We tried to make some plots for different values of A.
Here we set the parameters to be

$$\beta = 1$$
$$\omega_0 = 20$$
$$\omega = \sqrt{\omega_0^2 - \beta^2} = 19.97$$
$$\tau = \frac{2\pi}{\omega} = 0.314$$

We found that for three different values, x(t) behaves differently

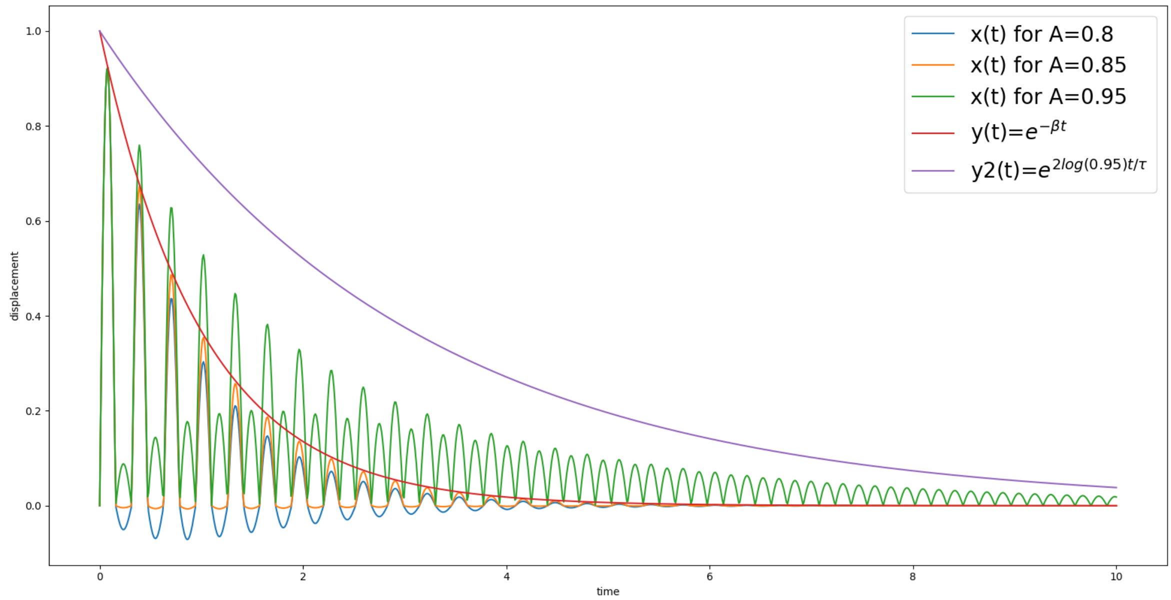


Fig. 2.1 Blue, orange, green lines are the x(t) for different A. Red line is a reference line for the damping rate $y(t) = e^{-\beta t}$, and the purple line is the reference line for $A^N(A = 0.95)$, or equivalently $y2(t) = e^{2\log(A=0.95)t/\tau}$.

We found that for A=0.95>0.85, $x(t) \geq 0$ for all t, and every half period $\tau/2$, the peak height changes a lot, from big to small, and from small to big interchangeably and repetitively, but all the peaks are above 0.

For A=0.85, there is half a period that x(t) almost equals to 0 and then at the next half period, peaks appear again.
Thus the critical value is at around 0.85.

The reason for the critical value of A, $A_{crit}$, is because of the interference between the new response and the old response.

For each n,

$$x_n(t) = A^n G\left(t, \frac{n}{2}\tau\right) = A^n e^{-\beta(t-\frac{n}{2}\tau)} \sin\left(\omega\left(t - \frac{n}{2}\tau\right)\right)\theta(t - \frac{n}{2}\tau)$$

And we can see easily that

$$x_n(t) + x_{n+1}(t) = A^n e^{-\beta(t-\frac{(n+1)}{2}\tau)} \sin\left(\omega\left(t - \frac{(n+1)}{2}\tau\right)\right)\left(-e^{-\beta\frac{1}{2}\tau} + A\right) =$$

$$\left(-e^{-\beta\frac{1}{2}\tau} + A\right) x_{n+1}(t) = (-b + A)x_{n+1}(t) \text{ at } t > \frac{n+1}{2}\tau$$

$$b = e^{-\beta\frac{1}{2}\tau}$$

Thus, if $b = A$, the interference between the n-th and (n+1)-th response function will cause half a period to have 0 value, and at next half period, the new response function will again give a positive peak.

Thus the critical value of A is

$$A_{crit} = b = e^{-\frac{\beta}{2}\tau} = 0.8544678930067565 \sim 0.85$$

in this case.

Moreover, it is also easy to see whether x(t) will be an exponential decrease or growth for $A > A_{crit} = b$.

We particularly look at the times where the peak may happen, i.e. when $t = \frac{N}{2}\tau + \frac{\tau}{4}$ .

$$x\left(t = \frac{N}{2}\tau + \frac{\tau}{4}\right) = \sum_{n=0}^{N} A^n G\left(t, \frac{n}{2}\tau\right) = \sum_{n=0}^{N} A^n e^{-\beta\left(t-\frac{n}{2}\tau\right)} \sin\left(\omega\left(t - \frac{n}{2}\tau\right)\right)$$

$$= \sum_{n=0}^{N} A^n b^{N-n} \sin\left[(N-n)\pi + \frac{\pi}{2}\right]\delta = \sum_{n=0}^{N} A^n (-b)^{N-n}\,\delta = a_N\delta$$

Where

$$\delta = e^{-\frac{\beta\tau}{4}}$$

$$N = t//(\frac{\tau}{2})$$

$$a_N = \sum_{n=0}^{N} A^n(-b)^{N-n}$$

And it has an iteration behavior

$$a_{N+1} = -ba_N + A^{N+1}$$

All we need to see is that whether $a_{N+1}$ converges or diverges as $N \to \infty$.
We can use mathematical induction rule to prove that $a_N$ will converge.
We assume that $0 < a_N \leq A^N$
For N=0,

$$a_0 = 1$$

And for N=1,

$$0 < a_1 = -b + A \leq A^1 \; (A > A_{crit} = b)$$

And if $0 < a_k \leq A^k$,
Then

$$a_{k+1} = -ba_N + A^{N+1} = \left(A - \frac{ba_N}{A^N}\right)A^N$$

Since $a_n > 0$, $a_{k+1} \leq A^{N+1}$, and since $\frac{a_n}{A^N} \leq 1$, $a_{k+1} = \left(A - \frac{ba_N}{A^N}\right)A^N \geq (A-b)A^N > 0$.
Thus if $0 < a_k \leq A^k$, $0 < a_{k+1} = -ba_N + A^{N+1} \leq A^{N+1}$
Thus we know by mathematical induction, $0 < a_N \leq A^N$ for all N.
Thus $a_N$ must converge as $N \to \infty$ and converge faster than $A^N$
(see Fig. 2.1's purple line).

$x\left(t = \frac{N}{2}\tau + \frac{\tau}{4}\right) = a_N\delta$ (the peak heights) will decrease exponentially.

Thus, Kay is right.
By the way, the mathematical induction proof was done by Kay's younger brother, who is a high school student, and knows nothing about the differential equations and Green functions.