Mathematical Analysis
Lecture

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12th October 2020

Contents

1 Rules

No textbook, so take notes.
Classes are mandatory

2 Requirements

During the classes we will start with a quiz, every practice. To pass the course you need 50% of points from the quizzed. A Quizes is 15min every quiz is worth 5 points. You get points from your top 10 quizes.

3 Notation

3.1 Number sets

1. Natural Numbers $N = \{1, 2, 3, ...\}$
2. Integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$
3. Rational $\mathbb{Q} = \{\frac{p}{q}; p, q \in Z, q \neq 0\}$
4. Irrational $ex. : \sqrt{2}, \pi, ...$
5. Real Numbers $\mathbb{R} = Rational + Irrational$

3.2 Sets notation

$(a, b) = x \in R : a < x < b$
$(a, b) = x \in R : a \leq x \leq b$
$(a, \infty) = x \in R : x > a$

$(a, b) - openinterval$
$[a, b] - closedinterval$

$A \subset B$ A is a subset of B
$x \in A$ X is an element of A, x belongs to A
$x \notin A$ X is not an element of A, x does not belong to A

3.3 Cartesian Product

Given two sets A and B, we can form the set consisting of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$. This set is called the Cartesian product of A and B and is denoted by AxB
$AxB\{(a, b) : a \in A, \in B\}$
If $A = B$, then AxA is denoted by A^2

3.4 Quantifiers

1. Existential $\exists$ "There exists x such that", "For at least one x"
2. Universal $\forall$ "For all x", "For each x", "For every x"

Example:
$\exists\, t > 0\ \forall\, x \in \mathbb{R}\ x^2 + 4x + 4 > t$
The statement above is false
The negation of the statement:
$\forall\, t > 0\ \exists\, x_0 \in \mathbb{R}\ x_0^2 + 4x_0 + 4 \leq t$

4 Functions

A function f is a rule that assigns to each element x in a set A exactly one element , called f(x), in a set B.
In our class $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ The set A is called the domain of the function f and will be denoted D_f.
The range of the function f is the set of all possible values of f(x) as x varies throughout the domain. The range of f will be denoted by R_f.
The most common method for visualizing a function is its.
If f is a function with domain D_f then its graph is the set of ordered pairs.
$\{(x, y) \in \mathbb{R}^2\ x \in D_f, y = f(x)\}$

Example:
Min function

$$
\begin{aligned}
f(x) &= min\{x, x^2\} \\
f(2) &= min\{2, 4\} = 2 \\
f(\frac{1}{2}) &= min\{\frac{1}{2}, \frac{1}{4}\} = \frac{1}{4}
\end{aligned} \tag{1}
$$

Absolute

$$
\begin{aligned}
f(x) = |x| &= \{x, if x \geq 0 \; or - x, if x - 2 < 0\} \\
f(x) = |x - 2| &= \{x - 2, if x \geq 0 \; or - (x - 2), if x - 2 < 0\}
\end{aligned} \tag{2}
$$

$|x - a|$ represents the distance between x and a

4.1 The Vertical Line Test

A curve is the XY plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

4.2 Classes of functions

1. Periodic functions
   We say that f is a periodic function if
   $\exists T > 0 \; \forall x \in D_f (x \pm T \in D_f$ and $f(x + T) = f(x))$
   A periodic function is a function that repeats its values after some determined period has been added to its independent variable.

2. Symmetric functions

   - Even
     A function f is called even if:
     $\forall x \in D_f \; (-x \in D_f)$ and $f(-x) = f(x)$
     The geometric significance of an even function is that its graph is symmetric with respect to the Y axis.
     If f is even D_f is symmetric about the Y Axis.
   - Odd
     A function f is called odd if:
     $\forall x \in D_f (-x \in D_f)$ and $f(-x) = -f(x)$

The graph of and odd function is symmetric about the origin.
If an odd function is defined at x=0 then f(0) must be 0!!

