

Problem 1

$$F(t) = F_0 \sum_{n=0}^{\infty} A^n \delta\left(t - \frac{n}{\omega_0} T\right)$$

To solve the full solution, we must first find the Green function of the equation of motion.

$$L\{x(t)\} = (\partial_t^2 + 2\beta\partial_t + \omega_0^2)x = F(t)/m$$

$$L\{G(t, t')\} = (\partial_t^2 + 2\beta\partial_t + \omega_0^2)G(t, t') = \delta(t - t')$$

We can solve it by Fourier transform:

$$\tilde{G}(w, t') = \int_{-\infty}^{\infty} dt e^{iwt} G(t, t')$$

$$\Rightarrow (-w^2 - 2i\beta w + \omega_0^2) \tilde{G}(w, t') = \int_{-\infty}^{\infty} dt e^{iwt} \delta(t - t')$$

$$\Rightarrow \tilde{G}(w, t') = \frac{e^{iwt'}}{-w^2 - 2i\beta w + \omega_0^2}$$

$$= \frac{-e^{iwt'}}{(w - w_+)(w - w_-)} \quad (w_{\pm} = -i\beta \pm \sqrt{\omega_0^2 - \beta^2})$$

$$\Rightarrow G(t, t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-e^{iwt'-it}}{(w - w_+)(w - w_-)} dw$$

Since the integrand contains two poles w_+ , w_- , which are both at the negative imaginary side of the complex frequency plane.

\Rightarrow By Jordan's Lemma, if $\begin{cases} t > t', G \text{ is finite} \\ t < t', G = 0 \end{cases}$

Finite value of G is

$$G(t, t') = 2\pi i \times (-1) \times \left[\frac{1}{2} \left(\frac{-e^{-i\omega_f(t-t')}}{\omega_+ - \omega_-} \right) + \frac{1}{2} \left(\frac{-e^{-i\omega_f(t-t')}}{\omega_- - \omega_+} \right) \right]$$

$$= i \left(\frac{e^{-i\omega_f(t-t')}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_f(t-t')}}{\omega_- - \omega_+} \right) \text{ if } t > t'$$

In total,

$$G(t, t') = i \left(\frac{e^{-i\omega_f(t-t')}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_f(t-t')}}{\omega_- - \omega_+} \right) \theta(t-t')$$

$$= i e^{-\beta(t-t')} \left(\frac{e^{i\omega_f(t-t')}}{2\omega} - \frac{e^{-i\omega_f(t-t')}}{2\omega} \right) \theta(t-t')$$

$$= e^{-\beta(t-t')} \sin[\omega_f(t-t')] \theta(t-t')$$

$(\omega := \sqrt{\omega_0^2 - \beta^2})$ ($\theta(t-t')$ is + Heaviside step function)

For a delta response function $\delta(t-t_0)$
The displacement is

$$x(t) = \int_{-\infty}^{\infty} G(t, t') \delta(t' - t_0) dt'$$

$$= G(t, t_0)$$

$$\therefore F(t) = \frac{F_0}{m} \sum_{n=0}^{\infty} A^n \delta(t - \frac{n}{2}\tau)$$

$$\Rightarrow x(t) = \frac{F_0}{m} \sum_{n=0}^{\infty} A^n G(t, \frac{n}{2}\tau)$$

We shall use python to plot $x(t)$ for different A ,

We tried to make some plots for different values of A.

Here we set the parameters to be

$$\begin{aligned}\beta &= 1 \\ \omega_0 &= 20 \\ \omega &= \sqrt{\omega_0^2 - \beta^2} = 19.97 \\ \tau &= \frac{2\pi}{\omega} = 0.314\end{aligned}$$

