HW 10

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Problem [1]

(a)

$$C_{el} = \frac{\pi^2}{2}\left(\frac{k_B T}{\varepsilon_F}\right) n k_B \quad (\text{A\&M } 2.81)$$

$$= \frac{\pi^2}{2}\left(\frac{1.38\times10^{-23}\times4}{3.24\times1.6\times10^{-19}}\right)\times\left(2.65\times10^{22}\ \#/\text{cm}^3\right)\times1.38\times10^{-23}\ \text{J/K}$$

$$= 1.92\times10^{-4}\ \text{J/K}\cdot\text{cm}^3$$

$(\text{Na},\ \varepsilon_F = 3.24\,\text{eV})$

(b)

$$C_{ph} = 234\left(\frac{T}{\Theta_D}\right)^3 n k_B$$

$$= 234\times\left(\frac{4\,\text{K}}{150\,\text{K}}\right)^3\times\left(2.65\times10^{22}\ \#/\text{cm}^3\right)\times1.38\times10^{-23}\ \text{J/K}$$

$$= 1.62\times10^{-3}\ \text{J/K}\cdot\text{cm}^3$$

(C)

When $C_{el} = C_{ph}$ $\Rightarrow$ $\frac{\pi^2}{2}\left(\frac{k_B T}{\varepsilon_F}\right) n k_B = 234\left(\frac{T}{\Theta_D}\right)^3 n k_B$

$$\Rightarrow \sqrt{\frac{\frac{\pi^2}{2}}{234}\,\frac{k_B\,\Theta_D^3}{\varepsilon_F}} = T = 1.38\ \text{K}$$

A: 1.38 K

Problem [2]

Total energy of electrons is

$$u = \int_{-\infty}^{\infty} \varepsilon\, g(\varepsilon) f(\varepsilon)\, d\varepsilon$$

$$\approx U_0 + \int_{\varepsilon -}^{\varepsilon + k_B T} (\varepsilon - \varepsilon_F) g(\varepsilon) f(\varepsilon)\, d\varepsilon$$

$$= U_0 + \int_{\varepsilon - k_B T}^{\varepsilon + k_B T} \frac{\alpha (\varepsilon - \varepsilon_F) |\varepsilon - \varepsilon_F|\, d\varepsilon}{e^{\frac{\varepsilon - \mu}{k_B T}} + 1}$$

$$(g(\varepsilon) \propto |\varepsilon - \varepsilon_F| := \alpha |\varepsilon - \varepsilon_F|)$$

$$\approx U_0 + \alpha (k_B T)^3 \int_{-1}^{1} \frac{\alpha s |s|\, ds}{e^{s} + 1} \qquad s := \frac{\varepsilon - \varepsilon_F}{k_B T}$$

$\mu \approx \varepsilon_F$ at low temp.

$$:= U_0 + A T^3$$

$$(A := \text{Const.})$$

$$\therefore C_{el} := \frac{\partial u}{\partial T} \propto A T^2 \propto T^2$$

Problem [3]

Total carrier density is fixed :

$$n = \int_0^\infty g(\varepsilon) f(\varepsilon) d\varepsilon$$

$$= \int_0^\infty \frac{m}{\pi\hbar^2} \frac{1}{e^{\frac{\varepsilon-\mu}{k_B T}}+1} d\varepsilon$$

$$= \frac{m k_B T}{\pi\hbar^2} \int_{-\frac{\mu}{k_B T}}^\infty \frac{ds}{e^s+1} \qquad s := \frac{\varepsilon-\mu}{k_B T}$$

$$= \frac{m k_B T}{\pi\hbar^2} \int_{-\frac{\mu}{k_B T}}^\infty \frac{e^{-s} ds}{e^{-s}+1}$$

$$= \frac{m k_B T}{\pi\hbar^2} \int_{-\frac{\mu}{k_B T}}^\infty \frac{e^{-s} ds}{e^{-s}+1}$$

$$= \frac{m k_B T}{\pi\hbar^2} \ln \left| e^{\frac{\mu}{k_B T}} + 1 \right|$$

$$\Rightarrow \mu = k_B T \ln \left| e^{\frac{n\pi\hbar^2}{m k_B T}} - 1 \right|$$

$$\mu(T\to 0) = \lim_{T\to 0} k_B T \ln \left| e^{\frac{n\pi\hbar^2}{m k_B T}} - 1 \right|$$

$$= \lim_{T\to 0} k_B T \cdot \frac{n\pi\hbar^2}{m k_B T} = \frac{n\pi\hbar^2}{m} = \varepsilon_F$$

Problem [4] A&M 23.2

(a) $g(\omega) = \sum_s \int \frac{d\vec{k}}{(2\pi)^3} \delta(\omega - \omega_s(\vec{k}))$

$= \sum_s \frac{1}{3} \int \frac{d\Omega}{4\pi} \int dk \frac{4\pi k^2}{(2\pi)^3} \cdot 3 \cdot \delta(\omega - \omega_s(\vec{k}))$