Example: Check if function is even or odd.
$f(x) = \frac{3^x - 3^{-x}}{x}$

(a) Check if domain is symmetric
 $D_f = \mathbb{R} \setminus \{0\}$
(b) Substitute -x for x

- Monotonicity:
  A function is monotonic if it is increasing, or decreasing, or non-decreasing, or non-increasing.
  - Increasing:
    A function f is called increasing on a set $I \subset D_f$,
    if $\forall x_1, x_2 \in I \; [(x_1 < x_2) \Rightarrow (f(x_1) < f(x_2))]$
  - Non-decreasing:
    A function f is called increasing on a set $I \subset D_f$,
    if $\forall x_1, x_2 \in I \; [(x_1 < x_2) \Rightarrow (f(x_1) \leq f(x_2))]$
  - Decreasing:
    A function f is called increasing on a set $I \subset D_f$,
    if $\forall x_1, x_2 \in I \; [(x_1 < x_2) \Rightarrow (f(x_1) > f(x_2))]$
  - Non-increasing:
    A function f is called increasing on a set $I \subset D_f$,
    if $\forall x_1, x_2 \in I \; [(x_1 < x_2) \Rightarrow (f(x_1) \geq f(x_2))]$

  Algebraic way to check monotonicity:
  let $f(x) = \frac{1}{1+x^2}$ and $I = (-\infty, 0]$
  Take any 2 points $x_1, x_2 \in I$ with $x_1 < x_2$.
  $f(x_2) - f(x_1) = \frac{1}{1+x_2^2} - \frac{1}{1+x_1^2} =$
  $\frac{1+x_1^2-1+x_2^2}{(1+x_1^2)(1+x_2^2)} =$
  $\frac{(x_1-x_2)(x_1+x_2)}{(1+x_2^2)(1+x_1^2)}$
  $x_1 - x_2 < 0$
  $x_1 + x_2 < 0$

4.3 New functions from Old functions

- Vertical and horizontal shifts:
  Suppose $c > 0$.

To obtain the graph of $y = f(x) + c$ shift the graph of y a distance of c units upwards. If $y = f(x) - c$ shift downwards.

To obtain the graph of $y = f(x - c)$ shift the graph y a distance of c units to the right.
To obtain the graph of $y = f(x + c)$ shift the graph y a distance of c units to the left.

- Vertical and horizontal stretching and reflecting:
  Suppose $c > 1$.
  To obtain the graph $y = c * f(x)$ stretch y vertically by a factor of x.
  To obtain the graph $y = f(c * x)$ compress the graph of y horizontally by a factor of c.
  To obtain the graph $y = -f(x)$ reflect the graph of y about the y axis.
  To obtain the graph $y = f(-x)$ reflect the graph of y about the x axis.

- Algebra of functions:
  let f and g be functions with domains D_f and D_g. Then the functions f_g, $f - g$, fg and $\frac{f}{g}$ are as follows:
  $(f \pm g)(x) = f(x) \pm g(x)$; $D_{f+g} = D_f \cap D_g$
  $(f * g)(x) = f(x) * g(x)$; $D_{f*g} = D_f \cap D_g$
  $(\frac{f}{g})(x) = \frac{f(x)}{f(g)}$; $D_{f+g} = D_f \cap D_g$

4.4 Composite Functions

Given 2 functions f and g the composite function is denoted by $f \circ g$ and is defined as $(f \circ g)(x) = f(g(x)))$.

Example:
If $f(x) = \sqrt{2 - x}$ and $g(x) = \sqrt{x}$, then:
$(f \circ g)(x) = f(g(x)) = \sqrt{2 - \sqrt{x}}$
$(g \circ f)(x) = g(f(x)) = \sqrt{\sqrt{2 - x}}$
For a $\sqrt{x}$ to be defined, we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have a $2 - \sqrt{x} \geq 0$ that is $\sqrt{x} \leq 2$ or $x \leq 4$. One can see that $D_{f \circ g} = [0, 4]$ therefore $D_{g \circ f} = (-\infty, 2]$

Let $h(x) = 3^{\sqrt{x+3}}$, write it as $f \circ g$:
$f(x) = 3^x$

$g(x) = \sqrt{x+3}$
$(f^{-1} \circ f)(x) = x, \forall x \in D_f$
$(f \circ f^{-1})(x) = x, \forall x \in R_f$

4.5 One-to-one functions

A function f is called an one-to-one function on a set $I \subset D_f$
$\forall x_1, x_2 \in I \ [(x_1 \neq x_2) \Rightarrow (f(x_1) \neq f(x_2))]$
Example:

