right] + V(x)A$$

$$\text{Imaginary Part: } \frac{\partial A}{\partial t} = -\frac{\hbar}{m}\left[\left(\frac{\partial A}{\partial x}\right)\left(\frac{\partial \varphi}{\partial x}\right) + \frac{1}{2}A\left(\frac{\partial^2 \varphi}{\partial x^2}\right)\right]$$

(b)

Let's look at Im. part first,

$$\frac{1}{A}\frac{\partial A}{\partial t} + \left(\frac{\hbar}{m}\frac{\partial \varphi}{\partial x}\right)\left(\frac{1}{A}\frac{\partial A}{\partial x}\right) + \frac{1}{2}\left(\frac{\hbar}{m}\frac{\partial^2 \varphi}{\partial x^2}\right) = 0$$

$$\Rightarrow \frac{\partial \ln|A|}{\partial t} \pm \frac{\partial\left(\frac{\hbar\phi}{m}\right)}{\partial x}\cdot\frac{\partial \ln|A|}{\partial x} + \frac{1}{2}\frac{\partial^2\left(\frac{\hbar\phi}{m}\right)}{\partial x^2} = 0 \quad (1.1)$$

Thus for finite value of $\frac{\partial \varphi}{\partial x}$ or $\frac{\partial^2 \varphi}{\partial x^2}$, $\frac{\partial \ln|A|}{\partial t}$, $\frac{\partial \ln|A|}{\partial x}$ will scale with $\frac{\hbar}{m}$, and as $\hbar \to 0$, the solution of the linear PDE (1.1) will be much smaller than $\frac{\partial \phi}{\partial x}$ or $\frac{\partial^2 \phi}{\partial x^2}$

$$\therefore \left|\frac{\partial \ln|A|}{\partial x}\right| \ll \left|\frac{\partial \phi}{\partial x}\right| \Rightarrow \left|\frac{\partial A}{\partial x}\right| \ll |A|\left|\frac{\partial \phi}{\partial x}\right|$$

And then at real part,

$$-\hbar\left(\frac{\partial \phi}{\partial t}\right) A = \frac{\hbar^2}{2m}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 A - \frac{\partial^2 A}{\partial x^2}\right] + V(x) A$$

and from previous argument, we know that the spatial variance of amplitude A will be much smaller than that of the phase $\phi \implies \frac{\partial^2 A}{\partial x^2} \to 0$

$$\therefore \; -\frac{\partial(\hbar\phi)}{\partial t} = \frac{1}{2m}\left[\frac{\partial(\hbar\phi)}{\partial x}\right]^2 + V(x)$$

Hamilton-Jacobi eq., $S := \hbar\phi$

so that $-\frac{\partial S}{\partial t} = H\left(S, \frac{\partial S}{\partial x}, x\right) = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x)$

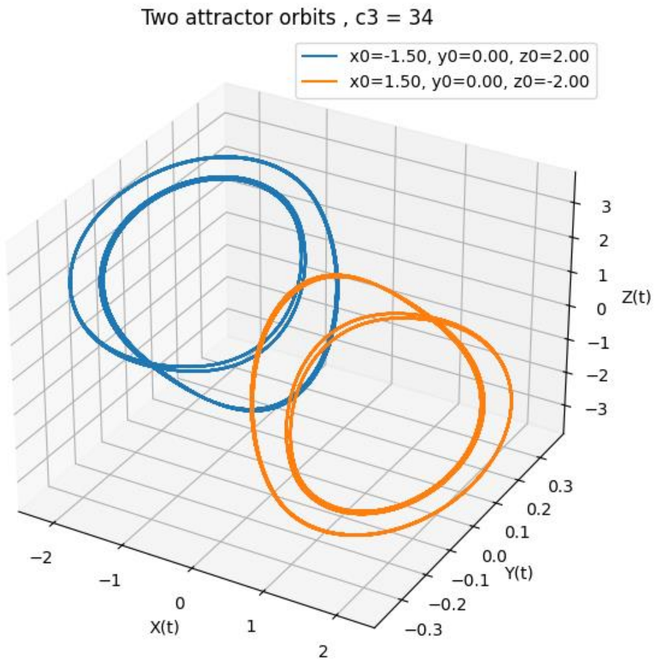
Problem2

(a)

