The first step in our research is to derive the general formula of the DM energy/momentum distribution when kicked by a neutrino with a given energy.

We assume that the DM particle was at rest first, and neutrino ν was incident on it with energy E and momentum $\boldsymbol{p}_i$, thus we can derive that in CM frame, the velocity of outgoing DM particle is:

$$\beta = \frac{|\boldsymbol{p_i}|}{E + m_{DM}} \ (from\ 48.4)$$
$$= \frac{\sqrt{E^2 - m_\nu^2}}{E + m_{DM}}$$

Where

$$|\boldsymbol{p_i}| = (E^2 - m_\nu^2)^{1/2}$$

Consider that the DM particle goes out along direction $\boldsymbol{\hat{n}}$, and the incident neutrino is going along $\widehat{\boldsymbol{n_0}}$, thus at the CM frame, the outgoing DM velocity is:

$$\boldsymbol{\beta}_{CM,out} = \beta\boldsymbol{\hat{n}}$$
$$= \beta cos\theta\widehat{\boldsymbol{n_0}} + \beta sin\theta\widehat{\boldsymbol{n_\perp}}$$

$(\theta\ is\ the\ angle\ between\ \boldsymbol{\hat{n}}\ and\ \widehat{\boldsymbol{n_0}})$

The $\beta cos\theta\widehat{n_0}$ term is parallel to the incident direction $\widehat{\boldsymbol{n_0}}$, while the $\beta sin\theta\widehat{\boldsymbol{n_\perp}}$ term is normal to $\widehat{\boldsymbol{n_0}}$**.** The two terms shall transfer differently from CM frame to lab frame, specifically:

$$\boldsymbol{\beta}_{lab,out,//} = \frac{\beta cos\theta\widehat{\boldsymbol{n_0}} + \beta\widehat{\boldsymbol{n_0}}}{1 + \beta^2 cos\theta}$$
$$= \frac{\beta(1 + cos\theta)}{1 + \beta^2 cos\theta}\widehat{\boldsymbol{n_0}}$$
$$\boldsymbol{\beta}_{lab,out,\perp} = \frac{\beta sin\theta\sqrt{1 - \beta^2}}{1 + \beta^2 cos\theta}\widehat{\boldsymbol{n_\perp}}$$
$$\boldsymbol{\beta}_{lab,out} = \boldsymbol{\beta}_{lab,out,//} + \boldsymbol{\beta}_{lab,out,\perp}$$

And

$$\gamma_{lab,out} = (1 - \left|\boldsymbol{\beta}_{lab,out,//}\right|^2 - |\boldsymbol{\beta}_{lab,out,\perp}|^2)^{-1/2}$$
$$= \frac{1 + \beta^2 cos\theta}{1 - \beta^2}$$

We integrate the energy over solid angle θ and φ to get the average energy for outgoing DM particle:

$$E_{lab,out,ave} = \overline{\gamma_{lab,out}} m_{DM}$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{m_{DM}(1+\beta^2 cos\theta)}{1-\beta^2} sin\theta d\theta d\varphi$$
$$= \frac{1+\frac{\beta^2}{2}}{1-\beta^2} m_{DM}$$

And the outgoing DM particle's momentum in lab frame is:

$$\boldsymbol{p}_{lab,out} = m_{DM}\gamma_{lab,out}\boldsymbol{\beta}_{lab,out}$$
$$= \frac{m_{DM}\beta(1+cos\theta)}{1-\beta^2}\widehat{\boldsymbol{n}_0} + \frac{m_{DM}\beta sin\theta\sqrt{1-\beta^2}}{1-\beta^2}\widehat{\boldsymbol{n}_\perp}$$

Since $\sqrt{1-\beta^2}$ is very small, we can neglect the normal component, and consider the parallel component only.
We take the average of momentum :

$$\boldsymbol{p}_{DM,out,ave} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{m_{DM}\beta(1+cos\theta)}{1-\beta^2}\widehat{\boldsymbol{n}_0}\, sin\theta d\theta d\varphi$$
$$= \frac{3}{2}\frac{m_{DM}\beta}{1-\beta^2}\widehat{\boldsymbol{n}_0}$$

And since the normal component will vanish after integration, we can still omit it when $\sqrt{1-\beta^2} \approx 0$
Here I assume that the outgoing direction is isotropic in CM frame, and DM particle only goes forward.

Basically, the momentum is proportional to $\frac{m_{DM}\beta}{1-\beta^2}$, where β is function of incident energy E and dark matter mass m_{DM}

$$\beta\,(E, m_{DM}) = \frac{\sqrt{E^2 - m_\nu^2}}{E + m_{DM}}$$

Here if $E \gg m_{DM}$, $\frac{m_{DM}\beta}{1-\beta^2} \approx \frac{E}{2}$, when $E \ll m_{DM}$, $\frac{m_{DM}\beta}{1-\beta^2} \approx E$

For the neutrino, their momentum after bouncing in lab frame is:

$$\boldsymbol{p}_{\nu,ave} = \boldsymbol{p}_{\boldsymbol{i}} - \boldsymbol{p}_{DM,ave}$$
$$= (\beta(E + m_{DM}) - \frac{3}{2}\frac{m_{DM}\beta}{1-\beta^2})\widehat{\boldsymbol{n}_0}$$

$$= \beta \left((E + m_{DM}) - \frac{3}{2} \frac{m_{DM}}{1 - \frac{E^2 - m_\nu^2}{(E + m_{DM})^2}} \right) \widehat{\boldsymbol{n_0}}$$

$$= \beta (E + m_{DM}) \left(\mathbf{1} - \frac{\frac{3}{2} (E + m_{DM}) m_{DM}}{2E m_{DM} + m_{DM}{}^2 - m_\nu^2} \right) \widehat{\boldsymbol{n_0}}$$

$$= \beta (E + m_{DM}) \left(\frac{\frac{1}{2} E m_{DM} - \frac{1}{2} m_{DM}{}^2 - m_\nu^2}{2E m_{DM} + m_{DM}{}^2 - m_\nu^2} \right) \widehat{\boldsymbol{n_0}}$$

$$\approx \frac{1}{2} \beta (E + m_{DM}) \left(\frac{E - m_{DM}}{2E + m_{DM}} \right) \widehat{\boldsymbol{n_0}}$$

Because m_ν^2 is very small

$$= \frac{1}{2} \beta \frac{E^2 - m_{DM}^2}{2E + m_{DM}} \widehat{\boldsymbol{n_0}}$$

We now then can review the energy distribution of neutrinos from supernovae, and therefore opens the discussion of next step: To compute the flux of DM arriving the Earth after being scattered by neutrinos emitted from a SN at a galactic location.