

Math Midterm Cheat Sheet

1 Week 1

1.1 Second Order Differential Equations (DE)

Formula	Description
$ay'' + by' + cy = 0$	Second order linear homogeneous DE
$r = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	Formula to determine 'r'
$y(x) = Ae^{r_1x} + Be^{r_2x}$	General solution if r_1 and r_2 two simple zeros of the characteristic eq
$y(x) = (A + Bx)e^{rx}$	General solution if $r_1 = r_2 = r$ is a double zero of the characteristic eq (multiplicity 2)
$y(x) = Ae^{r\alpha} \cos(\omega x) + Be^{r\alpha} \sin(\omega x)$	General solution If $r = \alpha \pm i\omega$ ($\alpha, \omega \in \mathbb{R}$) are the complex zeros of the characteristic equation

1.2 Guidelines for finding a particular solution

- If $f(t)$ is a polynomial in t , and $r = 0$ is not a solution of the characteristic polynomial, then try for $yp(t)$ a polynomial in t of the same degree as f . If $r = 0$ is a solution of the characteristic equation, then try for $yp(t)$ a polynomial of degree $\deg(f) + 1$, if $r = 0$ is a zero of multiplicity 1, and a polynomial of degree $\deg(f) + 2$ if $r = 0$ is a zero of multiplicity 2.
- If $f(t)$ is of the form $f(t) = \alpha \sin(\omega_0 t) + \beta \cos(\omega_0 t)$, and $\pm i\omega_0$ are not the zeros of the characteristic equation, then try $yp(t) = A \sin(\omega_0 t) + B \cos(\omega_0 t)$ as particular solution. If $\pm i\omega_0$ are the zeros of the characteristic equation, then try $yp(t) = At \sin(\omega_0 t) + Bt \cos(\omega_0 t)$ as particular solution
- If $f(t)$ is of the form $f(t) = p(t) * e^{\alpha t}$, with $p(t)$ a polynomial in t , and if α is not a zero of the characteristic equation, then try $yp(t) = q(t) * e^{\alpha t}$ as particular solution, with $q(t)$ a polynomial in t of the same degree as $p(t)$. If α is a zero of the characteristic polynomial of multiplicity k , then try $yp(t) = q(t) * t^k * e^{\alpha t}$ as particular solution, with $q(t)$ a polynomial in t , with $\deg(q) = \deg(p)$.

2 Week 2

2.1 Vectors in \mathbb{R}^2 and \mathbb{R}^3

2.1.1 Addition:

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 \text{ and } \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$$
$$\underline{a} + \underline{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

2.1.2 Scaler/multiplication:

$$\underline{a} \in \mathbb{R}^2 \text{ and } \alpha \in \mathbb{R}$$
$$\alpha * \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha * a_1 \\ \alpha * a_2 \end{bmatrix}$$

2.1.3 Cross products:

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 \text{ and } \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$$
$$\underline{a} \times \underline{b} = \begin{bmatrix} a_2 * b_3 - a_3 * b_2 \\ a_3 * b_1 - a_1 * b_3 \\ a_1 * b_2 - a_2 * b_1 \end{bmatrix}$$

2.1.4 Inner product \mathbb{R}^2 :

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2 \text{ and } \underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$$
$$(\underline{u}, \underline{v}) = u_1 * v_1 + u_2 * v_2$$

2.1.5 Inner product \mathbb{R}^3 :

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3 \text{ and } \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$$
$$(\underline{u}, \underline{v}) = u_1 * v_1 + u_2 * v_2 + u_3 * v_3$$

2.1.6 Geometric definition:

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 \text{ and } \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$$
$$(\underline{a}, \underline{b}) = ||\underline{a}|| * ||\underline{b}|| * \cos(\theta)$$
$$||\underline{a}|| = \sqrt{(\underline{a}, \underline{a})}$$

Where θ is the angle between the two vectors.

2.2 Parametric description

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \lambda \begin{bmatrix} d \\ e \\ f \end{bmatrix} + \mu \begin{bmatrix} g \\ h \\ i \end{bmatrix}$$

The vector containing a , b and c is the **supporting vector** and the vectors containing d , e , f and g , h , i are the **directional vectors**.