- (Strictly) increasing function are 1-1
- Exponential functions are 1-1

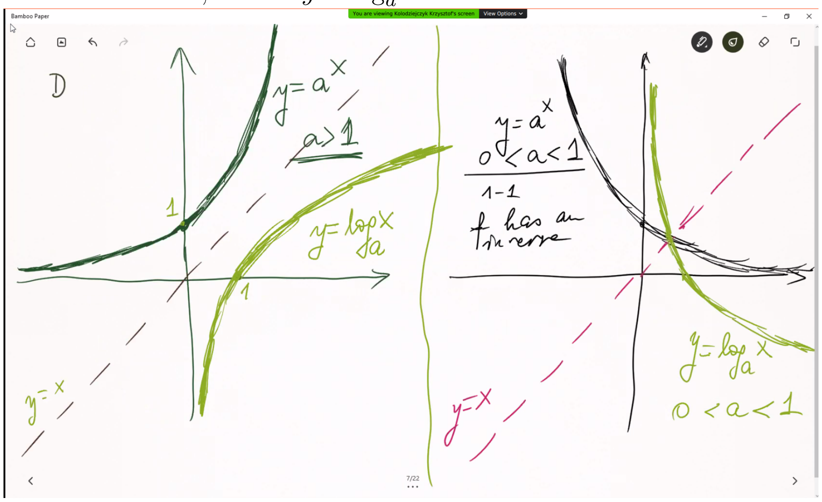
4.5.1 Horizontal Line Test

A function f in one-to-one if and only if no horizontal line intersects the graph at most once

Let f be a 1-1 function with a domain D_f and a range R_f. Then its inverse function f^{-1} has a domain $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$ and is defined by:
$(f^{-1})(y) = x \Leftrightarrow f(x) = y$
Example:
Let $g(x) = 3 + x + 2^x$. Is g invertible ? Yes, because its a strictly increasing function.
The inverse of $y = a^x$ is $y = \log_a x$ where $a > 1$.
The inverse of $y = a^x$ is $y = \log_a x$ where $0 < a < 1$.

5 Logarithms

The logarithm to the bare a is defined as the inverse function of the exponential function with bare a, that is $y = \log_a x$ means that $x = a^y$



5.1 Laws of logarithms

1. $\log_a(bc) = \log_a b + \log_a c$
2. $\log_a b^c = c * \log_a b$
3. $\log_a \frac{b}{c} = \log_a b - \log_a c$
4. $\log_a c = \log_a b * \log_b c$

6 Trigonometry

A standard position of an angle occurs when we place its vertex at the origin of a coordinate system and initial side on the positive x-axis.
A positive angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Negative angles are obtained by a clockwise rotation. Angles can be measured in radians. Mandatory for this class.
For an acute angle α the trigonometric functions are defined as ratios.
$\sin(\alpha) = \frac{opposite}{hypothenuse}$
$\cos(\alpha) = \frac{adjacent}{hypothenuse}$
$\tan(\alpha) = \frac{opposite}{adjacent}$
$\cot(\alpha) = \frac{adjacent}{opposite}$

The signs of the trig functions for angles in each of the quadrants can be remembered with: "**A**ll **S**tudents **T**ake **C**alculus"

6.1 Trigonometric identities

We have a angle α and a point $P(x, y)$

- $\forall \alpha \in \mathbb{R} \sin^2 \alpha + \cos^2 \alpha = 1$
- $\forall \alpha \in \mathbb{R} \sin(-\alpha) = -\sin \alpha$
- $\forall \alpha \in \mathbb{R}\ cos(-\alpha) = \cos \alpha$
- $\forall \alpha \in \mathbb{R} \sin(\alpha + 2\pi) = \sin(\alpha)$ and $\cos(\alpha + 2\pi) = \cos(\alpha)$
- $\sin(x + y) = \sin(x)\cos(y) + \cos(y)\sin(x)$
- $\sin(x - y) = \sin(x)\cos(y) - \cos(y)\sin(x)$
- $\sin(2x) = 2\sin(x)\cos(x)$
- If we denote $x + y = \alpha$ and $x - y = \beta$, then
  $\sin(\alpha) + \sin(\beta) = 2\sin(\frac{\alpha+\beta}{2}) * \cos(\frac{\alpha-\beta}{2})$
- $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
- $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
- $\cos(\alpha) + \cos(\beta) = 2\cos(\frac{\alpha+\beta}{2}) * \cos(\frac{\alpha-\beta}{2})$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$
- $\cos^2(x) = \frac{1+\cos(2x)}{2}$

