Mathematical Analysis
Lecture

Philip Policki

12th October 2020

Contents

1 Rules

No textbook, so take notes.
Classes are mandatory

2 Requirements

During the classes we will start with a quiz, every practice. To pass the course you need 50% of points from the quizzed. A Quizes is 15min every quiz is worth 5 points. You get points from your top 10 quizes.

3 Notation

3.1 Number sets

1. Natural Numbers $N = \{1, 2, 3, ...\}$
2. Integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$
3. Rational $\mathbb{Q} = \{\frac{p}{q}; p, q \in Z, q \neq 0\}$
4. Irrational $ex. : \sqrt{2}, \pi, ...$
5. Real Numbers $\mathbb{R} = Rational + Irrational$

3.2 Sets notation

$(a, b) = x \in R : a < x < b$
$(a, b) = x \in R : a \leq x \leq b$
$(a, \infty) = x \in R : x > a$

$(a, b) - openinterval$
$[a, b] - closedinterval$

$A \subset B$ A is a subset of B
$x \in A$ X is an element of A, x belongs to A
$x \notin A$ X is not an element of A, x does not belong to A

3.3 Cartesian Product

Given two sets A and B, we can form the set consisting of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$. This set is called the Cartesian product of A and B and is denoted by AxB
$AxB\{(a, b) : a \in A, \in B\}$
If $A = B$, then AxA is denoted by A^2

3.4 Quantifiers

1. Existential $\exists$ "There exists x such that", "For at least one x"

2. Universal $\forall$ "For all x", "For each x", "For every x"

Example:
$\exists\, t > 0\; \forall\, x \in \mathbb{R}\; x^2 + 4x + 4 > t$
The statement above is false
The negation of the statement:
$\forall\, t > 0\; \exists\, x_0 \in \mathbb{R}\; x_0^2 + 4x_0 + 4 \leq t$

4 Functions

A function f is a rule that assigns to each element x in a set A exactly one element , called f(x), in a set B.
In our class $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ The set A is called the domain of the function f and will be denoted D_f.
The range of the function f is the set of all possible values of f(x) as x varies throughout the domain. The range of f will be denoted by R_f.
The most common method for visualizing a function is its.
If f is a function with domain D_f then its graph is the set of ordered pairs.
$\{(x, y) \in \mathbb{R}^2\; x \in D_f, y = f(x)\}$

Example:
Min function

$$f(x) = min\{x, x^2\}$$
$$f(2) = min\{2, 4\} = 2$$
$$f(\frac{1}{2}) = min\{\frac{1}{2}, \frac{1}{4}\} = \frac{1}{4} \quad (1)$$

Absolute

$$f(x) = |x| = \{x, ifx \geq 0\ or - x, ifx - 2 < 0\}$$
$$f(x) = |x - 2| = \{x - 2, ifx \geq 0\ or - (x - 2), ifx - 2 < 0\} \quad (2)$$

$|x - a|$ represents the distance between x and a

4.1 The Vertical Line Test

A curve is the XY plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

4.2 Classes of functions

1. Periodic functions
   We say that f is a periodic function if
   $\exists T > 0\ \forall x \in D_f(x \pm T \in D_f$ and $f(x+T) = f(x))$
   A periodic function is a function that repeats its values after some determined period has been added to its independent variable.

2. Symmetric functions

   - Even
     A function f is called even if:
     $\forall x \in D_f\ (-x \in D_f)$ and $f(-x) = f(x)$
     The geometric significance of an even function is that its graph is symmetric with respect to the Y axis.
     If f is even D_f is symmetric about the Y Axis.
   - Odd
     A function f is called odd if:
     $\forall x \in D_f(-x \in D_f)$ and $f(-x) = -f(x)$

The graph of and odd function is symmetric about the origin.
If an odd function is defined at x=0 then f(0) must be 0!!

Example: Check if function is even or odd.
$f(x) = \frac{3^x - 3^{-x}}{x}$

(a) Check if domain is symmetric
$D_f = \mathbb{R} \setminus \{0\}$

(b) Substitute -x for x

- Monotonicity:
  A function is monotonic if it is increasing, or decreasing, or non-decreasing, or non-increasing.
  - Increasing:
    A function f is called increasing on a set $I \subset D_f$,
    if $\forall x_1, x_2 \in I \ [(x_1 < x_2) \Rightarrow (f(x_1) < f(x_2))]$
  - Non-decreasing:
    A function f is called increasing on a set $I \subset D_f$,
    if $\forall x_1, x_2 \in I \ [(x_1 < x_2) \Rightarrow (f(x_1) \leq f(x_2))]$
  - Decreasing:
    A function f is called increasing on a set $I \subset D_f$,
    if $\forall x_1, x_2 \in I \ [(x_1 < x_2) \Rightarrow (f(x_1) > f(x_2))]$
  - Non-increasing:
    A function f is called increasing on a set $I \subset D_f$,
    if $\forall x_1, x_2 \in I \ [(x_1 < x_2) \Rightarrow (f(x_1) \geq f(x_2))]$

