HW 10

Problem 1

(a)

$$H = \frac{p^2}{2m} + \sigma|x|$$

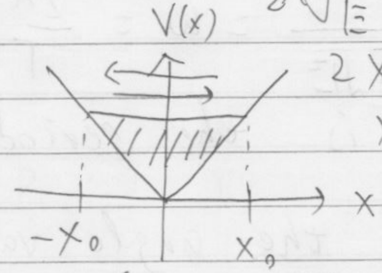
Since $\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$ and $H = E = T + V$

$\Rightarrow E$ is conserved

$\Rightarrow p(x, E) = \pm\sqrt{2m}\sqrt{E - \sigma|x|}$

Now we want to calculate for a constant energy E, the action variable $J(E)$

$$J(E) := \frac{1}{2\pi}\oint_E p\, dx$$



$2x_0$ is the range of the motion, $x_0(E) = \frac{E}{\sigma}$ (assume $E > 0$)

And we can determine the sign of $p(x, E)$:

$p(x, E) > 0$ for motion from $-x_0$ to $+x_0$

< 0 for motion from $+x_0$ to $-x_0$

And also the motion is symmetric

$\therefore$

$$J = \frac{1}{2\pi}\oint_E p\, dx = \frac{2}{2\pi}\int_{-x_0}^{x_0} +\sqrt{2m}\sqrt{E - \sigma|x|}\, dx$$

$$= \frac{4}{2\pi}\int_0^{x_0}\sqrt{2m}\sqrt{E - \sigma x}\, dx$$

$$= \frac{2\sqrt{2m}\sqrt{E}}{\pi}\int_0^1 \sqrt{1-u}\, du \cdot \frac{E}{\sigma} \quad \left(u := \frac{\sigma x}{E}\right)$$

$$= \frac{2\sqrt{2m}\, E\sqrt{E}}{\pi\sigma}\cdot\frac{2}{3} = \frac{4\sqrt{2m}\, E\sqrt{E}}{3\pi\sigma}$$

Action variable is

$$\therefore\ J(E) = \frac{4\sqrt{2m}}{3\pi}\ \frac{E\sqrt{E}}{\sigma}$$

And the corresponding angle variable is

$$\dot{\theta} = \frac{\partial H}{\partial J}$$

$$H = E = E(J) = \left(\frac{3\pi\sigma J}{4\sqrt{2m}}\right)^{\frac{2}{3}}$$

$$\therefore\ \dot{\theta} = \frac{\partial H(J)}{\partial J} = \frac{2}{3J}\cdot\left(\frac{3\pi\sigma J}{4\sqrt{2m}}\right)^{\frac{2}{3}}$$

$$= \frac{2}{3}\frac{E}{J} = \frac{2E}{3}\cdot\frac{3\pi\sigma}{4\sqrt{2m}\,E\sqrt{E}}$$

$$= \frac{\pi\sigma}{2\sqrt{2m}\sqrt{E}} = \omega = \frac{2\pi}{T}$$

$\Rightarrow$ $T = \dfrac{4\sqrt{2m}\sqrt{E}}{\sigma}$ is the period

$\theta = \dfrac{\pi\sigma t}{2\sqrt{2m}\sqrt{E}}$ is the angle variable

(b) In adiabatic process, $J(E)$ will not change

① Energy E

$\therefore J(E) = \frac{4\sqrt{2m}}{3\pi} \frac{E\sqrt{E}}{6}$ is conserved

$\therefore E \propto (6J)^{\frac{2}{3}} \propto 6^{\frac{2}{3}}$

$\therefore E = E_0 \cdot \left(\frac{6(t)}{6(t=0)}\right)^{\frac{2}{3}}$ will change with 6 accordingly.

