HW3

Problem 1.

(a)

Kinetic Energy:

$$T = T_1 + T_2$$

$$T_1 = \frac{1}{2} m_1 \dot{z}^2$$

$$T_2 = \frac{1}{2} m_2 \left[(\dot{z}\cos\alpha + l\dot{\theta}\cos\theta)^2 + (\dot{z}\sin\alpha - l\dot{\theta}\sin\theta)^2 \right]$$

$$\therefore T = \frac{1}{2}(m_1+m_2)\dot{z}^2 + m_2 l \dot{z}\dot{\theta}\cos(\alpha+\theta) + \frac{1}{2} m_2 l^2 \dot{\theta}^2$$

Potential Energy:

$$V = V_1 + V_2$$

$$V_1 = -m_1 g z \sin\alpha$$

$$V_2 = -m_2 g (z\sin\alpha + l\cos\theta)$$

$$\therefore V = -(m_1+m_2) g z\sin\alpha - m_2 g l \cos\theta$$

$$\therefore L = T - V$$

$$= \frac{1}{2}(m_1+m_2)\dot{z}^2 + m_2 l \dot{z}\dot{\theta}\cos(\alpha+\theta) + \frac{1}{2} m_2 l^2 \dot{\theta}^2$$

$$+ (m_1+m_2) g z \sin\alpha + m_2 g l \cos\theta$$

$$= \frac{1}{2} M \dot{z}^2 + M g \sin\alpha \, z + m_2 l \dot{z}\dot{\theta}\cos(\alpha+\theta)$$

$$+ \frac{1}{2} m_2 l^2 \dot{\theta}^2 + m_2 g l \cos\theta$$

$$M := m_1 + m_2$$

(b)

At small angles of α and θ,

$\sin\alpha \approx \alpha$

$\cos\theta \approx 1 - \frac{1}{2}\theta^2$

$\cos(\alpha+\theta) \approx 1 - \frac{1}{2}(\alpha+\theta)^2$

$$\therefore L \approx \frac{1}{2}M\dot{z}^2 + Mg\alpha z - \frac{1}{2}m_2 l \dot{z}\dot{\theta}(\alpha+\theta)^2 + \frac{1}{2}m_2 l^2\dot{\theta}^2 - \frac{1}{2}m_2 g l \theta^2 + \text{Const.}$$

The remaining const. terms we can neglect

Now the Euler-Lagrange eq.s are

$$\begin{cases} \frac{\partial L}{\partial z} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) \\ \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \end{cases} \Rightarrow \begin{cases} Mg\alpha = \frac{d}{dt}\left[M\dot{z} - m_2 l\dot{\theta}(\theta+\alpha)\right] \\ -m_2 g l\theta - m_2 l\dot{z}\dot{\theta}(\theta+\alpha) \\ = \frac{d}{dt}\left[-\frac{1}{2}m_2 l\dot{z}(\alpha+\theta)^2 + m_2 l^2\dot{\theta}\right] \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{z} - \left(\frac{m_2}{M}l\right)\dot{\theta}^2 - \left(\frac{m_2}{M}l\right)\ddot{\theta}(\theta+\alpha) = g\alpha \\ -g\theta = -\frac{1}{2}\ddot{z}(\alpha+\theta)^2 + l\ddot{\theta} \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{z} = g\alpha + \left(\frac{m_2}{M}l\right)\left[\dot{\theta}^2 + \ddot{\theta}(\theta+\alpha)\right] & \text{— (b.1)} \\ \ddot{\theta} = -\frac{g}{l}\theta + \frac{1}{2}\frac{\ddot{z}}{l}(\alpha+\theta)^2 & \text{— (b.2)} \end{cases}$$

Let's use the ansatz that $\ddot{z} \approx g\alpha$.

