# Mathematical Analysis
# Lecture

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## Contents

# 1 Rules

No textbook, so take notes.
Classes are mandatory

# 2 Requirements

During the classes we will start with a quiz, every practice. To pass the course you need 50% of points from the quizzed. A Quizes is 15min every quiz is worth 5 points. You get points from your top 10 quizes.

# 3 Notation

## 3.1 Number sets

1. Natural Numbers $N = \{1, 2, 3, ...\}$
2. Integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$
3. Rational $\mathbb{Q} = \{\frac{p}{q}; p, q \in Z, q \neq 0\}$
4. Irrational $ex. : \sqrt{2}, \pi, ...$
5. Real Numbers $\mathbb{R} = Rational + Irrational$

## 3.2 Sets notation

$(a, b) = x \in R : a < x < b$
$(a, b) = x \in R : a \leq x \leq b$
$(a, \infty) = x \in R : x > a$

$(a, b) - openinterval$
$[a, b] - closedinterval$

$A \subset B$ A is a subset of B
$x \in A$ X is an element of A, x belongs to A
$x \notin A$ X is not an element of A, x does not belong to A

### 3.3 Cartesian Product

Given two sets A and B, we can form the set consisting of all ordered pairs of the form $(a, b)$ where $a \in A$ and $b \in B$. This set is called the Cartesian product of A and B and is denoted by $AxB$
$AxB\{(a, b) : a \in A, \in B\}$
If $A = B$, then $AxA$ is denoted by $A^2$

### 3.4 Quantifiers

1. Existential $\exists$ "There exists x such that", "For at least one x"

2. Universal $\forall$ "For all x", "For each x", "For every x"

Example:
$\exists\, t > 0\ \forall\, x \in \mathbb{R}\ x^2 + 4x + 4 > t$
The statement above is false
The negation of the statement:
$\forall\, t > 0\ \exists\, x_0 \in \mathbb{R}\ x_0^2 + 4x_0 + 4 \leq t$

# 4 Functions

A function f is a rule that assigns to each element x in a set A exactly one element , called f(x), in a set B.
In our class $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ The set A is called the domain of the function f and will be denoted $D_f$.
The range of the function f is the set of all possible values of f(x) as x varies throughout the domain. The range of f will be denoted by $R_f$.
The most common method for visualizing a function is its.
If f is a function with domain $D_f$ then its graph is the set of ordered pairs.
$\{(x, y) \in \mathbb{R}^2\ x \in D_f, y = f(x)\}$

Example:
Min function

$$f(x) = min\{x, x^2\}$$
$$f(2) = min\{2, 4\} = 2$$
$$f(\frac{1}{2}) = min\{\frac{1}{2}, \frac{1}{4}\} = \frac{1}{4} \quad (1)$$

Absolute

$$f(x) = |x| = \{x, ifx \geq 0\ or - x, ifx - 2 < 0\}$$
$$f(x) = |x - 2| = \{x - 2, ifx \geq 0\ or - (x - 2), ifx - 2 < 0\} \quad (2)$$

$|x - a|$ represents the distance between x and a

## 4.1 The Vertical Line Test

A curve is the XY plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

## 4.2 Classes of functions

1. Periodic functions
   We say that f is a periodic function if
   $\exists T > 0\ \forall x \in D_f(x \pm T \in D_f$ and $f(x + T) = f(x))$
   A periodic function is a function that repeats its values after some determined period has been added to its independent variable.

2. Symmetric functions
   - Even
     A function f is called even if:
     $\forall x \in D_f\ (-x \in D_f)$ and $f(-x) = f(x)$
     The geometric significance of an even function is that its graph is symmetric with respect to the Y axis.
     If f is even $D_f$ is symmetric about the Y Axis.
   - Odd
     A function f is called odd if:
     $\forall x \in D_f(-x \in D_f)$ and $f(-x) = -f(x)$

The graph of and odd function is symmetric about the origin.
If an odd function is defined at x=0 then f(0) must be 0!!

Example: Check if function is even or odd.
$f(x) = \frac{3^x - 3^{-x}}{x}$

(a) Check if domain is symmetric
$D_f = \mathbb{R} \setminus \{0\}$
(b) Substitute -x for x

- Monotonicity:
A function is monotonic if it is increasing, or decreasing, or non-decreasing, or non-increasing.
  - Increasing:
  A function f is called increasing on a set $I \subset D_f$,
  if $\forall x_1, x_2 \in I\ [(x_1 < x_2) \Rightarrow (f(x_1) < f(x_2))]$
  - Non-decreasing:
  A function f is called increasing on a set $I \subset D_f$,
  if $\forall x_1, x_2 \in I\ [(x_1 < x_2) \Rightarrow (f(x_1) \leq f(x_2))]$
  - Decreasing:
  A function f is called increasing on a set $I \subset D_f$,
  if $\forall x_1, x_2 \in I\ [(x_1 < x_2) \Rightarrow (f(x_1) > f(x_2))]$
  - Non-increasing:
  A function f is called increasing on a set $I \subset D_f$,
  if $\forall x_1, x_2 \in I\ [(x_1 < x_2) \Rightarrow (f(x_1) \geq f(x_2))]$

  Algebraic way to check monotonicity:
  let $f(x) = \frac{1}{1+x^2}$ and $I = (-\infty, 0]$
  Take any 2 points $x_1, x_2 \in I$ with $x_1 < x_2$.
  $f(x_2) - f(x_1) = \frac{1}{1+x_2^2} - \frac{1}{1+x_1^2} =$
  $\frac{1+x_1^2-1+x_2^2}{(1+x_1^2)(1+x_2^2)} =$
  $\frac{(x_1-x_2)(x_1+x_2)}{(1+x_2^2)(1+x_1^2)}$
  $x_1 - x_2 < 0$
  $x_1 + x_2 < 0$

## 4.3 New functions from Old functions

- Vertical and horizontal shifts:
Suppose $c > 0$.

To obtain the graph of $y = f(x) + c$ shift the graph of y a distance of c units upwards. If $y = f(x) - c$ shift downwards.

To obtain the graph of $y = f(x - c)$ shift the graph y a distance of c units to the right.
To obtain the graph of $y = f(x + c)$ shift the graph y a distance of c units to the left.

- Vertical and horizontal stretching and reflecting:
  Suppose $c > 1$.
  To obtain the graph $y = c * f(x)$ stretch y vertically by a factor of x.
  To obtain the graph $y = f(c*x)$ compress the graph of y horizontally by a factor of c.
  To obtain the graph $y = -f(x)$ reflect the graph of y about the x axis.
  To obtain the graph $y = f(-x)$ reflect the graph of y about the y axis.

- Algebra of functions:
  let f and g be functions with domains $D_f$ and $D_g$. Then the functions $f_g$, $f - g$, $fg$ and $\frac{f}{g}$ are as follows:
  $(f \pm g)(x) = f(x) \pm g$; $D_{f+g} = D_f \cap D_g$
  $(f * g)(x) = f(x) * g(x)$; $D_{f*g} = D_f \cap D_g$
  $(\frac{f}{g})(x) = \frac{f(x)}{f(g)}$; $D_{f+g} = D_f \cap D_g$