We set c3=34, and we set two set of initial conditions:

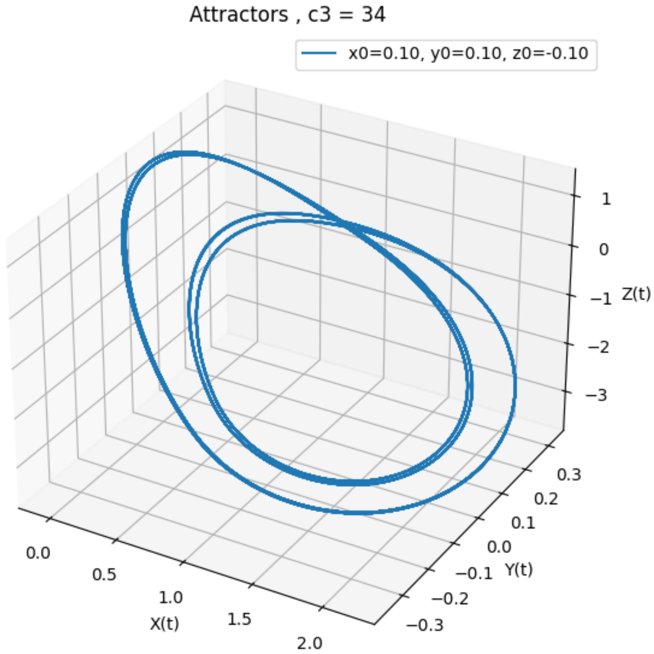
(x1=-1.50, y1 = 0.00, z1 = 2.00) and (x2=1.50, y2 = 0.00, z2 = -2.00)

Now we observe two sets of periodic orbits

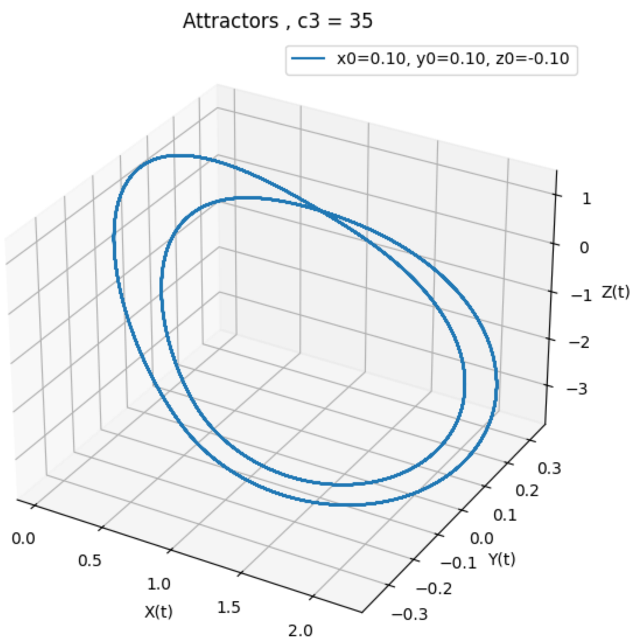


(b)

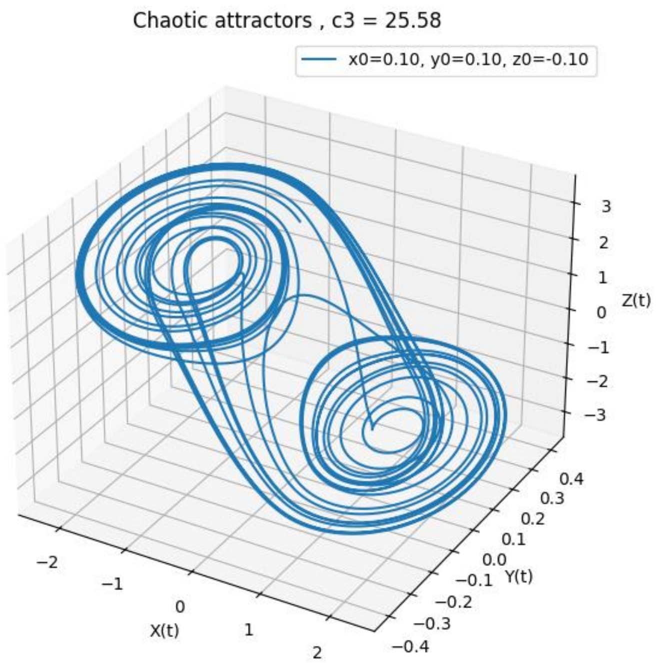
C3=34, we have period 4 solution.



