CSE 318
Assignment-02:
Solving the Max-Cut Problem by GRASP

April 28, 2025

Introduction

The maximum cut (MAX-CUT) problem involves an undirected graph G = (V, U), where V is the set of vertices and U is the set of edges with weights wuv. The goal of the MAX-CUT problem is to find a non-empty subset of vertices $S \subset V$ such that the weight of the cut $(S, \bar{S})$ is maximized. The cut weight is calculated as $w(S, \bar{S}) = \Sigma wuv$ for all edges (u, v) where $u \in S$ and $v \in \bar{S}$.

Algorithm 1: GRASP (Greedy Randomized Adaptive Search Procedure)

The GRASP algorithm consists of two phases: the construction phase, where a feasible solution is generated, and the local search phase, which aims to improve the solution by iterating and finding a locally optimal solution.

The procedure for GRASP is as follows:

1. procedure GRASP(MaxIterations)
2. for i = 1 to MaxIterations do
3. Build a greedy randomized solution x using semi-greedy heuristic
4. x ← LocalSearch(x)
5. if i = 1 then
6. x* ← x
7. else if w(x) > w(x*) then
8. x* ← x
9. end if
10. end for
11. return x*
12. end procedure

Algorithm 2: Randomized-1 Heuristic for MAX-CUT

The Randomized-1 heuristic generates a cut by randomly assigning vertices to two partitions (X and Y).

The procedure for the Randomized heuristic is as follows:

1. procedure RandomizedMaxCut(G = (V, E), w, n)
2. totalCutWeight ← 0
3. for i = 1 to n do
4. Initialize partitions: X ← ∅, Y ← ∅
5. for all vertex v ∈ V do
6. if Random choice with probability >= 1/2 then
7. X ← X ∪ {v}
8. else
9. Y ← Y ∪ {v}
10. end if
11. end for
12. cutWeight ← Σ (u,v) ∈ E, u ∈ X, v ∈ Y or u ∈ Y, v ∈ X wuv
13. totalCutWeight ← totalCutWeight + cutWeight
14. end for
15. averageCutWeight ← totalCutWeight / n
16. return averageCutWeight
17. end procedure

Algorithm 3: Greedy Heuristic for MAX-CUT

The Greedy Heuristic starts by placing the two vertices connected by the edge with the largest weight into different partitions, and then adds the remaining vertices in a greedy manner.

The procedure for the Greedy Heuristic is as follows:

1. procedure GreedyMaxCut(G = (V, E), w)
2. Initialize partitions: X ← ∅, Y ← ∅
3. Find edge (u, v) ∈ E with maximum weight wuv
4. X ← X ∪ {u}, Y ← Y ∪ {v}
5. U ← V \ {u, v} ▷ Unassigned vertices
6. for all vertex z ∈ U do
7. Compute weight if z ∈ X: wX = Σ y ∈ Y wzy
8. Compute weight if z ∈ Y: wY = Σ x ∈ X wzx
9. if wX > wY then
10. X ← X ∪ {z}
11. else
12. Y ← Y ∪ {z}
13. end if
14. end for
15. return partitions X, Y
16. end procedure

Algorithm 4: Semi-Greedy Heuristic for MAX-CUT

The Semi-Greedy Heuristic introduces randomness in the candidate selection process by evaluating the greedy function and selecting candidates from a Restricted Candidate List (RCL). The parameter α determines the randomness.

1. procedure SemiGreedyMaxCut(G = (V, E), w, α)

2. Initialize partitions: $X \leftarrow \emptyset$, $Y \leftarrow \emptyset$

3. Evaluate greedy function for each unassigned vertex v

4. Construct a Restricted Candidate List (RCL) based on the values

5. Randomly select a vertex from the RCL

6. Move the selected vertex to the partition that maximizes the cut value

7. Repeat until all vertices are assigned to partitions

8. return partitions X, Y

9. end procedure

I have tweaked the value of alpha to find a balance between randomness and greediness to find the sweet spot for it.