We found that for three different values, $x(t)$ behaves differently

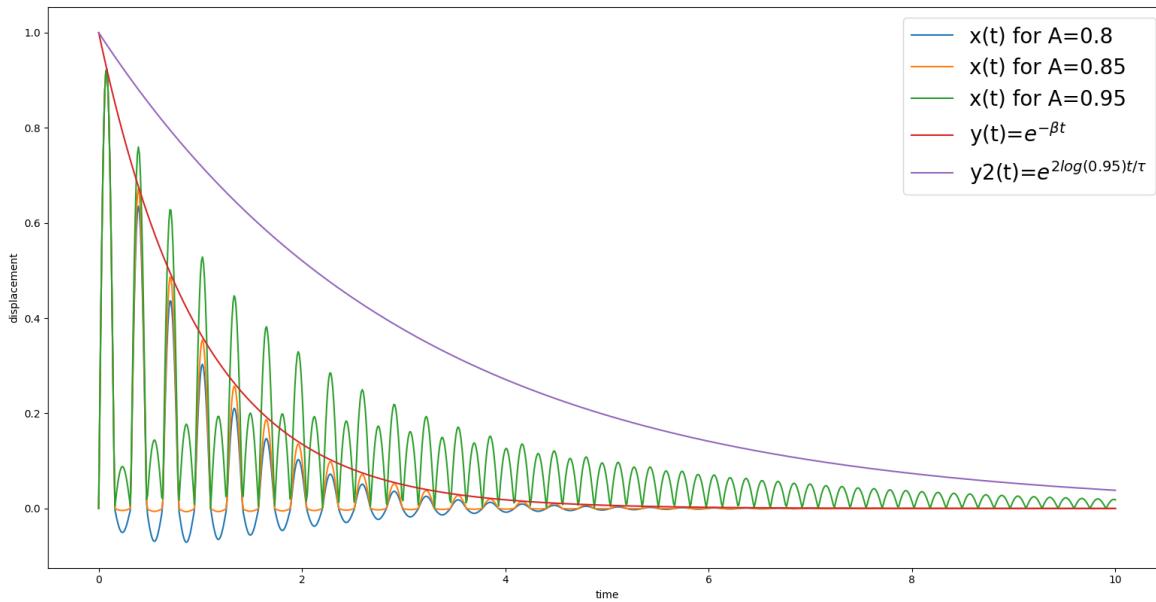


Fig. 2.1 Blue, orange, green lines are the $x(t)$ for different A. Red line is a reference line for the damping rate $y(t) = e^{-\beta t}$, and the purple line is the reference line for $A^N (A = 0.95)$, or equivalently $y2(t) = e^{2 \log(A=0.95)t/\tau}$.

We found that for $A=0.95>0.85$, $x(t) \geq 0$ for all t , and every half period $\tau/2$, the peak height changes a lot, from big to small, and from small to big interchangeably and repetitively, but all the peaks are above 0.

For $A=0.85$, there is half a period that $x(t)$ almost equals to 0 and then at the next half period, peaks appear again.
Thus the critical value is at around 0.85.

The reason for the critical value of A, A_{crit} , is because of the interference between the new response and the old response.

For each n,

$$x_n(t) = A^n G\left(t, \frac{n}{2}\tau\right) = A^n e^{-\beta(t-\frac{n}{2}\tau)} \sin(\omega(t - \frac{n}{2}\tau)) \theta(t - \frac{n}{2}\tau)$$

And we can see easily that

$$x_n(t) + x_{n+1}(t) = A^n e^{-\beta(t-\frac{(n+1)}{2}\tau)} \sin\left(\omega\left(t - \frac{(n+1)}{2}\tau\right)\right) \left(-e^{-\beta\frac{1}{2}\tau} + A\right) =$$

$$\left(-e^{-\beta\frac{1}{2}\tau} + A\right) x_{n+1}(t) = (-b + A)x_{n+1}(t) \text{ at } t > \frac{n+1}{2}\tau$$

$$b = e^{-\beta\frac{1}{2}\tau}$$

Thus, if $b = A$, the interference between the n-th and (n+1)-th response function will cause half a period to have 0 value, and at next half period, the new response function will again give a positive peak.

Thus the critical value of A is

$$A_{crit} = b = e^{-\frac{\beta}{2}\tau} = 0.8544678930067565 \sim 0.85$$

in this case.

Moreover, it is also easy to see whether $x(t)$ will be an exponential decrease or growth for $A > A_{crit} = b$.

We particularly look at the times where the peak may happen, i.e. when $t =$

$$\frac{N}{2}\tau + \frac{\tau}{4} .$$

$$\begin{aligned} x\left(t = \frac{N}{2}\tau + \frac{\tau}{4}\right) &= \sum_{n=0}^N A^n G\left(t, \frac{n}{2}\tau\right) = \sum_{n=0}^N A^n e^{-\beta(t-\frac{n}{2}\tau)} \sin\left(\omega\left(t - \frac{n}{2}\tau\right)\right) \\ &= \sum_{n=0}^N A^n b^{N-n} \sin\left[(N-n)\pi + \frac{\pi}{2}\right] \delta = \sum_{n=0}^N A^n (-b)^{N-n} \delta = a_N \delta \end{aligned}$$

Where

$$\delta = e^{-\frac{\beta\tau}{4}}$$

$$N = t // (\frac{\tau}{2})$$

$$a_N = \sum_{n=0}^N A^n (-b)^{N-n}$$

And it has an iteration behavior

$$a_{N+1} = -ba_N + A^{N+1}$$

All we need to see is that whether a_{N+1} converges or diverges as $N \rightarrow \infty$.

We can use mathematical induction rule to prove that a_N will converge.