(b.2) becomes

$$\ddot{\theta} = -\frac{g}{l}\left[\theta - \frac{1}{2}\alpha(\alpha+\theta)^2\right] \approx -\frac{g}{l}\theta$$

($\because \alpha, \theta$ are small)

$$\therefore \theta = \theta_m \sin(\omega t + \phi), \quad \omega = \sqrt{\frac{g}{l}}$$

(c) (b.2) becomes

$$\ddot{z} = g\alpha + \left(\frac{m_2}{M} l\right)\left\{\omega^2 \theta_m^2 \left[\cos^2(\omega t + \phi) - \omega^2 \sin^2(\omega t + \phi)\right] - \omega^2 \theta_m \alpha \cdot \sin(\omega t + \phi)\right\}$$

$$= g\alpha + \frac{m_2}{M} g \left[\theta_m^2 \cos(2\omega t + 2\phi) - \theta_m \alpha \sin(\omega t + \phi)\right]$$

$$\Rightarrow \dot{z} = g\alpha t + \frac{m_2}{M} g \left[\frac{\theta_m^2}{2\omega} \sin(2\omega t + 2\phi) + \frac{\theta_m \alpha}{\omega}\left(\cos(\omega t + \phi) - \cos\phi\right)\right]$$

$(\because \dot{z}(t=0) = 0)$ — (1.c.1)

$$\Rightarrow z = \frac{1}{2} g\alpha t^2 + \frac{m_2}{M} g \left\{-\frac{\theta_m^2}{4\omega^2}\left[\cos(2\omega t + 2\phi) - \cos\phi\right] + \frac{\theta_m \alpha}{\omega^2}\left[\sin(\omega t + \phi) - \omega t \cdot \cos\phi\right]\right\} \quad (\because z(t=0) = 0)$$

From (1.c.1), we can intuitively think that at small times t if

$$t < \frac{m_2}{M} \frac{\theta_m^2}{2\omega\alpha} \quad \text{— (1.c.2)}$$

Then it is possible that $V_z(t) < 0$

$\therefore$ The condition that negative V_z appears is that $m_2 \gg m_1$, and $\theta_m \gg \alpha$

We have given example of $V_z(t)$ in the next page.

In this simulation, we set g = 10m/s^2, $\alpha = 10^{-4}\, rad, \theta_m = 10^{-1} rad, m_1 = 0kg, m_2 = 1kg, l = 1m, \omega = \sqrt{\frac{g}{l}} = 3.16\frac{rad}{s}$,

And the initial phase $\phi = 0\, rad$.

And we plotted the formula of $\dot{z} = v_z(t)$ explicitly.

We can see that in Fig. 1, after before the 16 th period (period is defined as the pendulum oscillation period $T = 2\pi/\omega$), $v_z(t)$ can still be negative, but after around 16 periods, $v_z(t)$ is totally above 0. This can be understood that as t becomes larger, the $g\alpha t$ term becomes more dominant than the oscillating terms, and $v_z(t)$ can no longer be negative.

And we can calculate through the inequality (1.c.2), that the maximum time for $v_z(t)$ to be possible to be negative is:

$$T_{max} = \frac{\frac{m_2}{M}\theta_m^2}{2\omega\alpha} \approx 16T = 32\pi/\omega$$

Which is around 16 periods, consistent our observation in Fig.1.

