First let us investigate the solutions of Pell's equation $x^2-7y^2=1$ in positive integers.

Lemma 1 *There are no solutions in positive integers for $x^2-7y^2=1$ satisfying $x+\sqrt{7}y<8+3\sqrt{7}$.*

This Lemma can be verified by checking for all positive integers (x,y) satisfying $x+\sqrt{7}y<8+3\sqrt{7}$ as there are only finitely many such (x,y).

Lemma 2 *(x,y) is a solution in positive integers to $x^2-7y^2=1$ if and only if $x+\sqrt{7}y=(8+3\sqrt{7})^k$ for some positive integer k.*

Proof: If $x+\sqrt{7}y=(8+3\sqrt{7})^k$, then $x-\sqrt{7}y=(8-3\sqrt{7})^k$ and $x^2-7y^2=(8+3\sqrt{7})^k(8-3\sqrt{7})^k=1$, hence (x,y) is a solution.

Now we have a family of solutions (x',y') given by, $x'+\sqrt{7}y'=(8+3\sqrt{7})^k$ for each natural number k.We will show that all the solutions are of this form by using proof by contradiction. If there is a solution (x_2,y_2) not of this form, then there exists a non negative integer k such that

$$(8+3\sqrt{7})^k<(x_2+y_2\sqrt{7})<(8+3\sqrt{7})^{k+1}=(x_1+\sqrt{7}y_1) \tag{1}$$

then as (x,y) given by

$$x+\sqrt{7}y=(x_1x_2-7y_1y_2)+\sqrt{7}(x_2y_1-x_1y_2)=\frac{x_1+\sqrt{7}y_1}{x_2+\sqrt{7}y_2}<8+3\sqrt{7}$$

is also a solution in positive integers, inequality has been obtained from (1), but from Lemma 1 there are no solutions for which $x+\sqrt{7}y<8+3\sqrt{7}$. Hence all the solutions are obtained from $x+\sqrt{7}y=(8+3\sqrt{7})^k$. This completes the proof of Lemma 2.

Coming back to our original problem $f(N)=2+2\sqrt{28N^2+1}$ is an integer, without loss of generality we can assume $N>0$(for $N=0$, $2+2\sqrt{28N^2+1}$ is a perfect square, and $f(-N)=f(N)$), which implies $\sqrt{28N^2+1}$ is rational, which implies $\sqrt{28N^2+1}$ is positive integer(As intersection of algebraic integers and rationals is integers).

Let $x=\sqrt{28N^2+1}$ then $x^2-7(2N)^2=1$ and from Lemma 2, $x+2\sqrt{7}N=(8+3\sqrt{7})^k$, which implies

$$2N=\sum_{i=0,(k-i)\equiv 1 \bmod 2}^{k}\binom{k}{i}8^i3^{k-i}7^{\frac{k-i-1}{2}}.$$

The right hand side of above expression is even if and only if k is even. Hence, $k=2s$ for natural s, and $x+2\sqrt{7}N=(8+3\sqrt{7})^{2s}$ which implies

$$x=\frac{((8+3\sqrt{7})^{2s}+(8-3\sqrt{7})^{2s})}{2},$$

and

$$2+2\sqrt{28N^2+1}=2x+2=(8+3\sqrt{7})^{2s}+(8-3\sqrt{7})^{2s}+2=((8+3\sqrt{7})^s+(8-3\sqrt{7})^s)^2,$$

as $(8+3\sqrt{7})^s+(8-3\sqrt{7})^s$ is an integer, $2+2\sqrt{28N^2+1}$ is a perfect square.