Manuscript

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February 2021

1 Kinematics

The first step in our research is to derive the general formula of the dark matter (DM) energy/momentum distribution when kicked by a neutrino with a given energy.

We assume that the DM particle χ was at rest first, and neutrino ν was incident on it with energy E and momentum, thus we can derive that in centre of mass (CM) frame, the velocity of outgoing χ is:

$$\begin{aligned}\beta &= \frac{|\boldsymbol{p_i}|}{E + m_\chi} \\ &= \frac{\sqrt{E^2 - m_\nu^2}}{E + m_\chi}\end{aligned} \tag{1}$$

In vector form, we have:

$$\begin{aligned}\boldsymbol{\beta_{CM,out}} &= \beta \widehat{\boldsymbol{n}} \\ &= \beta \cos\theta \widehat{\boldsymbol{n_0}} + \beta \sin\theta \widehat{\boldsymbol{n_\perp}}\end{aligned} \tag{2}$$

And in lab frame,

$$\boldsymbol{\beta_{lab,out}} = \frac{\beta(1+\cos\theta)\widehat{\boldsymbol{n_0}} + \beta\sin\theta\sqrt{1-\beta^2}\widehat{\boldsymbol{n_\perp}}}{1+\beta^2\cos\theta} \tag{3}$$

Here $\widehat{\boldsymbol{n_0}}$ is the unit vector along the incident χ direction, and $\widehat{\boldsymbol{n_\perp}}$ is the unit vector perpendicular to it, and θ is the angle between the outgoing direction and $\widehat{\boldsymbol{n_0}}$.

The gamma value can be obtained:

$$\gamma_{lab,out} = \frac{1+\beta^2\cos\theta}{1-\beta^2}$$

If we calculate the energy of χ undergoing isotropic upscattering in CM frame, we can obtain it through taking the average of gamma on solid angle:

$$\begin{aligned}\overline{\gamma_{lab,out}} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{1+\beta^2\cos\theta}{1-\beta^2}\sin\theta\, d\theta\, d\phi \\ &= \frac{1}{1-\beta^2}\end{aligned} \tag{4}$$

Thus the energy of outgoing χ is:

$$\begin{aligned}E_{lab,out,ave} &= m_\chi\overline{\gamma_{lab,out}} \\ &= \frac{m_\chi}{1-\beta^2}\end{aligned} \tag{5}$$

Momentum of outgoing χ is:

$$\begin{aligned}\boldsymbol{p_{lab,out}} &= m_\chi\gamma_{lab,out}\boldsymbol{\beta_{lab,out}} \\ &= m_\chi\frac{\beta(1+\cos\theta)\widehat{\boldsymbol{n_0}}+\beta\sin\theta\sqrt{1-\beta^2}\widehat{\boldsymbol{n_\perp}}}{1-\beta^2} \\ &= m_\chi\frac{\beta(1+\cos\theta)\widehat{\boldsymbol{n_0}}+\beta\sin\theta\sqrt{1-\beta^2}(\cos\phi\widehat{\boldsymbol{n_{x\perp}}}+\sin\phi\widehat{\boldsymbol{n_{y\perp}}})}{1-\beta^2}\end{aligned} \tag{6}$$

The average momentum in an isotropic upscattering case is:

$$\begin{aligned}\boldsymbol{p_{lab,out,ave}} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}m_\chi\frac{\beta(1+\cos\theta)\widehat{\boldsymbol{n_0}}+\beta\sin\theta\sqrt{1-\beta^2}\widehat{\boldsymbol{n_\perp}}}{1-\beta^2}\sin\theta\, d\theta\, d\phi \\ &= \frac{m_\chi\beta}{1-\beta^2}\widehat{\boldsymbol{n_0}}\end{aligned} \tag{7}$$

The perpendicular part is averaged out. We can also write the average momentum/energy in terms of masses of ν and χ and incident kinetic energy of neutrino T_ν:

$$\begin{aligned}T_{lab,out,ave} &= \frac{T_\nu^2+2m_\nu T_\nu}{T_\nu+\dfrac{(m_\nu+m_\chi)^2}{2m_\chi}} \\ \boldsymbol{p_{lab,out,ave}} &= m_\chi\frac{T_\nu+m_\nu+m_\chi}{2m_\chi T_\nu+(m_\nu+m_\chi)^2}\sqrt{T_\nu(T_\nu+2T_\nu m_\nu)}\widehat{\boldsymbol{n_0}}\end{aligned} \tag{8}$$