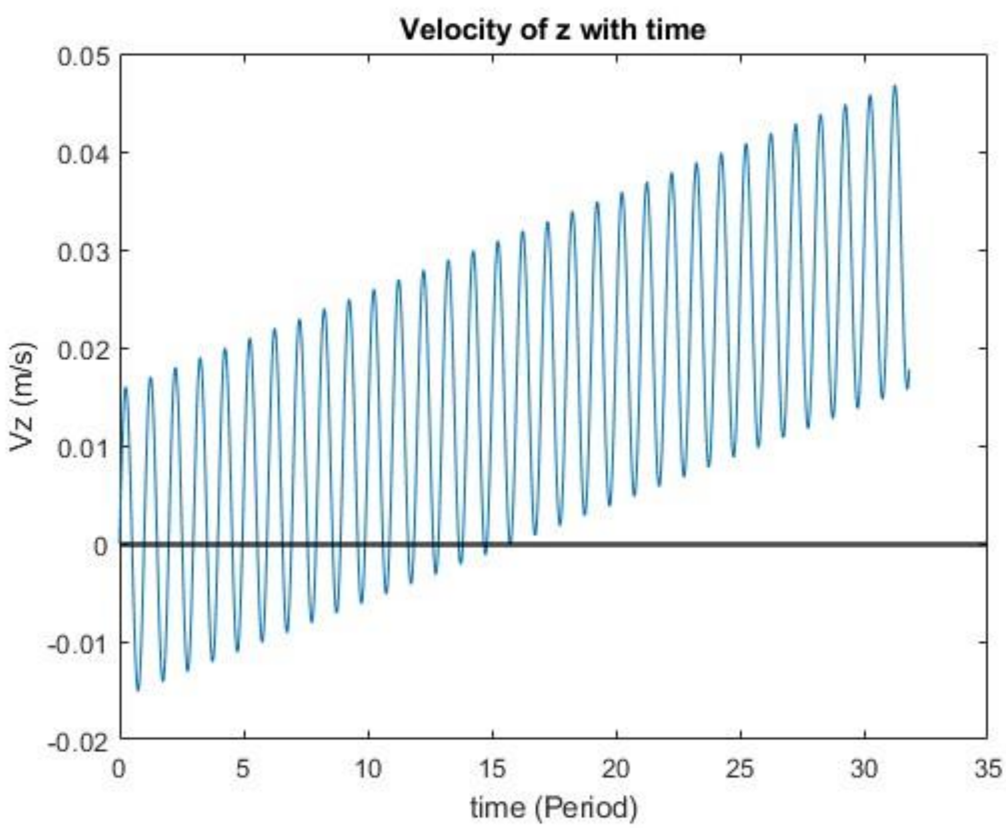
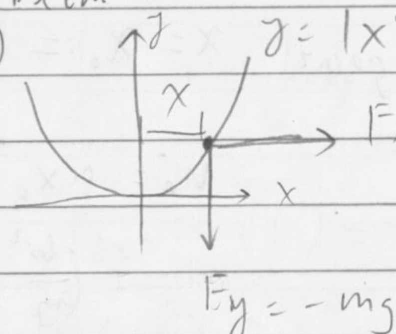
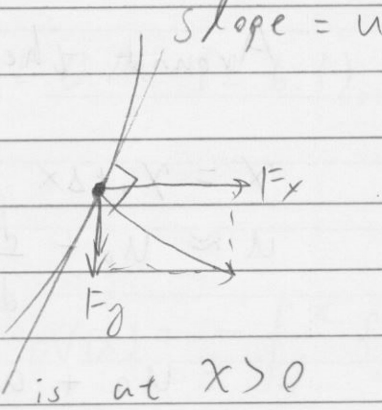


Fig.1 $v_z(t)$

Problem 2.

(a)





Here we assume the equilibrium is at $x > 0$

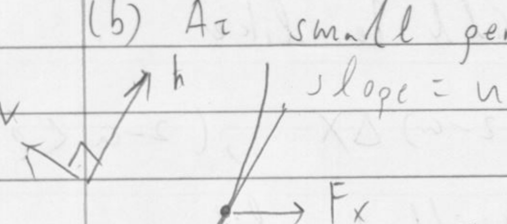
At equilibrium

$$\frac{F_x}{F_y} = -u_{eq} = \frac{m\omega^2 x_{eq}}{-mg} \Rightarrow x_{eq} = \frac{g\, u_{eq}}{\omega^2}$$

$$\Rightarrow x_{eq} = \frac{g}{\omega^2} \cdot a x_{eq}^{a-1} \Rightarrow x_{eq} = \sqrt[a-2]{\frac{\omega^2}{ag}}$$

This can also be for $x < 0$, $x_{eq} = -\sqrt[a-2]{\frac{\omega^2}{ag}}$

(b) At small perturbation



$$m a_h = F_x \cos\alpha - F_y \sin\alpha$$

$$= m\omega^2 \frac{x}{\sqrt{1+u^2}} - mg\frac{u}{\sqrt{1+u^2}}$$

$\cos\alpha = \frac{1}{\sqrt{1+u^2}}$

$$\Rightarrow m \cdot \frac{a_x}{\cos\alpha} = (m\omega^2 x - mgu)\frac{1}{\sqrt{1+u^2}}$$

$\sin\alpha = \frac{u}{\sqrt{1+u^2}}$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{(\omega^2 x - gu)}{1+u^2} \quad -(2.b.1)$$

$u = a x^{a-1}$

for $x > 0$

Around the equilibrium point $x = x_0 := x_{eq} = \sqrt[a-2]{\frac{w^2}{ag}} > 0$