C3 = 35, we have period 2 solution

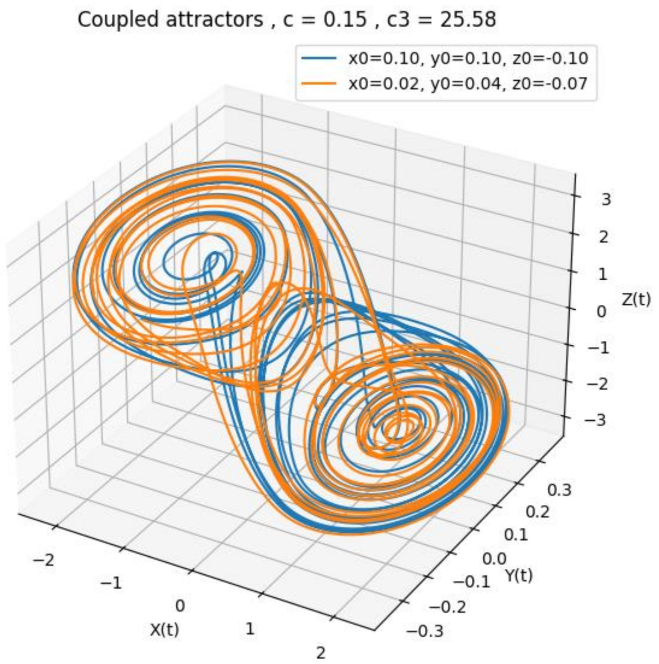


(c) At c3 = 25.58, we can see a chaotic attractor

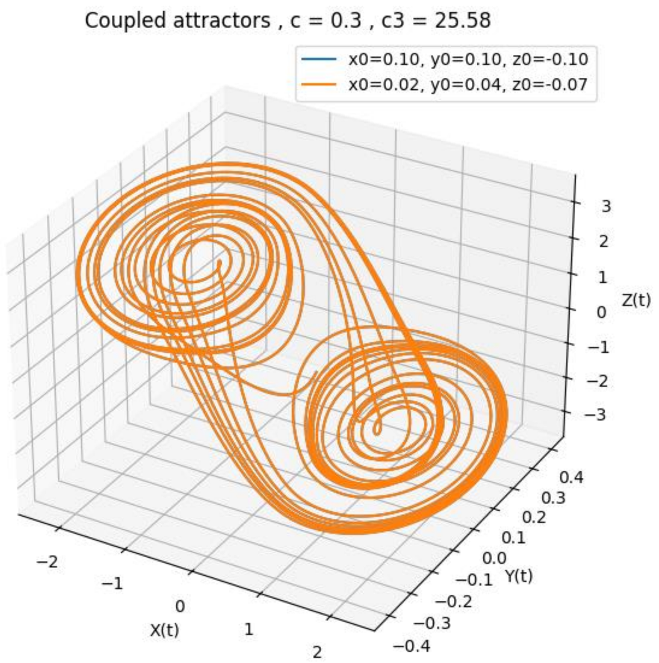


(d)

If we set coupled coefficient c=0.15, we have two uncorrelated attractors



However, if c=0.3, the two attractor orbits start to synchronize (orbits overlap)



Problem [3]

(a) Assume $\psi = \psi_r + i\psi_i$, where ψ_r = real part, ψ_i = imaginary part.