6.2 Reduction formulas

- $\sin(\alpha + k\frac{\pi}{2}) = \sin(\alpha)\cos(k\frac{pi}{2}) + \cos(\alpha)\sin(k\frac{\pi}{2})$

  when k is even $\pm\sin(\alpha)$

  when k is odd $\pm\cos(\alpha)$

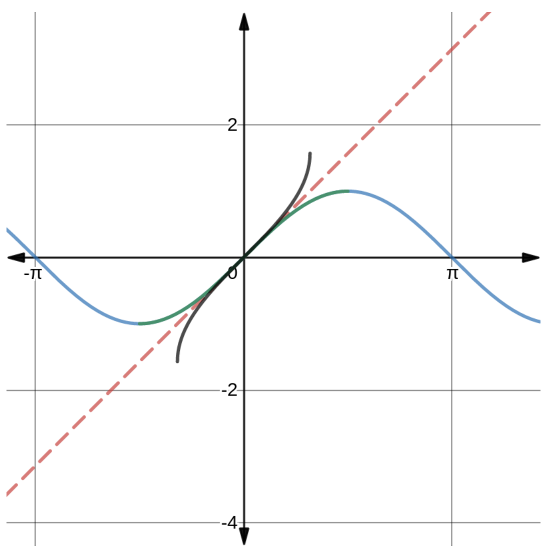
- $\cos(\alpha + l\frac{\pi}{2}) = \cos(\alpha)\cos(k\frac{\pi}{2}) - \sin(\alpha)\sin(k\frac{\pi}{2})$

  when k is even $\cos(\alpha)$

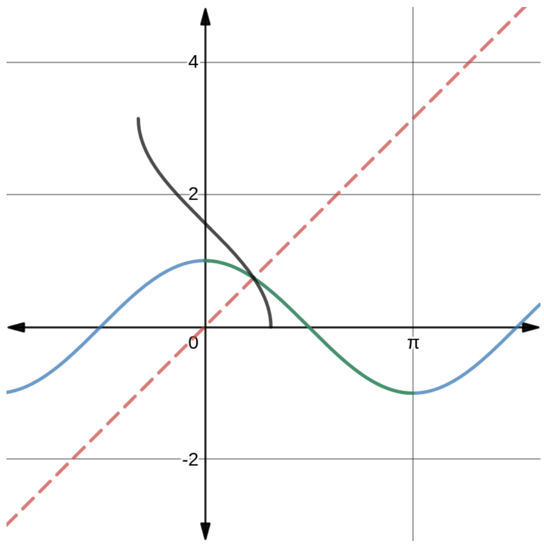
  when k is odd $\pm\sin(\alpha)$

6.3 Inverse of trigonometric functions / Cyclometric functions

The function $\sin(x)$ is not 1-1 but if we consider $f(x) = \sin(x)$ for $x \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ as a 1-1 function The existing inverse is called arcsine $\arcsin(x) = y \Leftrightarrow \sin(y) = x$ and $\frac{-\pi}{2} \leq \frac{\pi}{2}$



The function $\cos(x)$ is not 1-1 but if we consider $f(x) = \cos(x)$ for $x \in [0, \pi]$ as a 1-1 function The existing inverse is called arc-cosine $\arccos(x) = y \Leftrightarrow \sin(y) = x$ and $0 \leq \pi$



Example:

Show that $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$
$\arcsin(x) = \alpha \Leftrightarrow \sin(\alpha) = x$ and $\frac{-\pi}{2} \leq \alpha \leq \frac{\pi}{2}$
$\arccos(x) = \beta \Leftrightarrow \cos(\beta) = x$ and $0 \leq \beta \leq \pi$

7 Sequences

A sequence can be thought of as a list of numbers written in a definite order ex. $a_1, a_2, ..., a_n$. The numbers have special names
a_1 is called the first term of the sequence. (a_n) infinite sequence of numbers, $\{a_n\} = \{a_n : n \in \mathbb{N}\}$.
A sequence can be defined as a function whose domain is is the set $\mathbb{N}$. But we usually write a_n instead of the function notation f(n).
Sequences can be defined by:

