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Title: Homework 5 - Solutions

Question 1: Power function, over natural numbers

We would like to show that:

$$\forall n : \ power\ n\ x\ =\ x^n$$

The Principle of Induction for natural numbers is:

$$\forall n,\ P(n)\ if\ P(0)\ and\ P(n-1) \Rightarrow\ P(n)$$

Our property P is:

$$P(n,x)\ is\ power\ n\ x\ =\ x^n$$

Our inductive proof would have 2 cases:
Base case: $P(0,x)$: show that $power\ (0,x)\ =\ 1$

$power\ 0\ x\ =\ 1.0$ (by definition of power)

Step: $P(n,x)$: show that $power\ n\ x\ =\ x^n$
Our Inductive Hypothesis is where $power\ (n\ -1)\ x\ = x^{(n-1)} *. x^{(n-2)}\ *. x^{(n-3)}\ *. .....\ *. x^0$ holds.

$power\ n\ x\ =\ x\ *. power\ (n-1)\ x$ (by definition of *power*)
$=\ x\ *. x^{(n-1)} *. x^{(n-2)}\ *. x^{(n-3)}\ *. .....\ *. x^0$ (by inductive hypothesis)
$=\ x^n$ (by properties of multiplication)

Hence we have proven $\forall n :\ power\ n\ x\ =\ x^n$

Question 2: Power over structured numbers

We would like to show that:

$$\forall n \in nat,\ power\ n\ x\ =\ x^{\,to\text{-}Int\ (n)}$$

The Principle of Induction for type *nat* is:

$$\forall n,\ P(n)\ if\ P(Zero)\ and\ P(Succ\ n)\ when\ \ P(n)\ holds.$$

Our property P (for type *nat*) is:

$$P(n,x)\ is\ power\ n\ x\ =\ x^{\,to\text{-}Int\ (n)}$$

Our inductive proof would have 2 cases:

P(Zero,x): show that $power\ zero\ x\ \ =\ x^{to\text{-}Int\ (zero)}$

$$\begin{aligned} power\ Zero\ x\ &=\ 1.0 && \text{(by definition of power)} \\ &=\ x^0 && \text{(by mathematical definition of power)} \\ &=\ x^{\,to\text{-}Int\ (Zero)} && \text{(by definition of to\_Int)} \end{aligned}$$

P(n, x) : show that $power\ (Succ\ n)\ x\ =\ x^{\,to\text{-}Int\ (Succ\ n)}$

Our inductive hypothesis is $power\ n\ x\ =\ x^{\,to\text{-}Int\ (n)}$

$$\begin{aligned} power\ (Succ\ n)\ x\ &=\ x\ *.\ power\ n\ \ x && \text{(by definition of \textit{power})} \\ &=\ x\ *.\ x^{\,to\text{-}Int\ n} && \text{(by inductive hypothesis)} \\ &=\ x^{\,to\text{-}Int\ n\ +\ 1} && \text{(by mathematical power)} \\ &=\ x^{\,to\text{-}Int\ (Succ\ n)} && \text{(by definition of to\_Int)} \end{aligned}$$

Hence we have proven $power\ n\ x\ =\ x^{\,to\text{-}Int\ (n)}$

Question 3: Length of Lists

The principle of induction for list type is:

$$\forall l,\ P(l)\ if\ P([\ ])\ and\ P(v\ ::\ l'),\ if\ P(l')$$

We need to show the following property P:

$$P\ (l, r)\ =\ length\ (l\ @\ r)\ =\ length\ l\ +\ length\ r$$

Our base case is: $P\ (\ [\ ],\ r)\ :\ show\ that\ \ length\ ([\ ]\ @\ r\ )\ =\ length\ [\ ]\ +\ length\ r$

from definition of function, we know that $P[\ ]\ \Rightarrow\ 0$

$length\ (\ [\ ]\ @\ r\ )\ =\ length\ r$ (by the function definition and properties of list: $[\ ]\ @\ l\ =\ l$)

$=\ 0\ +\ length\ r$ (by the properties of addition)

$=\ length\ [\ ]\ +\ length\ r$ (by definition of length)

Step: $P\ (x :: xs\ ,\ r)\ :\ show\ that\ length\ (x : xs\ @\ r)\ =\ length\ (x :: xs)\ +\ length\ r$

Our inductive hypothesis is $length\ (\ xs\ @\ r\ )\ =\ length\ xs\ +\ length\ r$

$length\ (x : xs\ @\ r)\ =\ length\ (x :: xs\ @\ r)$ (by properties of lists)

$=\ 1\ +\ length\ (xs\ @\ r)$ (by definition of sum)

$=\ 1\ +\ length\ xs\ +\ length\ r$ (by inductive hypothesis)

$=\ length\ (x : xs\ )\ +\ length\ r$ (by definition of sum)

Hence we have proven that $length\ (l\ @\ r)\ =\ length\ l\ +\ length\ r$.

