Kinematics

Wen-Hua Wu

December 2020

The first step in our research is to derive the general formula of the dark matter (DM) energy/momentum distribution when kicked by a neutrino with a given energy.

We assume that the DM particle χ was at rest first, and neutrino ν was incident on it with energy E and momentum, thus we can derive that in centre of mass (CM) frame, the velocity of outgoing χ is:

$$\begin{aligned} \beta &= \frac{|\boldsymbol{p_i}|}{E + m_\chi} \\ &= \frac{\sqrt{E^2 - m_\nu^2}}{E + m_\chi} \end{aligned} \tag{1}$$

In vector form, we have:

$$\begin{aligned} \boldsymbol{\beta_{CM,out}} &= \beta \widehat{\boldsymbol{n}} \\ &= \beta \cos\theta \widehat{\boldsymbol{n_0}} + \beta \sin\theta \widehat{\boldsymbol{n_\perp}} \end{aligned} \tag{2}$$

And in lab frame,

$$\boldsymbol{\beta_{lab,out}} = \frac{\beta(1 + \cos\theta)\widehat{\boldsymbol{n_0}} + \beta \sin\theta \sqrt{1 - \beta^2}\widehat{\boldsymbol{n_\perp}}}{1 + \beta^2 \cos\theta} \tag{3}$$

Here $\widehat{\boldsymbol{n_0}}$ is the unit vector along the incident χ direction, and $\widehat{\boldsymbol{n_\perp}}$ is the unit vector perpendicular to it, and θ is the angle between the outgoing direction and $\widehat{\boldsymbol{n_0}}$.

The gamma value can be obtained:

$$\gamma_{lab,out} = \frac{1 + \beta^2 \cos\theta}{1 - \beta^2}$$

If we calculate the energy of χ undergoing isotropic upscattering in CM frame, we can obtain it through taking the average of gamma on solid angle:

$$\begin{aligned} \overline{\gamma_{lab,out}} &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{1 + \beta^2 \cos\theta}{1 - \beta^2} \sin\theta \, d\theta \, d\phi \\ &= \frac{1}{1 - \beta^2} \end{aligned} \tag{4}$$

Thus the energy of outgoing χ is:

$$\begin{aligned} E_{lab,out,ave} &= m_\chi \overline{\gamma_{lab,out}} \\ &= \frac{m_\chi}{1-\beta^2} \end{aligned} \tag{5}$$

Momentum of outgoing χ is:

$$\begin{aligned} \boldsymbol{p_{lab,out}} &= m_\chi \gamma_{lab,out} \boldsymbol{\beta_{lab,out}} \\ &= m_\chi \frac{\beta(1+\cos\theta)\widehat{\boldsymbol{n_0}} + \beta \sin\theta \sqrt{1-\beta^2}\widehat{\boldsymbol{n}_\perp}}{1-\beta^2} \\ &= m_\chi \frac{\beta(1+\cos\theta)\widehat{\boldsymbol{n_0}} + \beta \sin\theta \sqrt{1-\beta^2}(\cos\phi \widehat{\boldsymbol{n_{x}}_\perp} + \sin\phi \widehat{\boldsymbol{n_{y}}_\perp})}{1-\beta^2} \end{aligned} \tag{6}$$

The average momentum in an isotropic upscattering case is:

$$\begin{aligned} \boldsymbol{p_{lab,out,ave}} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} m_\chi \frac{\beta(1+\cos\theta)\widehat{\boldsymbol{n_0}} + \beta \sin\theta \sqrt{1-\beta^2}\widehat{\boldsymbol{n}_\perp}}{1-\beta^2} \sin\theta \, d\theta \, d\phi \\ &= \frac{m_\chi \beta}{1-\beta^2}\widehat{\boldsymbol{n_0}} \end{aligned} \tag{7}$$

The perpendicular part is averaged out. We can also write the average momentum/energy in terms of masses of ν and χ and incident kinetic energy of neutrino T_ν:

$$\begin{aligned} T_{lab,out,ave} &= \frac{T_\nu^2 + 2m_\nu T_\nu}{T_\nu + \dfrac{(m_\nu + m_\chi)^2}{2m_\chi}} \\ \boldsymbol{p_{lab,out,ave}} &= m_\chi \frac{T_\nu + m_\nu + m_\chi}{2m_\chi T_\nu + (m_\nu + m_\chi)^2}\sqrt{T_\nu(T_\nu + 2T_\nu m_\nu)}\widehat{\boldsymbol{n_0}} \end{aligned} \tag{8}$$