1. By giving the formula for the nth term: $a_n = \frac{(-1)^{n+1}(n+\sqrt[n]{n})}{2^n}$
2. By a description: Let a_n be the digit in the n-th decimal place of the number $\sqrt{2}$
3. By a recursive relation: $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2} \; \forall \; n \geq 3$

7.1 Limit Laws for convergent sequences

If (a_n) and (b_n) are convergent sequences and c is a constant then:

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(c * a_n) = c \lim_{n\to\infty}(a_n)$
3. $\lim_{n\to\infty}(a_n * b_n) = \lim_{n\to\infty}(a_n) * \lim_{n\to\infty}(b_n)$
4. $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$ if $\lim_{n\to\infty}(b_n) \neq 0$
5. $\lim_{n\to\infty}(a_n)^p = (\lim_{n\to\infty} a_n)^p$
6. $\lim_{n\to\infty}(\sqrt[n]{a_n}) = \sqrt[k]{\lim_{n\to\infty}(a_n)}$
7. $\lim_{n\to\infty}(\sqrt[n]{a}) = 1$
8. $\lim_{n\to\infty}(\sqrt[n]{n}) = 1$

7.1.1 Squeeze Theorem for sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$.
and
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

A sequence is bounded above if there is a number such that $a_n \leq M$

7.2 Monotonic sequence theorem

Every bounded and monotonic sequence is convergent (has a limit).

7.3 Infinite Limits

$\lim\limits_{n\to\infty} a_n = \infty$. This means that for every positive number M there is an integer n_0 such that $a_n > M$ whenever $n > n_0$.
$\lim\limits_{n\to\infty} a_n = \infty \Leftrightarrow \forall_{M>0} \ \exists_{n_0 \in \mathbb{N}} \quad \sqrt{n} > M$

7.3.1 Infinity Theorem

1. $a + \infty = \infty$
2. $a\infty =$

8 Limits Involving Infinity

Let f be a function defined on both sides of a, except possibly at a itself. Then
$\lim_{x \to a} f(x) = \infty$
means that the value of f(x) can be made arbitrarily large by taking x sufficiently close to a (but not equal to a
$\forall_{M>0} \exists_{\delta>0} \forall x((a < |x-a| < \delta) = \rightarrow (f(x) > M))$

1. $p + \infty = \infty$
2. $p * \infty = \pm\infty$
3. $\frac{p}{\infty} = 0$
4. $p^{\infty} = \infty || 0$

9 Asymptotes

The line x=a is called a vertical asymptote if the curve y=f(x) if at least one of the following statements is true:

1. $\lim_{x \to a} f(x) = \infty$
2. $\lim_{x \to a} f(x) = -\infty$

10 Continuity

A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.
A function f is continuous at a if and only id it is continuous from the right and continuous from the left at a.

A function f is continuous on a open interval (a, b) if it is continuous at every number in the interval.
We say that f is continuous on [a, b] if it is continuous on (a, b) and if f is continuous from the right at a and continuous from the left at b. Example:

- f(x) = x is continuous on $\mathbb{R}$
- f(x) = sign(x) is continuous on $(0, \infty)$ and $(-\infty)$, is discontinuous at 0
- f(x) = tg(x) is continuous on $(\frac{-\pi}{2}, \frac{\pi}{2})$

- f(x) = arcsin(x) is continuous on [-1, 1]
- f(x) = arctan(x) is continuous on $\mathbb{R}$

10.1 Theorem

If f and g are continuous at a and $c \in \mathbb{R}$, then the following functions are also continuous functions at a.

- f + g
- f - g
- c*f
- f*g
- $\frac{f}{g}$ if $g(a) \neq 0$

<u>Corollary</u>

- Any polynomial is continuous everywhere
- Any rational function is continuous whenever it is defined, that is, it is continuous on its domain

10.2 Theorem

If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$
In other words $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$.

10.3 Theorem

If g is continuous at a and f is continuous at g(a) then (f comp g) is continuous at a.

10.4 Theorem

Suppose that f is continuous on the closed interval [a, b] and let α be any number between f(a) and f(b). Then there exists a number $c \in (a, b)$ such that f(c) = α

11 Derivatives

The derivative of a function f at a number a, denoted f'(a) is $f'(a) = \lim_{s \to 0} \frac{f(a+s)-f(a)}{s}$ or $f'(a) = \lim_{t \to x} \frac{f(t)-f(x)}{t-x}$.
A derivative is the slope of a tangent line.
A function f is differentiable at $x_0 iff f'(x_0) exists.$

11.1 Theorem

If f is differentiable at x_0, then f is continuous at x_0

How can a function fail to be differentiable?