$$\therefore \mathcal{L} = |\partial_t \psi|^2 - \mu^2 |\psi|^2 - c^2 |\nabla\psi|^2$$

$$= |\partial_t \psi_r + i \partial_t \psi_i|^2 - \mu^2 |\psi_r + i\psi_i|^2$$

$$- c^2 |\nabla\psi_r + i\nabla\psi_i|^2$$

$$= (\partial_t \psi_r)^2 - \mu^2 \psi_r^2 - c^2 (\nabla\psi_r)^2$$

$$+ (\partial_t \psi_i)^2 - \mu^2 \psi_i^2 - c^2 (\nabla\psi_i)^2$$

E-L eq.:

$$\begin{cases} \frac{\partial L}{\partial \psi_r} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\psi}_r}\right) - \nabla\cdot\left(\frac{\partial L}{\partial \nabla\psi_r}\right) = 0 \\ \frac{\partial L}{\partial \psi_i} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\psi}_i}\right) - \nabla\cdot\left(\frac{\partial L}{\partial \nabla\psi_i}\right) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -\mu^2 \psi_r - \partial_t^2 \psi_r + c^2 \nabla^2 \psi_r = 0 \\ -\mu^2 \psi_i - \partial_t^2 \psi_i + c^2 \nabla^2 \psi_i = 0 \end{cases}$$

For $\mu \to 0$ just becomes normal wave eq.

$$\partial_t^2 \psi = c^2 \nabla^2 \psi$$

(b) From (a), we know

$$\mu^2 \psi + \partial_t^2 \psi = c^2 \nabla^2 \psi$$

$$\psi(\vec{r}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \, e^{+i\vec{k}\cdot\vec{r}} \tilde{\psi}(\vec{k}, t)$$

$\therefore$ The E.M in Fourier transform to k-space becomes

$$\mu^2 \tilde{\psi} + \partial_t^2 \tilde{\psi} = -c^2 |\vec{k}|^2 \tilde{\psi}$$

$$\Rightarrow -\frac{\partial^2}{\partial t^2} \tilde{\psi}(\vec{k}, t) = (\mu^2 + c^2 |\vec{k}|^2) \tilde{\psi}(\vec{k}, t)$$

We may assume $\tilde{\psi}(\vec{k}, t) = \phi(\vec{k}) e^{-i\omega t}$

$$\therefore \quad \omega^2 \phi = (\mu^2 + c^2 |\vec{k}|^2) \phi(\vec{k})$$

$$\Rightarrow \quad \omega = \pm \sqrt{\mu^2 + c^2 |\vec{k}|^2}$$

$\therefore$ The wavefunction in k-space has the time-dependence:

$$\tilde{\psi}(\vec{k}, t) = \phi(\vec{k}) \, e^{\pm i \sqrt{\mu^2 + c^2 |\vec{k}|^2} \, t}$$

And we can see that as $k \to 0$

$$\omega(\vec{k}) \to \pm |\mu| \neq 0 \text{ for finite } \mu$$

(C)

$E = \hbar \omega_{\vec{k}} = \hbar \sqrt{\mu^2 + c^2 |\vec{k}|^2}$ (Now we just restrict to positive energy case)

We can see that

$$E^2 = (\hbar\mu)^2 + c^2 p^2 \qquad p^2 := \hbar^2 |k|^2$$

$\therefore$ Corresponding to the relativitistic $E-p$ dispersion, and rest mass m is

$$m = \frac{\hbar \mu}{c^2}, \quad \text{so that}$$

$$E^2 = (mc^2)^2 + p^2 c^2$$

(d)

$$L = |\partial_t \psi|^2 + \mu^2 |\psi|^2 - c^2 |\nabla \psi|^2 - u|\psi|^4$$

$$\begin{aligned} &= (\partial_t \psi_r)^2 + \mu^2 \psi_r^2 - c^2 (\nabla \psi_r)^2 \\ &\quad + (\partial_t \psi_i)^2 + \mu^2 \psi_i^2 - c^2 (\nabla \psi_i)^2 \\ &\quad - u (\psi_r^2 + \psi_i^2)^2 \end{aligned}$$

