Problems in Algebra and Analytic Geometry. Part I.

Problem 1 *Determine true statements:*
a) it is not true that $3 \cdot 8 = 25$; *b)* $3 > 4$ *or* $5 > 3$;
c) $3^3 = 27$ *and* $\sqrt{9999} = 99$; *d)* $2 \geq 1 \Leftrightarrow 1 \geq 0$.

Problem 2 *Consider the following statements:*
p: "John passed math test";
q: "John passed physics test';
r: "John passed computer science test";
Suppose that the statement:
"If John passed math test and did not pass physics, then he did not pass computer science test"
is false. Which of the tests did John pass for sure?

Problem 3 *Show that the below are laws of logic:*
1. $\neg(\neg p) \Leftrightarrow p$ *(Rule of Double Negation);*
2. $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge (\neg q))$ *(Rule of Conditional);*
3. $p \vee (\neg p)$ *(Rule of the Excluded Middle);*
4. $[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$ *(Distributive Law I);*
5. $[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$ *(Distributive Law II);*
6. $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$ *(Rule of Hypothetical Syllogism);*
7. $[\neg(p \vee q)] \Leftrightarrow [(\neg p) \wedge (\neg q)]$ *(De Morgan's Law I);*
8. $[\neg(p \wedge q)] \Leftrightarrow [(\neg p) \vee (\neg q)]$ *(De Morgan's Law II);*
9. $[(\neg p) \Rightarrow p] \Rightarrow p$ *(Law of Clavius);*
10. $(p \Rightarrow q) \Leftrightarrow [(\neg q) \Rightarrow (\neg p)]$ *(Rule of Contraposition);*
11. $[p \wedge (p \Rightarrow q)] \Rightarrow q$ *(Rule of Detachment).*

Problem 4 *Are the following statements laws of logic?*
a) $p \Rightarrow (p \vee q)$; *b)* $[(p \Rightarrow q) \wedge (q \Rightarrow p)] \Rightarrow (p \vee q)$;
c) $[p \vee (\neg q)] \Rightarrow (p \wedge q)$; *d)* $(p \Rightarrow q) \Leftrightarrow [(\neg q) \Rightarrow (\neg p)]$.

Problem 5 *Write the following statements with the use of quantifiers:*
1. *Every real number is positive;*
2. *Equation* $\sqrt{x} = 1$ *has real solutions;*
3. *Set of natural numbers is bounded from above;*
4. *Set* $A \subset R$ *has the greatest element;*
5. *Set* $B \subset R$ *does not contain the smallest element;*
6. *Every natural number is even;*
7. *Equation* $x^2 + x + 1 = 0$ *has no solution.*

Problem 6 *Verify if the given statements are true:*

a) $\bigvee_{x\in R} \sin x = \frac{1}{2}$;

b) $\bigwedge_{x\in R} x^2 + 4x + 3 > 0$;

c) $\bigwedge_{x\in R} \bigvee_{y\in R} x^2 - y^2 = 0$;

d) $\bigvee_{y\in R} \bigwedge_{x\in R} xy = 0$;

e) $\bigvee_{x\in R} \bigwedge_{n\in N} \bigvee_{m\in N} \frac{m}{n} + \frac{n}{m} = x$.

Problem 7 *Prove that:*

a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;

b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

c) $A \cup B = A \cap B$ *if and only if* $A = B$.

Problem 8 *Using the Principle of Mathematical Induction show that:*

a) $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$;

b) $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$;

c) $(1 + \frac{1}{1})^1 \cdot (1 + \frac{1}{2})^2 \cdot ... \cdot (1 + \frac{1}{n})^n = \frac{(n+1)^n}{n!}$;

d) $2^n < n!$ *for* $n > 4$;

e) $(1 + x)^n \geq 1 + nx$ *for* $x \geq -1$ *and* $n \in N$;

f) 5 *divides* $n^5 - n$;

g) $a + ar + ar^2 + ... + ar^{n-1} = a\frac{1-r^n}{1-r}$ *for* $r \neq 1$.

Problem 9 *Show that for every pair* $n, k \in N$ *where* $1 \leq k \leq n$ *we have:*

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Problem 10 *Use the binomial theorem to expand the following:*

a) $(2x + y)^4$; b) $(3x - 2y)^5$; c) $(x - \frac{1}{x})^6$.

Problem 11 *Find the coefficient of the indicated term:*

a) x^{15} *in the expansion of* $(x + x^2)^{10}$;

b) x^{-2} *in the expansion of* $(2x^5 - \frac{1}{x^2})^8$;

c) a^5 *in the expansion of* $(a^3 + \frac{1}{a^2})^{15}$.