Algorithm 5: Local Search for MAX-CUT

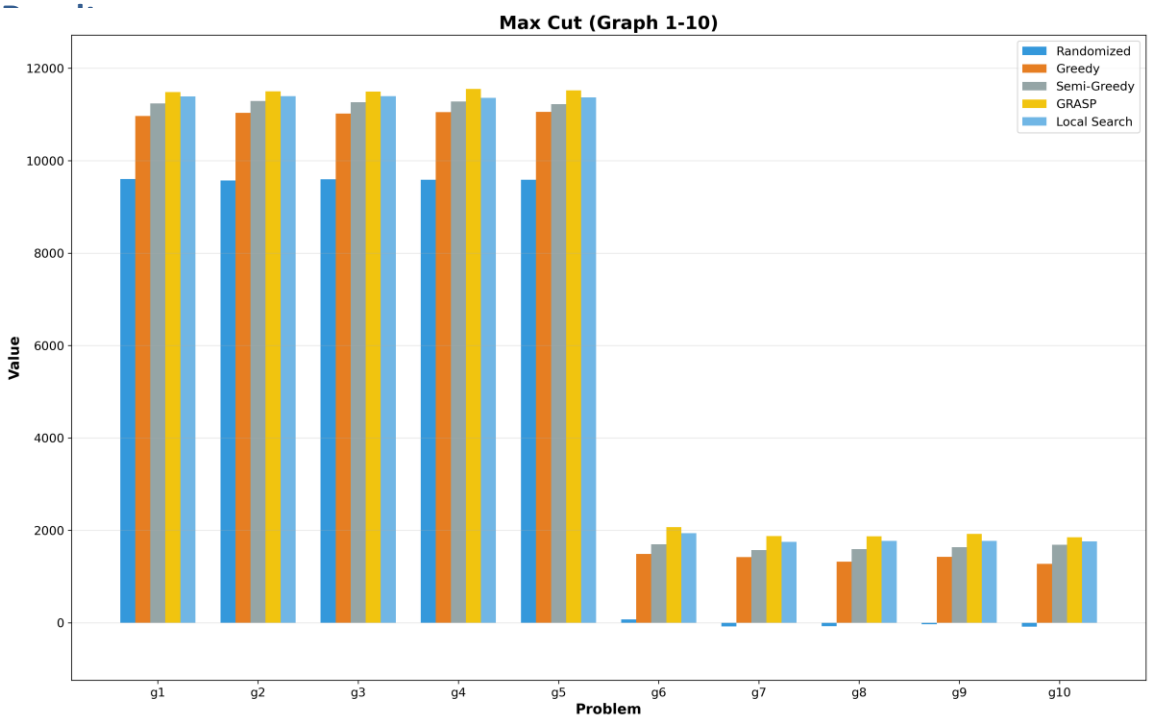
The local search phase explores neighboring solutions by moving vertices between the two partitions, improving the solution until no better solution is found.

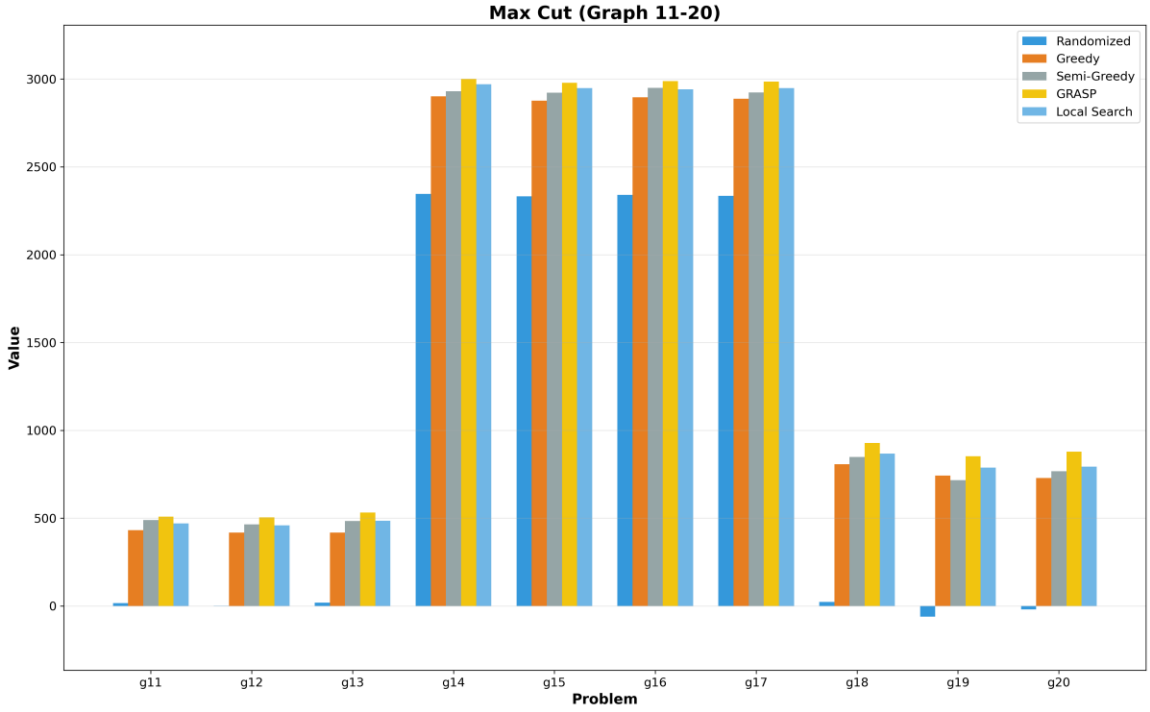
The procedure for local search is as follows:

