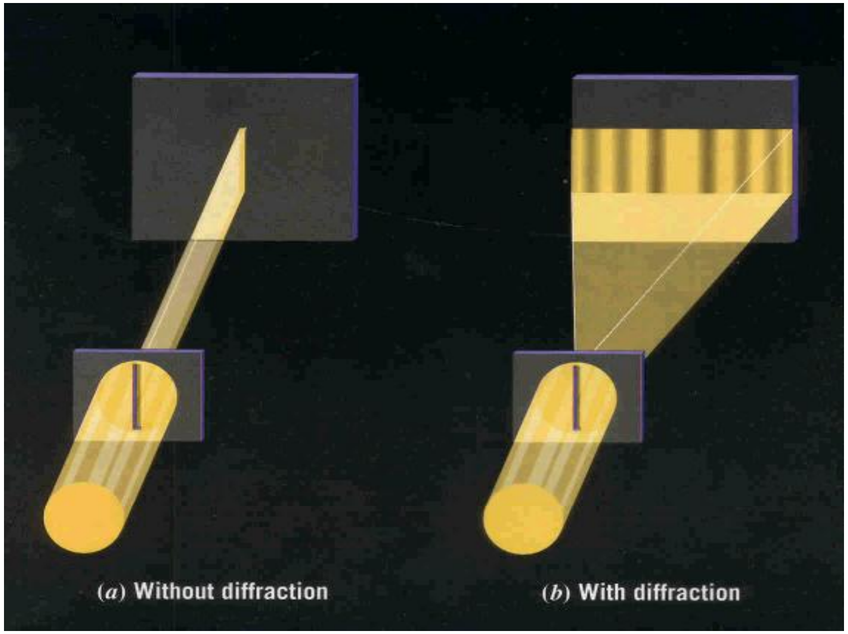
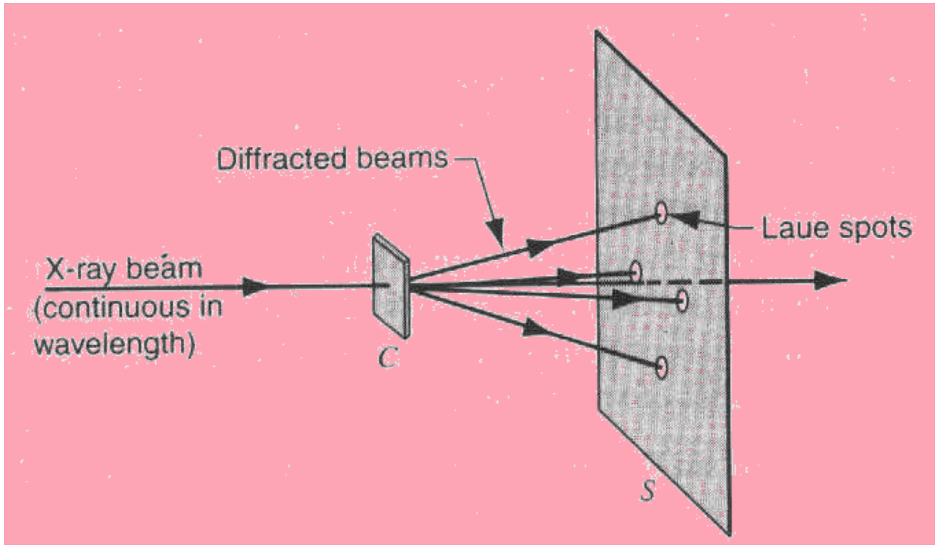
Chapter 42 (43) Wave Optics (2)

Diffraction and Gratings
(衍射与光栅)

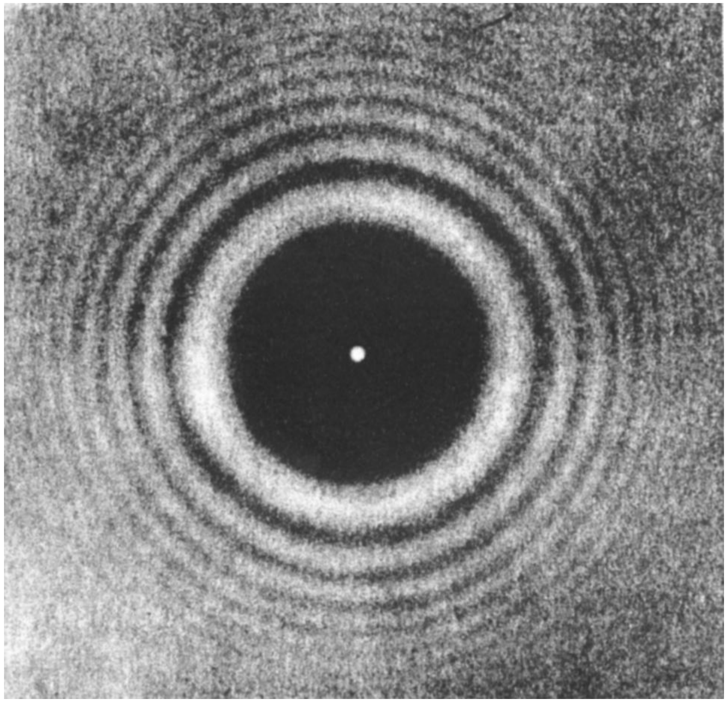




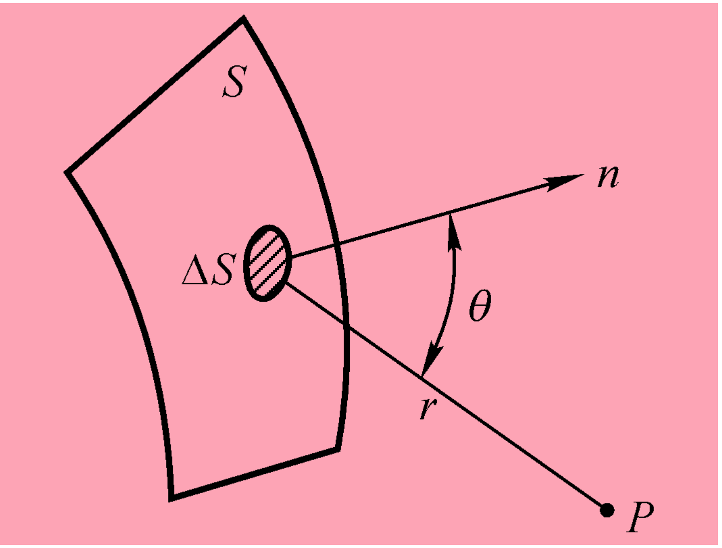
42-1 Diffraction and the wave theory of light

- When a light passes through a hole, or meets a disk etc., with dimensions comparable to the wavelength, we can see patterns on the screen behind them

- Evidence for the wave nature of light



Fresnel's(菲涅耳) theory of light (1788-1827)



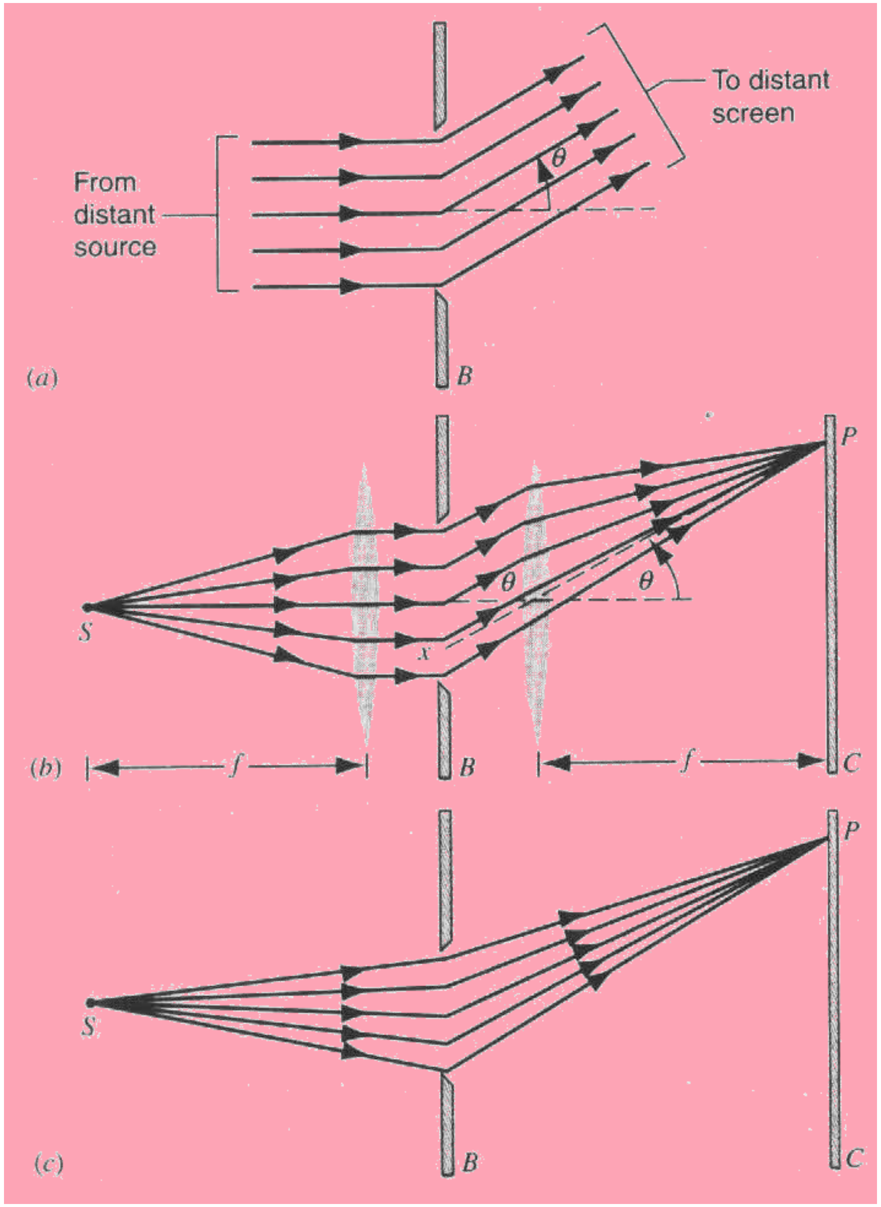
$$dE_P = C\frac{dS}{r}K(\theta)\cos(\frac{2\pi}{\lambda}r - \omega t + \varphi_0)$$

$$E_P = \int_S C\frac{K(\theta)}{r}\cos(\frac{2\pi}{\lambda}r - \omega t + \varphi_0)dS$$

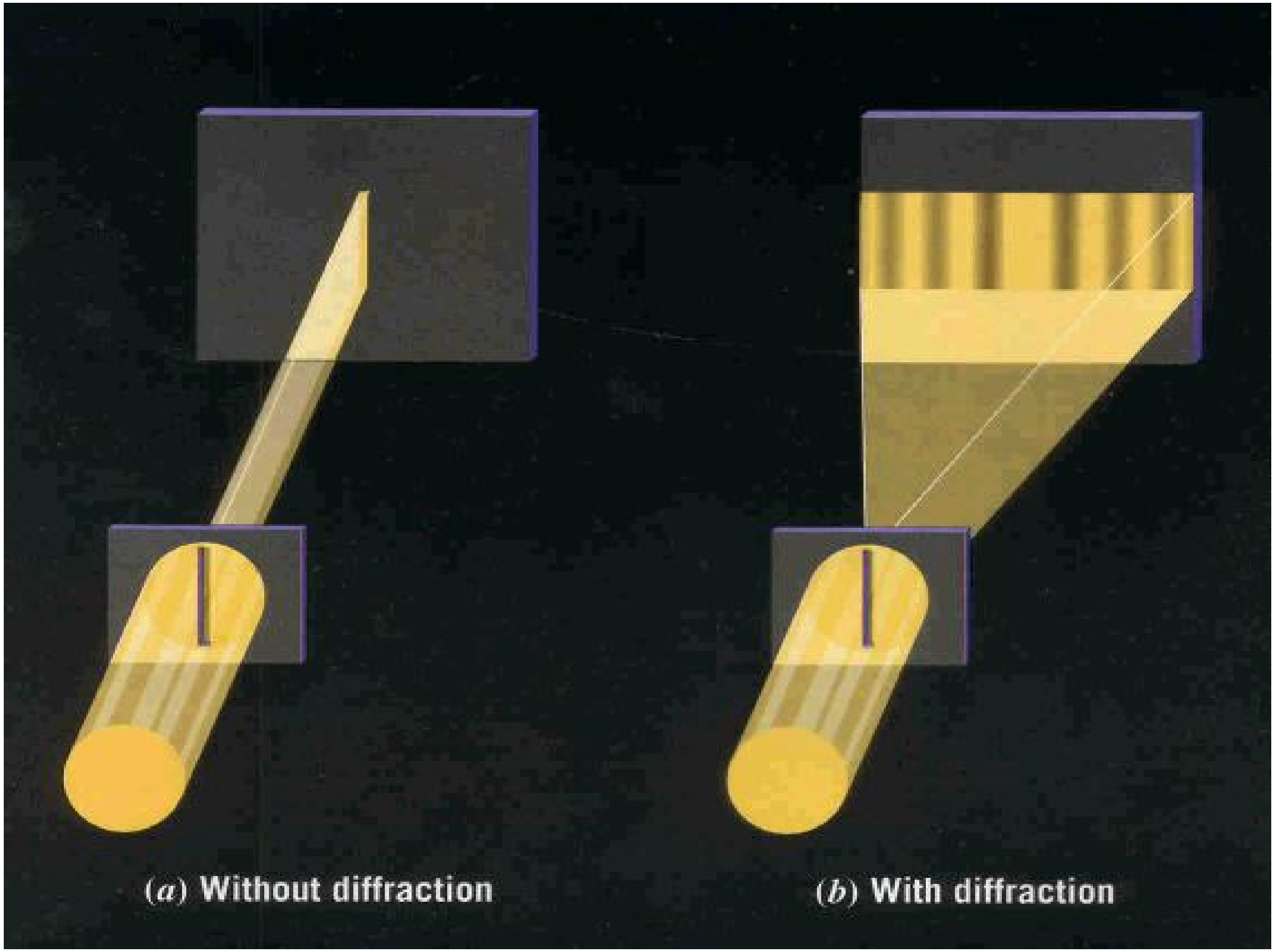
Makes it feasible for calculation of intensity at any point on screen

Fraunhofer(弗朗和夫) diffraction
—infinite separation

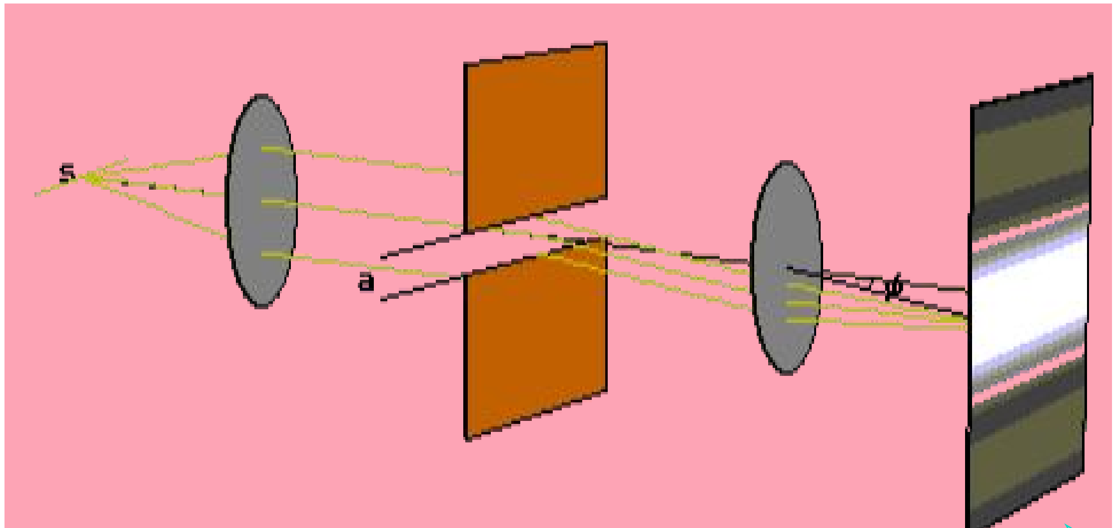
Fresnel's (菲涅尔) diffraction
—finite separation



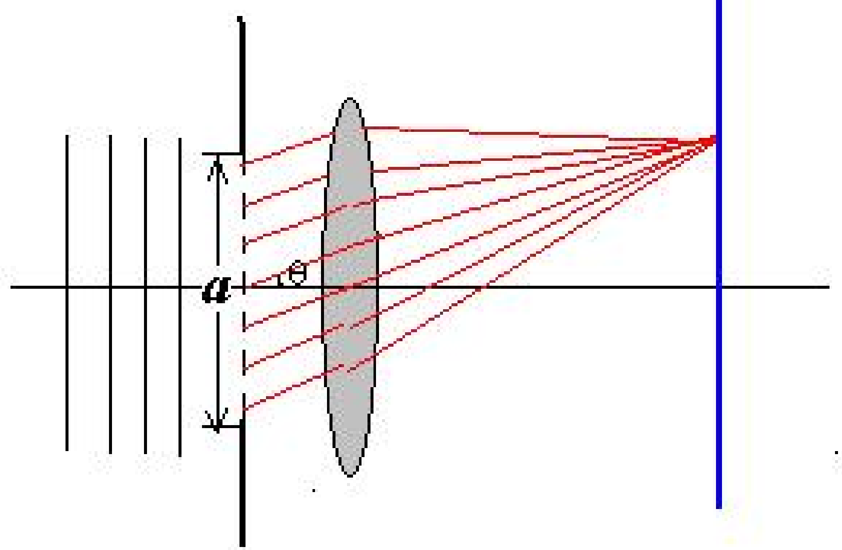
42-2 Single-slit diffraction
(单缝衍射)



