Algebra and Analytic Geometry lecture

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Contents

Chapter 1

Complex Numbers

1.1 Sets of numbers

- Natural Numbers $N = \{1, 2, 3, 4, ...\}$
- Integers $Z = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
- Rational Numbers $Q = \{\frac{p}{q}; q, p \subset Z, q \neq 0\}$
- Real Numbers R*Any distance from 0 on the number line*

1.2 Operation law on numbers

1. Commutative law
   $a + b = b + a$
   $ab = ba$
2. Associative law
   $(a + b) + c = a + (b + c)$ $(ab)c = a(cb)$ $a(b + c) = ab + bc$

For $\{N, Z, Q, R\}$ all the laws listed above are true

1.3 Divisibility

$A|B$ if there is a $c \subset N$

1.4 Prime Numbers

A natural number $\neq 1$ is called a prime if it has only two divisors, namely 1 and itself.
$Examples : 2, 3, 5, 7, ...$

1.4.1 Theorem

Every natural number can be uniquely (up to orders of factors) described as a product of prime numbers.
$Example : 24 = 2 * 2 * 2 * 3$

1.5 Principle of Mathematical Induction

Law for proving statements
If $p_1, p_2, ..., p_k$ are statements and:

1. p_1 is true
2. if p_k is true, then p_{k+1} is also true

Example:

1.6 Binomial formula

Newtons formula for expanding powers of sums
$(x + y)^1 = x + y$
$(x + y)^2 = x^2 + 2xy + y^2$
$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + x^2$
$(x + y)^4 = \textit{To difficult to remember or to expand}$

1.6.1 Theorem

$(a + b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}a^{n-n}b^n$

This can be simplified to:
$\sum\limits_{k=0}^{n} \binom{n}{k}a^{n-k}b^k$

1.6.2 Find a given coefficient of a given equation

Question: Find the coefficient of x^4 in $(2x - \frac{1}{x})^6$

$(2x - \frac{1}{x})^6 = (2x + \frac{-1}{x})^6$

$\sum\limits_{k=0}^{6} \binom{6}{k} (2x)^{6-k} (\frac{-1}{x})^k$

To calculate 'k' to substitute it into the equation we have to have only one x so:

$2^{6-k}x^{6-k} - x^{-k} -> x^{6-k-k} -> x^{6-2k}$

Based on that and the fact that we want to get the coefficient of x^4 we do the following:

$6 - 2k = 4$

$k = 1$

So we substitute and we get:

$\binom{6}{1} 2x^{6-1} * (\frac{-1}{x})^1 = \frac{6!}{5!} 2x^5 * \frac{-1}{x} = 6 * 2x^5 * \frac{-1}{x} = -12x^4$

Therefore the answer is -12

1.7 Complex numbers

Def: A complex number is a pair of real numbers.

Example: (2, 3) they are traditionally denoted by z, u, w .. .

The set of all complex numbers will be denoted by $\mathbb{C}$ so:

$\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\}$

if z = (x, y) $(x, y) \in \mathbb{R}$, then x is called the real part of z and denoted as $R_e z$ and y is the imaginary part.

Geometric interpretation:

A complex number z=(x, y) can be viewed as a point on a Cartesian plane. Such plane will be called the complex plane.

As Vectors on the plane with a initial point at the origin of the plane

1.7.1 Operations on complex numbers

1. Addition:
   f $z_1 = (x_1, y_1),\ z_2 = (x_2, y_2) \in \mathbb{C}$, then their sum is defined to be $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$
   $(1, 0) + (b, 0) = (a + b, 0)$

2. Multiplication:
   if $z_1 = (x_1, y_1), z_2 = (x_1, y_1)$ then their product is defined to be the number $z_1 z_2 = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$
   Example: $(1, 2) * (3, -1) = (1 * 3 - 2(-1),\ 1(-1), 3 * 2) = (5, 5)$
   Properties: Commutative and Associative
   $(a, 0)(b, 0) = (ab, 0)$
   $(0, 1)(0, 1) = (-1, 0)$
   $z = (x, y) = (x, 0) + (0, y) = (x, 0) + (0, 1)(y, 0) = x + iy$