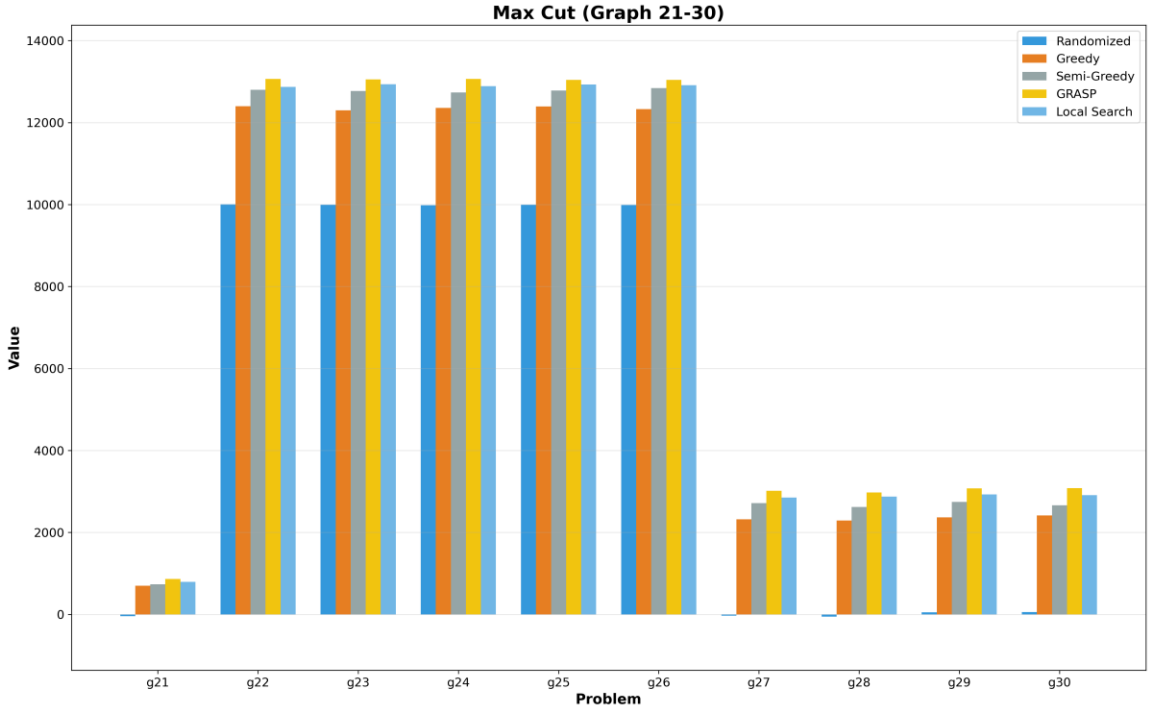
1. For each vertex $v \in V$, calculate $\delta(v)$ based on the current solution
2. Move the vertex to the partition that improves the cut weight
3. Repeat until no improving move is found