Fraunhofer's Single-slit Diffraction
(弗朗和夫单缝衍射)

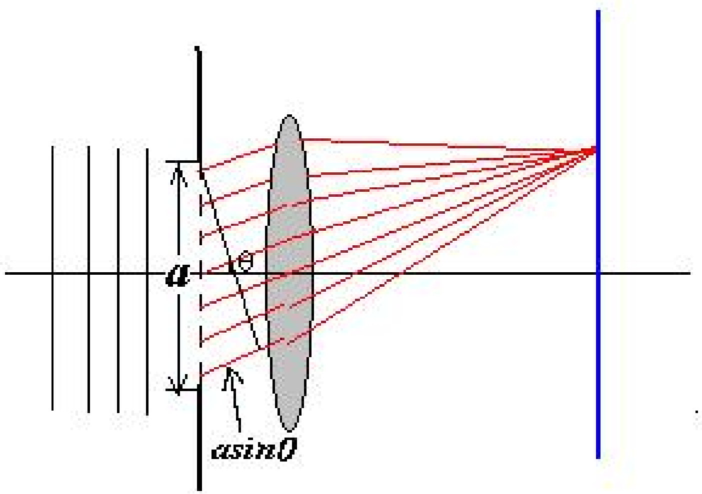


- Dispersion of the light

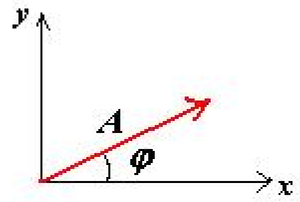


Intensity in single-slit diffraction

The whole slit is divided into *N* strips with *Δ x* = *a / N*, which can be regarded as Huygen's wavelet, therefore,

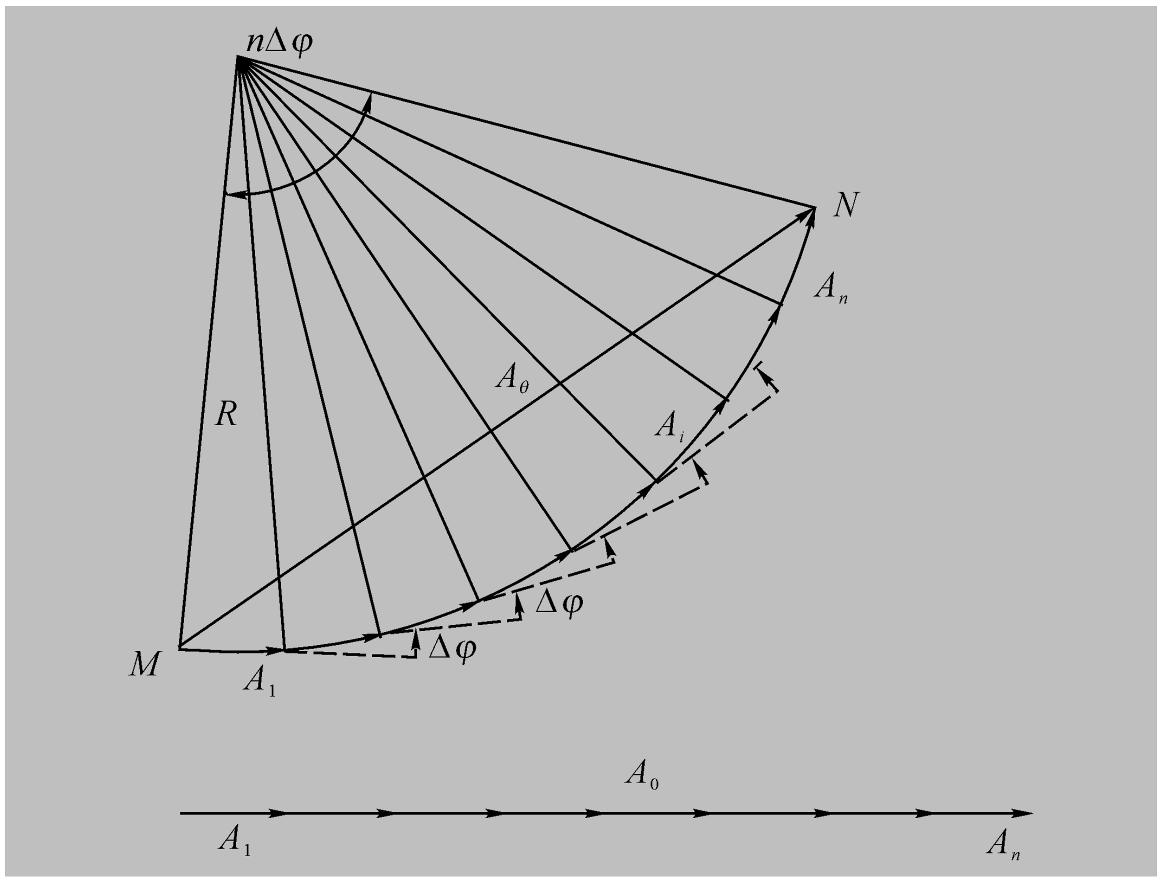


$$\widetilde{U}(P) = Ae^{i\varphi(P)}$$

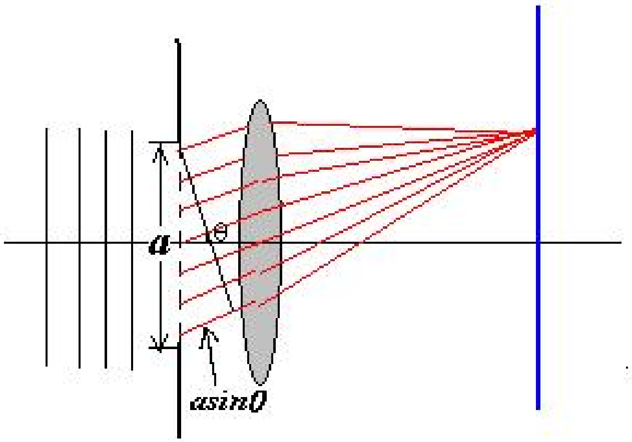


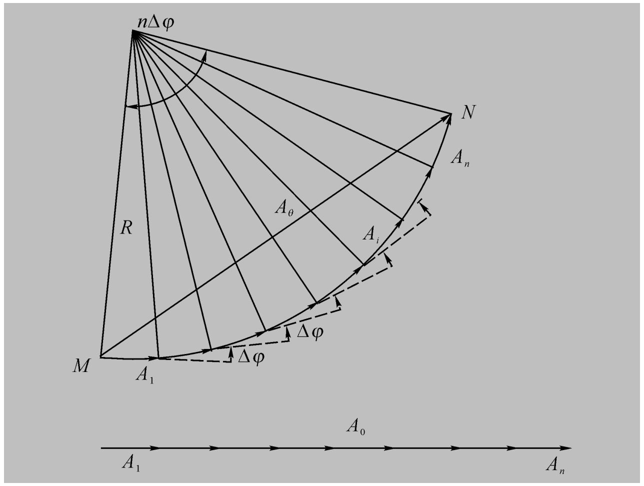
$$E = E_m e^{i\varphi_m}$$

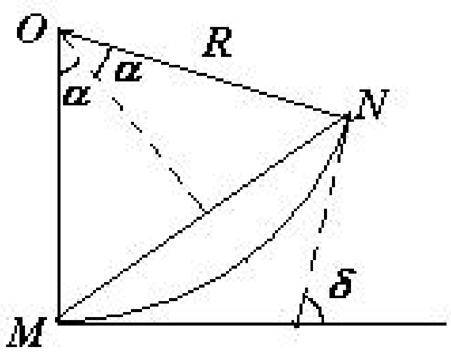
$$\Delta\varphi = \frac{2\pi}{\lambda}\Delta x \sin\theta$$



Intensity in single-slit diffraction (con.)







$$E_1 = E_0 e^{i0}$$
$$E_2 = E_0 e^{i\Delta\varphi}$$
$$E_3 = E_0 e^{i2\Delta\varphi}$$
....
$$E_N = E_0 e^{i(N-1)\Delta\varphi}$$

$$\Delta\varphi = \frac{2\pi}{\lambda}\cdot\frac{a}{N}\sin\theta$$

$$E_\theta = \overline{MN} = 2R\sin\alpha$$

$$R = \frac{\widehat{MN}}{2\alpha}$$

$$\therefore E_\theta = \widehat{MN}\,\frac{\sin\alpha}{\alpha}$$

$\widehat{MN} = E_m$ The electric field at the center ($\theta = 0$).

$$\therefore E_\theta = E_m \frac{\sin\alpha}{\alpha}$$

$$\delta = N\Delta\varphi = \frac{2\pi}{\lambda} a\sin\theta$$

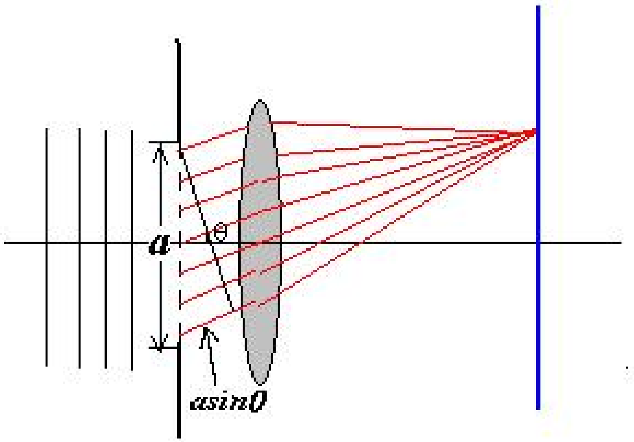
$$\alpha = \frac{\delta}{2} = \frac{\pi a\sin\theta}{\lambda}$$

The intensity at P:

$$I_\theta = E_\theta^2 = E_m^2\left(\frac{\sin\alpha}{\alpha}\right)^2 = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2$$

$$\alpha = \frac{\pi a\sin\theta}{\lambda}$$

Discussions



$$I_\theta = E_\theta^2 = E_m^2\left(\frac{\sin\alpha}{\alpha}\right)^2 = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2$$

$$\alpha = \frac{\pi a \sin\theta}{\lambda}$$

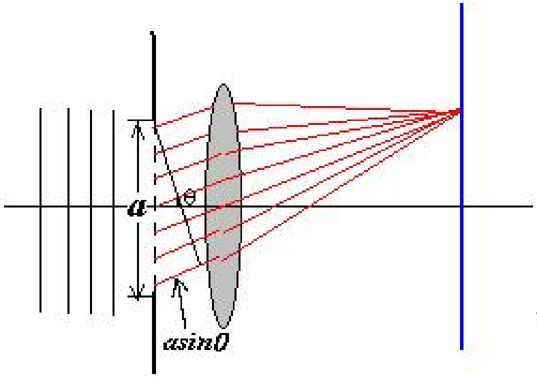
(1) If $\alpha = \dfrac{\pi a \sin\theta}{\lambda} = m\pi \quad (m = \pm 1, \pm 2, ...)$

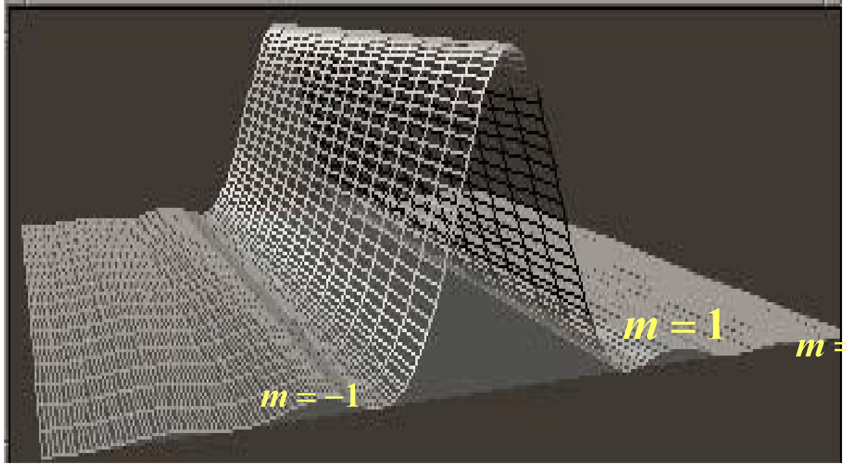
$$I_\theta = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2 = 0 \quad \text{------minina}$$

$$a\sin\theta = m\lambda$$

(2) $\theta = 0,\ \alpha = \dfrac{\pi a \sin\theta}{\lambda} \to 0,\ \lim\limits_{\alpha\to 0}\left(\dfrac{\sin\alpha}{\alpha}\right) = 1$

$$I_{\theta=0} = I_m$$





(3) $\alpha = \dfrac{\pi a \sin\theta}{\lambda} = (m + \dfrac{1}{2})\pi$, $a\sin\theta = (m+\dfrac{1}{2})\lambda$, I_θ Maximum

$$\frac{I_1}{I_m} = 0.045,\quad \frac{I_2}{I_m} = 0.016,\quad \frac{I_3}{I_m} = 0.0083$$

$$\frac{d}{d\alpha}\left(\frac{\sin\alpha}{\alpha}\right) = 0,\quad \alpha = \mathrm{tg}\alpha$$

$\alpha = \pm 1.43\pi,\ \pm 2.46\pi,\ \pm 3.47\pi.....$ The second Maximum

$$I_\theta = E_\theta^2 = E_m^2\left(\frac{\sin\alpha}{\alpha}\right)^2 = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2$$

$$\alpha = \frac{\pi a \sin\theta}{\lambda}$$

(4) The half-angle width (半角宽度) for the bright fringe (主极大) at the center.

For the paraxial rays: $a\sin\theta = \lambda$, $\sin\theta \approx \Delta\theta = \dfrac{\lambda}{a}$

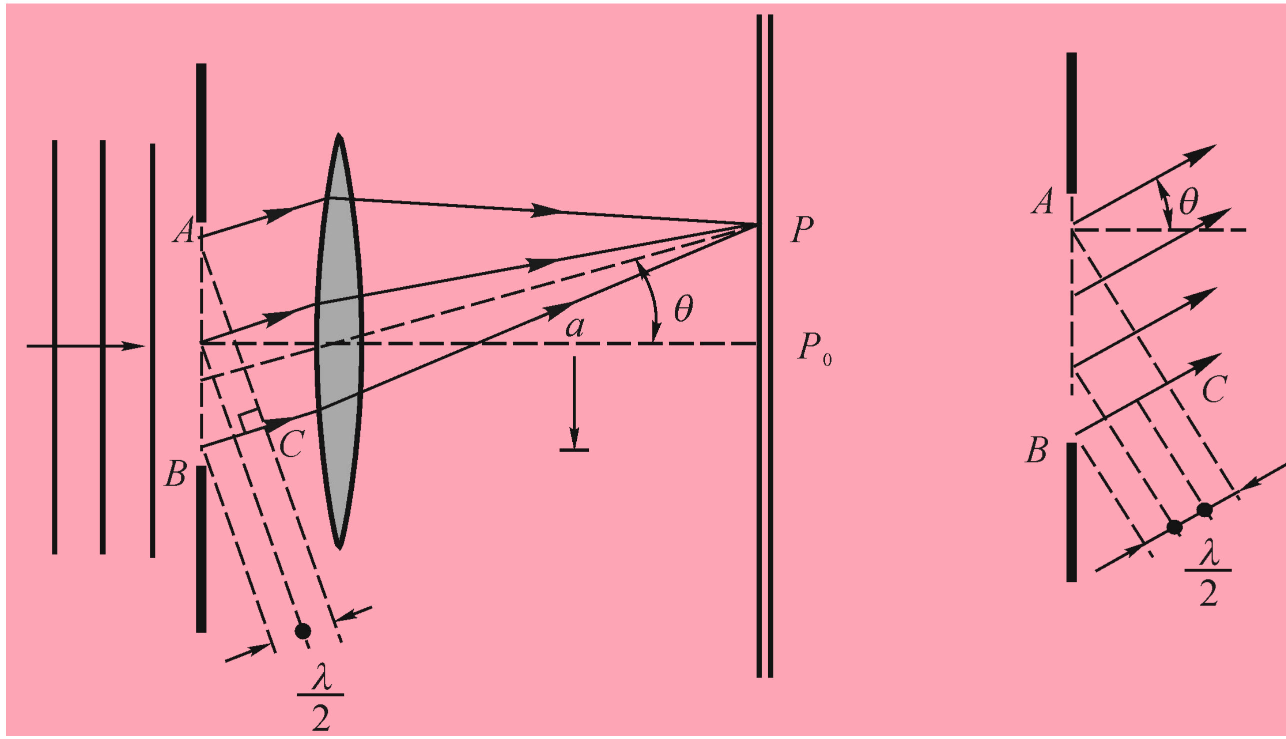
$$\Delta y_m \approx f \cdot \Delta\theta = f \cdot \frac{\lambda}{a}$$

a bigger, $\Delta\theta$ smaller

a smaller, $\Delta\theta$ bigger

Half-wavelength strip (半波片):

The whole slit is divided into N strips so that the adjacent rays have path difference of a half-wavelength:



$$\frac{a}{N}\sin\theta = \lambda/2, \quad \text{for adjacent rays}$$

$$a\sin\theta = N\lambda/2, \quad \text{whole } \Delta L_0 = BC \text{ for top and bottom rays}$$

