Math Midterm Cheat Sheet

1 Week 1

1.1 Second Order Differential Equations (DE)

Formula	Description
$ay'' + by' + cy = 0$	Second order linear homogeneous DE
$r = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$	Formula to determine 'r'
$y(x) = Ae^{r_1x} + Be^{r_2x}$	General solution if $r1$ and $r2$ two simple zeros of the characteristic eq
$y(x) = (A + Bx)e^{rx}$	General solution if $r1 = r2 = r$ is a double zero of the characteristic eq (multiplicity 2)
$y(x) = Ae^{r\alpha}\cos(\omega x) + Be^{r\alpha}\sin(\omega x)$	General solution If $r = \alpha \pm i\omega(\alpha, \omega \in \mathbb{R})$ are the complex zeros of the characteristic equation

1.2 Guidelines for finding a particular solution

- If f (t) is a polynomial in t, and r = 0 is not a solution of the characteristic polynomial, then try for yp(t) a polynomial in t of the same degree as f . If r = 0 is a solution of the characteristic equation, then try for yp(t) a polynomial of degree deg(f) + 1, if r = 0 is a zero of multiplicity 1, and a polynomial of degree deg(f) + 2 if r = 0 is a zero of multiplicity 2.

- If f (t) is of the form $f(t) = \alpha sin(\omega_0 t) + \beta cos(\omega_0 t)$, and $\pm i\omega_0$ are not the zeros of the characteristic equation, then try $yp(t) = Asin(\omega_0 t) + Bcos(\omega_0 t)$ as particular solution. If $\pm i\omega_0$ are the zeros of the characteristic equation, then try $yp(t) = Atsin(\omega_0 t) + Btcos(\omega_0 t)$ as particular solution

- If f (t) is of the form $f(t) = p(t) * e^{\alpha t}$, with p(t) a polynomial in t, and if α is not a zero of the characteristic equation, then try $yp(t) = q(t) * e^{\alpha t}$ as particular solution, with q(t) a polynomial in t of the same degree as p(t). If α is a zero of the characteristic polynomial of multiplicity k, then try $yp(t) = q(t) * t^k * e^{\alpha t}$ as particular solution, with q(t) a polynomial in t, with deg(q) = deg(p).

2 Week 2

2.1 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

2.1.1 Addition:

$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$ and $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$
$\underline{a} + \underline{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$

2.1.2 Scaler/multiplication:

$\underline{a} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$
$\alpha * \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha * a_1 \\ \alpha * a_2 \end{bmatrix}$

2.1.3 Cross products:

$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$ and $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$
$\underline{a} \times \underline{b} = \begin{bmatrix} a_2 * b_3 - a_3 * b_2 \\ a_3 * b_1 - a_1 * b_3 \\ a_1 * b_2 - a_2 * b_1 \end{bmatrix}$

2.1.4 Inner product $\mathbb{R}^2$:

$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$
$(\underline{u}, \underline{v}) = u_1 * v_1 + u_2 * v_2$

2.1.5 Inner product $\mathbb{R}^3$:

$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3$ and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$
$(\underline{u}, \underline{v}) = u_1 * v_1 + u_2 * v_2 + u_3 * v_3$

2.1.6 Geometric definition:

$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$ and $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$
$(\underline{a}, \underline{b}) = ||\underline{a}|| * ||\underline{b}|| * cos(\theta)$
$||\underline{a}|| = \sqrt{(\underline{a}, \underline{a})}$
Where θ is the angle between the two vectors.

2.2 Parametric description

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \lambda \begin{bmatrix} d \\ e \\ f \end{bmatrix} + \mu \begin{bmatrix} g \\ h \\ i \end{bmatrix}$$

The vector containing *a, b and c* is the **supporting vector** and the vectors containing *d, e, f* and *g, h, i* are the **directional vectors**.

Example:

Let the following points P(2,2, 0), Q(6, 0, 1) and R(3, 3, 1) be given. Find the parametric description of the plane containing P, Q and R.

Solution:

1. Determine supporting vector: $\vec{OP} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

2. Determine directional vectors:

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} 6-2 \\ 0-2 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{PR} = \vec{OR} - \vec{OP} = \begin{bmatrix} 3-2 \\ 3-2 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3. Parametric description:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example:

Let the following points P(1, 1, 0), Q(2, 3, -1) and R(5, 0, 1) be given. Find the parametric description of the plane containing P, Q and R. Find the line l through S(6, -12, -16) perpendicular to V. Calculate the distance from S to V.