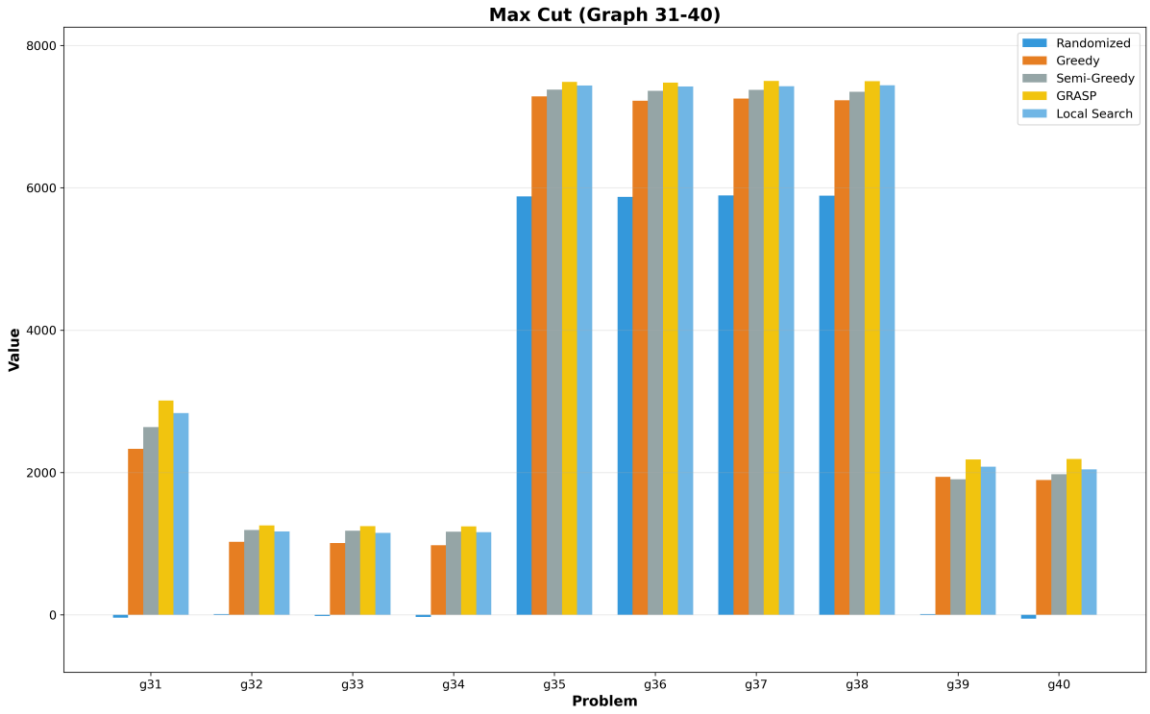
For the sake of implementation, I have chosen the semi-greedy algorithm for the construction phase and then ran the Local Search algorithm.