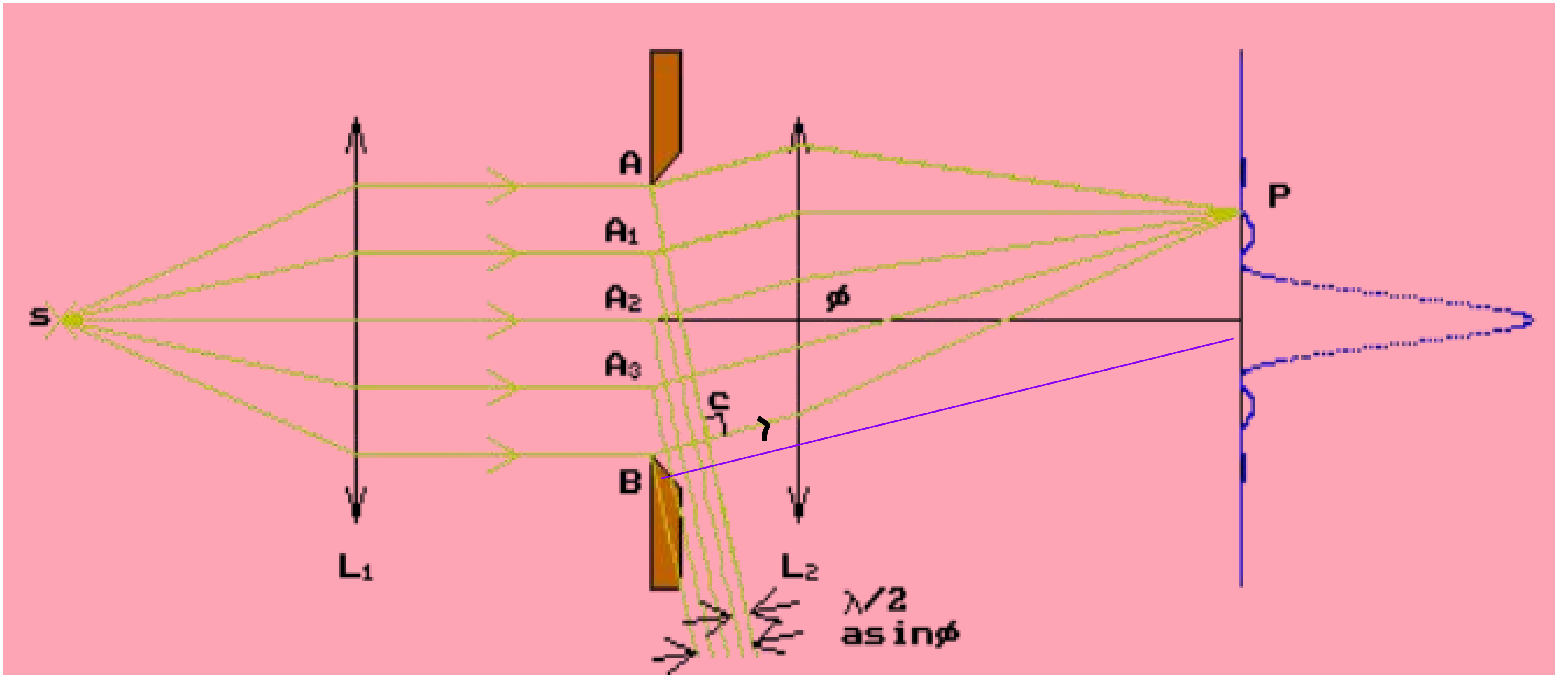
Diffraction pattern

The angle θ (the position on the screen) for the maxima and minima can be determined as following:

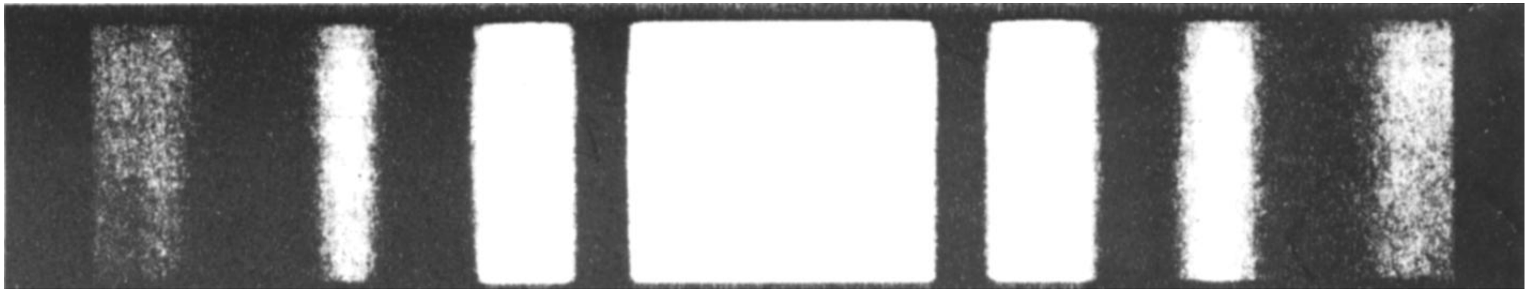
$$\delta = a\sin\theta = \begin{cases} 0 & \textbf{Bright fringe at the center} \\ 2m\dfrac{\lambda}{2} = m\lambda & m = \pm1, \pm2, \pm3\ldots \text{min ima} \\ (2m+1)\dfrac{\lambda}{2} & m = \pm1, \pm2, \pm3\ldots \text{max ima} \end{cases}$$

N : even

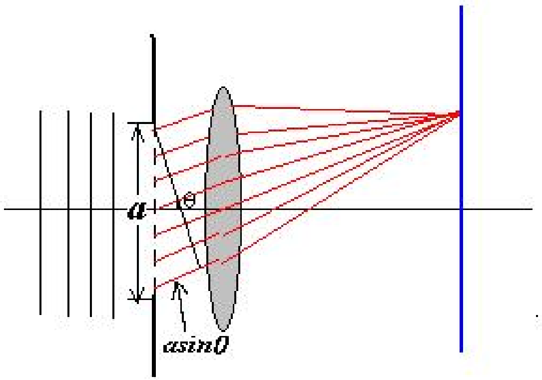
N : odd



Qualitative intensity distribution **(定性强度分布)**



Example: A slit of width a = 0.5 mm is illuminated by a monochromic light. Behind the slit there placed a lens (*f* = 100 cm), and the first maximum fringe (一级最大) is observed at a distance of 1.5 mm from the central bright fringe on the focal plane (screen). Find the wavelength of the light, and the width of the central bright fringe (中心零级最大).

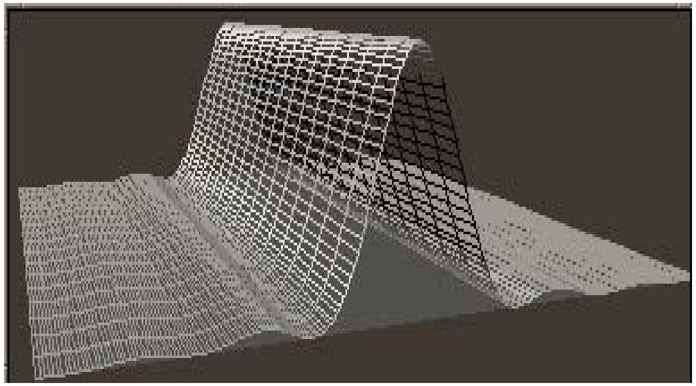


$$\text{maxima:} \quad a\sin\theta = (2m+1)\frac{\lambda}{2}, \quad \text{and} \quad \sin\theta \approx \theta = \frac{x}{f}$$

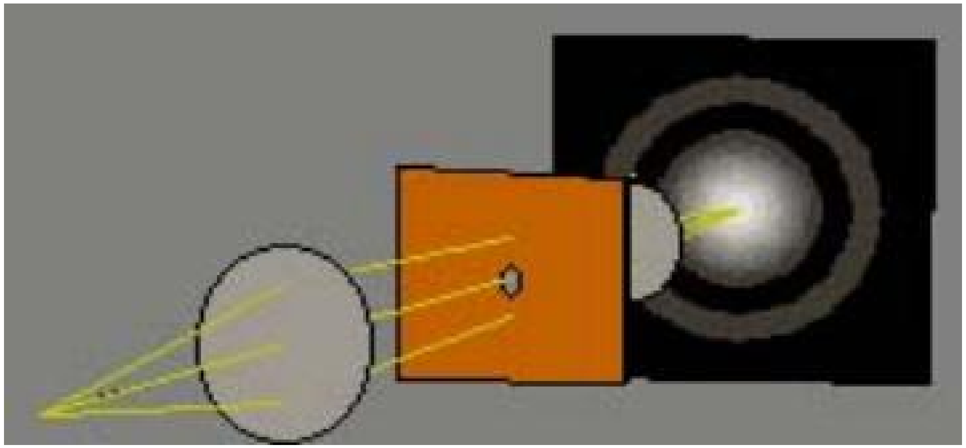
$$\lambda = \frac{2ax}{(2m+1)f} = \frac{1500\text{nm}}{3} = 500\text{nm}$$

$$\text{minima:} \quad a\sin\theta = m\lambda, \quad \Delta\theta = \frac{\lambda}{a}$$

$$\Delta y_0 = 2f\frac{\lambda}{a} = 2\text{mm}$$



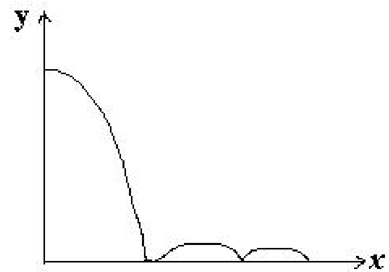
42-3 Fraunhofer Diffraction at Circular Aperture（弗朗和夫园孔衍射）



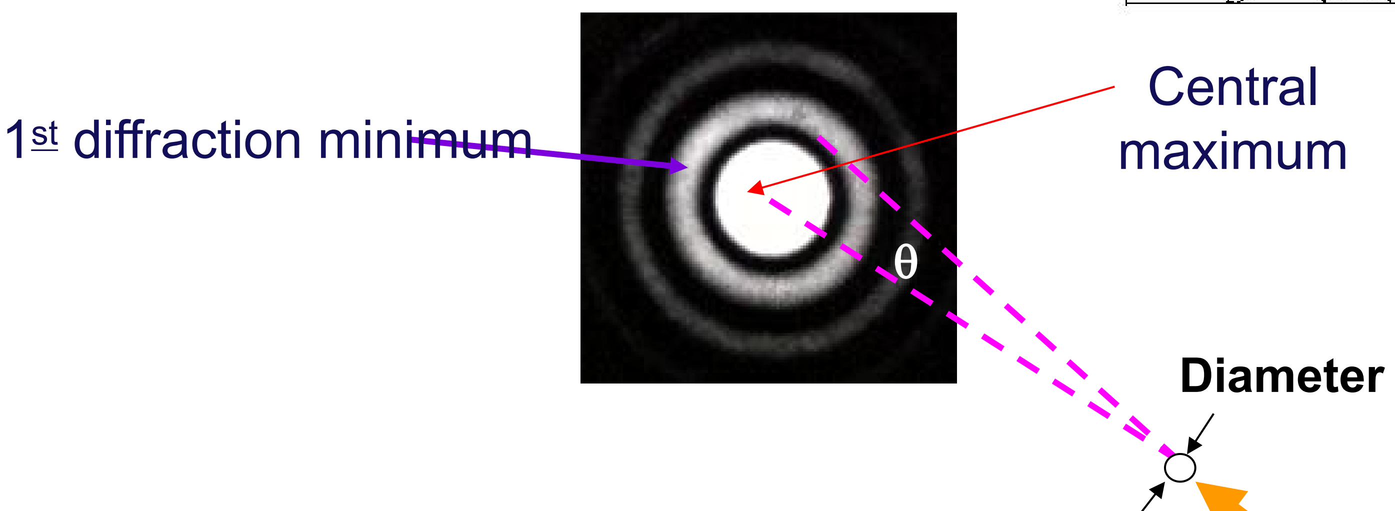
$$I(\theta)=I_0\left[\frac{2J_1(x)}{x}\right]^2,\quad x=\frac{2\pi a\sin\theta}{\lambda}$$

$J_1(x)$ The first order Bessel Function

a The radius of Circular Aperture



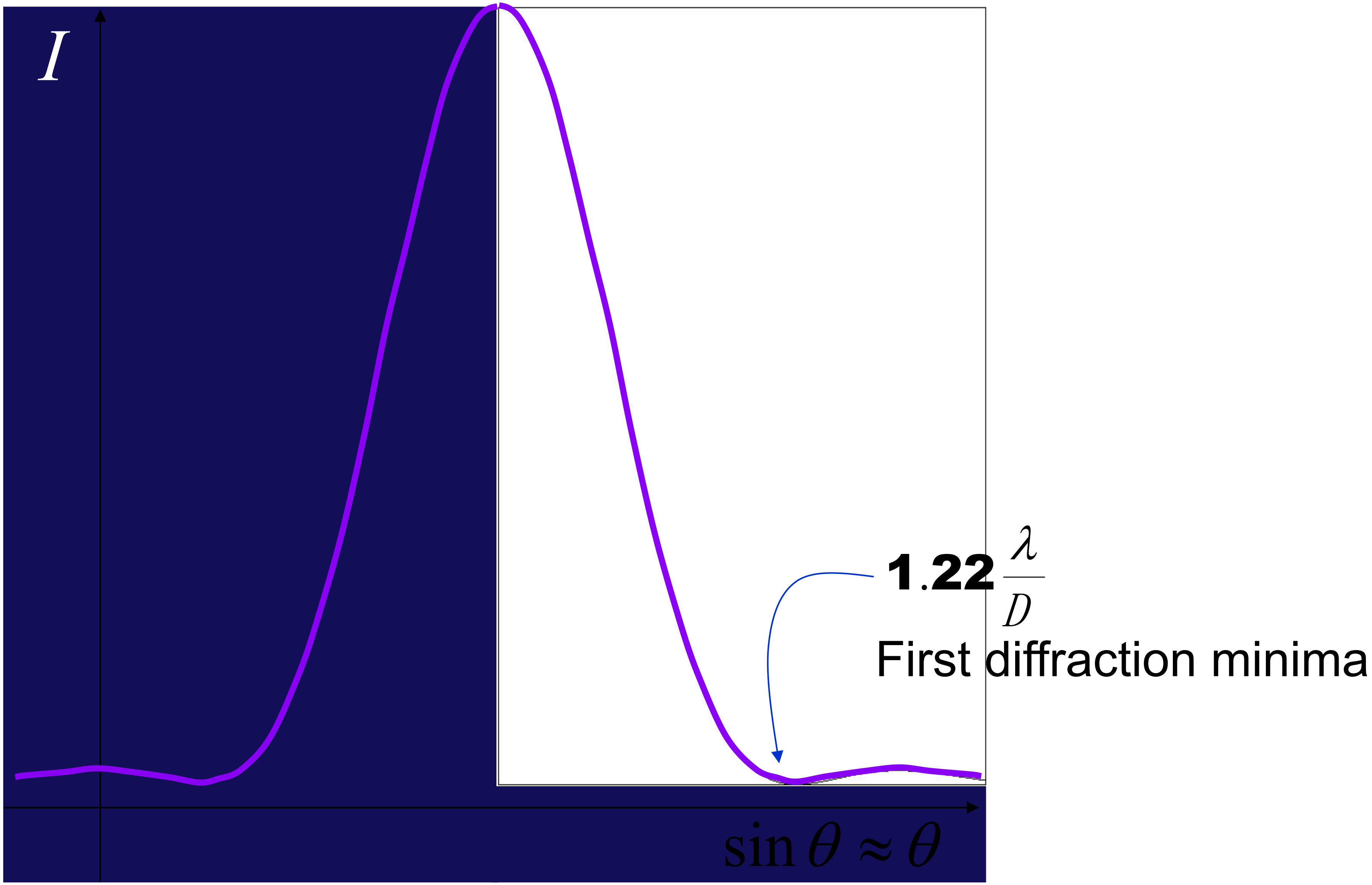
$$y=\left[\frac{2J_1(x)}{x}\right]^2$$



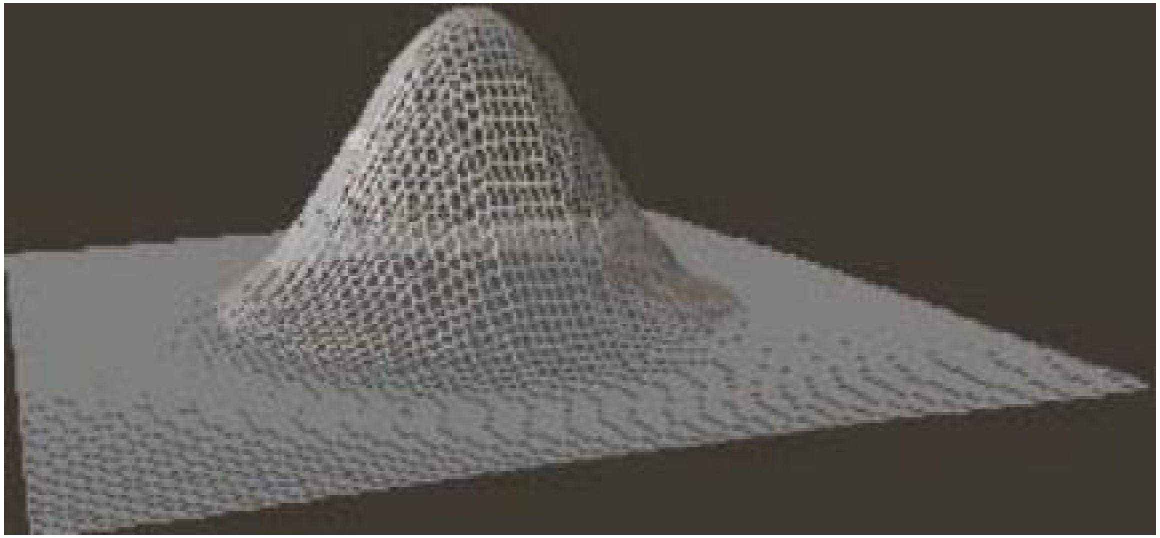
$$\Delta\theta=0.61\frac{\lambda}{a}$$
$$=1.22\frac{\lambda}{D}$$