E-L eq. becomes:

$$\left\{ \begin{aligned} &\frac{\partial L}{\partial \psi_r} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\psi}_r}\right) - \nabla \cdot \left(\frac{\partial L}{\partial \nabla \psi_r}\right) = 0 \\ &\frac{\partial L}{\partial \psi_i} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\psi}_i}\right) - \nabla \cdot \left(\frac{\partial L}{\partial \nabla \psi_i}\right) = 0 \end{aligned} \right.$$

$$\Rightarrow \left\{ \begin{aligned} &\mu^2 \psi_r - 2u (\psi_r^2 + \psi_i^2) \psi_r - \partial_t^2 \psi_r + c^2 \nabla^2 \psi_r = 0 \\ &\mu^2 \psi_i - 2u (\psi_r^2 + \psi_i^2) \psi_i - \partial_t^2 \psi_i + c^2 \nabla^2 \psi_i = 0 \end{aligned} \right.$$

$$\Rightarrow \mu^2 \psi - 2u |\psi|^2 \psi - \partial_t^2 \psi + c^2 \nabla^2 \psi = 0$$

For lowest energy, we assume that $\psi(\vec{r}, t) = \bar{\psi}$ = const.

$$\therefore \mu^2 \bar{\psi} - 2u |\bar{\psi}|^2 \bar{\psi} = 0$$

$\therefore$ This problem has infinite solutions

because as long as

$$|\bar{\Psi}|^2 = \frac{\mu^2}{2u}, \quad \bar{\Psi} = \sqrt{\frac{\mu^2}{2u}}\, e^{i\theta} \quad \forall\, \theta \in R$$

is a solution.

(e) $\psi := (\bar{\Psi} + \phi)\, e^{i\theta}$ $\quad \phi :=$ real

$$L = |\partial_t \psi|^2 + \mu^2 |\psi|^2 - c^2 |\nabla \psi|^2 - u|\psi|^4$$

$$= |\partial_t \phi + i\bar{\Psi}\partial_t \theta|^2 - c^2 |\nabla\phi + i\bar{\Psi}\nabla\theta|^2$$

$$+ [\mu^2 - u(\bar{\Psi}+\phi)^2](\bar{\Psi}+\phi)^2$$

$$= (\partial_t \phi)^2 - c^2 (\nabla\phi)^2 + (\mu^2 \bar{\Psi}^2 - u\bar{\Psi}^4)$$

$$+ (2\mu^2 \phi \bar{\Psi} - 4u\phi\bar{\Psi}^3)\ (=0)$$

To Quardratic terms $+ (\mu^2 \phi^2 - 6u\phi^2 \bar{\Psi}^2)$

$$+ \bar{\Psi}^2 [(\partial_t \theta)^2 - c^2 \nabla\theta]$$

$$= (\partial_t \phi)^2 - c^2 (\nabla\phi)^2 + \underbrace{\left(\frac{\mu^4}{4u}\right)}_{\text{const.}} - 2\mu^2 \phi^2$$

const. $\rightarrow$ eliminate

$$+ \bar{\Psi}^2 [(\partial_t \theta)^2 - c^2 (\nabla\theta)^2]$$

$$:= L1(\partial_t \phi, \nabla\phi, \phi, t, \vec{r}) \quad (:= (\partial_t \phi)^2 - c^2(\nabla\phi)^2 - 2\mu^2\phi^2)$$

$$+ L_2(\partial_t \theta, \nabla\theta, \theta, t, \vec{r}) \quad (:= [(\partial_t \theta)^2 - c^2(\nabla\theta)^2]\bar{\Psi})$$

(f) L1 will give the E.M of ϕ, which we have known how to solve from (a), (b), (C), and it will give

$$W_\phi(\vec{k}) = \sqrt{c^2 |\vec{k}|^2 + 2\mu^2}$$

L^2, however, will give a gapless dispersion:

$$\omega_\theta(\vec{k}) = c|\vec{k}|$$

$$\Rightarrow \tilde{\phi}(\vec{k}, t) = \tilde{\phi}(\vec{k}) \cdot e^{i\omega_\phi(\vec{k})t}$$

$$\tilde{\theta}(\vec{k}, t) = \tilde{\theta}(\vec{k})\, e^{i\omega_\theta(\vec{k})t}$$