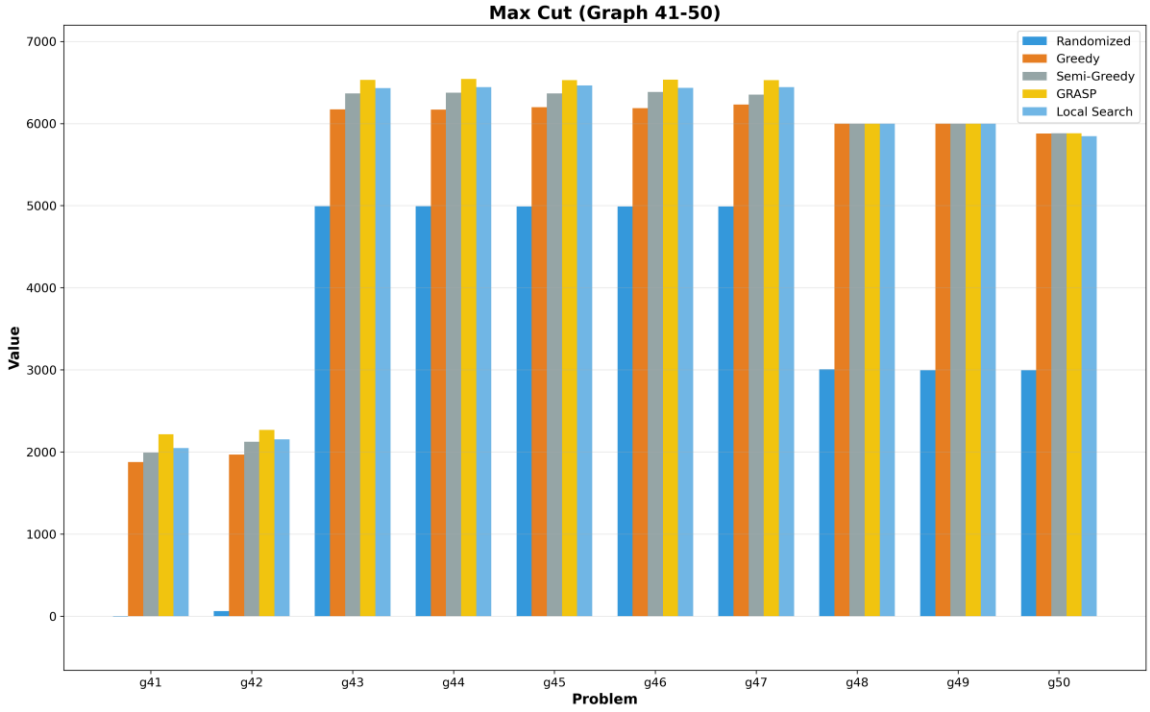
Graph Plots for Benchmark Problems

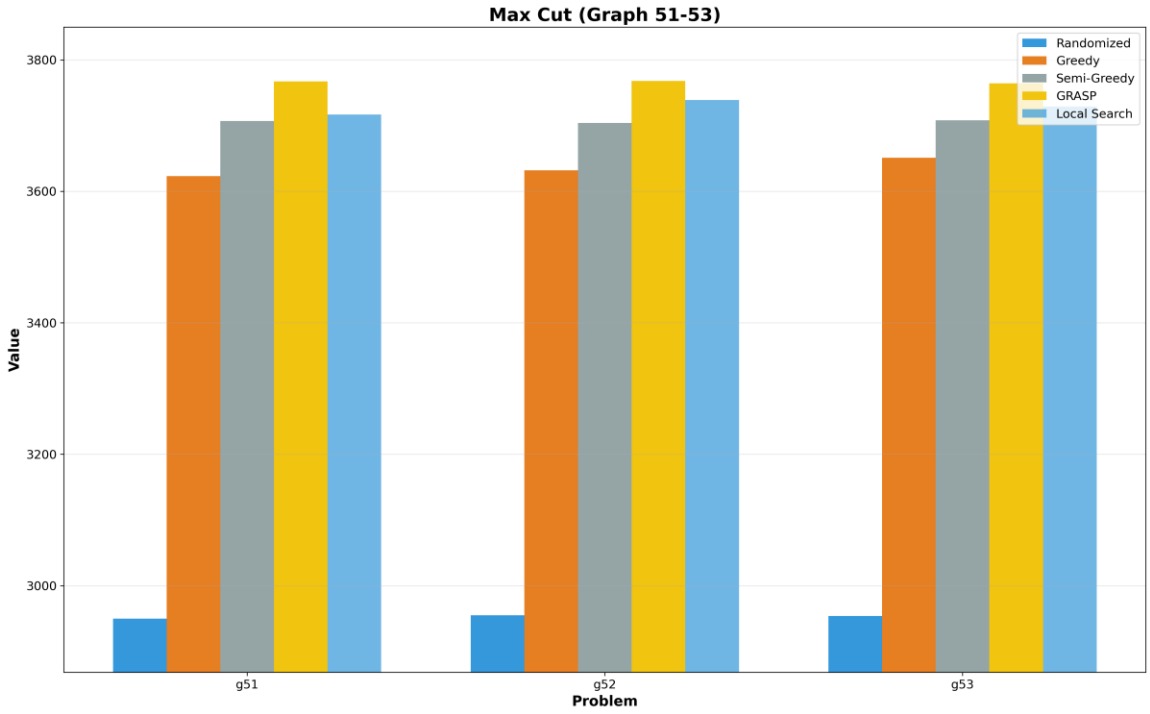

Max Cut (Graph 1-10)
Randomized
Greedy
Semi-Greedy
GRASP
Local Search
Value
Problem
g1
g2
g3
g4
g5
g6
g7
g8
g9
g10


Max Cut (Graph 11-20)
Randomized
Greedy
Semi-Greedy
GRASP
Local Search
Value
Problem
g11
g12
g13
g14
g15
g16
g17
g18
g19
g20


Max Cut (Graph 21-30)
Randomized
Greedy
Semi-Greedy
GRASP
Local Search
Value
14000
12000
10000
8000
6000
4000
2000
0
g21
g22
g23
g24
g25
g26
g27
g28
g29
g30
Problem


Max Cut (Graph 31-40)
Randomized
Greedy
Semi-Greedy
GRASP
Local Search
Value
8000
6000
4000
2000
0
g31
g32
g33
g34
g35
g36
g37
g38
g39
g40
Problem


Max Cut (Graph 41-50)
Randomized
Greedy
Semi-Greedy
GRASP
Local Search
Value
7000
6000
5000
4000
3000
2000
1000
0
g41
g42
g43
g44
g45
g46
g47
g48
g49
g50
Problem