Example:

Let the following points P(2, 2, 0), Q(6, 0, 1) and R(3, 3, 1) be given. Find the parametric description of the plane containing P, Q and R.

Solution:

1. Determine supporting vector: $\vec{OP} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

2. Determine directional vectors:

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} 6-2 \\ 0-2 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{PR} = \vec{OR} - \vec{OP} = \begin{bmatrix} 3-2 \\ 3-2 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3. Parametric description:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example:

Let the following points P(1, 1, 0), Q(2, 3, -1) and R(5, 0, 1) be given. Find the parametric description of the plane containing P, Q and R. Find the line l through S(6, -12, -16) perpendicular to V. Calculate the distance from S to V.

Solution:

1. Determine parametric description of the plane containing P, Q and R:

$$\vec{OP} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{PQ} = \begin{bmatrix} 2-1 \\ 3-1 \\ -1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{PR} = \begin{bmatrix} 5-1 \\ 0-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Parametric description: of V: } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}, (\lambda, \mu \in \mathbb{R})$$

2. Determine the normal vector of the plane:

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -9 \end{bmatrix}$$

3. Determine equation of the plane V:

$$x - 5y - 9z = d$$

4. Determine d:

$$(\vec{OP}, \vec{n}) = 1 * 1 + 1 * -5 + 0 * -9 = 1 - 5 = -4, \text{ so the equation of the plane is: } x - 5y - 9z = -4$$

5. Determine the line l through S(6, -12, -16) perpendicular to V:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ -16 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -5 \\ -9 \end{bmatrix} = \begin{bmatrix} 6 + \lambda \\ -12 - 5\lambda \\ -16 - 9\lambda \end{bmatrix}$$

6. Determine point of intersection of l and V:

$$x - 5y - 9z = -4$$

$$(6 + \lambda) - 5(-12 - 5\lambda) - 9(-16 - 9\lambda) = -4$$

$$6 + \lambda + 60 + 25\lambda + 144 + 81\lambda = -4$$

$$210 + 107\lambda = -4$$

$$\lambda = -\frac{214}{107} = -2$$

7. Determine the point of intersection:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ -16 \end{bmatrix} + -2 \begin{bmatrix} 1 \\ -5 \\ -9 \end{bmatrix} = \begin{bmatrix} 6 - 2 \\ -12 + 10 \\ -16 + 18 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

8. Determine the distance from S to V:

$$\left\| \begin{bmatrix} 6 \\ -12 \\ -16 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2 \\ -10 \\ -18 \end{bmatrix} \right\| = \sqrt{2^2 + (-10)^2 + (-18)^2} = \sqrt{4 + 100 + 324} = \sqrt{428} = 2\sqrt{107}$$

Conclusion:

The distance from S to V is $2\sqrt{107}$

3 Week 3

3.1 System of Linear Equations

General form of a system of linear equations:

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m\end{aligned}\tag{1}$$

3.2 Row Operations

- **Row Interchange:** Interchange two rows.
- **Row Scaling:** Multiply a row by a non-zero scalar.
- **Row Replacement:** Replace a row by the sum of that row and a multiple of another row.
- **Elementary Row Operations:** Row Interchange, Row Scaling, Row Replacement.

Example:

Solve the following system of linear equations using row operations:

$$\begin{aligned}x - 2y + 4z &= 16 \\x + 3y + z &= 24 \\x + y + 2z &= 7\end{aligned}\tag{2}$$

Solution:

$$\begin{bmatrix} 1 & -2 & 4 & 16 \\ 1 & 3 & 1 & 24 \\ 1 & 1 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 16 \\ 0 & 7 & -7 & -28 \\ 0 & 3 & -2 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 16 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 16 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Therefore, the solution is $x = 2, y = -1, z = 3$.

3.2.1 Definitions

The **coefficient matrix** is the matrix of coefficients of the variables in the system of linear equations.

The **augmented coefficient matrix** is the coefficient matrix with the constants on the right side of the vertical line.

$$\begin{bmatrix} 1 & -2 & 4 & | & 16 \\ 1 & 3 & 1 & | & 24 \\ 1 & 1 & 2 & | & 7 \end{bmatrix}$$

Basic variables are the variables that have a leading 1 in their column, see first and second column. **Free variables** are the variables that do not have a leading 1 in their column, see third column.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1/4 & -5/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$