1. Has a 'corner'
2. Is discontinuous as x_0
3. Has a vertical tangent

11.2 Derivative rules

- $(cf)'(x) = c * f'(x)$
- $(f \pm g)'(x) = f'(x) \pm g'(x)$
- $f(x) = \sin x \; f'(x) = \cos x$
- $f(x) = \cos x \; f'(x) = -\sin x$
- $(f * g)'(x) = f'(x) * g(x) + f(x) * g'(x)$
- $f(x) = c^x \; f'(x) = c^x \ln c$
- $(\frac{f}{g})'(x) = \frac{f'(x)*g(x)-f(x)*g'(x)}{(g(x))^2}$
- $(f \circ g)'(x) = f'(g(x)) * g'(x)$
- $(f \circ g \circ h)'(x) = f'(g(h(x)) * g'(h(x)) * h'(x)$
- $(log_a x)' = \frac{1}{x \ln a}$

11.3 Derivatives of Inverse Functions

Let f be a strictly monotonic function such that:

1. f is continuous near x_0
2. f is differentiable at x_0 and $f'(x_0) \neq 0$

Then $f^{-1}(y_0) = \frac{1}{f'(x_0)}$

11.4 Other

$h(x) = f(x)^{g(x)} = [e^{\ln f(x)}]^{g(x)}$
$h'(x) = e^{g(x)*\ln f(x)} * [g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}]$

12 Indeterminate Forms

12.1 L'Hospital

Suppose g and f are differentiable and g'(x) = 0 near x_0 (except possibly at x_0)
Suppose that $\lim_{x\to x_0} f(x)$ and $\lim_{x\to\infty} g(x) = 0$ or $\lim_{x\to x_0} f(x) \neq \pm$ and $\lim_{x\to x_0} g(x) = \pm\infty$
Then if the limit $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$

12.2 Indeterminate Powers

0^0, ∞^0, 1^∞
$f(x)^{g(x)} = e^{g(x)*\ln f(x)}$, $\lim_{x\to x_0} e^{g(x)*\ln f(x)} = e^{\lim_{x\to x_0} g(x)*\ln f(x)}$

13 Maxima and Minima

A function has a local maximum (minimum) at x_0 if $f(x_0) \geq f(x)$ ($f(x_0) \leq f(x)$) when x is near x_0

13.1 Fermat's Theorem

If f has a local maximum or minimum at x_0, and if f'(x) exists, then f'(x_0) = 0. The converse of Fermat's theorem is false in general when f'(x_0) = 0, f does not necessarily have a max on min at x_0. For example f(x) = x^3.

13.2 Mean value Theorem

If f is continuous on [a, b] and differentiable on [a,b], then there exists a number c $\in$[a,b] such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

13.2.1 Increasing / decreasing Test

1. If f'(x) $>$ 0 on an interval, then f is increasing on that interval.
2. If f'(x) $>$ 0 on an interval, then f is decreasing on that interval.

14 The Second Derivative Test

Suppose f"(x) is continuous near c.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

15 Indefinite Integrals

15.1 Anti derivative

A function F is called anti-derivative of f on an interval I, if the derivative F'(x) = f(x) for all x in I.
The notation $\int f(x)dx$ is traditionally used to denote the family of all anti derivatives of f.
$\int f(x)dx = F(x) + C$
where F is an anti derivative of f.

15.2 Integral rules

1. $\int cf(x)dx = c\int f(x)dx$
2. $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

15.3 Table of Indefinite Integrals

1. $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c$ where $\alpha \neq 0$
2. $\int \frac{1}{x} dx = \ln|x| + c$ where $\alpha = 0$
3. $\int a^x dx = \frac{a^x}{\ln a} + c$
4. $\int \frac{1}{\cos^2 x} dx = \tan x + c$
5. $\int \frac{dx}{\sqrt{1-x^2}} = \pm \arcsin x + c$

15.4 The substitution rule

If n = g(x) is a differentiable function whose range is an interval I and f in continuous on I, then $\int f(g(x))) * g'(x)dx = \int f(u)du$

15.5 Integration by parts

If f' and g' are continuous on I, then $\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$