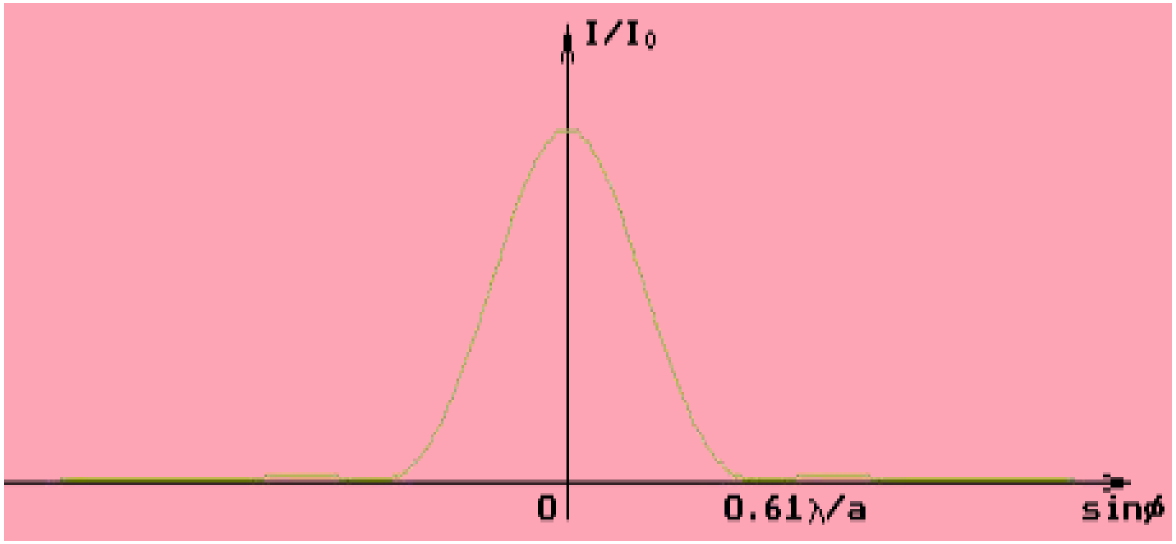
First diffraction minimum is at $\sin\theta=1.22\frac{\lambda}{D}$

Intensity from Circular Aperture



Intensity Distribution





Examples

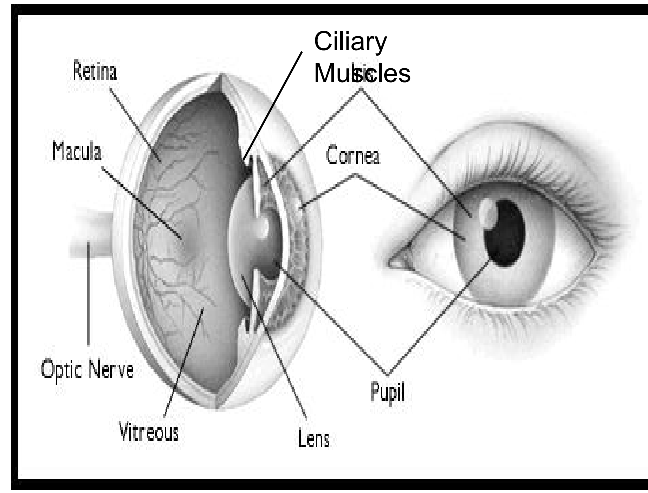
- **Eyes**

$D = 2mm - 8mm$

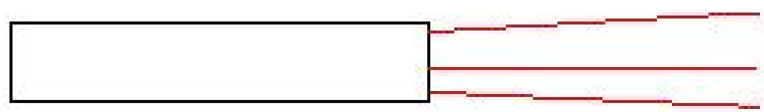
$\lambda = 550nm, \quad D = 2mm, \quad \Delta\theta = 1.22\dfrac{\lambda}{D} = 3.4\times10^{-4} rad = 1'$

$f = 20mm, \quad$ Airy disk: $d = f\cdot 2\Delta\theta \approx 14\mu m$

on the $1mm^2$ cornea, there are 5400 Airy disk (爱利斑点).

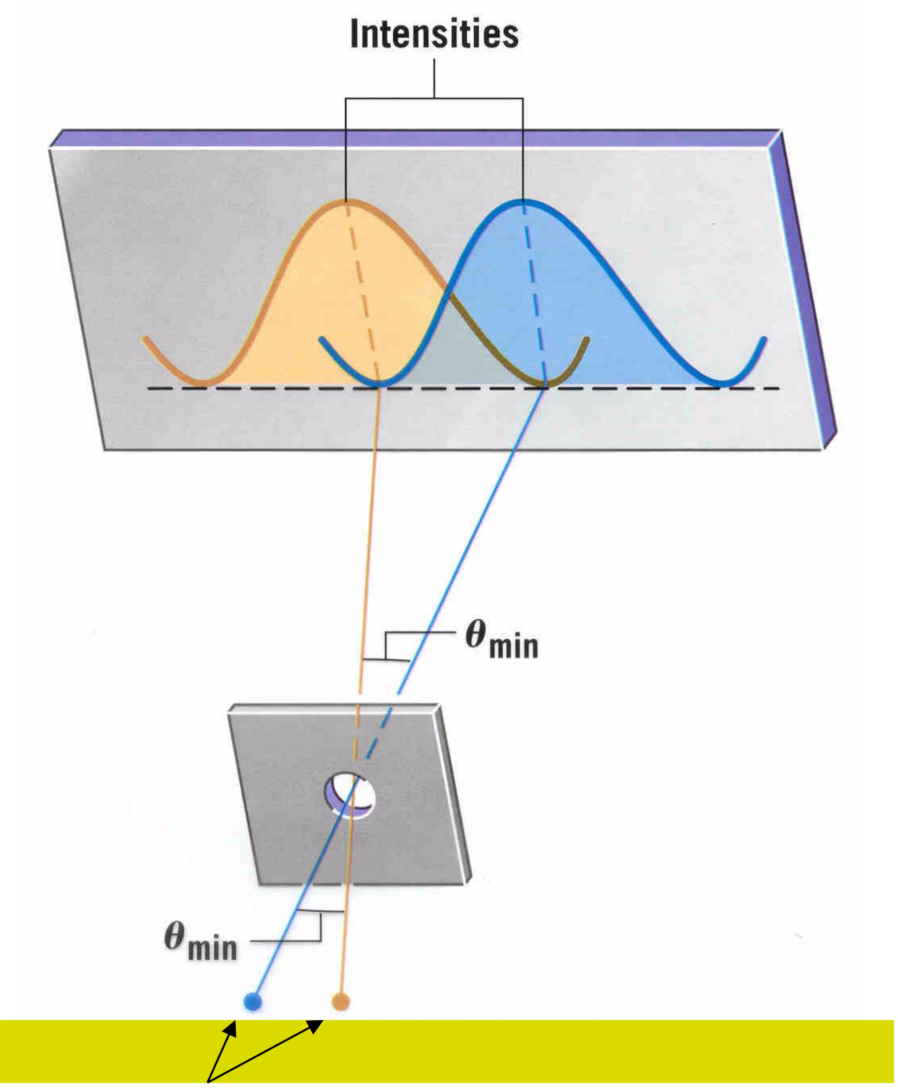


- **He-Ne Laser**



$\lambda = 632.8nm, \quad \Delta\theta = 1.22\dfrac{\lambda}{D} = 7.7\times10^{-4} rad = 2.7'$

$10km$ apart, the diameter of light disk: 7.7m

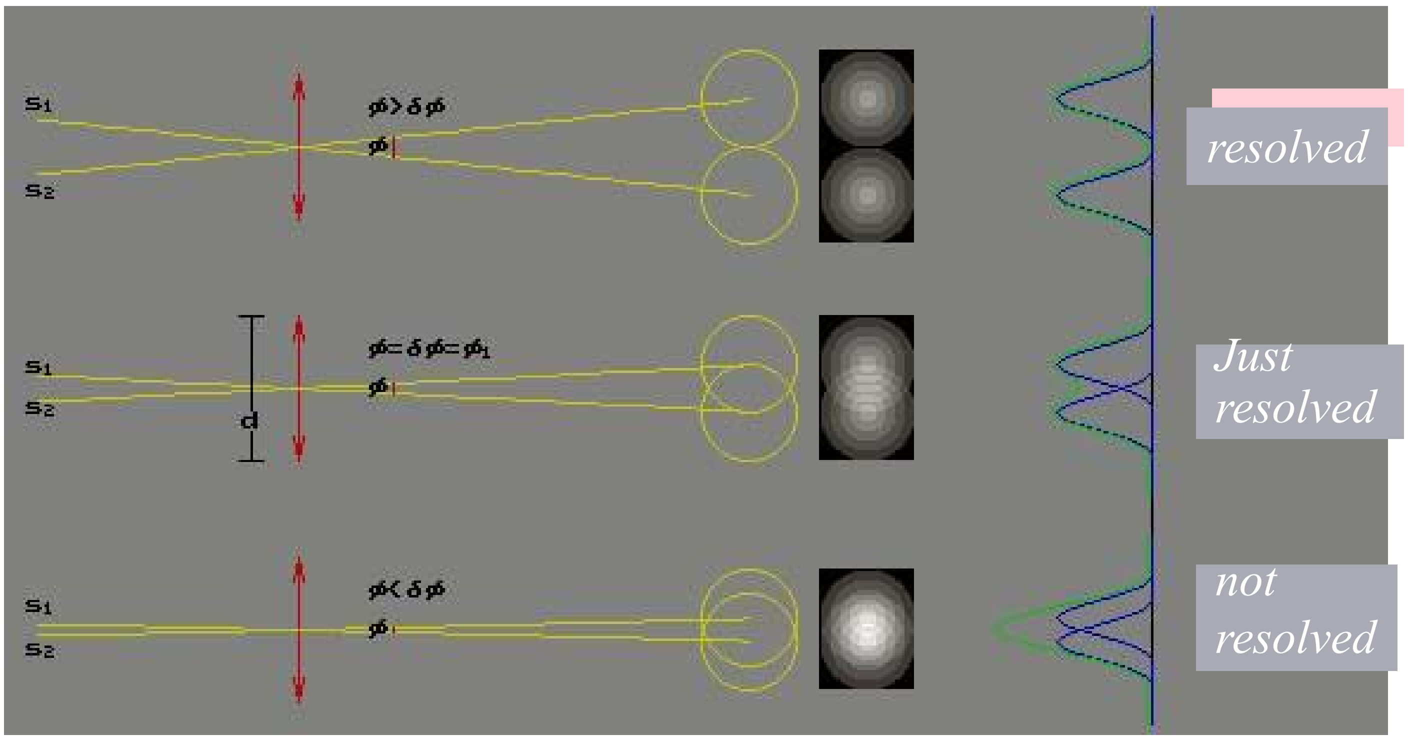


These objects are *just* resolved

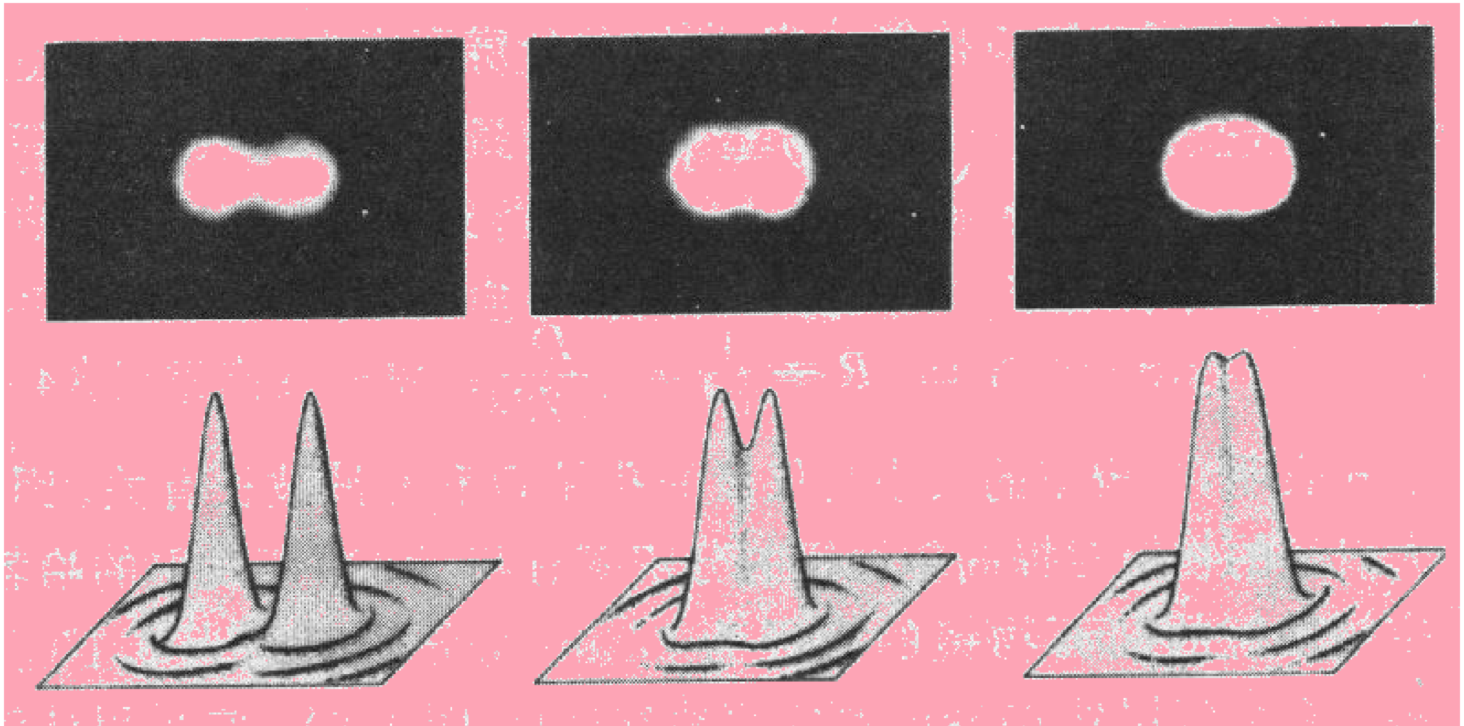
Two objects are just resolved when the maximum of one is at the minimum of the other.

Rayleigh's criterion (瑞利判据)

Two objects are just resolved when the maximum of one is at the minimum of the other



$$\theta_R = \theta_{\min} = 1.22\frac{\lambda}{D}, \quad R = 1/\theta_R \Rightarrow \text{resolution ability (分辨能力)}$$



(a) (b) (c)

Resolving Power (分辨本领)

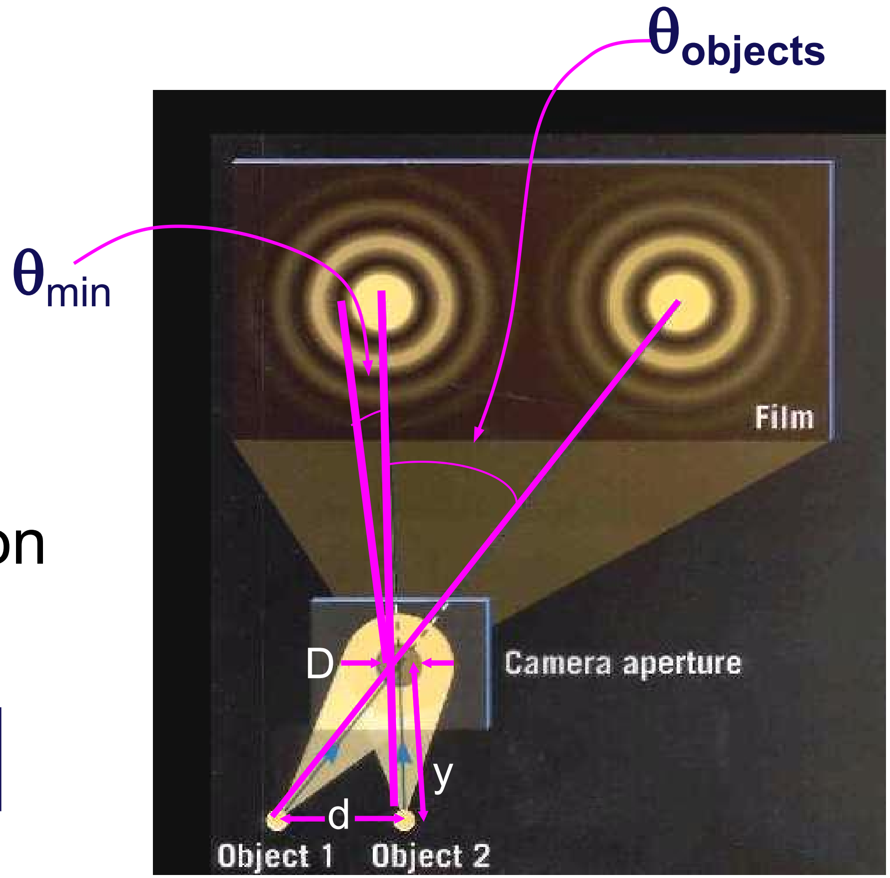
To see two objects distinctly, need $\theta_{\text{objects}} > \theta_{\text{min}}$

$\theta_{\textbf{objects}}$ is angle between objects and aperture:

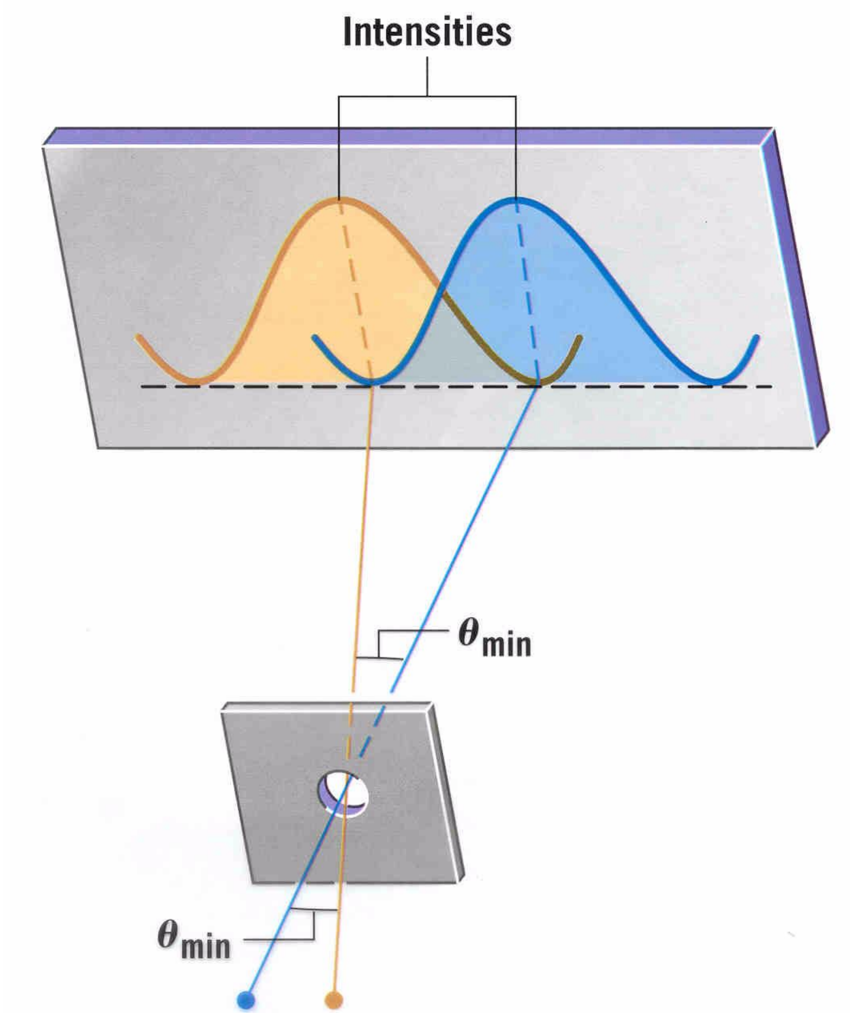
$$\boxed{\theta_{\textbf{objects}} \approx \tan(d/y)}$$

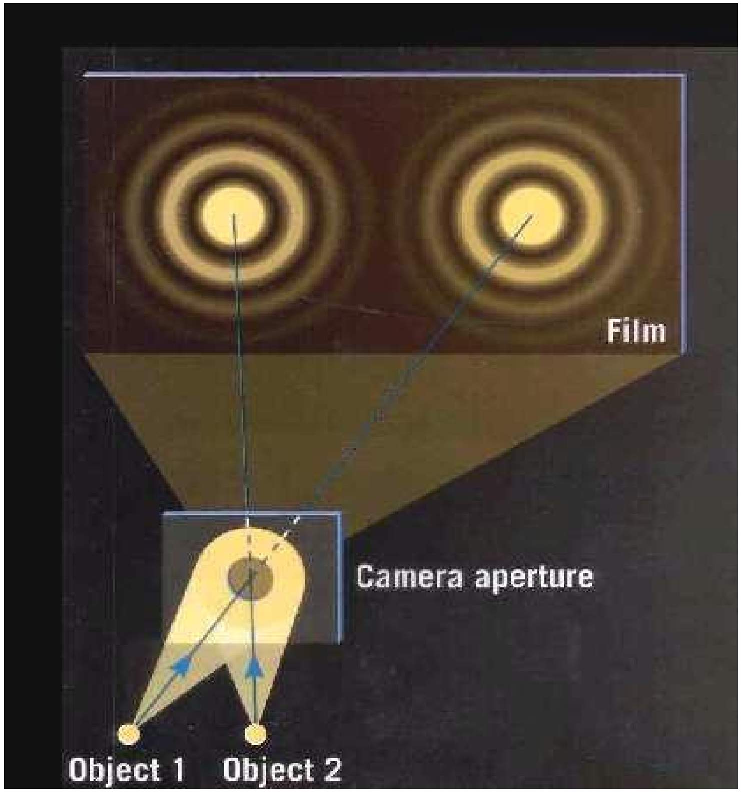
θ_{min} is minimum angular separation that aperture can resolve:

$$\boxed{\textbf{sin}\ \theta_{\text{min}} \approx \theta_{\text{min}} = 1.22\ \lambda/D}$$



Improve resolution by increasing $\theta_{\textbf{objects}}$ or decreasing θ_{min}

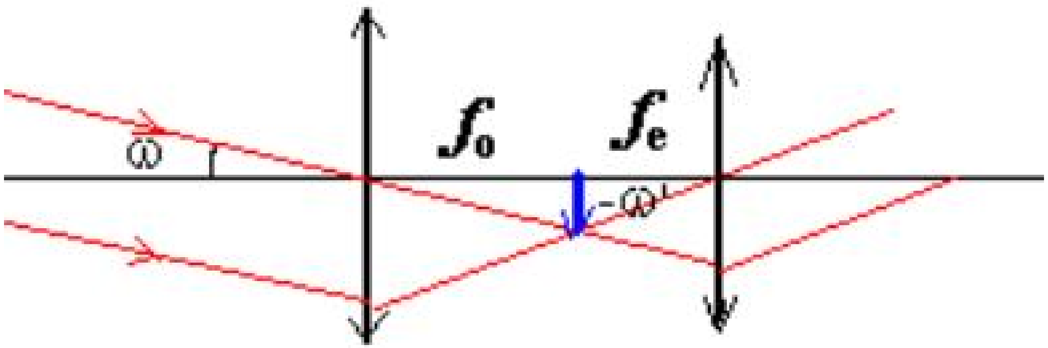




[Demonstration!]

These objects are *just* resolved

Resolution of Telescopes



$$M = \frac{\omega'}{\omega} = -\frac{f_o}{f_e},$$

The smallest resolution angle:

$$\theta_R = 1.22\frac{\lambda}{D}$$

In order to decrease θ_R

$$D\uparrow, \text{ or } \lambda\downarrow$$

- **Hubble Space Telescope:**

  D = 5 m

- **Electron Microscope.**

  $$\lambda = \frac{h}{p} \approx 0.01nm$$

- **Ultraviolet light Micro.**

$$D = 5.0cm,\ \lambda = 550\text{nm},\ \theta_{\text{R}} = 1.22\frac{\lambda}{D} = 1.3\times10^{-5}rad$$

$$D = 50cm,\ \lambda = 550\text{nm},\ \theta_{\text{R}} = 1.22\frac{\lambda}{D} = 1.3\times10^{-6}rad$$

For eyes, the smallest resoled angle : $\theta_{\text{e}} = 1' = 2.9\times10^{-4}rad$

for D = 5.0cm, the mag. $M = \frac{\theta_e}{\theta_R} = \frac{2.9\times10^{-4}}{1.3\times10^{-5}} = 22.4$

for D = 50cm, the mag. $M = \frac{\theta_e}{\theta_R} = \frac{2.9\times10^{-4}}{1.3\times10^{-6}} = 224$

Homework:

Page 977 (Exercises)

20

27

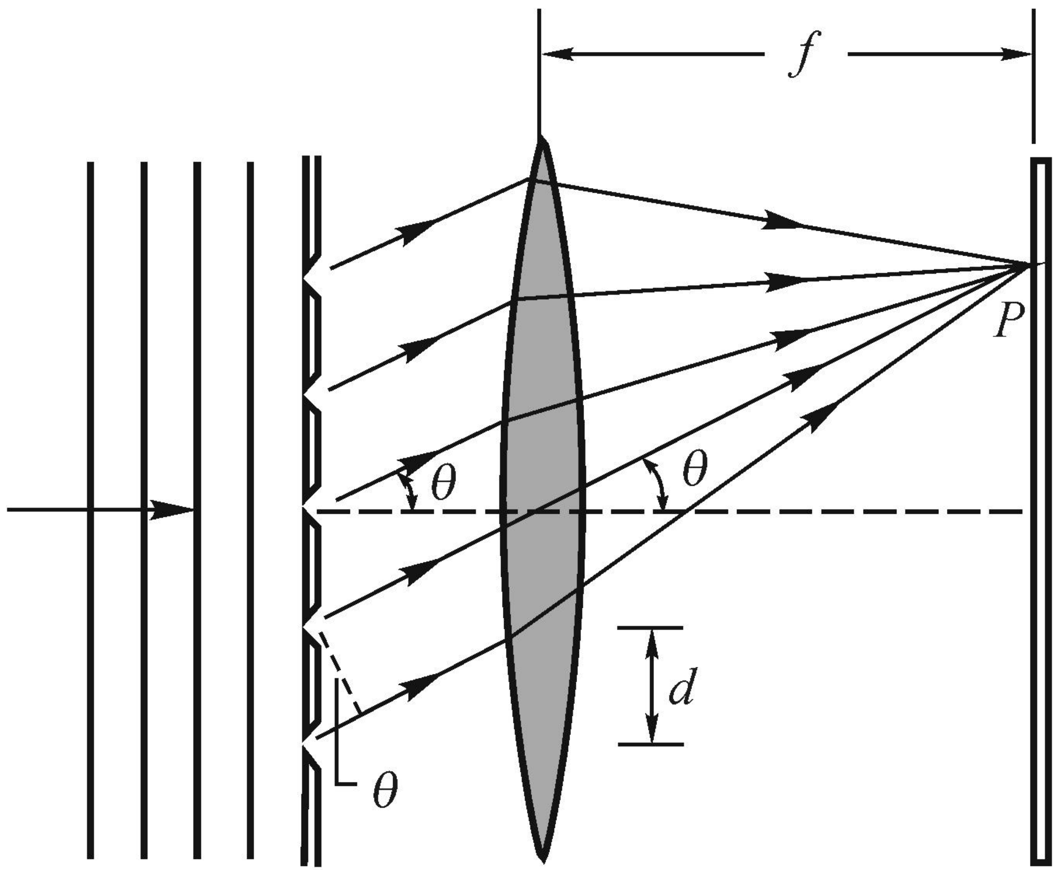
Page 978 Problems

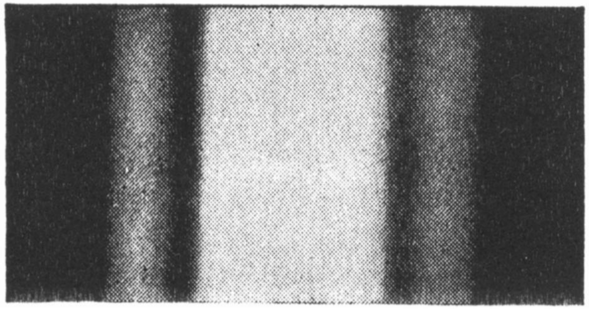
1

3

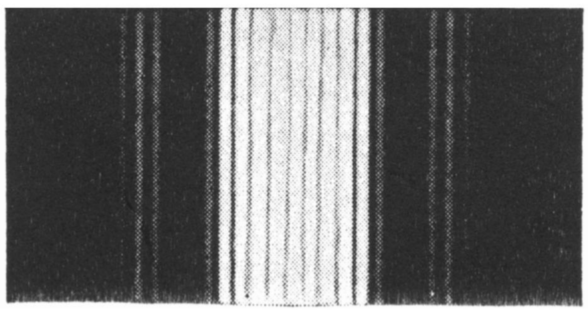
42-4 Gratings (光栅) and Spectra (光谱)

1. Fraunhofer diffraction by multiple slits(多缝)

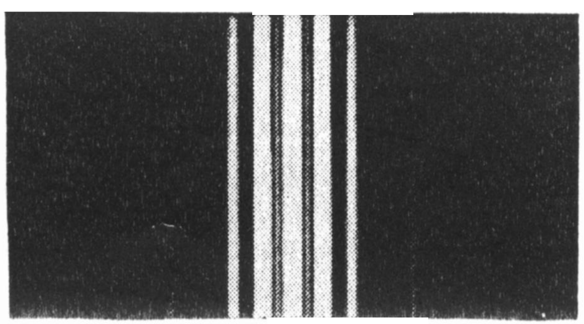




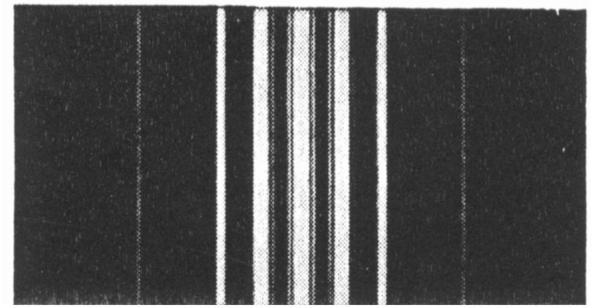
(*a*) **1 slit**



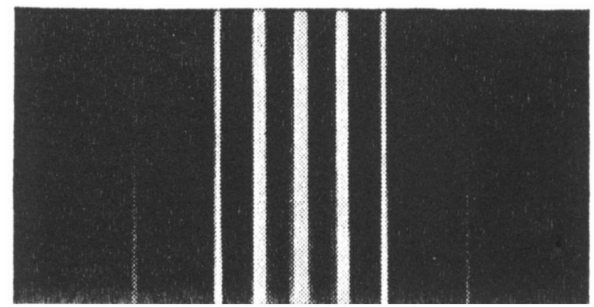
(*b*) **2 slits**



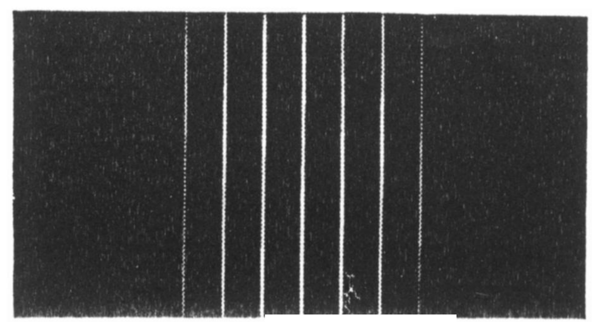
(*c*) **3 slits**



(*d*) **5 slits**

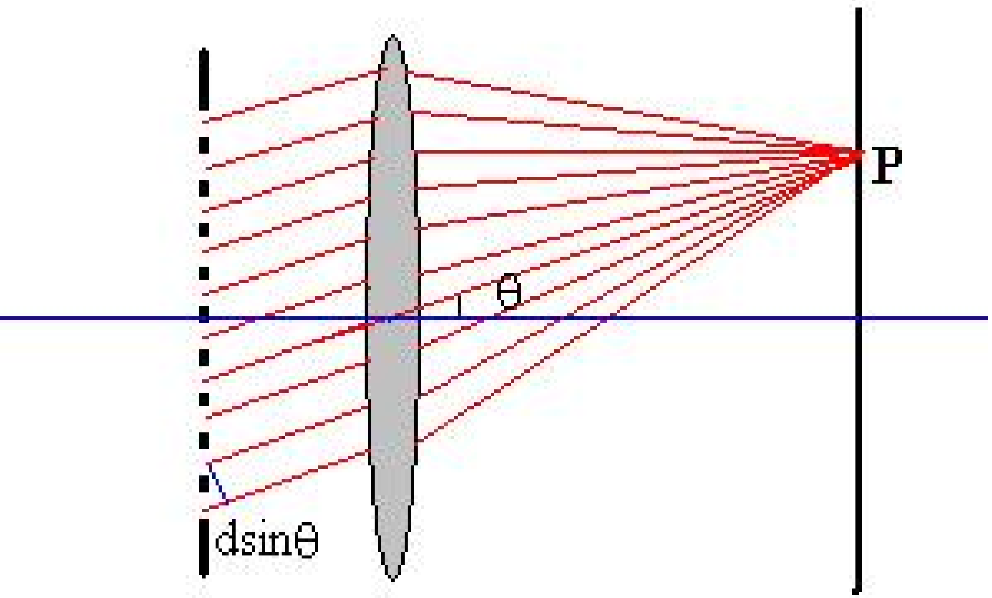


(*e*) 6 slits



(*f*) **20 slits**

The intensity of diffraction for N slits



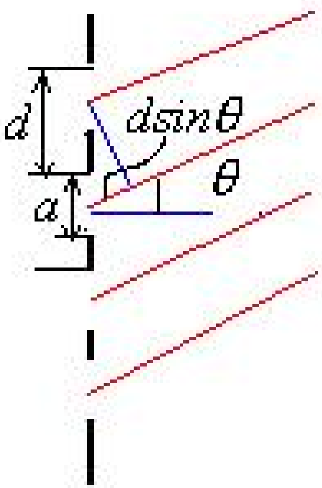
The width of slit : a

The distance between slits : d

$$d = a + b$$

- **At First, there is only one slit opened.**

Diffraction due to simple slit

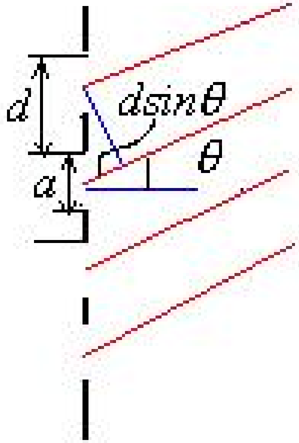


$$E_\theta = E_m \frac{\sin\alpha}{\alpha}, \quad \alpha = \frac{\pi a \sin\theta}{\lambda}$$

$$I = I_\theta = E_m^2 \left(\frac{\sin\alpha}{\alpha}\right)^2 = I_m \left(\frac{\sin\alpha}{\alpha}\right)^2$$

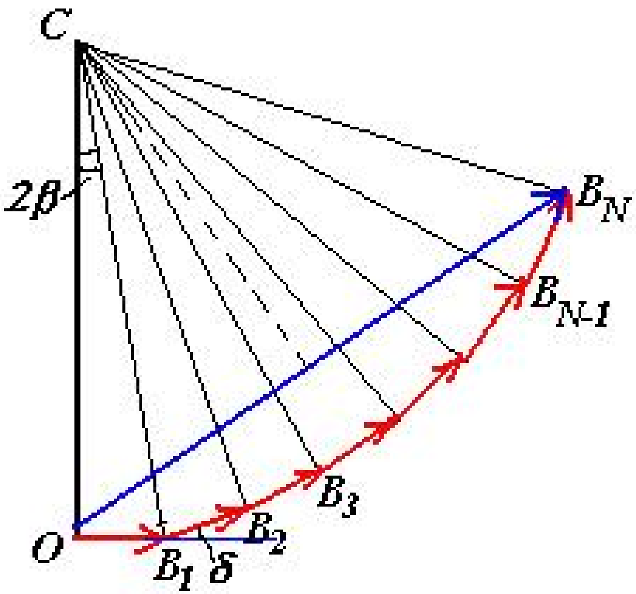
The intensity of diffraction for *N* slits (con.)