$u_0 = a x_0^{a-1}$

$= \left(\frac{w^2}{ag}\right)^{\frac{a-1}{a-2}} \cdot a$

$x = x_0 + \Delta x$

$$u \approx u_0 + \left.\frac{du}{dx}\right|_{x=x_0} \cdot \Delta x$$

$$= u_0 + a(a-1)x_0^{a-2}\,\Delta x$$

$$= u_0 + \frac{(a-1)u_0}{x_0}\Delta x = u_0\left(1 + \frac{(a-1)\Delta x}{x_0}\right)$$

$$\therefore \frac{d^2x}{dt^2} \approx \frac{1}{1+u_0^2}\left\{w^2(x_0+\Delta x) - g u_0\left[1 + \frac{(a-1)\Delta x}{x_0}\right]\right\}$$

$(\because w^2 x_0 = u_0 g)$

$$= \frac{1}{1+u_0^2}\left[w^2 - \frac{u_0 g}{x_0}(a-1)\right]\Delta x$$

$$= \frac{w^2}{1+u_0^2}(2-a)\,\Delta x$$

$\Rightarrow$ Small oscillation will be like

$$\frac{d^2x}{dt^2} = \frac{d^2\Delta x}{dt^2} \approx \frac{w^2}{1+u_0^2}(2-a)\,\Delta x, \quad 2-a < 0$$

$\Rightarrow$ Oscillation frequency will be

$$w_0 = \sqrt{\frac{w^2}{1+u_0^2}(a-2)}$$

(c) If we keep the original E.O.M (2.b.1),
we can write it as

$$\frac{d^2x}{dt^2} = f(x) = \frac{w^2 x - u(x) g}{1+u^2(x)}$$

$$\Rightarrow \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x) = \varepsilon_0 \quad , \quad V(x) = -\int_{x_0}^{x} f(y)\,dy$$

$$\Rightarrow \quad \frac{dx}{dt} = \pm\sqrt{2(\varepsilon_0 - V(x))}$$

We have to solve it numerically.

We first show the potential diagram of $V(x)$ in Fig. 2. In this simulation, we set the parameters as follows:

$$a = 4, \qquad rotational\ angular\ velocity\ \omega = 3\frac{rad}{s}, g = \frac{10m}{s^2}$$

From above, we can calculate the equilibrium position of x is

$$x_0 = \sqrt[a-2]{\frac{\omega^2}{ag}} = 0.47m$$

We can see in Fig.2, the potential indeed has a local & global minimum at $x = x_0 = 0.47m$, confirming our calculation in problem 2.a.