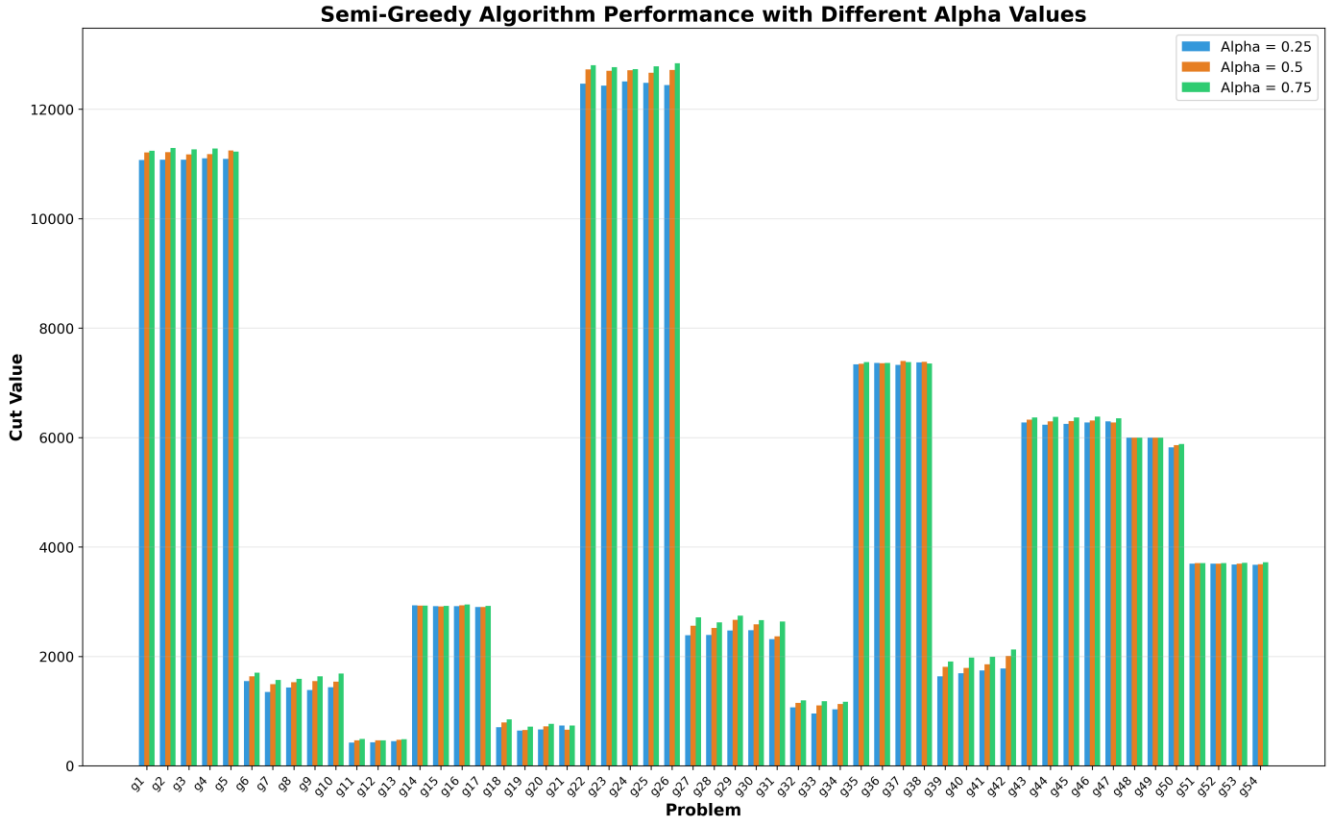

Max Cut (Graph 51-53)
Randomized
Greedy
Semi-Greedy
GRASP
Local Search
Value
3800
3600
3400
3200
3000
g51
g52
g53
Problem

Comparison of Algorithms

From the graphs mentioned we can easily compare the performance of various algorithms and say confidently various facts about it.

Algorithm	Performance	Reliability	Time Complexity	Use Case
Randomized	Good in simpler problems, inconsistent in complex ones	Low reliability (randomness leads to variability)	Fast, but suboptimal	Suitable for positive graphs or when quick approximate solutions are needed but underperforms in negative edged graphs.
Greedy	Good for small to medium graphs, starts to lose in complex ones	Moderate reliability (prone to local optima)	Fast	Best for simple problems with good local optima, fast and simple
Semi-Greedy	Generally, provides better performance than Greedy and Randomized	More reliable than Greedy and Randomized	Moderate (due to RCL construction)	Balanced between exploration and exploitation, works well on complex graphs
Local Search	2nd best as it is not stuck in the a higher value but optimized to the local optima.	More reliable than semi-greedy	Moderately slow (due to iterations)	Better performance in a large graph as it has a lot of maxima as we can see by the plots.
GRASP	Best performance across most problem sets	Highly reliable, stable, robust	Slow (due to multiple iterations)	Best for high-quality solutions, especially on large and complex graphs