The interference between slits.



$$\delta = \frac{2\pi}{\lambda}\cdot d\sin\theta = 2\beta$$

$$\therefore \beta = \frac{\pi d\sin\theta}{\lambda}$$



$$\angle OCB_N = N\delta = 2N\beta$$

$$\therefore E_\theta = \overline{O}\,\overline{B}_N = 2\overline{O}\,\overline{C}\sin N\beta$$

$$= 2\cdot\frac{E_1}{2\sin\beta}\cdot\sin N\beta$$

$$= E_1\frac{\sin N\beta}{\sin\beta}$$

$$2\overline{O}\,\overline{C}\sin\beta = \overline{O}\,\overline{B}_1 = E_1$$

$$\therefore \overline{O}\,\overline{C} = \frac{E_1}{2\sin\beta}$$

$$E_1 = E_m(\frac{\sin\alpha}{\alpha})e^{i0}$$

$$E_2 = E_m(\frac{\sin\alpha}{\alpha})e^{i\delta}$$

$$E_3 = E_m(\frac{\sin\alpha}{\alpha})e^{i2\delta}$$

....

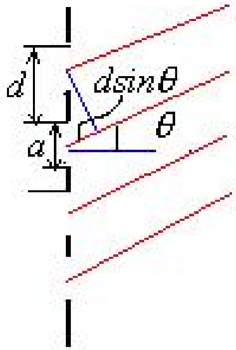
$$E_N = E_m(\frac{\sin\alpha}{\alpha})e^{i(N-1)\delta}$$

$$\alpha = \frac{\pi a\sin\theta}{\lambda}$$

$$\beta = \frac{\pi d\sin\theta}{\lambda}$$

$$I_\theta = I_m(\frac{\sin\alpha}{\alpha})^2(\frac{\sin N\beta}{\sin\beta})^2$$

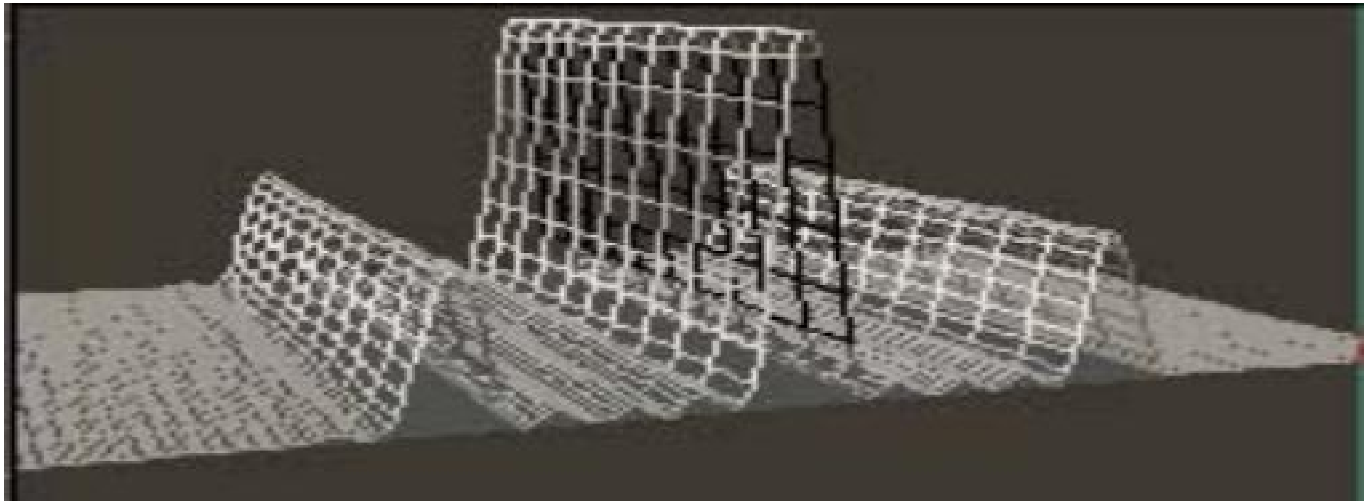
Discussions



$$I_\theta = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2\left(\frac{\sin N\beta}{\sin\beta}\right)^2 \qquad \alpha = \frac{\pi a\sin\theta}{\lambda} \qquad \beta = \frac{\pi d\sin\theta}{\lambda}$$

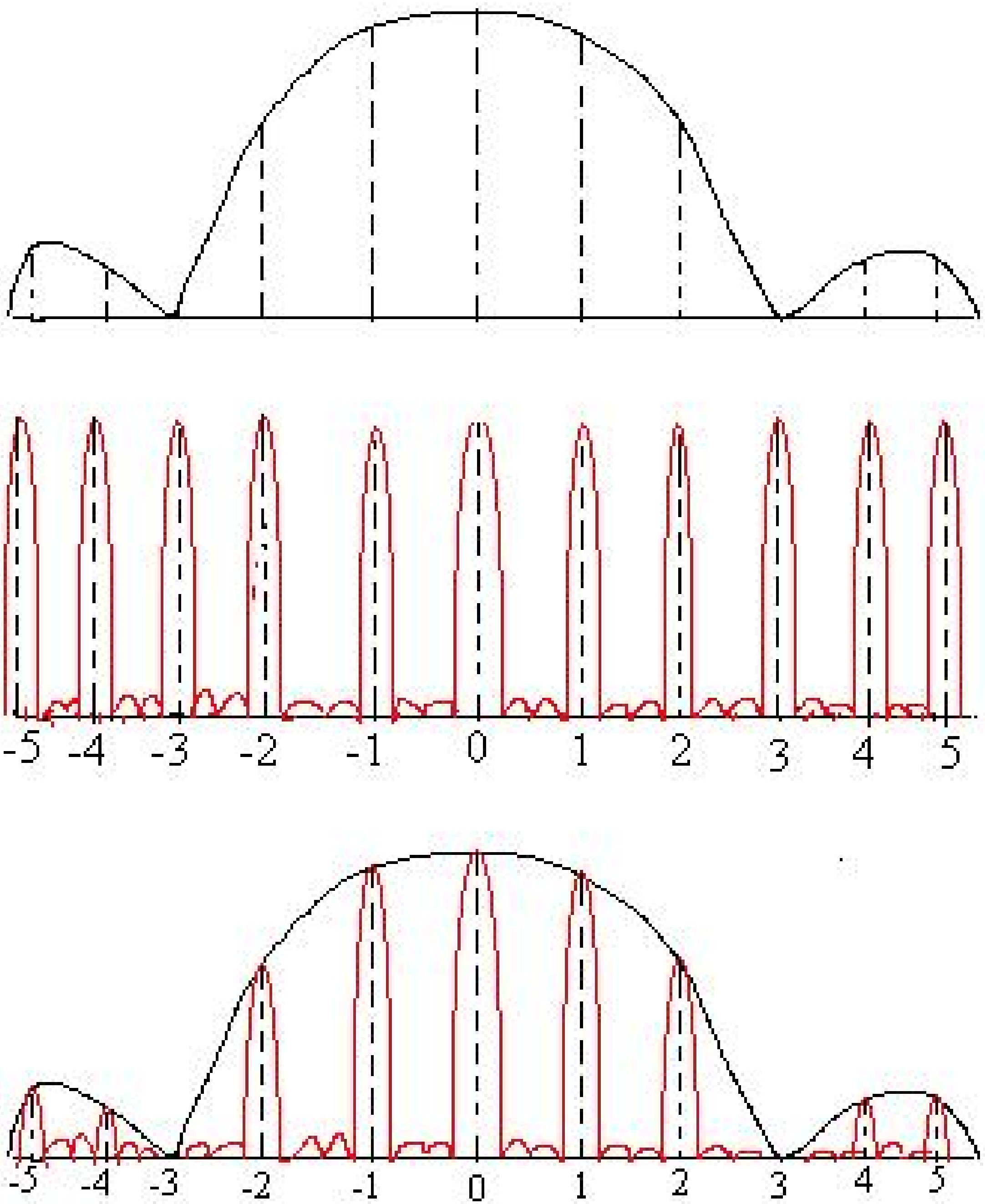
Single-slit diffraction

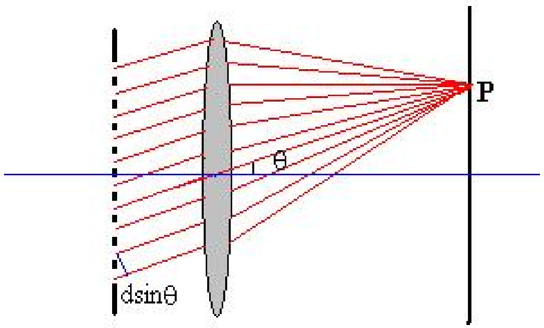
The interference between slits



Between 2 principal maxima (主极大):

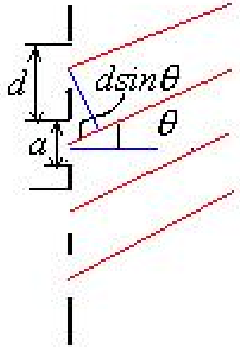
N-1 minima (极小), N-2 maxima (次极大)





$$N = 4$$

$$I_\theta = I_m \left(\frac{\sin\alpha}{\alpha}\right)^2 \left(\frac{\sin N\beta}{\sin\beta}\right)^2$$



$$I_\theta = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2\left(\frac{\sin N\beta}{\sin\beta}\right)^2$$

A. $\left(\frac{\sin N\beta}{\sin\beta}\right)^2$

The interference between slits

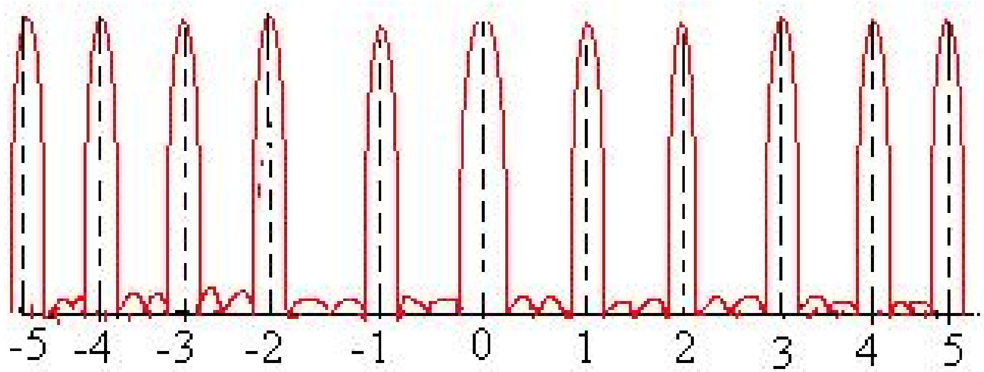
$$a.\quad \beta = m\pi \quad (m = 0, \pm 1, \pm 2.....),\quad \beta = \frac{\pi d\sin\theta}{\lambda}$$

principal maximum (主极大)

$$\sin N\beta = 0,\quad \sin\beta = 0,\quad \lim_{\sin\beta\to 0}\frac{\sin N\beta}{\sin\beta} = N$$

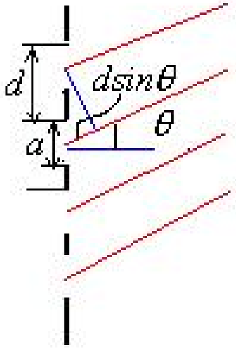
$$I_\theta = N^2 I_m$$

$$d\sin\theta = m\lambda$$



$$\because \theta < 90,\ |\sin\theta| < 1,\ \Rightarrow |m_{\max}| < \frac{d}{\lambda}$$

$$\text{If } \lambda > d,\ m = 0$$



$$I_\theta = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2\left(\frac{\sin N\beta}{\sin\beta}\right)^2$$

A. $$\left(\frac{\sin N\beta}{\sin\beta}\right)^2$$

The interference between slits

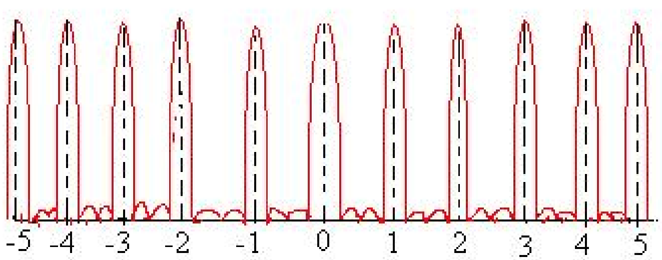
b. The position of the zero point (零点位置)
The number of the second maximum (次极大的数目)
The half-angle width of a main maximum (主极大的半角宽度)

If $\sin N\beta = 0$ But $\sin\beta \neq 0$ $I_\theta = 0$

$$\Rightarrow \quad \beta = (m + \frac{n}{N})\pi, \quad \sin\theta = \frac{\lambda}{d}(m + \frac{n}{N})$$

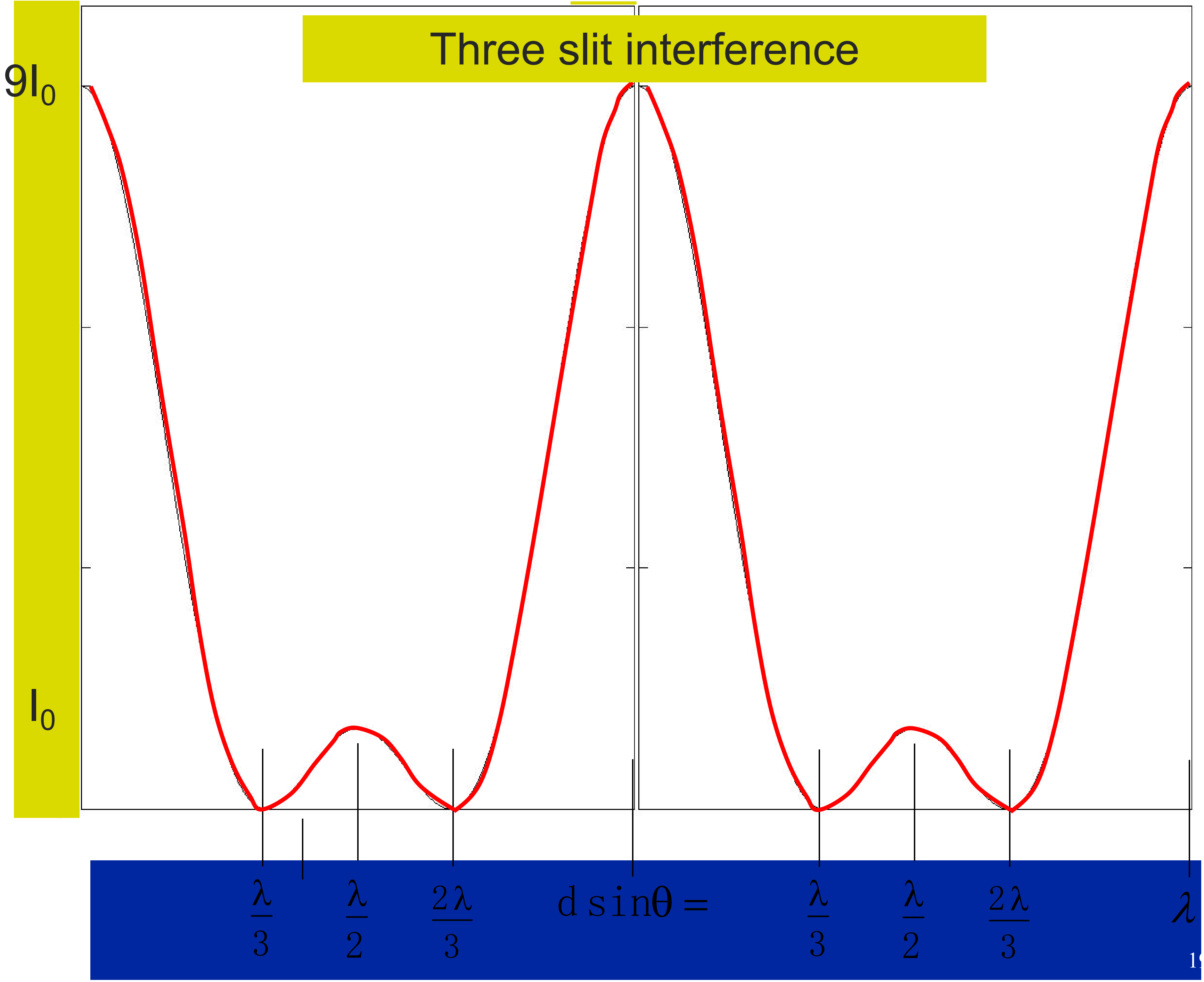
$m = 0, \pm 1, \pm 2........; \quad n = 1, 2, 3......\mathrm{N}-1$