3. Better multiplication:
   $(3+2i)(1-2i) = (3+2i)1+(3+2i)(2i) = 3+2i+6i-4 = -1+8i$

4. Inverse:
   $z = x + iy$
   $w = \frac{x}{x^2+y^2} = i\frac{-y}{x^2+y^2}$
   $w = \frac{1}{z}$

5. Conjugate:
   Properties:
   $\overline{z} = x + i(-y) = x - iy$

   (a) $\overline{z + w} = \overline{z} + \overline{w}$
   (b) $\overline{z * w} = \overline{z} * \overline{w}$
   (c) $\overline{z^n} = \overline{z}^n$

6. Modulus:
   if $z = x + iy$ the modulus is $|z| = \sqrt{x^2 + y^2}$ Properties:

   - $|z| \geq 0$ and $|z| = 0 \equiv z = 0$
   - $|\bar{z}| = |z|$
   - $|zw| = |z| * |w|$
   - $|z^n| = |z|^n \; n \in \mathbb{N}$
   - $|z + w| \leq |z| + |w|$
   - $z\bar{z} = |z|^2$

1.7.2 Polar form

A polar form of a complex number $z \neq 0$ is the form $z = r(\cos(\alpha) + i\sin(\alpha))$, where $r > 0$ Example:
$z = 3(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}))$
If $z = r(\cos\alpha + i\sin\alpha),\ w = rR(\cos\alpha + i\sin\alpha)$

- $zw = rR[\cos(\alpha + \beta) + i\sin(\alpha + \beta)]$
- $\frac{z}{w} = \frac{r}{R}[\cos(\alpha - \beta) + i\sin(\alpha - \beta)]$
- $z^n = r^n[\cos(n\alpha) + i\sin(n\alpha)]$

1.7.3 Geometry of algebra of complex numbers

Geometry of algebra of complex numbers.

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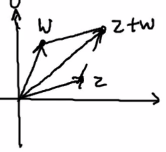
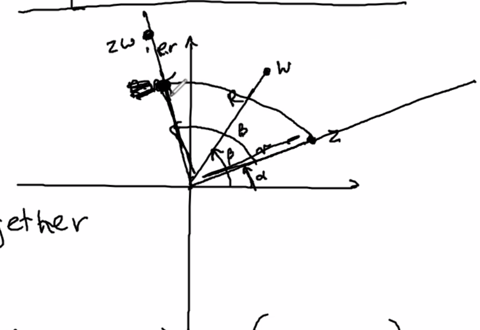


- $zw = \underbrace{r(\cos\alpha + i\sin\alpha)}_{z} \cdot \underbrace{R(\cos\beta + i\sin\beta)}_{w} =$
$= rR\left[\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right]$



So, multiplication of z by w is the rotation of z by β (about the origin) together with multiplication by R

In particular, if $w = (\cos\beta + i\sin\beta)$ $\quad (|w| = 1)$

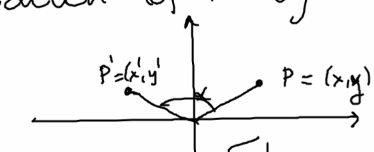
the product zw is simply the rotation of z by β.

Examples:

1) iz is the rotation of z by $\frac{\pi}{2}$ (counterclockwise)
(since $i = 1(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$)

Notice that we knew before that $|zw| = |z| \cdot |w|$

2) Consider the following problem:
Let $P = (x, y) \in \mathbb{R}^2$. Find coordinates of $P' = (x', y')$, which is the result of rotation of P by angle α about the origin.



$x' =$
$y' =$

1.7.4 Roots

Consider equation $z^n = W$.
The n^{th} root of a complex number w is the set $\{z : z^n = w\}$. We denote it by $\sqrt[n]{w}$. Properties:

- If $w = 0$, then $\sqrt[n]{0} = \{0\}$
- if $w \neq 0$, then $\sqrt[n]{0}$ has exactly n elements

Examples:

- $\sqrt{-1} = \{i, -i\}$
- $\sqrt{4}^{(\mathbb{C})} = \{2, -2\}$ it may be a real or complex root

If $w = r(\cos\alpha + i\sin\alpha)$, then $z^n = w$ has exactly n solutions given by $z_k = \sqrt[n]{r}[\cos(\frac{\alpha+2k\pi}{n}) + i\sin(\frac{\alpha+2k\pi}{k})]$.