Problem 4

(a)

$$L = |\partial_t \psi|^2 + \mu^2|\psi|^2 - u|\psi|^4 - c^2|(\vec{\nabla} - iq\vec{A})\psi|^2 + \frac{\gamma}{2}\left[\frac{(\partial_t \vec{A})^2}{c^2} - (\nabla\times\vec{A})^2\right]$$

$$= (\partial_t\psi)(\partial_t\psi^*) + (\mu^2 - c^2q^2A^2 - u\psi\psi^*)\psi\psi^* - c^2(\vec{\nabla}\psi)\cdot(\vec{\nabla}\psi^*) - ic^2q[(\vec{\nabla}\psi)\cdot(\vec{A}\psi^*) - (\vec{\nabla}\psi^*)\cdot(\vec{A}\psi)] - \frac{\gamma}{4}F_{\mu\nu}{}^2$$ (See Appendix 4(a) for details)

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu)$$

① E - L on ψ^* will be

$$\frac{\partial L}{\partial\psi^*} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{\psi}^*}\right) - \frac{\partial}{\partial x_\mu}\left(\frac{\partial L}{\partial\partial_{x_\mu}\psi^*}\right) = 0$$

$$\Rightarrow (\mu^2 - c^2q^2A^2 - 2u|\psi|^2)\psi - \partial_t^2\psi + c^2(\nabla^2\psi) - 2ic^2q(\vec{\nabla}\psi)\cdot\vec{A} = 0$$

Insere solution $\psi = \bar{\psi}$ and $\vec{A} = 0$ inside we found

$$(\mu^2 - 2u|\bar{\psi}|^2)\bar{\psi} = 0 \Rightarrow$$ the same as Problem[3]

② To be assured, we look at E - L on A_ν

$$2c^2q^2|\psi|^2A^\nu - ic^2q[(\partial^\nu\psi)\psi^* - (\partial^\nu\psi^*)\psi]$$ (see appendix 4(a))

$$+\gamma[\underbrace{+\partial_\mu(\partial^\mu A^\nu)}_{\text{Gives wave eq. of}} - \underbrace{\partial^\nu(\partial_\mu A^\mu)}_{\text{Choose Lorentz gauge} = 0}] = 0$$

$$\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A}$$

For $\vec{A} = 0$, $\psi = \bar{\psi}$, ② also satisfies

Appendix 4 (a)

① If we define electromagnetic tensor $F_{\mu\nu}$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{bmatrix} 0 & +\frac{E_1}{c} & +\frac{E_2}{c} & +\frac{E_3}{c} \\ -\frac{E_1}{c} & 0 & -B_3 & B_2 \\ -\frac{E_2}{c} & B_3 & 0 & -B_1 \\ -\frac{E_3}{c} & -B_2 & B_1 & 0 \end{bmatrix}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -\frac{E_1}{c} & -\frac{E_2}{c} & -\frac{E_3}{c} \\ +\frac{E_1}{c} & 0 & -B_3 & B_2 \\ +\frac{E_2}{c} & B_3 & 0 & -B_1 \\ +\frac{E_3}{c} & -B_2 & B_1 & 0 \end{bmatrix}$$

(+ − − − convention)

$$\therefore F_{\mu\nu}^2 := F_{\mu\nu}F^{\mu\nu} = 2B^2 - \frac{2E^2}{c^2} \Rightarrow -\frac{\gamma}{4}F_{\mu\nu}^2 = \frac{\gamma}{2}\left[\frac{(\partial_t \vec{A})^2}{c^2} - (\nabla\times\vec{A})^2\right]$$

$$= L_{\text{Gauge field}}$$

② E-L eq. regarding A_ν:

$$L = -c^2q^2\vec{A}^2 - ic^2q\left[(\vec{\nabla}\psi)\cdot(\vec{A}\psi^*) - (\vec{\nabla}\psi^*)\cdot(\vec{A}\psi)\right]$$

$$-\frac{\gamma}{4}F_{\mu\nu}^2 \text{ (We only include terms related to } A_\nu)$$

$$= c^2q^2A_\nu A^\nu - ic^2q(\partial^\nu\psi A_\nu\psi^* - \partial^\nu\psi^* A_\nu\psi) - \frac{\gamma}{4}F_{\mu\nu}^2$$