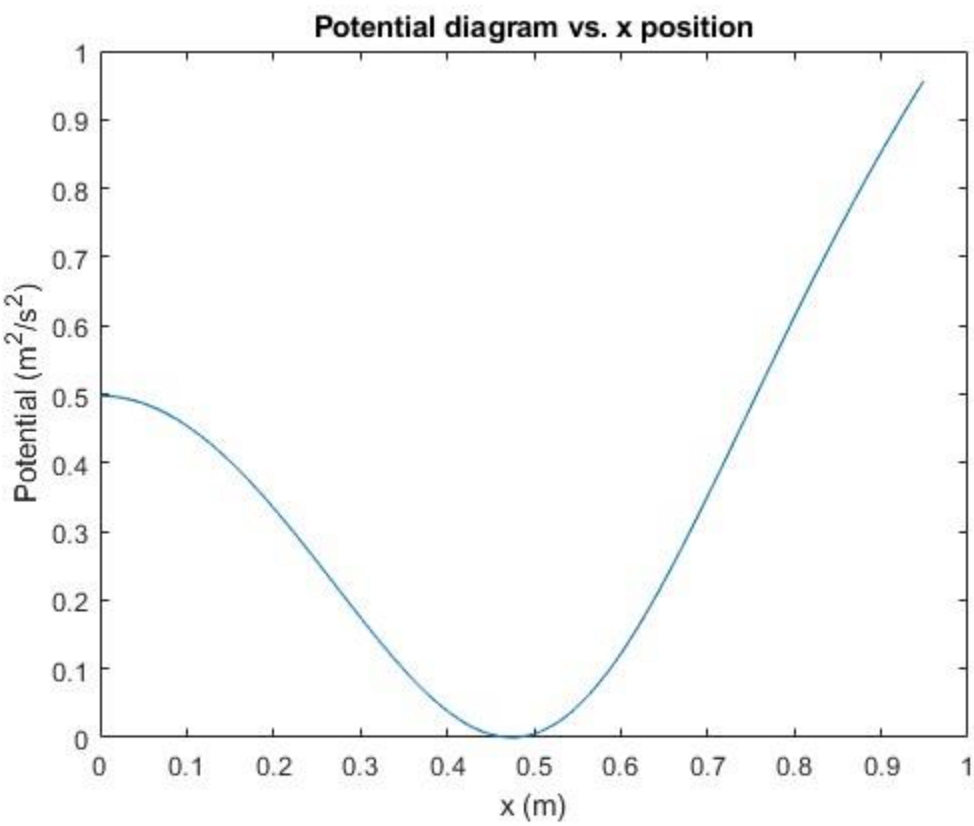


Fig.2 Potential diagram V(x) for $0 < x < 1m$.

And the small oscillation angular velocity is

$$\omega_0 = \sqrt{\frac{\omega^2}{1+u_0^2}(a-2)} = 3.9\, rad/s, \quad where\; u_0 = \; a x_0^{a-1}$$

Small oscillation period is

$$T_0 = \frac{2\pi}{\omega_0} = 1.61\, sec$$

We plotted out the diagram of oscillation with time, $x(t)$, for many different energies $\epsilon_0 =$

We normalized the oscillation amplitude of each oscillation for comparison of their periods.

We can see that if the energies are small (small oscillation cases), the periods converge to the small oscillation period $T_0 = 1.6\, sec$. On the other hand, if the energies becomes larger, the period began growing longer, deviating the estimation of small oscillation period $T_0 = 1.6\, sec$.

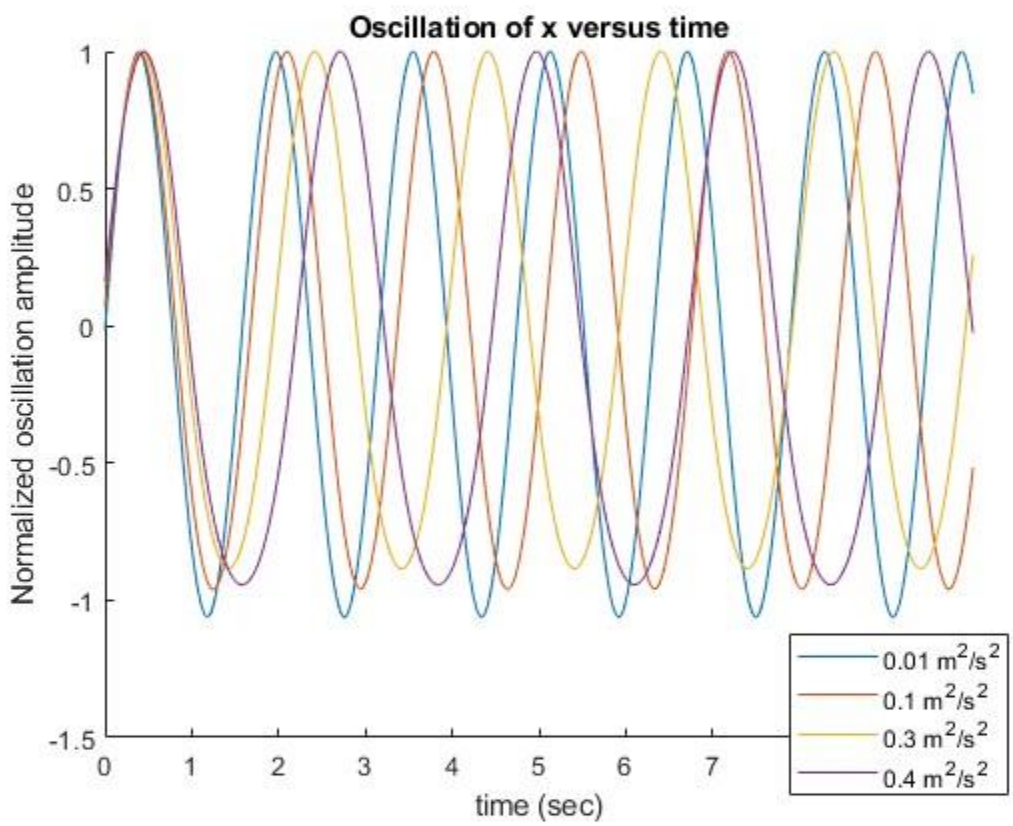


Fig.3 Oscillation vs. time x(t) for different energies

Problem 3

(a) $\vec{r}' = \vec{r} + \vec{V} t$

$\Rightarrow \dot{\vec{r}}' = \dot{\vec{r}} + \vec{V}$

$\therefore \quad L' = \frac{1}{2} m |\dot{\vec{r}}'|^2 = \frac{1}{2} m |\dot{\vec{r}} + \vec{V}|^2$