Solution:

1. Determine parametric description of the plane containing P, Q and R:

$$\vec{OP} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{PQ} = \begin{bmatrix}2-1\\3-1\\-1-0\end{bmatrix} = \begin{bmatrix}1\\2\\-1\end{bmatrix}, \vec{PR} = \begin{bmatrix}5-1\\0-1\\1-0\end{bmatrix} = \begin{bmatrix}4\\-1\\1\end{bmatrix}$$

Parametric description: of V: $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} + \lambda\begin{bmatrix}1\\2\\-1\end{bmatrix} + \mu\begin{bmatrix}4\\-1\\1\end{bmatrix}$, $(\lambda, \mu \in \mathbb{R})$

2. Determine the normal vector of the plane:

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{bmatrix}1\\2\\-1\end{bmatrix} \times \begin{bmatrix}4\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\-5\\-9\end{bmatrix}$$

3. Determine equation of the plane V:

x - 5y - 9z = d

4. Determine d:

$(\vec{OP}, \vec{n}) = 1*1 + 1*-5 + 0*-9 = 1 - 5 = -4$, so the equation of the plane is: x-5y-9z = -4

5. Determine the line l through S(6, -12, -16) perpendicular to V:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\-12\\-16\end{bmatrix} + \lambda\begin{bmatrix}1\\-5\\-9\end{bmatrix} = \begin{bmatrix}6+\lambda\\-12-5\lambda\\-16-9\lambda\end{bmatrix}$$

6. Determine point of intersection of l and V:

x - 5y - 9z = -4

$(6+\lambda) - 5(-12-5\lambda) - 9(-16-9\lambda) = -4$

$6 + \lambda + 60 + 25\lambda + 144 + 81\lambda = -4$

$210 + 107\lambda = -4$

$\lambda = -\frac{214}{107} = -2$

7. Determine the point of intersection:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\-12\\-16\end{bmatrix} + -2\begin{bmatrix}1\\-5\\-9\end{bmatrix} = \begin{bmatrix}6-2\\-12+10\\-16+18\end{bmatrix} = \begin{bmatrix}4\\-2\\2\end{bmatrix}$$

8. Determine the distance from S to V:

$$|| \begin{bmatrix}6\\-12\\-16\end{bmatrix} - \begin{bmatrix}4\\-2\\2\end{bmatrix} || = || \begin{bmatrix}2\\-10\\-18\end{bmatrix} || = \sqrt{2^2 + (-10)^2 + (-18)^2} = \sqrt{4 + 100 + 324} = \sqrt{428} = 2\sqrt{107}$$

Conclusion:

The distance from S to V is $2\sqrt{107}$

3 Week 3

3.1 System of Linear Equations

General form of a system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

3.2 Row Operations

- **Row Interchange:** Interchange two rows.
- **Row Scaling:** Multiply a row by a non-zero scalar.
- **Row Replacement:** Replace a row by the sum of that row and a multiple of another row.
- **Elementary Row Operations:** Row Interchange, Row Scaling, Row Replacement.

Example:
Solve the following system of linear equations using row operations:

$$\begin{aligned} x - 2y + 4z &= 16 \\ x + 3y + z &= 24 \\ x + y + 2z &= 7 \end{aligned} \tag{2}$$

Solution:
$\begin{bmatrix} 1 & -2 & 4 & 16 \\ 1 & 3 & 1 & 24 \\ 1 & 1 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 16 \\ 0 & 7 & -7 & -28 \\ 0 & 3 & -2 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 16 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 16 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$
Therefore, the solution is $x = 2, y = -1, z = 3$.

3.2.1 Definitions

The **coefficient matrix** is the matrix of coefficients of the variables in the system of linear equations. The **augmented coefficient matrix** is the coefficient matrix with the constants on the right side of the vertical line.
$\begin{bmatrix} 1 & -2 & 4| & 16 \\ 1 & 3 & 1| & 24 \\ 1 & 1 & 2| & 7 \end{bmatrix}$

Basic variables are the variables that have a leading 1 in their column, see first and second column. **Free variables** are the variables that do not have a leading 1 in their column, see third column.
$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1/4 & -5/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$