Comparison of Alpha

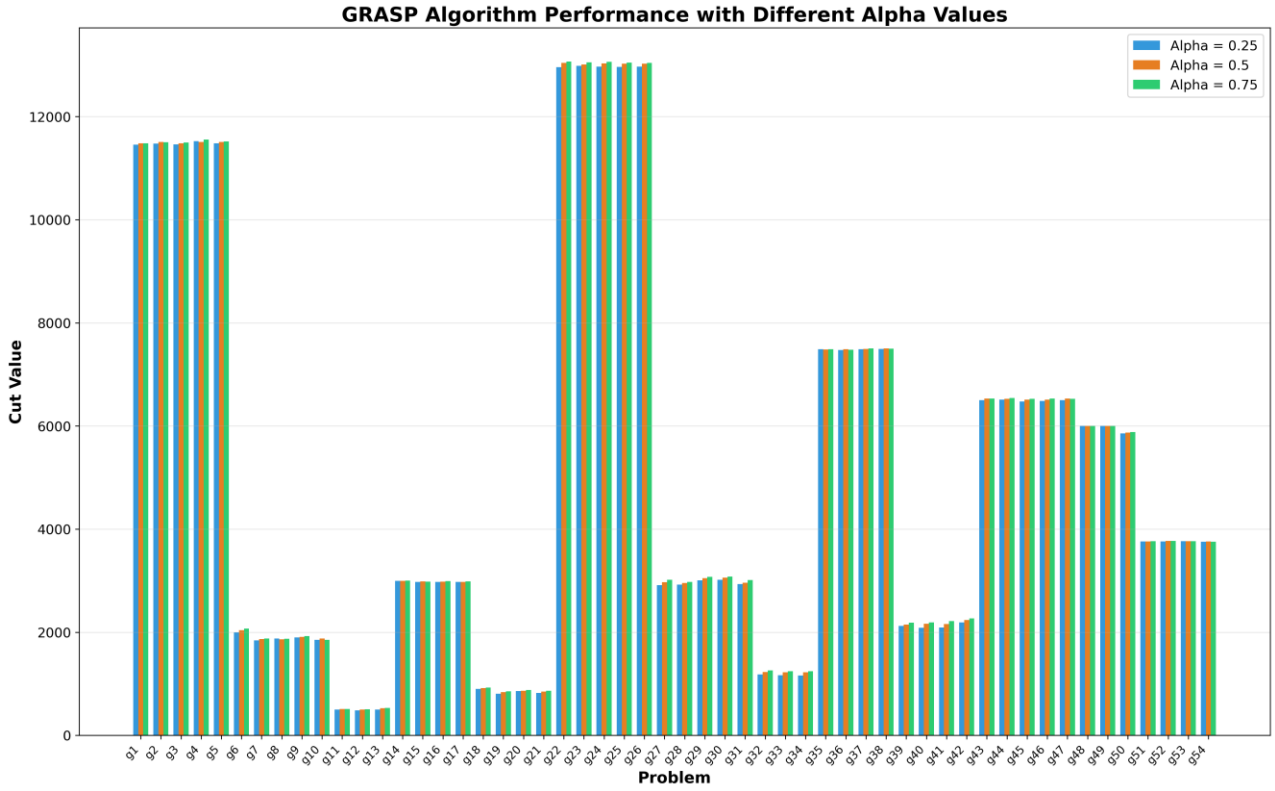


Effect of Different α Values:

- **When α = 0.25**:
  - **RCL is large**: α = 0.25 implies that the threshold μ is much closer to **w_min**, meaning the RCL will contain a **large number of candidates**.
  - **High randomness**: Because the RCL is large, the algorithm is more random in selecting vertices. While this provides **more exploration**, it also means that many of the candidates may not contribute much to the cut value, and the algorithm often selects **poor candidates**.
  - **Result**: The algorithm struggles to improve the cut, as it introduces too much randomness and selects suboptimal vertices.
- **When α = 0.50**:
  - **Moderate RCL size**: α = 0.50 implies μ is closer to the middle of the greedy function values, leading to an **intermediate-sized RCL**.
  - **Balanced exploration and exploitation**: With α = 0.50, the algorithm strikes a balance between exploration (randomness) and exploitation (greedy), but it's not fully focusing on the best candidates, so it can still include **suboptimal candidates** that dilute the quality of the cut.
  - **Result**: The algorithm gives **moderate performance**, but some of the exploration still leads to suboptimal cuts, and it doesn't fully exploit the best options available.
- **When α = 0.75**:

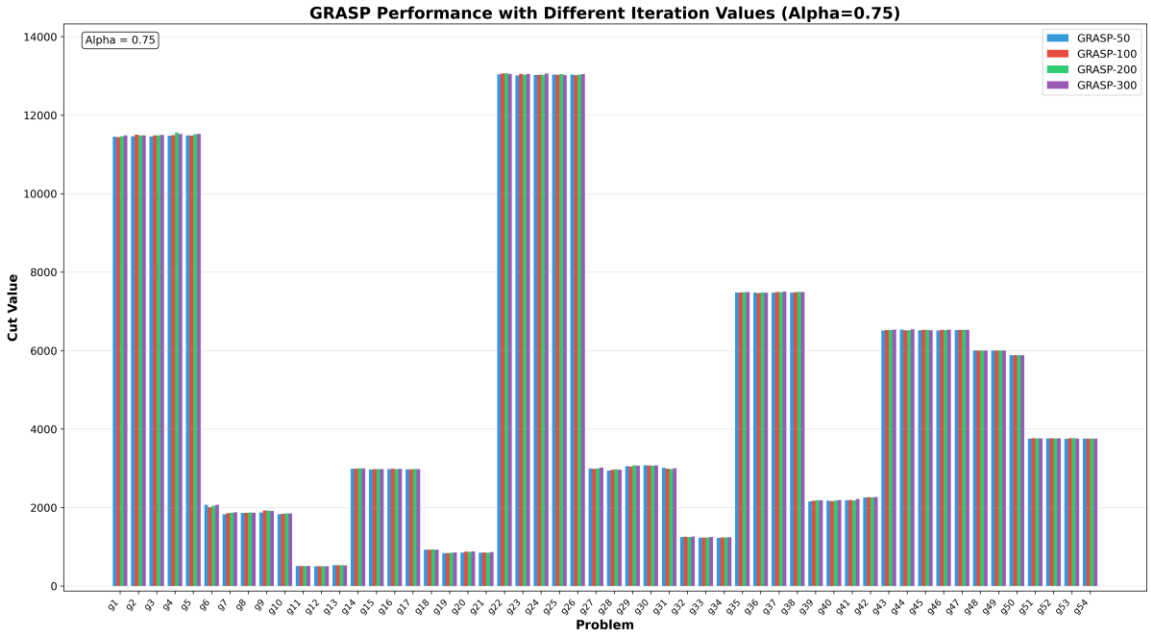
- **Smaller RCL, focused selection**: α = 0.75 brings the threshold μ closer to **w_max**, so the RCL becomes smaller, focusing only on the **best candidates** (those closer to **w_max**). This means that the algorithm **prioritizes better choices** for the cut.
- **Less randomness**: While there's still some randomness in selecting candidates from the smaller RCL, the algorithm is now more focused on high-quality vertices that will contribute significantly to the cut.
- **Result**: The algorithm performs **better** because it focuses on better candidates while still retaining a bit of randomness to avoid getting stuck in a local optimum. It **finds better cuts** by avoiding suboptimal vertices and concentrating on those that maximize the cut value.

This ensures a better grasp value when used in the construction phase



A slight increase in the performance for a higher alpha value but as we move from **α = 0.25** to **α = 0.75**, we're gradually improving the **quality of the initial solution**, but the **improvement in the final solution** may not be very large. At a certain point, the **local search** might make up for any initial shortcomings from the construction phase, **diluting the benefit** of having a larger α in the case of GRASP.

Comparison of Number of Iterations



We can see that, not much change or improvements happen for the number of iterations of GRASP algorithm as it has similar solution ranges so variation in the iterations doesn't guarantee a different value. Moreover after certain iterations it will get stuck in the best value so much improvement is not possible and for a reasonable small iteration number the solution is guaranteed to be optimal.

Conclusion

The GRASP algorithm and its components (Greedy, Randomized, Semi-Greedy, and Local Search) provide a robust method for solving the MAX-CUT problem. The results and comparisons indicate the effectiveness of GRASP in achieving high-quality solutions.

References

1. Feo, T. A., & Resende, M. G. (1995). Greedy Randomized Adaptive Search Procedures. Journal of Global Optimization, 6(2), 109-133.
2. Fischetti, M., & Lodi, A. (2003). Local search algorithms for the max-cut problem. INFORMS Journal on Computing, 15(4), 349-366.