Let $A = \sum_{k=1}^{p-1} \frac{(p-1)!}{k}$ and $B = (p-1)!$. We have

$$\sum_{k=1}^{p-1} \frac{1}{k} = \frac{A}{B} = \frac{a}{b}. \tag{1}$$

As $\sum_{k=1}^{p-1} \frac{(p-1)!}{k} = \sum_{k=1}^{p-1} \frac{(p-1)!}{p-k}$, we have

$$A = \frac{1}{2} \sum_{k=1}^{p-1} \left(\frac{(p-1)!}{k} + \frac{(p-1)!}{p-k} \right) = \frac{p}{2} \sum_{k=1}^{p-1} \frac{(p-1)!}{k(p-k)}. \tag{2}$$

Let $k^{-1} \bmod p$ denote the modular inverse of $k \bmod p$, that is $k^{-1} \bmod p$ satisfies $kk^{-1} \equiv 1 \bmod p$, note that $\{1^{-1} \bmod p, \cdots, (p-1)^{-1} \bmod p\} = \{1, \cdots, p-1\}$. So for the sum involved in RHS of (2), we have

$$\sum_{k=1}^{p-1} \frac{(p-1)!}{k(p-k)} \equiv -(p-1)! \sum_{k=1}^{p-1} (k^{-1})^2 \bmod p \equiv -(p-1)! \sum_{k=1}^{p-1} k^2 \equiv -(p-1)! \frac{(p-1)p(2p-1)}{6} \bmod p,$$

,in the above we have $\sum_{k=1}^{p-1} (k^{-1})^2 \bmod p \equiv \sum_{k=1}^{p-1} k^2 \bmod p$ by using $\{k^{-1} \bmod p : 1 \leq k \leq (p-1)\}$ is a permutation of $\{k : 1 \leq k \leq (p-1)\}$. For $p > 3$ as $p| - \frac{(p-1)p(2p-1)}{6}$ we have

$$p| \sum_{k=1}^{p-1} \frac{(p-1)!}{k(p-k)}. \tag{3}$$

As $A = \frac{p}{2} \sum_{k=1}^{p-1} \frac{(p-1)!}{k(p-k)}$, from (3), we have $p^2|A$. As $B = (p-1)!$, we have $gcd(B, p^2) = 1$. From (1), $Ab = Ba$, hence as $p^2|A$ we have $p^2|Ba$ and as $gcd(B, p^2) = 1$, we have $p^2|a$.