2 DM Flux

We can obtain the dark matter (DM) flux Φ_χ kicked by neutrinos emitted from a supernova (SN) with known positions, onto the earth by the following formula:

$$\Phi_\chi = L_\nu n_\chi c\sigma_{\nu-\chi}$$

n_χ and L_ν are number density of DM per volume on the line of sight from SN to earth and number density per area of neutrino, $\sigma_{\nu-\chi}$ is the cross section between DM and neutrino, which is about 10^{-30} 10^{-40}. c is light speed. We adopt the NFW model for DM distribution:

$$n_\chi = \frac{\rho_s}{m_\chi \frac{r}{r_s}(1+\frac{r}{r_s})^2}$$

$\rho_s = 0.184 \quad GeV/cm^3, r_s = 24.42 \quad kpc$ (M. Cirelli, G. Corcella, A. Hektor, G. Hutsi, M. Kadastik, et al., JCAP 1103, 051 (2011), 1012.4515.)

And L_ν is:

$$L_\nu = \frac{N_\nu}{4\pi R^2}$$

N_ν is the total number of neutrino in the shock (roughly 10^{58}), R is the distance between shock and SN. And if we want to calculate the actual flux observed on the earth, we must first figure out the relation between traveling time of neutrino t, and the receiving time of DM flux t'. Suppose the total traveling time of neutrino flux is T, and set $t = 0$ when neutrino flux begins from the SN, and $t' = 0$ when neutrino flux reaches the earth. And if we take the velocity of DM particle to be v, neutrino's velocity to be c, then it's easy to obtain that:

$$\begin{aligned} t' &= \frac{(c-v)(T-t)}{v} \\ \frac{dt}{dt'} &= \frac{-v}{c-v} \end{aligned} \tag{9}$$

Then the DM flux observed on the earth is:

$$\Phi'_\chi(t') = \Phi_\chi(t)\frac{dt}{dt'} = \frac{-v}{c-v}\frac{N_\nu}{4\pi R^2} n_\chi c \sigma_{\nu-\chi} \tag{10}$$

Where $n'_\chi(t') = n_\chi(t)$

If you notice the negative sign, that's because the nearer the DM particle is kicked the sooner it can reach the earth, thus the DM flux observed is sort of time reversal. Furthermore, we can see that the flux is actually proportional to the passed DM Halo density along LOS. Thus if we can observe some significant DM flux caused by a SN, we can use the observation to examine the density distribution of DM Halo along the LOS. And if we suppose $m_\chi = 1keV$, the SN is on the line from Galactic center (GC) to the earth, and is $0.1kpc$ away from GC, while the earth is roughly $10kpc$ from GC, and the cross section is $10^{-30}cm^2$. Then the total number of DM received on earth is 8×10^{10} DM particles per square centimeter. If $m_\chi = 10MeV$ the total number of DM received on earth is 8×10^6 DM particles per square centimeter. It can be easily seen that the particle number is inversely proportional to DM mass.

We can also calculate the time elapse during which we receive the DM particles.

We call the time elapse to be "delay time" since there is a delay when we receive DM signal after we receive the SN neutrino signal. The delay time is $16102sec$ (about 4 hours) when $m_\chi = 1keV$ and is $4 \times 10^{11} sec$ (about 13071 years). The delay time is approximately proportional to the square of DM mass. Thus, from above, we can know that DM flux is inversely proportional to the cubic of DM mass because DM flux is approximately DM number divided by the delay time.

3 Angular Effect

Consider center of mass (CM) frame outgoing angle for an elastic scattering to be η, and the lab frame outgoing angle to be θ. Then consider the formula of outgoing momentum in (3), we can easily obtain that:

$$\tan\theta = \tan\frac{\eta}{2}\sqrt{1-\beta^2}$$

Where

$$\beta = \frac{\sqrt{E^2 - m_\nu^2}}{E + m_\chi}(See(1)inStep1)$$

Then for an elastic scattering with isotropic upscattering distribution, we consider the formula of the number density outgoing particle for a given η is:

$$\frac{dn}{d\eta} = \frac{1}{2}\sin\eta$$

Thus, the number density for a given θ is:

$$\begin{aligned}\frac{dn}{d\theta} &= \frac{1}{2}\sin\eta\frac{d\eta}{d\theta} \\ &= \frac{2\tan\theta}{\sec\theta^2 - \beta^2}\sqrt{1-\beta^2}\frac{\sec\theta^2}{\sec^2\theta - \beta^2}\sqrt{1-\beta^2} \\ &= \frac{2\tan\theta\sec^2\theta}{(\sec\theta^2 - \beta^2)^2}(1-\beta^2)\end{aligned} \tag{11}$$

