Hidden Golden Ratio Integral (@solvingtogether)

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1 Question

If $I =$

$$\int_0^1 \frac{1-x^2}{x^2+(x^2+1)^2}\,dx$$

Then find $tan(I) + sec(I)$

2 Solution

Let us take substitution $x = tan\theta$ so that $dx \;=\; \sec^2\theta\; d\theta$, thus our integral becomes,

$$\int_0^{\pi/4} \frac{1-\tan^2\theta}{\tan^2\theta \;+\; \sec^4\theta}\sec^2\theta\; d\theta$$

Now converting $tan\theta$ and $sec\theta$ to respective sin and cos functions, the integral becomes,

$$\int_0^{\pi/4} \frac{\cos^2\theta-\sin^2\theta}{\sin^2\theta\;\;\cos^2\theta\;+\;1}\; d\theta$$

Using double angle properties, the integral can be rewritten as:

$$\int_0^{\pi/4} \frac{\cos 2\theta}{\frac{\sin^2 2\theta}{4}+1}d\theta$$

Taking a final substitution $\sin 2\theta = u$ so that $2\cos 2\theta\; d\theta = du$,

$$\int_0^1 \frac{2}{4+u^2}\; du$$

Plugging in the limits, we get,

$$I = \tan^{-1}\left(\frac{1}{2}\right)$$

For the final answer,

$$\tan I + \sec I = \frac{1}{2}+\frac{\sqrt{5}}{2}$$

$$\boxed{\tan I + \sec I = \phi}$$

Where ϕ is the Golden Ratio.