We assume that $0 < a_N \leq A^N$

For N=0,

$$a_0 = 1$$

And for N=1,

$$0 < a_1 = -b + A \leq A^1 (A > A_{crit} = b)$$

And if $0 < a_k \leq A^k$,

Then

$$a_{k+1} = -ba_N + A^{N+1} = \left(A - \frac{ba_N}{A^N}\right) A^N$$

Since $a_n > 0$, $a_{k+1} \leq A^{N+1}$, and since $\frac{a_n}{A^N} \leq 1$, $a_{k+1} = \left(A - \frac{ba_N}{A^N}\right) A^N \geq (A - b)A^N > 0$.

Thus if $0 < a_k \leq A^k$, $0 < a_{k+1} = -ba_N + A^{N+1} \leq A^{N+1}$

Thus we know by mathematical induction, $0 < a_N \leq A^N$ for all N.

Thus a_N must converge as $N \rightarrow \infty$ and converge faster than A^N (see Fig. 2.1's purple line).

$x(t = \frac{N}{2}\tau + \frac{\tau}{4}) = a_N \delta$ (the peak heights) will decrease exponentially.

Thus, Kay is right.

By the way, the mathematical induction proof was done by Kay's younger brother, who is a high school student, and knows nothing about the differential equations and Green functions.

Problem 2

(a)

$$\text{For transformation } \delta \vec{q}_i = F_i \delta \eta$$

$$\begin{aligned} (\delta L)_q &= \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial q_i} \delta q_i \\ &= \sum_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta \dot{q}_i + \frac{\partial L}{\partial q_i} \delta q_i \\ &= \frac{d}{dt} \left(\sum_i p_i F_i \right) \delta \eta \quad p_i := \frac{\partial L}{\partial \dot{q}_i} \\ &= \frac{d}{dt} \left(\sum_i p_i F_i \right) \delta \eta \end{aligned}$$

$$\text{For transformation } \delta t = G \delta \eta$$

$$(\delta L)_t = \frac{\partial L}{\partial t} \delta t = - \frac{dH}{dt} G \delta \eta \quad \left(\because - \frac{\partial L}{\partial t} = \frac{dH}{dt} \right)$$

\Rightarrow For total transformation

$$\begin{cases} \delta \vec{q} = \vec{F} \delta \eta \\ \delta t = G \delta \eta \end{cases} \Rightarrow (\delta L)_{\text{total}} = \frac{d}{dt} (\vec{p} \cdot \vec{F} - GH) \delta \eta$$

$\vec{p} \cdot \vec{F} - GH$ is the conserved quantity

Here we assume G is a constant of time; if otherwise, the conserved quantity will have a more complex form.

$$(b) \text{ For } L = L(r, \theta - \omega t, \dot{r}, \dot{\theta})$$

The transformation has symmetry
 $\delta\theta = \delta\eta$

$$\delta t = \frac{1}{\omega} \delta\eta$$

$$\text{Since } \delta(\theta - \omega t) = (1 - \frac{1}{\omega} \cdot \omega) \delta\eta = 0 \\ \Rightarrow \delta L = 0$$

\therefore The conserved quantity is

$$Q := I \cdot \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{1}{\omega} H = \frac{1}{\omega} (wL_z - H)$$

Q is conserved.

To understand more specifically, we assume the Lagrangian is

$$L(\theta - \omega t, \dot{\theta}) = \frac{1}{2} I (\dot{\theta} - \omega)^2 - \frac{1}{2} k (\theta - \omega t)^2$$

Which is a rotational harmonic oscillator in the constant rotational frame.

$$\Rightarrow L_z = I(\theta - \omega)$$

$$H = \dot{\theta} L_z - L = \frac{1}{2} I (\dot{\theta}^2 - \omega^2) + \frac{1}{2} k (\theta - \omega t)^2$$

Which looks quite unnatural and probably not the expression for energy E .

Then

$$\begin{aligned} wL_z - H &= Iw(\dot{\theta} - \omega) - \frac{1}{2} I (\dot{\theta}^2 - \omega^2) - \frac{1}{2} k (\theta - \omega t)^2 \\ &= -\frac{1}{2} I (\dot{\theta} - \omega)^2 - \frac{1}{2} k (\theta - \omega t)^2 \\ &= -\left[\frac{1}{2} I (\dot{\theta} - \omega)^2 + \frac{1}{2} k (\theta - \omega t)^2 \right] \end{aligned}$$

We can see that

$$E := \frac{1}{2} I (\dot{\theta} - \omega)^2 + \frac{1}{2} k (\theta - \omega t)^2$$

is a conserved charge, and is the total energy before the rotational transformation
 $\begin{cases} \theta \rightarrow \theta - \omega t \\ \dot{\theta} \rightarrow \dot{\theta} - \omega \end{cases}$

Noether's theorem tells us the conserved quantity before and after coordinate transformation, also including the effect of Galilean boost (for both translational and rotational)