1.8 Polynomials

A (complex) polynomial is a function of the form: $W(z) = a_0 + a_1z + ... + a_nz^n$, where $a_0, a_1, ..., a_n \in \mathbb{C}$ and are called coefficients. A degree of a nonzero polynomial is the largest n such that $a_n \neq 0$. For zero polynomial $W(z) = 0$ we agree that its degree is $-\infty$.
Examples:

1. $W(z) = (i + 2)z^2 + z - i$
2. $W(z) = i + z^3$

If $a_0, a_1, ..., a_n \in \mathbb{R}$, then W(z) is called a real polynomial. We can add and multiply polynomials as functions. Moreover if W_1, W_2, W_3 are polynomials, then

- $W_1 + W_2 = W_2 + W_1$
- $(W_1 + W_2) + W_3 = W_1 + (W_2 + W_3)$
- $W_1W_2 = W_2W_1$

- $W_1(W_2W_3) = (W_1W_2)W_3 = W_1W_2W_3$
- $W_1(W_2 + W_3) = W_1W_2 + W_1W_3$

We can always divide two polynomials: $W(Z) : P(Z) = Q(z)$. If $W(Z), P(Z)$ are polynomials, then there are polynomials $Q(Z), R(Z)$ such that $W(Z) = P(Z)Q(Z) + R(Z)$

A complex number z_0 is a root of a polynomial $W(Z)$ if $W(z_0) = 0$.

1.8.1 Bezout Theorem

z_0 is a root of $W(z) \Leftrightarrow W(z)$ is divisible by z_0

1.8.2 Fundamental Theorem of Algebra

Every polynomial of degree ≥ 1 has at least one complex root. If $W(z)$ is a polynomial of degree $n \geq 1$, then there are complex numbers $z_1, z_2, ..., z_n$ such that $W(z) = A(z - z_1)(z - z_2)...(z - z_n)$

1.8.3 Rational Functions

A rational function is a function of the form $f(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials.
$f(x) = \frac{4x^2 - x + 2}{x - 3ix^3}$.
If both P(x), Q(x) are real polynomials, then $f(x) = \frac{P(x)}{Q(x)}$ is called a <u>real rational function</u>.
A rational function $f(x) = \frac{P(x)}{Q(x)}$ is called a <u>proper rational function</u> if the degP < degQ.

Theorem

If $f(x) = \frac{P(x)}{Q(x)}$ is a rational function, then it can be written as a sum of a polynomial and a proper rational function.
Every real rational function is a sum of a real polynomial and a real proper rational function.

Partial Fraction

A partial fraction is a (proper) rational function of the form:

- $\frac{A}{(x-a)^n}$, $A \in \mathbb{R}$, $n \in \mathbb{N}$ 1st type
- $\frac{Ax+B}{(ax^2+bx+c)^m}$, $A, B, a, b, c \in \mathbb{R}$, $m \in \mathbb{N}$ 2nd Type

Partial fractions decomposition

Every proper rational function can be written as a sum of partial fractions. Moreover if $\frac{P(x)}{Q(x)}$ is a proper rational function then

- If $(x-a)^n$ is a factor of $Q(x)$ and $(x-a)^{n+1}$ is not, then in partial fractions decomposition we will get terms, $\frac{A_1}{(x-a)^1}$, $\frac{A_2}{(x-a)^2}$, ..., $\frac{A_n}{(x-a)^n}$
- If $ax^2 + bx + c$ with $\Delta < 0$ appears in decomposition of $Q(x)$ exactly n times
- No other terms will appear in the decomposition of $\frac{P(x)}{Q(x)}$

If $W_1(x) = a_0, a_1 x + ... + a_n x^n$, $W_2(x) = b_0, b_1 x + ... + b_n x^n$, then $W_1 = W_2 \leftrightarrow$
$\forall_{x \in \mathbb{R}} W_1(x) = W_2(x) \leftrightarrow$
$\forall_{k=0,1,2,...,n} a_k = b_k \leftrightarrow$
$\forall_{z \in \mathbb{C}} W_1(z) = W_2(x)$

Chapter 2

Matrices

2.1 Algebra of Matrices

A matrix is an array consisting of numbers (real or complex). The whole theory is identical foe both real and complex numbers. We will develop it in case of real numbers. Elements of a matrix are called entries. If a matrix consist of m rows and n columns than we say that it is of size m x n.
Matrices 1 x 2 can be identified with points on a plane (with coordinate system)

2.1.1 Special Types of Matrices

1. Zero matrices will be denoted by 0
2. Square matrices, they have size of n x n