Question 4: List length and reverse

The principle of induction for list type is:

$$\forall l,\ P(l)\ if\ P([\ ])\ and\ P(v\ ::\ l'),\ if\ P(l')$$

We need to show the following property P:

$$P\ (l, r)\ =\ length\ (reverse\ l@r)\ =\ length\ l : r$$

Base: $P\ (\ [\ ],\ r)\ :\ show\ that\ \ length\ (reverse\ [\ ]\ @\ r\ )\ =\ length\ ([\ ]@r)$

$length\ (reverse\ [\ ]\ @\ r)$	$= length\ (r\ @\ [\ ])$	(by definition of reverse)
	$=\ length\ r$	(by list properties)
	$=\ 0\ +\ length\ r$	(by mathematical operation)
	$=\ length\ [\ ]\ +\ length\ r$	(by proof from Question 3)
	$=\ length\ ([\ ]\ @\ r)$	(by definition of length)

Step: $P(x :: xs)\ :\ show\ that\ length\ (reverse\ x :: xs)\ = length\ (x : xs)$

The inductive hypothesis is $length\ (reverse\ xs)\ =\ length\ xs$

$length\ (reverse\ x :: xs)$	$=\ length\ (reverse\ xs\ @\ [x])$	(by definition of reverse)
	$=\ length\ (reverse\ xs\ )\ +\ length\ ([x])$	(by definition of length)
	$=\ length\ (reverse\ xs)\ +\ 1$	(by definition of length)
	$= 1 +\ length\ (xs)$	(by inductive hypothesis)
	$=\ length\ (x : xs)$	(by definition of length)

Therefore, we have proven that $length\ (reverse\ l@r)\ =\ length\ l : r$

Question 5: List length and reverse

The principle of induction for list type is:

$$\forall l,\ P(l)\ if\ P([\,])\ and\ P(v\ ::\ l'),\ if\ P(l')$$

We need to show the following property P:

$$P\ (l1, l2)\ =\ \ reverse\ (l1 @ l2)\ =\ reverse\ l2\ @\ reverse\ l1$$

Base: $P\ (\,[\,],\ l2)\ :\ show\ that\ \ reverse\ ([\,]\ @\ l2\ )\ =\ reverse\ l2\ @\ reverse\ [\,]$

$reverse\ (\,[\,]\ @\ l2\ )$	$=\ reverse\ l2$	(by properties of lists)
	$=\ reverse\ l2\ @\ [\,]$	(by properties of lists)
	$=\ reverse\ l2\ @\ reverse\ [\,]$	(by definition of reverse)

Step: $P(x : xs,\ r)\ :\ show\ that\ reverse\ (x : xs\ @\ r\ )\ =\ reverse\ r\ @\ reverse\ (x : xs)$

By Induction Hypothesis we have $reverse\ (xs\ @\ r)\ =\ reverse\ r\ @\ reverse\ xs$

$reverse\ (x : xs\ @\ r)$	$=\ reverse\ (x :: xs\ @\ r\ )$	(by properties of lists)
	$=\ reverse\ (x\ ::\ (xs\ @\ r)\ )$	(by properties of lists)
	$=\ reverse\ \ (\ xs\ @\ \ r\ )\ @\ [x]$	(by definition of reverse)
	$=\ reverse\ r\ @\ reverse\ xs\ @\ [x]$	(by inductive hypothesis)
	$=\ reverse\ r\ @\ reverse\ (xs\ @\ [x])$	(by properties of lists)
	$=\ reverse\ r\ @\ reverse\ (x\ ::\ xs)$	(by definition of reverse)
	$=\ reverse\ r\ @\ reverse\ (x\ :\ xs)$	

Thus we have proven that $reverse\ (l1\ @\ l2)\ =\ reverse\ l2\ @\ reverse\ l1$