$$\beta = \frac{\pi d \sin\theta}{\lambda}$$

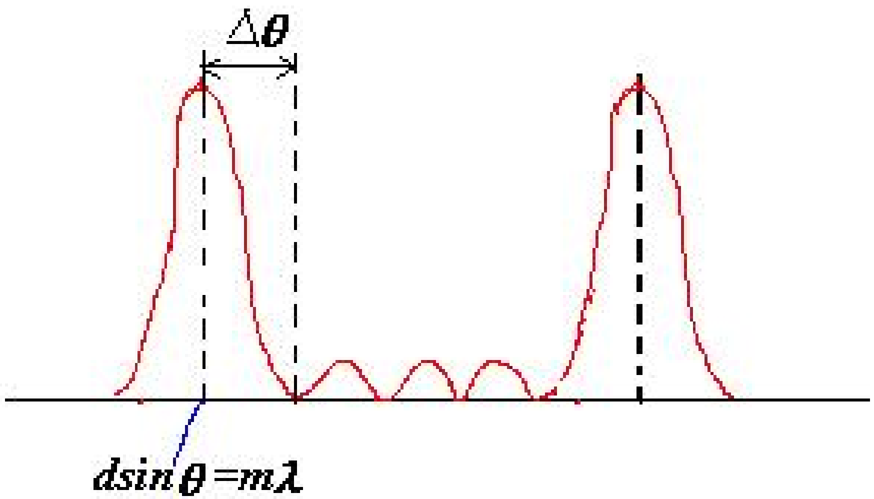


Between two principal maximum:
N-1 minima, N-2 the second maximum

Three slit interference



The half-angle width of a main maximum (主极大的半角宽度)



$$d\sin\theta = m\lambda$$

$$\theta \text{ small}, \quad \sin\theta \approx \theta$$

$$\theta_{\mathrm{m}} \approx \frac{m\lambda}{d}$$

$$\theta_m + \Delta\theta \approx \left(m + \frac{1}{N}\right)\frac{\lambda}{d}$$

$$\therefore \Delta\theta = \frac{\lambda}{Nd}$$

If θ is not small, $\sin\theta \neq \theta$

$$d\sin\theta = m\lambda$$

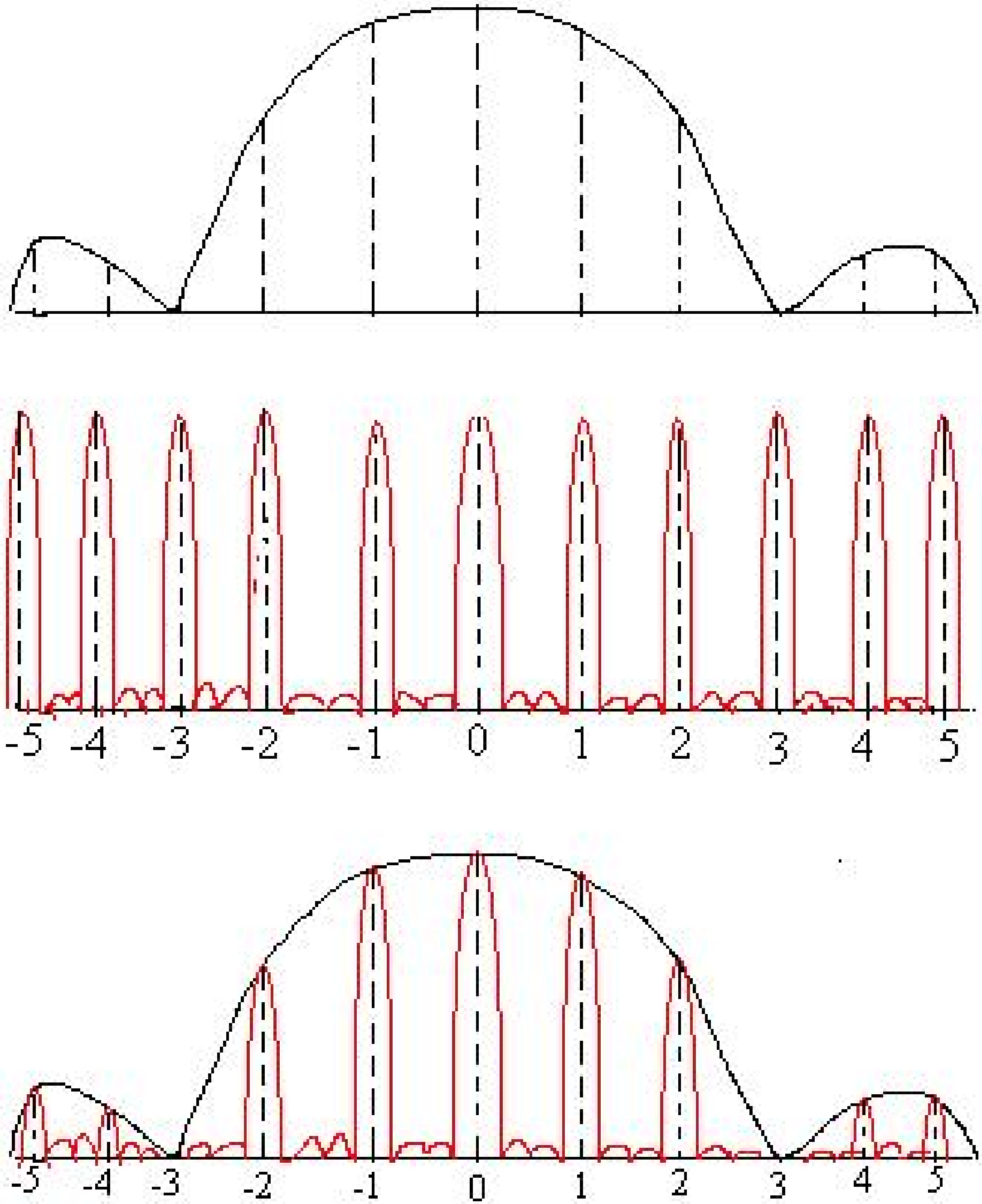
$$d\cos\theta \cdot \Delta\theta = \frac{1}{\mathrm{N}}\lambda$$

$$\Delta\theta = \frac{\lambda}{Nd\cos\theta}$$

Nd is bigger,

$\Delta\theta$ becomes smaller.

grating, $N \approx 10^5$, $10^3 / mm$, $d \approx 10^{-6} m$



B. $\left(\frac{\sin\alpha}{\alpha}\right)^2$

$N = 4$

$$\alpha = \frac{\pi}{\lambda} a \sin\theta = \pi$$

$$\beta = \frac{\pi}{\lambda} d \sin\theta = 3\pi$$

$$d = 3a$$

Missing maximum

Example:A grating (*N=5000*) is illuminated by two monochromic lights with wave lengths of *600* and *400* nm respectively. The *m* th principal maximum (主极大) of the former light is meet the m+1 th principal maximum of the later at *3 cm* from the central fringe on the screen. The focus length of the lens is *50 cm*. Find the grating constant (光栅常数) *d*, and the typical width of the principle fringes.

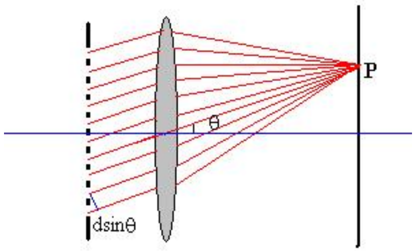
Equation for principal maxima:

$$d\sin\theta = m\lambda$$

$$d\sin\theta = m\lambda_1 = (m+1)\lambda_2$$

$$600\times m = 400\times(m+1)$$

$$\Rightarrow m = 2$$



$$d\sin\theta \approx d\frac{y}{f} = m\lambda_1$$

$$d = \frac{mf\lambda_1}{y} = 2\times10^{-5}\,m = 20\ \mu m$$

Angle width:

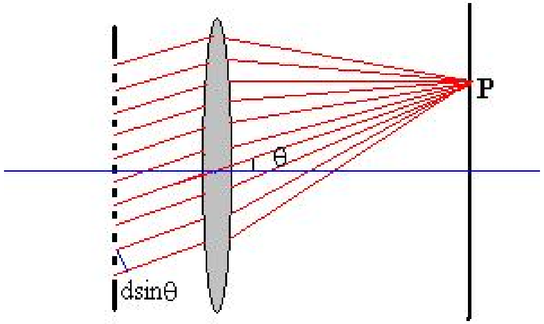
$$\Delta\theta = \frac{\lambda}{Nd\cos\theta}$$

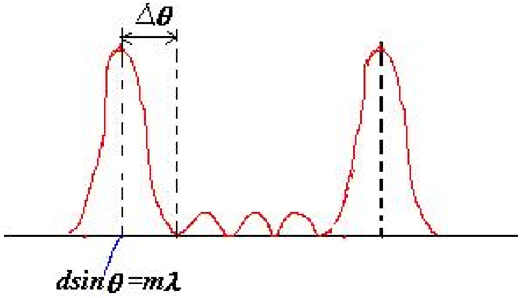
take $\quad m = 2, \quad d\sin\theta = m\lambda_1$

$$\therefore \quad \theta \approx \frac{2\times 6\times 10^{-7}}{2\times 10^{-5}} = 0.06, \quad \cos\theta \approx 1$$

$$\Delta\theta_{m=2} = \frac{600\times 10^{-9}}{5000\times 2\times 10^{-5}} = 6\times 10^{-6}$$

$$\Delta y_{m=2} = f\,\Delta\theta_{m=2} = 6\times 10^{-6}\,\mathrm{m} = 6\,\mu\mathrm{m}$$



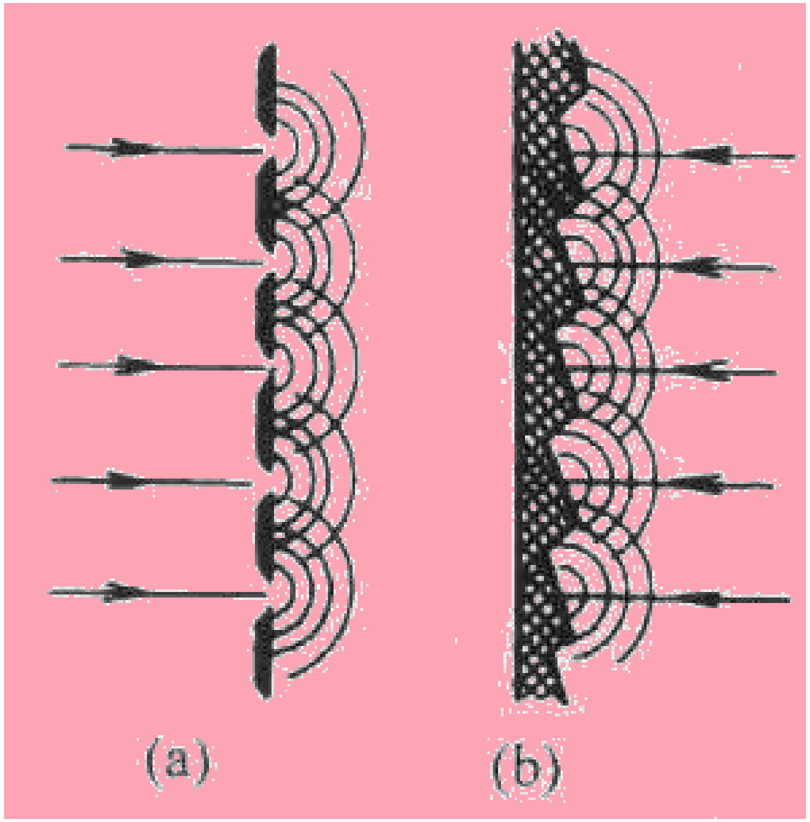


Diffraction gratings

Spectragraphs:

$$d \sin \theta = m\lambda$$

$$m = 0, \pm 1, \pm 2, \ldots$$

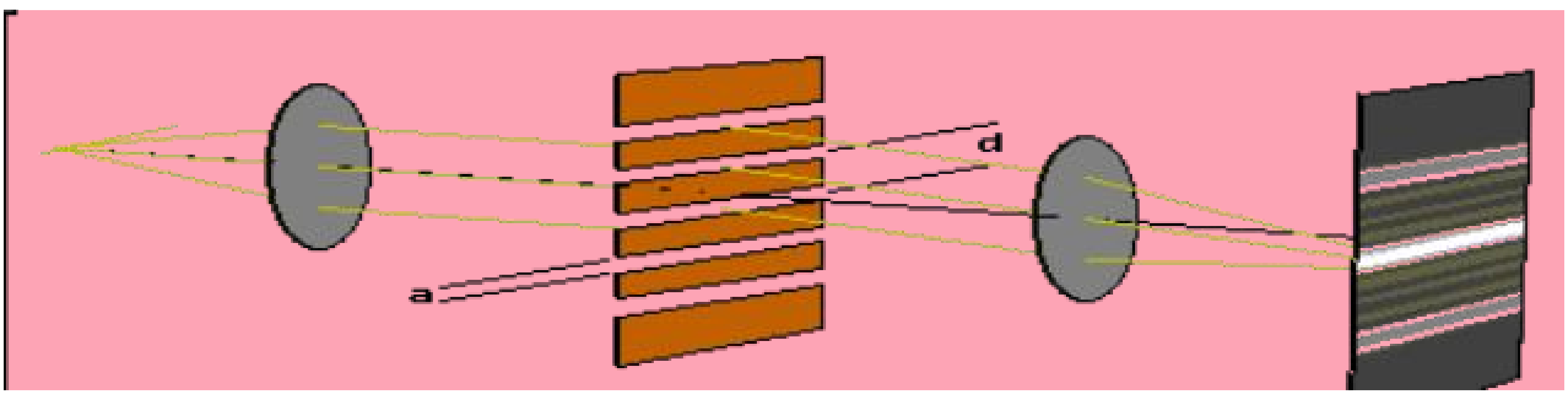


3. Dispersion and resolving power

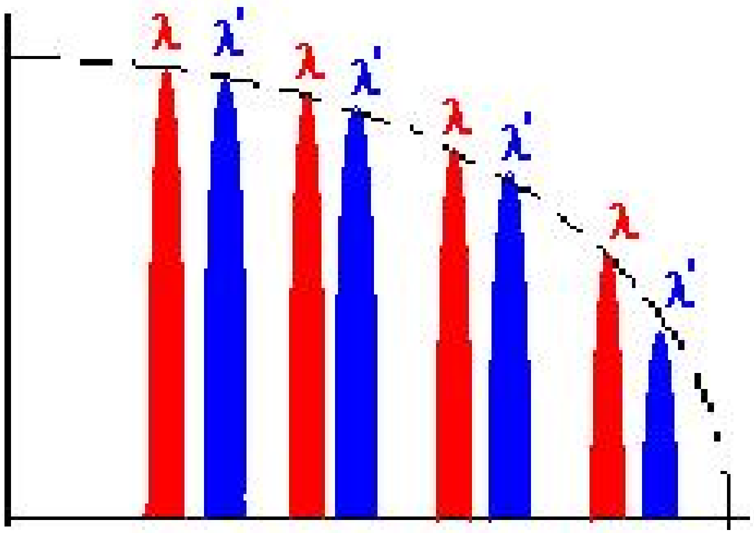
(色散和分辨本领)

- **Dispersion（色散）：**

$$d\sin\theta = m\lambda \quad m = 0, \pm 1, \pm 2, \ldots$$



Dispersion power (色散本领)



$$D = \frac{\Delta\theta}{\Delta\lambda}$$ (The angular seperation $\Delta\theta$ per unit wavelength internal $\Delta\lambda$)

$$d\sin\theta = m\lambda$$

$$d\cos\theta \cdot \Delta\theta = m\Delta\lambda$$

$$\therefore D = \frac{\Delta\theta}{\Delta\lambda} = \frac{m}{d\cos\theta}$$

$d\downarrow,\ \ m\uparrow,\ D$ becomes bigger

And D is independent of N.

For modern grating (现代光栅)

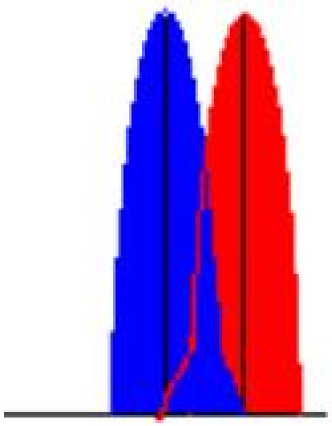
$d \approx 10^{-2} - 10^{-3}\,mm$

$m = 1,\ D_\theta \approx 0.1'/\mathring{A} - 1'/\mathring{A}$

$f = 1m,\ \ D_l = 0.1 - 1mm/\mathring{A}$

Resolving power (分辨本领)

Rayleigh's Criterion (瑞利判据)



If the maximum of one line falls on the first minimum its neighbor, we should be able to resolve the lines.