$= \sum_s \frac{1}{3} \int \frac{d\Omega}{4\pi} \int dk \frac{3k^2}{2\pi^2} \delta(\omega - \omega_s(\vec{k}))$

For low-freq, the integrand of dk can be taken approximately to k_D $\forall \Omega$

$\therefore g(\omega) \approx \sum_s \frac{1}{3} \int \frac{d\Omega}{4\pi} \int_0^{k_D} dk \frac{3k^2}{2\pi^2} \delta(\omega - k c_s(\hat{k}))$

(Where $\omega_s(\vec{k}) := k \cdot c_s(\hat{k})$)

$= \sum_s \frac{1}{3} \int \frac{d\Omega}{4\pi} \frac{3}{2\pi^2} \cdot \left(\frac{\omega^2}{c_s(\hat{k})^3}\right)$

$= \frac{3}{2\pi^2} \omega^2 \sum_s \frac{1}{3} \int \frac{d\Omega}{4\pi} \frac{1}{c_s(\hat{k})^3}$

$= \frac{3}{2\pi^2} \frac{\omega^2}{c^3}$

Where $\frac{1}{c^3} = \sum_s \frac{1}{3} \int \frac{d\Omega}{4\pi} \frac{1}{c_s(\hat{k})^3}$ (A&M 23.18)

$\therefore$ Gives the same form of result as (23.36)

(b) In d-dim crystal,

$$\int \frac{d\vec{k}}{(2\pi)^d} = \int \frac{d\Omega}{\alpha(d)} \int \frac{\alpha(d)}{(2\pi)^d} k^{d-1} \, dk$$

For low-freq, ($\alpha(d)$ is some constant)

$$g(\omega) \approx \sum_s \frac{1}{3} \int \frac{d\Omega}{\alpha(d)} \int_0^{k_D} dk \frac{3\alpha(d)}{(2\pi)^d} k^{d-1} \delta(\omega - k c_s(\hat{k}))$$

$$= \sum_s \frac{1}{3} \int \frac{d\Omega}{\alpha(d)} \frac{3\alpha(d)}{(2\pi)^d} \cdot \frac{\omega^{d-1}}{c_s(\hat{k})^d}$$

$$= \frac{3\alpha(d)}{(2\pi)^d} \cdot \frac{\omega^{d-1}}{c^d} \propto \omega^{d-1}$$

(c is the average phase velocity)

(c)

$$u \approx \int_0^\infty d\omega \cdot \frac{\hbar\omega \, g(\omega)}{e^{\frac{\hbar\omega}{k_B T}} - 1}$$

$$= A \int_0^\infty d\omega \frac{\omega^d}{e^{\frac{\hbar\omega}{k_B T}} - 1} \quad (\because g(\omega) \propto \omega^{d-1})$$

(A, B are just some constant)

$$= \left(\frac{k_B T}{\hbar}\right)^{d+1} A \int_0^\infty \frac{x^d}{e^x - 1}$$

$$= B T^{d+1}$$

$$\Rightarrow c_v = \left(\frac{\partial u}{\partial T}\right)_v = (d+1) B T^d \propto T^d$$

(d)

Assume that $\omega = a k^{\nu}$

Then we can reformulate the $g(\omega)$ as:

$$g_D(\omega) = 3\int_0^{k_D} \frac{\alpha(d) k^{d-1} dk}{(2\pi)^d} \cdot \delta(\omega - a k^{\nu})$$

$$= \frac{3\alpha(d)}{(2\pi)^d} \int_0^{k_D} k^{d-1} dk \, \delta(\omega - a k^{\nu})$$

$$= \frac{3\alpha(d)}{(2\pi)^d} \left[\left(\frac{\omega}{a}\right)^{\frac{1}{\nu}}\right]^{d-1} \frac{1}{(a\nu)\left(\frac{\omega}{a}\right)^{\frac{\nu-1}{\nu}}}$$

$$= \frac{3\alpha(d)}{(2\pi)^d} \frac{1}{a\nu} \cdot \left(\frac{\omega}{a}\right)^{\frac{d-\nu}{\nu}} \propto \omega^{\frac{d}{\nu}-1}$$

$$\therefore \quad C_V = \frac{\partial}{\partial T}\bigg|_V \int_0^{\infty} \frac{g(\omega)\hbar\omega \, d\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1}$$

$$= \frac{\partial}{\partial T}\bigg|_V A \int_0^{\infty} \frac{\omega^{\frac{d}{\nu}} d\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1}$$

$$= \frac{\partial}{\partial T}\bigg|_V A \left(\frac{k_B T}{\hbar}\right)^{\frac{d}{\nu}+1} \int_0^{\infty} \frac{x^{\frac{d}{\nu}} dx}{e^{x} - 1}$$

$$\propto T^{\frac{d}{\nu}}.$$