$= \frac{1}{2} m |\dot{\vec{r}}|^2 + m \dot{\vec{r}} \cdot \vec{V} + \frac{1}{2} m |\vec{V}|^2$

$= L + m\left(\frac{d}{dt}\vec{r}\right) \cdot \vec{V} + \frac{1}{2} m V^2 \neq L$

And the additional term A, $L' = L + A$

$A := m\left(\frac{d}{dt}\vec{r} \cdot \vec{V}\right) + \frac{1}{2} m V^2 = \frac{d}{dt}\left(m \vec{r} \cdot \vec{V} + \frac{1}{2} m V^2 t\right)$

is a total derivative of a function of t and position $\vec{r}$, $A = \frac{d}{dt} D(\vec{r}, t)$

$\Rightarrow$ Will not affect the action if with fixed endpoints $(t_1, \vec{r}_1)$, $(t_2, \vec{r}_2)$

$\Rightarrow$ Will not affect the equations of motion

(b) If the new Lagrangian is

$$L' = L + \frac{dD(q,\dot{q})}{dt} := L + A(q,\dot{q})$$

Since the EoM does not change with the total derivative, thus

$$\frac{\partial L'}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) \Rightarrow \frac{\partial L}{\partial q_i} + \frac{\partial A}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d}{dt}\left(\frac{\partial A}{\partial \dot{q}_i}\right)$$

$$\Rightarrow \frac{\partial A}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial A}{\partial \dot{q}_i}\right) \quad \left(\because \frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)$$

Now we look at the transformation

$$\frac{\partial L'}{\partial s} = \frac{\partial L}{\partial s} + \frac{\partial A}{\partial s} = \frac{\partial A}{\partial s} \quad \left(\because \frac{\partial L}{\partial s} = 0 \text{ for the original invariance}\right)$$

$$\Rightarrow 0 = \frac{\partial L'}{\partial s} - \frac{\partial A}{\partial s}$$

$$= \left(\frac{\partial L'}{\partial Q_i}\frac{\partial Q_i}{\partial s} + \frac{\partial L'}{\partial \dot{Q}_i}\frac{\partial \dot{Q}_i}{\partial s}\right) - \left(\frac{\partial A}{\partial Q_i}\frac{\partial Q_i}{\partial s} + \frac{\partial A}{\partial \dot{Q}_i}\frac{\partial \dot{Q}_i}{\partial s}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\frac{\partial Q_i}{\partial s}\right) - \frac{d}{dt}\left(\frac{\partial A}{\partial \dot{q}_i}\frac{\partial Q_i}{\partial s}\right)$$

$$= \frac{d}{dt}\left[\left(\frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial A}{\partial \dot{q}_i}\right)\cdot C_i\right] \quad \left(C_i := \frac{\partial Q_i}{\partial s}\right)$$

$$= 0$$

$$P = \underline{P'} - \frac{\partial A}{\partial \dot{q}_i}\cdot C_i \quad \text{is conserved}$$

$$:= \frac{\partial L'}{\partial \dot{q}_i} C_i$$

(c) For the Galileen transform

$$L' = L + m\vec{V}\cdot\dot{\vec{r}} + \frac{1}{2}mV^2$$

$$A = m\vec{V}\cdot\vec{r} + \frac{1}{2}mV^2$$

$$\frac{\partial A}{\partial \dot{r}} = m\vec{V}$$

$$\Rightarrow \frac{\partial L'}{\partial \dot{q}_i} C_i - \frac{\partial A}{\partial \dot{q}_i} \cdot C_i$$

$$= m(\dot{\vec{r}}' - \vec{V}) \quad \text{is conserved}$$

, which is obvious that if we boost the coordinates, the conserved momentum is boosted by the negative velocity $-\vec{V}$.