HW 2

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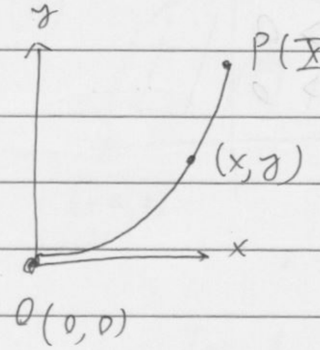
Problem 1



By conservation of energy, $v = \sqrt{2gx}$ at any point (x, y).

At endpoint P, $\begin{cases} x(\tau = T) = X \\ y(\tau = T) = Y \end{cases}$

$$J = \int \sqrt{(dt^2 + \gamma\, ds^2)}$$

$$= \int \sqrt{\left(\frac{ds}{v}\right)^2 + \gamma\, ds^2} \qquad \dot{x} = \frac{dx}{d\tau}, \quad \dot{y} = \frac{dy}{d\tau}$$

$$= \int \sqrt{\left(\frac{1}{v^2} + \gamma\right)}\, \sqrt{\dot{x}^2 + \dot{y}^2}\; d\tau$$

$$= \int \sqrt{\left(\frac{1}{2gx} + \gamma\right)(\dot{x}^2 + \dot{y}^2)}\; d\tau, \quad L = \sqrt{\left(\frac{1}{2gx} + \gamma\right)(\dot{x} + \dot{y}^2)}$$

$$\frac{\partial L}{\partial y} = \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$$

$$\therefore \frac{\partial L}{\partial \dot{y}} = \sqrt{\frac{1}{2gx} + \gamma}\; \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \text{Const.} := C$$

$$\Rightarrow \frac{u^2}{u^2+1} = \frac{C}{\gamma + \frac{1}{2gx}} = \frac{2Cgx}{2\gamma gx + 1} \quad \left(u := \frac{dy}{dx}\right)$$

$$\Rightarrow u = \frac{dy}{dx} = \sqrt{\frac{2Cgx}{2(\gamma - C)gx + 1}}$$

$\therefore$ Things will be different for three conditions,

1. $\gamma > C$

2. $\gamma = C$

3. $\gamma < C$

-4, solution has the correct form, but what is a,b?

① $\gamma > c$

$$\frac{dy}{dx} = \sqrt{b} \cdot \sqrt{\frac{x}{x+2a}} \qquad \begin{matrix} a > 0 \\ b > 0 \end{matrix}$$

$$\Rightarrow y = a(-\phi + \sinh\phi) \cdot \sqrt{b}$$

$$x = a(-1 + \cosh\phi)$$

② $\gamma = c$

$$\frac{dy}{dx} = \sqrt{2cgx}, \quad y = \sqrt{2cg} \cdot \frac{2}{3} x^{\frac{3}{2}}$$

③ $\gamma < c$

$$\frac{dy}{dx} = \sqrt{b} \cdot \sqrt{\frac{x}{2a-x}} \qquad \begin{matrix} a > 0 \\ b > 0 \end{matrix}$$

$$\Rightarrow y = a(\phi - \sin\phi)$$

$$x = a(1 - \cos\phi)$$

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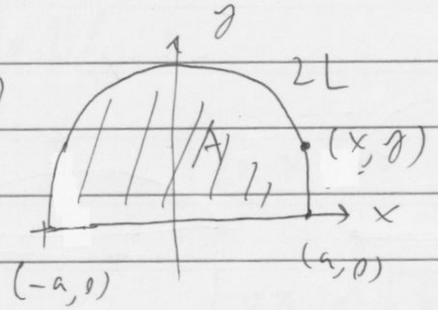
(a)



$$A = \int y\,dx = \int y\,\dot{x}\,d\tau$$

Constraint:

$$g = \int ds - 2L$$

$$x = x(\tau),\ y = y(\tau) \qquad = \int \sqrt{\dot{x}^2+\dot{y}^2}\,d\tau - 2L$$

$\therefore$ Total action

$$J = A - \lambda g$$

$$= \int \underbrace{\left(y\dot{x} - \lambda\sqrt{\dot{x}^2+\dot{y}^2}\right)}_{:= \mathcal{L}(x,y,\dot{x},\dot{y},\tau)} d\tau + \underbrace{2\lambda L}_{\text{Const.}}$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} = 0$$

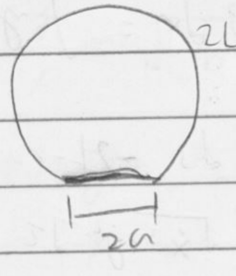
$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}} = y - \lambda\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = \text{Const.} = C$$

