Infinite Radical Limit

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1 Question

Find x for which

$$\lim_{x \to \infty} \sqrt{1 + \sqrt{x + \sqrt{x^2 + ... \ + \ \sqrt{x^n}}}} = 2$$

2 Solution

The key observation here is that

$$
\begin{aligned}
t + 1 &= \sqrt{t^2 + 2t + 1} \\
&= \sqrt{t^2 + \sqrt{(2t+1)^2}} \\
&= \sqrt{t^2 + \sqrt{4t^2 + 4t + 1}} \\
&= \sqrt{t^2 + \sqrt{4t^2 + \sqrt{(4t+1)^2}}} \\
&= \sqrt{t^2 + \sqrt{4t^2 + \sqrt{16t^2 + \sqrt{64t^2 + ...}}}}
\end{aligned}
$$

Now putting $t = 1$, we get,

$$2 = \sqrt{1 + \sqrt{4 + \sqrt{16 + \sqrt{64 + ...}}}}$$

Hence, the answer is

$$\boxed{x = 4}$$