② Amplitude $x_0(E)$

Since $x_0 = x_0(E) = \frac{E(6)}{6}$, and $E \propto 6^{\frac{2}{3}}$

$\therefore x_0 \propto 6^{\frac{-1}{3}} \Rightarrow x_0 = x_0(6)$

$= x_0(6_0) \cdot \left(\frac{6(t)}{6(t=0)}\right)^{\frac{-1}{3}}$

$6_0 = 6(t=0)$

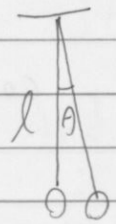
③ Period T,

$T = T(E) = 4\sqrt{2m}\frac{\sqrt{E}}{6} \propto \frac{\sqrt{E}}{6} \propto 6^{\frac{-2}{3}}$

$\therefore T(6) = T(6(t=0)) \cdot \left(\frac{6(t)}{6(t=0)}\right)^{\frac{-2}{3}}$

Problem 2

$$I = ml^2$$



$$H = \frac{p_\theta^2}{2I} + mgl(1-\cos\theta)$$

$$\simeq \frac{p_\theta^2}{2I} + \frac{mgl}{2}\theta^2 \quad (\because \theta \text{ is small})$$

$H = E$ is conserved ($\because \frac{\partial H}{\partial t} = 0$)

$$\therefore \quad p_\theta = \pm\sqrt{2I\left(E - \frac{mgl}{2}\theta^2\right)}$$

$$= \pm\sqrt{2EI - mglI\theta^2}$$

Action is

$$J(E) = \frac{1}{2\pi}\oint_E p_\theta \, d\theta = \frac{4}{2\pi}\int_0^{\theta_{max}} \sqrt{2EI - mglI\theta^2}\, d\theta$$

$$= \frac{2}{\pi}\sqrt{2EI}\int_0^{b_{max}} \sqrt{1-b^2}\, db \sqrt{\frac{2E}{mgl}}$$

$$\left(b := \sqrt{\frac{mgl}{2E}}\,\theta,\ b_{max} = \sqrt{\frac{mgl}{2E}}\,\theta_{max} = \sqrt{\frac{mgl}{2E}}\cdot\sqrt{\frac{2E}{mgl}} = 1\right)$$

$$= \frac{4E\sqrt{I}}{\pi\sqrt{mgl}}\int_0^1 \sqrt{1-b^2}\, db$$

$$= \frac{4}{\pi}\sqrt{\frac{l}{g}}\,E \cdot \frac{\pi}{4} = \sqrt{\frac{l}{g}}\,E$$

$$\Rightarrow H = E = \sqrt{\frac{g}{l}}\,J(E)$$

$\therefore$ Corresponding angle variable ϕ is

$$\dot{\phi} = \frac{\partial H}{\partial J} = \sqrt{\frac{g}{l}} = \omega = \frac{2\pi}{T}$$

$\Rightarrow T = 2\pi\sqrt{\frac{l}{g}}$ is the period which is the same as the period from our understanding of small angle pendulum.

Amplitud $\theta_{max}(E) = \sqrt{\frac{2E}{mgl}} \propto \sqrt{\frac{E}{l}}$

Now let's look at adiabatic process.

$J(E) = \sqrt{\frac{l}{g}}\, E$ will be conserved

$\Rightarrow E = \sqrt{\frac{g}{l}}\, J \propto \sqrt{\frac{1}{l}}$ as l changes slowly

① Amplitude θ_{max}

$\theta_{max}(E) \propto \sqrt{\frac{E}{l}} \propto \frac{1}{l}$ as l changes

② Energy E

$E \propto \sqrt{\frac{1}{l}}$

③ frequency $\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{l}} \propto \frac{1}{\sqrt{l}}$

Problem 3

(a)

$$L = T - V$$

$$= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\Rightarrow H = p_r \cdot \dot{r} + p_\theta \dot{\theta} - L$$

$$= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r} = T + V$$

Since

$H = T + V = E$, and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \Rightarrow \frac{dE}{dt} = 0,\ E \text{ is conserved}$$

And $\frac{dp_\theta}{dt} = \frac{-\partial H}{\partial \theta} = 0 \Rightarrow p_\theta$ is also conserved

let's make $p_\theta = l$ = angular momentum afterwards

(b) Now let's look at the Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} - \frac{GMm}{r} := \frac{p_r^2}{2m} + \underbrace{V_{eff}(r)}_{:= \frac{l^2}{2mr^2} - \frac{GMm}{r}}$$

$$\therefore\ p_r(r, E) = \pm 2m\sqrt{E - V_{eff}(r)}$$

$\therefore$ Action variable

$$I_r(E, p_\theta = l, M) = \oint \frac{p_r}{2\pi}\, dr = \frac{2}{2\pi}\int_{r_{min}}^{r_{max}} dr \sqrt{2m\left(E - \frac{l^2}{2mr^2} + \frac{GMm}{r}\right)}$$

$$= \frac{l}{\pi}\int_{r_{min}}^{r_{max}} \sqrt{\left(\frac{1}{r_{min}} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_{max}}\right)}\, dr$$

($\because$ $E - \frac{l^2}{2mr^2} + \frac{GMm}{r} = 0$ has two roots of $\frac{1}{r_{min}}$ and $\frac{1}{r_{max}}$, at which $p_r = 0$ because r will have min / max and start increase / decrease.)