Then E-L of A_ν is

$$\frac{\partial L}{\partial A_\nu} - \partial_\mu\left[\frac{\partial L}{\partial(\partial_\mu A_\nu)}\right] = 2c^2q^2A^\nu - ic^2q\left[(\partial^\nu\psi)\psi^* - (\partial^\nu\psi^*)\psi\right] + \frac{\gamma}{4}\partial_\mu\left[\frac{\partial F_{\mu\nu}^2}{\partial(\partial_\mu A_\nu)}\right]$$

Where

$$\frac{\gamma}{4}\partial_\mu\left[\frac{\partial F_{\mu\nu}^2}{\partial(\partial_\mu A_\nu)}\right] = \frac{\gamma}{4}\partial_\mu\left\{\frac{\partial[(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)]}{\partial(\partial_\mu A_\nu)}\right\}$$

$$= \frac{\gamma}{2}\partial_\mu\left[\frac{\partial(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\mu\partial^\nu A^\nu)}{\partial(\partial_\mu A_\nu)}\right]$$

$$= \gamma\,\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \gamma\,\partial_\mu F^{\mu\nu}$$

(b) $\psi = (\bar{\Psi} + \phi)\, e^{i\theta}$, $\vec{A} = \vec{a}$

$$\therefore L = |\partial_t \phi + i \bar{\Psi} \partial_t \theta|^2 - c^2 |\vec{\nabla}\phi + \bar{\Psi}\vec{\nabla}\theta - iq\vec{a}\bar{\Psi}|^2$$
$$+ [\mu^2 - u(\bar{\Psi}+\theta)^2](\bar{\Psi}+\theta)^2 \quad - \frac{\gamma}{4} \quad f_{\mu\nu}^2$$

$(f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu)$

$$= \left.\begin{matrix} [(\partial_t\phi)^2 - c^2(\nabla\phi)^2] + \bar{\Psi}^2[(\partial_t\theta)^2 - c^2(\nabla\theta)^2] \\ + \left(\frac{\mu^4}{4u}\right) - 2\mu^2\phi^2 \end{matrix}\right] \to L_1(\phi,\theta)$$
$$- c^2 q^2 \bar{\Psi}^2 |\vec{a}|^2 \quad - \frac{\gamma}{4} \quad f_{\mu\nu}^2 \quad \to L_2(a_\mu)$$

Where the $L_2(\vec{a})$ part gives the wave eq. of gauge field a_μ

(c)

From the E-L eq. in ② in 4(a), we can solve the E-L eq. in $L_2(\vec{a})$ in (b):

$$2c^2 q^2 |\bar{\psi}|^2 a^\nu + \gamma \partial_\mu (\partial^\mu a^\nu) = 0$$

(Here we already chose Lorentz gauge)

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2 \vec{A} + \frac{2c^2q^2}{\gamma}|\bar{\psi}|^2 \vec{A} = 0$$

$$\Rightarrow -\omega_A^2 + c^2 k_A^2 + \left(\frac{\sqrt{2}c^2 q}{\sqrt{\gamma}}\right)^2 = 0$$

$$\Rightarrow \hbar\omega_A = \sqrt{[c(\hbar k_A)]^2 + \left(\frac{\sqrt{2}\hbar c^2 q}{\sqrt{\gamma}}\right)^2}$$

$$E_A = \sqrt{(p_A c)^2 + (m_A c^2)^2}$$

$m_A = \frac{\sqrt{2}\hbar q}{\sqrt{\gamma}}$ is the rest mass of gauge field

For $\theta(\vec{k},t)$ and $\phi(\vec{k},t)$, the answers are the same as in Problem 3 (f)

$$\omega_\phi(\vec{k}_\phi) = \sqrt{c^2|\vec{k}_\phi|^2 + 2\mu^2}$$

$$\omega_\theta(\vec{k}_\theta) = c|\vec{k}_\theta|$$