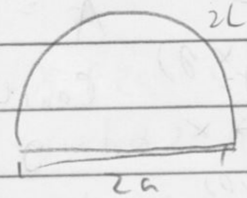
$$\Rightarrow \frac{\dot{x}^2}{\dot{x}^2+\dot{y}^2} = \frac{(y-C)^2}{\lambda^2}$$

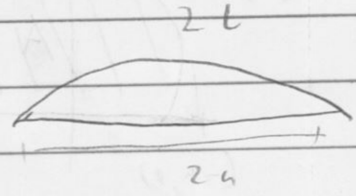
$$\Rightarrow \begin{cases} x = \lambda\cos\theta + B \\ y = \lambda\sin\theta + C \end{cases} \quad B, C \text{ are consts.}$$

$\therefore$ The solution is an arch of a circle with radius λ, and the arch length is $2L$.

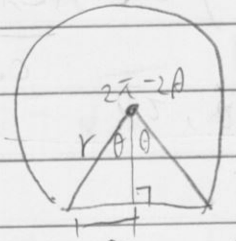
(b) Small a Middle a Large a







(c)



$r\sin\theta = a$, $r = \frac{a}{\sin\theta}$

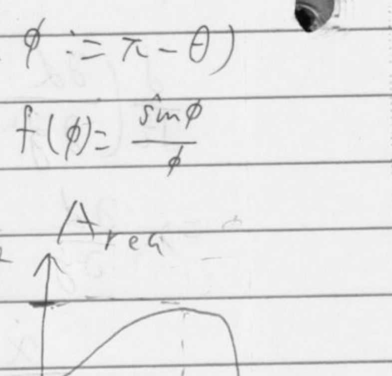
$r \cdot (2\pi - 2\theta) = 2L \Rightarrow \frac{a}{\sin\theta}(\pi - \theta) = L$

$\Rightarrow \frac{\sin\phi}{\phi} = \frac{a}{L} \quad (\phi := \pi - \theta)$

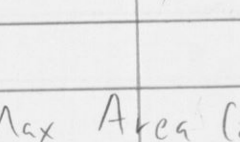
$\Rightarrow \phi = f^{-1}\left(\frac{a}{L}\right)$, $f(\phi) = \frac{\sin\phi}{\phi}$

Now calculate area

$$A = \frac{1}{2} r^2 (2\phi + \sin 2\theta)$$



Max Area Case



$$= r^2(\phi + \sin\theta\cos\theta)$$

$$= \frac{a^2}{\sin^2\phi}(\phi - \sin\phi\cos\phi)$$

$$= \frac{L^2}{\phi}\left(1 - \frac{a}{L}\cos\phi\right)$$

$$= \frac{L^2}{f^{-1}\left(\frac{a}{L}\right)}\left[1 - \frac{a}{L}\cos\left(f^{-1}\left(\frac{a}{L}\right)\right)\right]$$

By matlab, we found at $\frac{2L}{a} = \pi$, $a = \frac{2L}{\pi}$,

We have maximum area

$$A_{max} = \frac{1}{2}a^2\pi = \frac{\pi}{2}\left(\frac{2L}{\pi}\right)^2 = \frac{2}{\pi}L^2, \quad \theta = \phi = \frac{\pi}{2}$$

Problem 3.

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(a)

$$J \equiv \int 2\pi x \, ds = \int 2\pi x \sqrt{1+y'^2} \, dx$$

$$= \int \underbrace{2\pi x \sqrt{\dot{x}^2+\dot{y}^2}}_{:= L(x,y,\dot{x},\dot{y})} \, d\tau \qquad \dot{x} = \frac{dx}{d\tau}, \ \dot{y} = \frac{dy}{d\tau}$$

$$\frac{\partial L}{\partial y} = \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{y}}\right)$$

$$\Rightarrow \frac{\partial L}{\partial \dot{y}} = x \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = \text{const.} = C$$

$$\Rightarrow \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{C}{x} \Rightarrow \frac{\dot{y}^2}{\dot{x}^2+\dot{y}^2} = \frac{C^2}{x^2}$$

$$\Rightarrow \frac{u^2}{1+u^2} = \left(\frac{C}{x}\right)^2, \quad u = \sqrt{\frac{\left(\frac{C}{x}\right)^2}{1-\left(\frac{C}{x}\right)^2}} = \sqrt{\frac{C^2}{x^2-C^2}} = \frac{dy}{dx}$$

We define

$$x = C\cosh\phi$$

$$\Rightarrow dy = \sqrt{\frac{C^2}{x^2-C^2}} \, dx = C\sinh\phi \, d\phi \frac{1}{\sqrt{\sinh^2\phi}} = C \, d\phi$$

$$\Rightarrow y = C\phi + y_0$$

$$\Rightarrow x = C\cdot\cosh\left(\frac{y-y_0}{C}\right) = a\cosh\left(\frac{y-b}{a}\right)$$

a, b are const.