Make $r_{min} = \alpha$, $r_{max} = \beta$

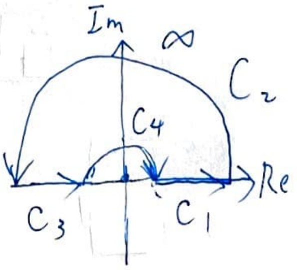
$$I_r = \frac{l}{\pi}\int_\alpha^\beta \sqrt{\left(\frac{1}{\alpha} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{\beta}\right)}\, dr = \frac{l}{\pi}\int_\beta^\alpha \sqrt{\left(\frac{r}{\alpha} - 1\right)\left(1 - \frac{r}{\beta}\right)}\, dr$$

$$= \frac{l}{\pi\sqrt{\alpha\beta}}\int_\alpha^\beta \sqrt{(r-\alpha)(\beta - r)}\left(\frac{dr}{r}\right)$$

We can use Contour Integral to solve.

$$I_r = \frac{l}{\pi\sqrt{\alpha\beta}}\int_\alpha^\beta \sqrt{(r-\alpha)(r-\beta)}\,\frac{dr}{r} = \mathrm{Re}\left\{ P\int_{-\infty}^{\infty} \sqrt{(r-\alpha)(\beta - r)}\,\frac{dr}{r} \right\}\frac{l}{\pi\sqrt{\alpha\beta}}$$

$$= \mathrm{Re}\left\{ \int_{C_1 + C_3} \sqrt{(z-\alpha)(\beta - z)}\,\frac{dz}{z} \right\}\frac{l}{\pi\sqrt{\alpha\beta}}$$



$$= -\mathrm{Re}\left\{ \int_{C_2 + C_4} \sqrt{(z-\alpha)(\beta - z)}\,\frac{dz}{z} \right\}\frac{l}{\pi\sqrt{\alpha\beta}}$$

① At C_4 (Around 0 pole)

$$-\mathrm{Re}\left\{\oint_{C_4}\sqrt{(z-\alpha)(\beta-z)}\,\frac{dz}{z}\right\} = \lim_{\varepsilon\to 0^+}\mathrm{Re}\left\{\int_0^{\pi}\sqrt{(\varepsilon e^{i\theta}-\alpha)(\beta-\varepsilon e^{i\theta})}\,\frac{d(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\right\}$$

$$= \sqrt{-\alpha\beta}\; i\pi = -\sqrt{\alpha\beta}\,\pi$$

$$\therefore I_r(C_4) = \frac{l}{\pi\sqrt{\alpha\beta}}\left(-\sqrt{\alpha\beta}\,\pi\right) = -l$$

② At C_2 (Around ∞ pole)

$$-\mathrm{Re}\left\{\oint_{C_2}\sqrt{(z-\alpha)(\beta-z)}\,\frac{dz}{z}\right\} = -\mathrm{Re}\left\{\oint_{C_2}\sqrt{-z^2+(\alpha+\beta)z-\alpha\beta}\,\frac{dz}{z}\right\}$$

$$= -\mathrm{Re}\left\{\oint_{C_2}\left(1-\frac{\alpha}{2z}\right)\left(1-\frac{\beta}{2z}\right) i\,dz\right\} = -\left(\frac{\alpha+\beta}{2}\right)(-i)\cdot\underbrace{(-\pi i)}$$

$$= \left(\frac{\alpha+\beta}{2}\right)\pi$$

Residue at ∞ is opposite in sign compared to residue at 0

$$\Rightarrow I_r(C_2) = \frac{l}{\pi\sqrt{\alpha\beta}}\left(\frac{\alpha+\beta}{2}\right)\cdot\pi = \frac{l(\alpha+\beta)}{2\sqrt{\alpha\beta}}$$