And you can obtain the solid angle distribution:

$$\begin{aligned}\frac{dn}{d\Omega} &= \frac{1}{2\pi\sin\theta}\frac{dn}{d\theta} \\ &= \frac{\sec^3\theta}{\pi(\sec^2\theta - \beta^2)^2}(1-\beta^2)\end{aligned} \tag{12}$$

If you integrate this number density from $\theta = 0$ to $\theta = \theta_{max} = \pi/2$ (Because $\eta_{max} = \pi$), you can get 1.

$$\int_0^{2\pi}\int_0^{\theta_{max}} \frac{\sec^3\theta}{\pi(\sec^2\theta - \beta^2)^2}(1-\beta^2)\sin\theta\, d\theta\, d\omega = 1 \tag{13}$$

The physical meaning of θ_{max} is that dark matter particles can be upscattered to at most the direction perpendicular to the incident neutrinos, it can never be upscattered backwards.
We can recall an identity which can be derived easily from divergence law:

$$\int_0^{2\pi}\int_0^{\pi}\frac{\cos{(\theta-\psi)}}{R^2+r^2-2Rr\cos{\psi}}R^2\sin{\psi}\,d\psi\,d\omega=1 \tag{14}$$

Where

$$\begin{gathered}\frac{r}{\sin{(\theta-\psi)}}=\frac{R}{\sin{\theta}}\\ \psi=\theta-\arcsin{(\frac{r}{R}\sin{\theta})}\end{gathered} \tag{15}$$

Then if we insert $\frac{dn}{d\Omega}$ into it and change the integration domain of ψ from $[0,\pi]$ to $[0,\psi_{max}]$, we can also obtain 1:

$$\int_0^{2\pi}\int_0^{\psi_{max}}\frac{dn}{d\Omega}\frac{\cos{(\theta-\psi)}}{R^2+r^2-2Rr\cos{\psi}}R^2\sin{\psi}\,d\psi\,d\omega=1 \tag{16}$$

Where

$$\begin{gathered}\psi_{max}=\theta_{max}-\arcsin{(\frac{r}{R}\sin{\theta_{max}})}\\ \\ \theta_{max}=\frac{\pi}{2}\end{gathered} \tag{17}$$

We can define ψ to be the angle between the line of sight from supernova(SN) to Earth and the outgoing direction of a neutrino emitted by SN. Then we can see that (15) has certain physical meanings concerning the upscattering number of dark matter(DM) particle numbers from different angles (not just the direction of LOS from SN to Earth) due to the different scattering angles of DM particles. We call the effect of different scattering angles to be angular effect. And we can see from (15) that if DM volume density is homogeneous through all the space, angular effect itself produces no additional upscattering DM particles onto Earth. However, in the real process of upscattering, we must take the density distribution of DM particles into account. Therefore, if we consider NFW distribution for DM Halo in Milky Way, we can recalculate the total DM particles upscattered onto Earth when we take angular effect into account.
We can now compute the total upscattered DM particle number density per area L_χ through LOS integration combined with angular effect. We can compare the one-direction(the direction of LOS from SN to Earth) LOS integration with the

multiple-direction(considering angular effect) LOS integration.
One-direction:

$$L_\chi = \int_0^R \frac{N_\nu}{4\pi R^2}\sigma_{\chi\nu}\frac{\rho_\chi}{m_\chi}dr \tag{18}$$

Multiple-direction:

$$L_\chi = \int_0^R \int_0^{2\pi} \int_0^{\psi_{max}} \frac{N_\nu}{4\pi r^2}\sigma_{\chi\nu}\frac{\rho_\chi}{m_\chi}\frac{dn}{d\Omega}\frac{\cos(\theta-\psi)}{R^2+r^2-2Rr\cos\psi}r^2\sin\psi\, d\psi\, d\omega\, dr \tag{19}$$

Where $\dfrac{dn}{d\Omega}$ is in the form of (12). And if you look closely into (12), you will find that as $\beta \to 1$, $\dfrac{dn}{d\Omega}$ will become a delta function in θ, and that (18) is the limit form of (19) as $\beta \to 1$.

4 Integrated DM Flux

Knowing that the dark matter (DM) masses may vary in a large range. We want to see how much delayed time there is for a DM flux accelerated by supernova (SN) neutrino shock to arrive the Earth, after the neutrino shock's arriving. We then need to go back to our discussion about neutrino-DM colliding kinematics. In it, we know that the energy of DM particle after collision is:

$$E_\chi = \gamma_\chi m_\chi = \frac{1}{1-\beta^2}m_\chi$$

Where

$$\beta(E, m_\chi) = \frac{\sqrt{E^2 - m_\nu^2}}{E + m_\chi}$$