(b) $\because$ cosh is an even function,

$$\cosh(t) = \cos(-t)$$

$\Rightarrow$ $b = 0$ for

$$x = R = x(y=l) = x(y=-l)$$

$$\Rightarrow R = a\cosh\left(\frac{l}{a}\right)$$

$$\Rightarrow \frac{R}{a} = \cosh\left(\frac{l}{a}\right) = \frac{e^{\frac{l}{a}} + e^{-\frac{l}{a}}}{2}$$

$$\Rightarrow a = a(R, l)$$

(c) $a = \frac{l}{\gamma}$

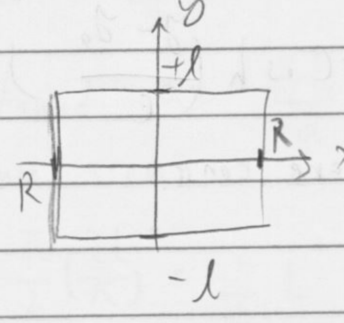
$$\Rightarrow \frac{R\gamma}{l} = \cosh\gamma$$

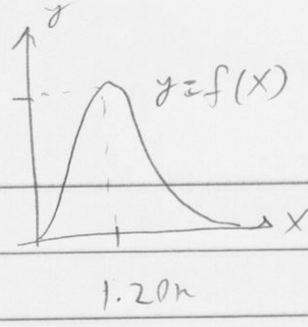
$$\Rightarrow \gamma = \frac{l}{R}\cosh\gamma$$

(d) At $\frac{l}{R} \ll 1$, γ is small, $\cosh\gamma \simeq 1$

$\therefore$ $\gamma \approx \frac{l}{R}$

$a \approx R$

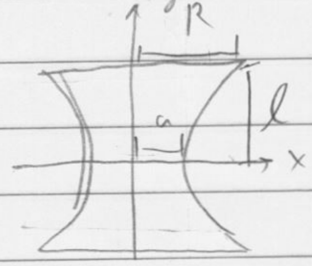




(e) $\gamma = \frac{l}{R}\cosh\gamma$

$$\Rightarrow \frac{l}{R} = \frac{\gamma}{\cosh\gamma} = \frac{2\gamma}{e^{\gamma}+e^{-\gamma}} = f(\gamma)$$

$f(\gamma)$ has a maximum value $0.6627 = \left(\frac{l}{R}\right)_{max}$ at $\gamma_{max} = 1.2012$, obtained numerically by Matlab.

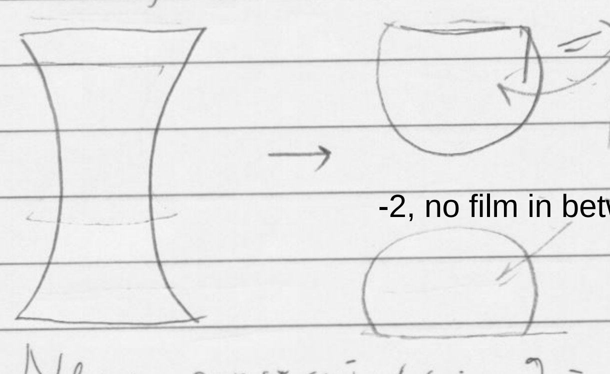


$\therefore \frac{l}{R}$ must not exceed 0.6627

At $\frac{l}{R} = 0.6627$, $a = \frac{l}{\cancel{1.2012}} = \frac{0.6627}{1.2012}R$

$\approx 0.55R$

(f) The soap film breaks, and split into two new films of spheres



$\Rightarrow$ Bubbles are spheres

Fixed volume

-2, no film in between for this case

$\Rightarrow$ New constraint

New constraints: $J = \int \pi x^2\, dy - V$

$$= \int \pi x^2 \dot{y}\, d\tau - V$$

$$\Rightarrow \int \big(\underbrace{2\pi x\sqrt{\dot{x}^2+\dot{y}^2} - \lambda\pi x^2\dot{y}}_{\mathcal{L}}\big)\, d\tau + \lambda V \equiv J$$

$$\frac{\partial\mathcal{L}}{\partial\dot{y}} = 2\pi x\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} - \lambda\pi x^2 = 0 \qquad \frac{\dot{y}^2}{\dot{x}^2+\dot{y}^2} = \left(\frac{x}{R}\right)^2$$

$$\Rightarrow \frac{\dot{y}^2}{\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{x}{R} \Rightarrow \quad \begin{array}{l} x = R\cos\theta \\ y = R\sin\theta + y_0 \end{array} \Rightarrow \text{Sphere}$$