$$\therefore I_r = -l + \frac{l(\alpha+\beta)}{2\sqrt{\alpha\beta}}$$

And from previous knowledge of Kepler's motion, we know that

$$\alpha = \frac{a}{1+\varepsilon},\quad \beta = \frac{a}{1-\varepsilon},\quad a = \frac{l^2}{m(GMm)},\quad \varepsilon = \sqrt{1-\frac{2|E|\,l^2}{G^2M^2m^3}}$$

$$\therefore I_r = -l + \frac{l}{\sqrt{1-\varepsilon^2}} = -l + \frac{GMm\sqrt{m}}{\sqrt{2|E|}}$$

(c)

The adiabatic invariant for angle motion is

$$I_\theta(E) = \frac{1}{2\pi}\oint p_\theta \, d\theta = \ell$$

$\Rightarrow$ ℓ (angular momentum) will be conserved for adiabatic process

And since $I_r(E, p_\theta = \ell, M)$ will also be conserved

$$\Rightarrow I_r = -\ell + \frac{GMm\sqrt{m}}{\sqrt{2}\sqrt{|E|}}$$

will not change when M changes

$\therefore$ The ratio $\frac{M}{\sqrt{|E|}}$ will not change

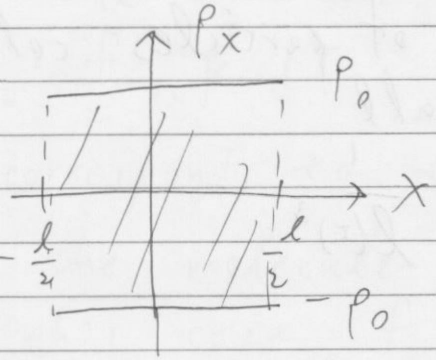
$\Rightarrow \sqrt{|E|} \propto M$

$|E| \propto M^2$ over long times

Problem 4

(a)

In the motion of a single particle over one period T;



The adiabatic invariant

$$I(E) = \frac{1}{2\pi}\oint_E p_x\,dx = \frac{1}{2\pi} p_0 \cdot 2l = \frac{p_0 l}{\pi}$$

In an adiabatic process $l = l(t)$,

$I(E)$ will be conserved

$$\Rightarrow \quad p_0 l = \text{const.} \Rightarrow p_0 \propto \frac{1}{l(t)}$$

Moreover, the period to complete one cycle also changes with $l(t)$,

$$T = \frac{2l}{(p_0/m)} \propto l^2(t)$$

We know that pressure P is the average rate of exchange of momentum at the boundary, i.e.

$$P \propto \underbrace{\frac{N}{T}}_{\text{rate of particles colliding with the wall}} \cdot p_0 \quad \rightarrow \text{average momentum}$$

rate of particles colliding with the wall

$$\therefore P \propto \frac{p_0}{T} \propto \frac{1}{l(t)^3}$$

(b) We assume that temperature Temp will be proportional to the average kinetic energy

$$\text{Temp}(\bar{E}) = \frac{p_0^2}{2m} \cdot C \propto \frac{1}{l(t)^2}$$

(c) From (a), we know

$$P \propto \frac{1}{l^3}, \text{ and since } V \propto l$$

$\therefore PV^3$ is a constant in this case

And from our understanding of classical thermodynamics, the adiabatic expansion / compression has the following formula:

$PV^{\gamma} = \text{const.}$, $\gamma := \frac{f+2}{f}$

and the internal energy will be

$dU = \frac{f}{2} nR\,dT$

f corresponds to the degree of freedom of the molecule motion.

In this case, $f = 1 \Rightarrow \gamma = \frac{f+2}{f} = 3$

$\therefore \gamma = 3$ is consistent.

(d) Since the walls expand too fast, it has no time to do work on the particles.

$\Rightarrow$ Kinetic energy of gas particles are approximately conserved

$\Rightarrow T = \text{Temp}$ is conserved

$\therefore \text{Temp} = \text{Temp}$ (E)

V definitely increases.

$P \propto \frac{P_0}{T(\text{Period})} = \frac{P_0}{\frac{2l}{(P_0/m)}} \propto \frac{1}{l}$ ($\because P_0$ conserved)

reduces