

# Golden Ratio Integral

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Question:

$$\int_1^{\phi} \frac{(x^2 + 1)}{(x^4 - x^2 + 1)} \ln \left( 1 + x - \frac{1}{x} \right) dx$$

Answer: Dividing numerator and denominator of the fraction by  $x^2$ ,

$$\int_1^{\phi} \frac{(1 + \frac{1}{x^2})}{(x^2 - 1 + \frac{1}{x^2})} \ln \left( x - \frac{1}{x} + 1 \right) dx$$

Now taking a substitution,  $x - \frac{1}{x}$  as  $t$ , so that,

$$\left( 1 + \frac{1}{x^2} \right) dx = dt$$

Thus, the original integral (with new limits) can be rewritten as:

$$\int_0^1 \frac{\ln(1+t)}{1+t^2} dx$$

Taking the trigonometric substitution

$$\tan \theta = t$$

such that

$$(\sec^2 \theta) d\theta = dt$$

The integral can be re-written with new limits as (let us denote it by (i))

$$\int_0^{\pi/4} \ln(1 + \tan \theta) d\theta$$

By applying King's rule of sum of limits, the same integral can be re-written as

$$\int_0^{\pi/4} \ln \left( 1 + \tan \left( \frac{\pi}{4} - \theta \right) \right) d\theta$$

Applying expansion for tangent function,

$$\int_0^{\pi/4} \ln \left( 1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right) d\theta$$

This can be simplified as (let us denote it by (ii)):

$$\int_0^{\pi/4} (\ln 2 - \ln(1 + \tan \theta)) d\theta$$

Now adding (i) and (ii), we get twice of the value of the required definite integral as:

$$\int_0^{\pi/4} \ln 2 d\theta$$

Now substituting the limits and dividing by 2 (because we added two integrals in last step), we get our required answer as:

$$\frac{\pi}{8} \ln 2$$

The approximate value of which is: 0.272198261288