For a given grating, the half-angle width:

$$d\sin\theta = m\lambda$$

$$D_\theta = \frac{\Delta\theta}{\Delta\lambda} = \frac{m}{d\cos\theta}$$

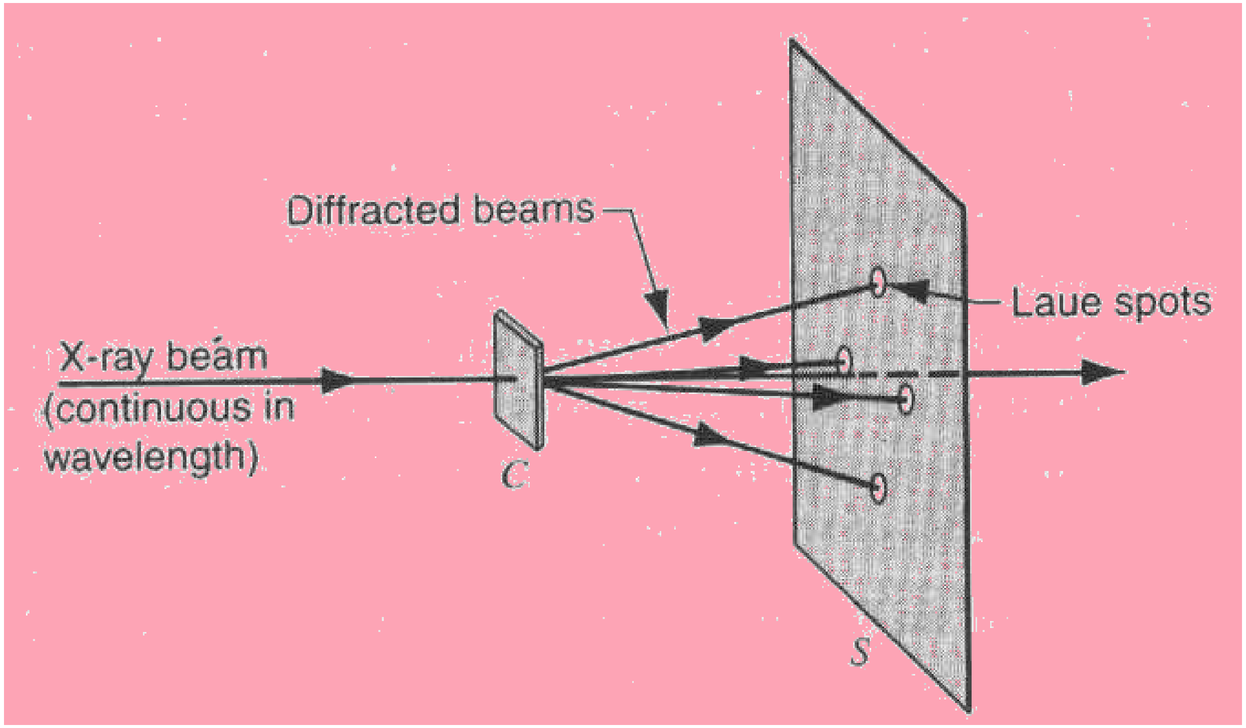
$$\Delta\theta_w = \frac{\lambda}{Nd\cos\theta}$$

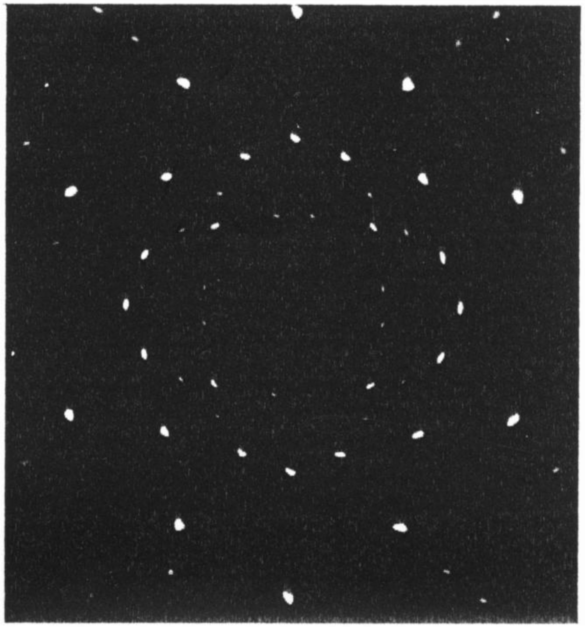
$$\Delta\lambda = \frac{\Delta\theta_w}{D_\theta} = \frac{d\cos\theta}{m}\cdot\frac{\lambda}{Nd\cos\theta} = \frac{\lambda}{Nm}$$

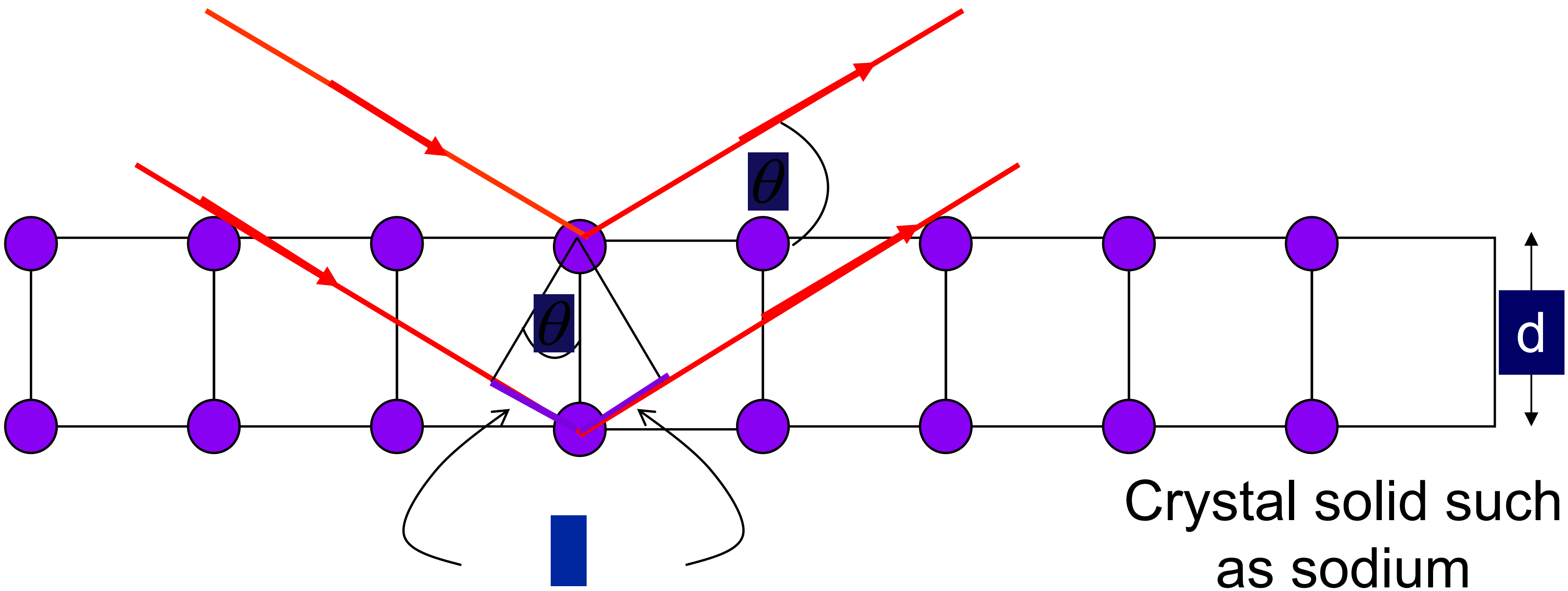
$$\text{Resolving power:}\quad R = \frac{\lambda}{\Delta\lambda} = Nm$$

It depends on N, m; independent of d.

4. X-ray Diffraction (x光衍射)







Constructive interference: $$2\,d\sin\theta = m\lambda$$

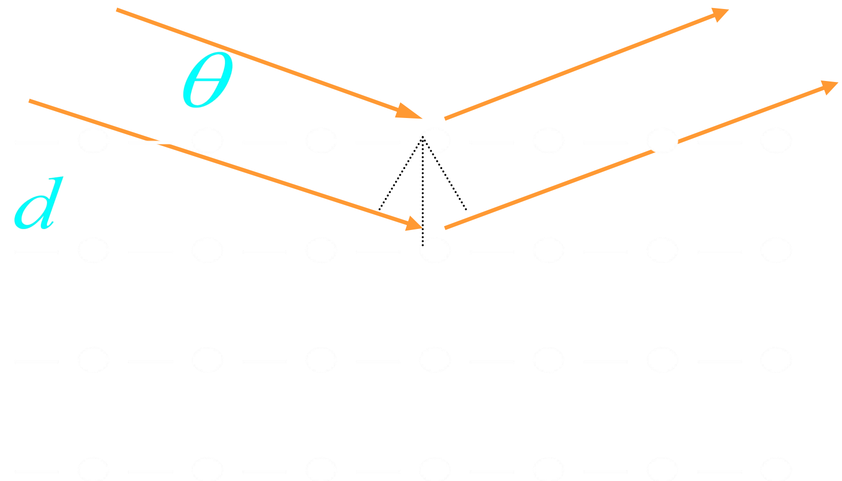
in NaCl

$1^{\underline{st}}$ maximum will be at 10^0

- **Wavelength**:

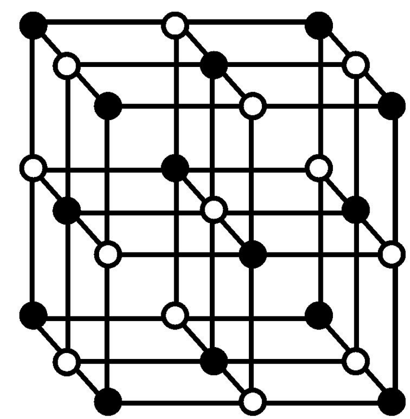
  Typical value of 0.1 nm

- **Bragg's law**

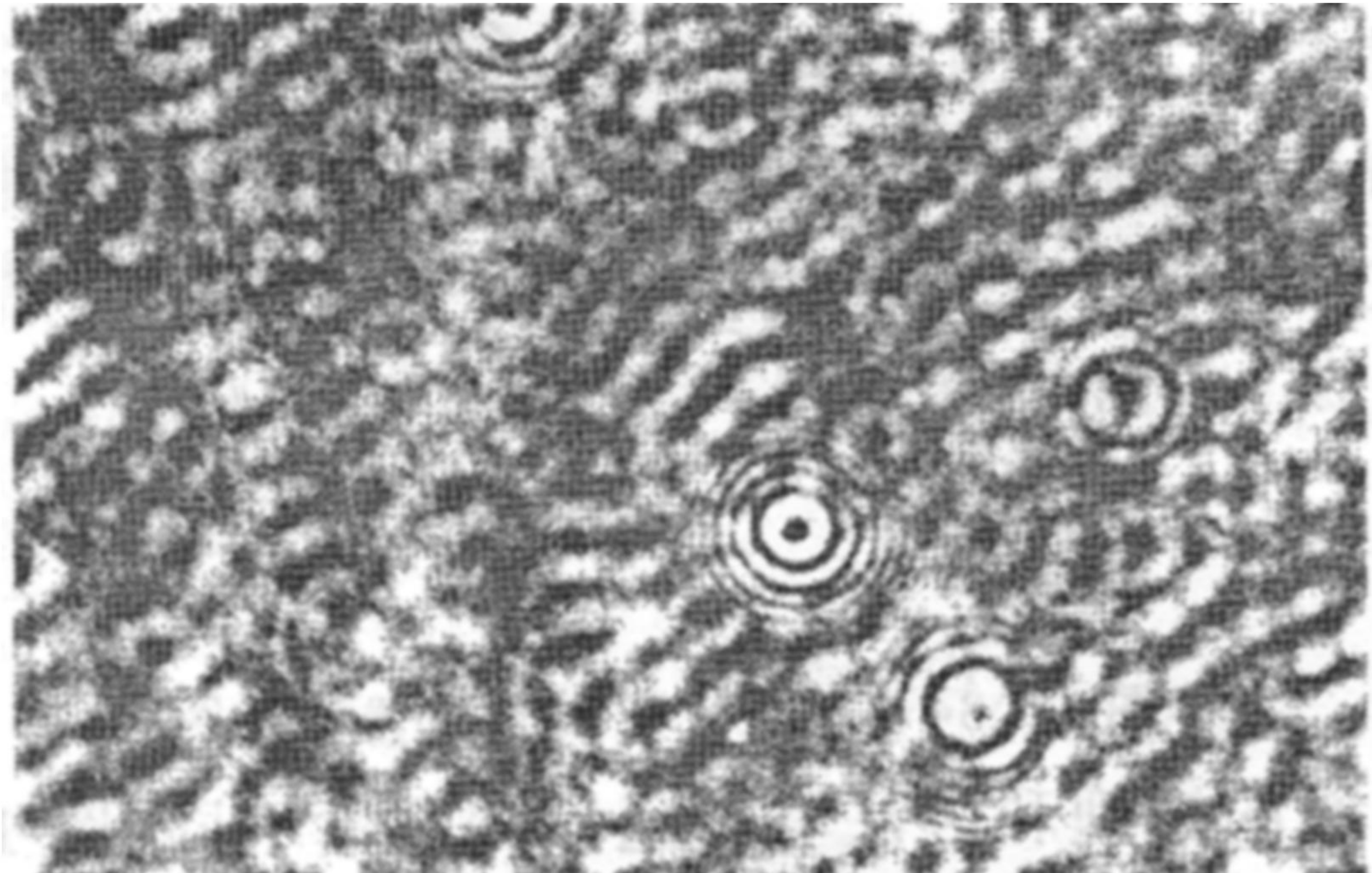


$$\boxed{\begin{aligned}&2d\sin\theta = m\lambda\\&m = 1,2,3,\ldots\end{aligned}}$$

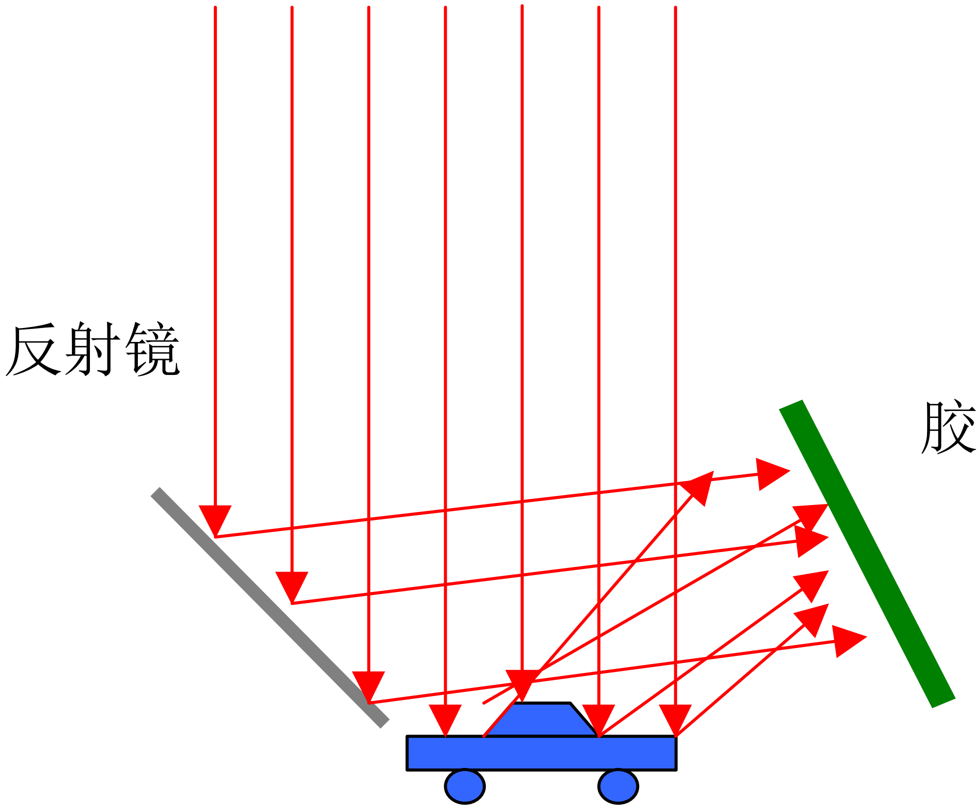
Determination of crystalline structures



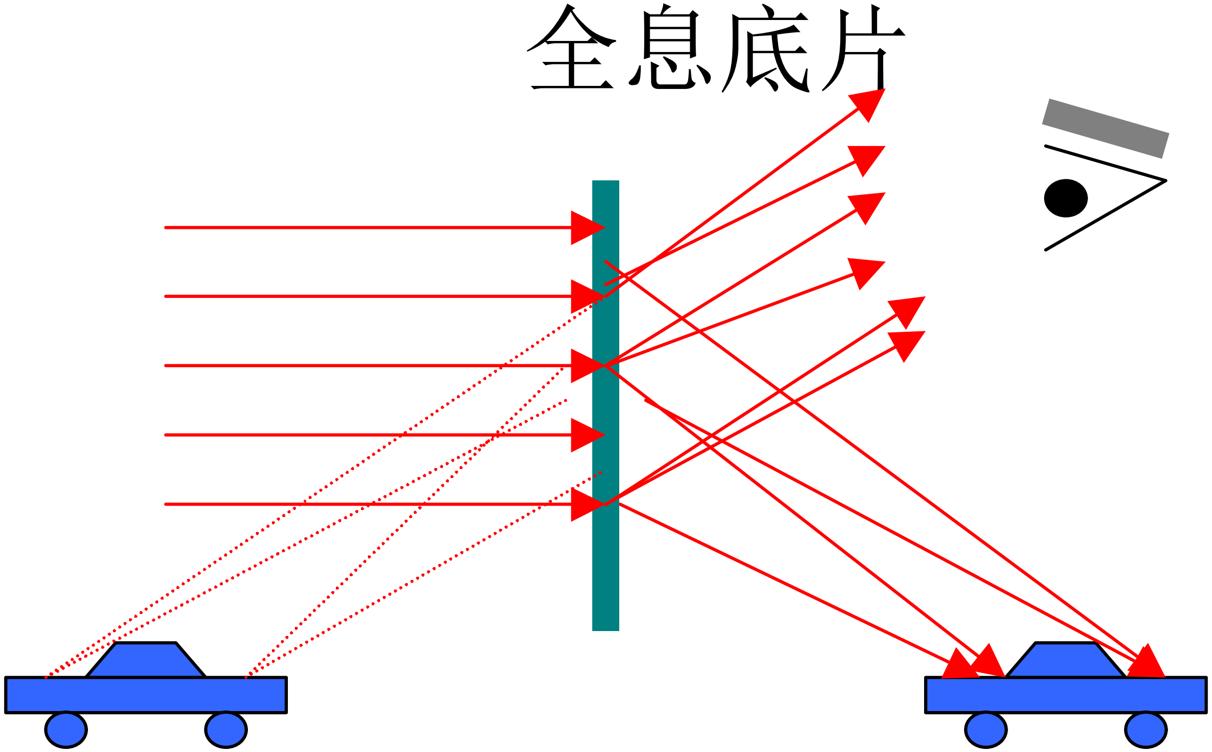
5 Holography (全息照相)



- 干涉记录（拍摄）：



- 衍射再现：



- **Normally pictures (photographs) record light intensities from the object.**

- **Holography:**

recording both the intensity and phase of the waves (light) from object.

Three dimensional image of the original object is reproducible by using the reference light.

- **D. Gabor got the Nobel price in 1971 by this discovery in 1948.**

Homework:

Page 996 Exercises

12

19

Page 998 Problems

1

7