  Algebraic way to check monotonicity:
  let $f(x) = \frac{1}{1+x^2}$ and $I = (-\infty, 0]$
  Take any 2 points $x_1, x_2 \in I$ with $x_1 < x_2$.
  $f(x_2) - f(x_1) = \frac{1}{1+x_2^2} - \frac{1}{1+x_1^2} =$
  $\frac{1+x_1^2-1+x_2^2}{(1+x_1^2)(1+x_2^2)} =$
  $\frac{(x_1-x_2)(x_1+x_2)}{(1+x_2^2)(1+x_1^2)}$
  $x_1 - x_2 < 0$
  $x_1 + x_2 < 0$

4.3 New functions from Old functions

- Vertical and horizontal shifts:
  Suppose $c > 0$.

To obtain the graph of $y = f(x) + c$ shift the graph of y a distance of c units upwards. If $y = f(x) - c$ shift downwards.

To obtain the graph of $y = f(x - c)$ shift the graph y a distance of c units to the right.
To obtain the graph of $y = f(x + c)$ shift the graph y a distance of c units to the left.

- Vertical and horizontal stretching and reflecting:
  Suppose $c > 1$.
  To obtain the graph $y = c * f(x)$ stretch y vertically by a factor of x.
  To obtain the graph $y = f(c * x)$ compress the graph of y horizontally by a factor of c.
  To obtain the graph $y = -f(x)$ reflect the graph of y about the y axis.
  To obtain the graph $y = f(-x)$ reflect the graph of y about the x axis.

- Algebra of functions:
  let f and g be functions with domains D_f and D_g. Then the functions f_g, $f - g$, fg and $\frac{f}{g}$ are as follows:
  $(f \pm g)(x) = f(x) \pm g(x)$; $D_{f+g} = D_f \cap D_g$
  $(f * g)(x) = f(x) * g(x)$; $D_{f*g} = D_f \cap D_g$
  $(\frac{f}{g})(x) = \frac{f(x)}{f(g)}$; $D_{f+g} = D_f \cap D_g$

4.4 Composite Functions

Given 2 functions f and g the composite function is denoted by $f \circ g$ and is defined as $(f \circ g)(x) = f(g(x)))$.

Example:
If $f(x) = \sqrt{2-x}$ and $g(x) = \sqrt{x}$, then:
$(f \circ g)(x) = f(g(x)) = \sqrt{2 - \sqrt{x}}$
$(g \circ f)(x) = g(f(x)) = \sqrt{\sqrt{2-x}}$
For a $\sqrt{x}$ to be defined, we must have $x \geq 0$. For $\sqrt{2-\sqrt{x}}$ to be defined we must have a $2 - \sqrt{x} \geq 0$ that is $\sqrt{x} \leq 2$ or $x \leq 4$. One can see that $D_{f \circ g} = [0, 4]$ therefore $D_{g \circ f} = (-\infty, 2]$

Let $h(x) = 3^{\sqrt{x+3}}$, write it as $f \circ g$:
$f(x) = 3^x$

$g(x) = \sqrt{x+3}$
$(f^{-1} \circ f)(x) = x, \forall x \in D_f$
$(f \circ f^{-1})(x) = x, \forall x \in R_f$

4.5 One-to-one functions

A function f is called an one-to-one function on a set $I \subset D_f$
$\forall x_1, x_2 \in I \ [(x_1 \neq x_2) \Rightarrow (f(x_1) \neq f(x_2))]$
Example:

- (Strictly) increasing function are 1-1
- Exponential functions are 1-1

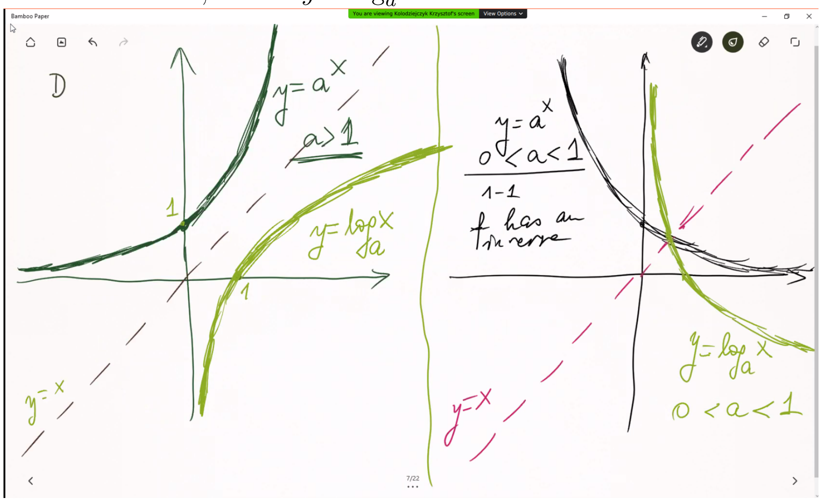
4.5.1 Horizontal Line Test

A function f in one-to-one if and only if no horizontal line intersects the graph at most once

Let f be a 1-1 function with a domain D_f and a range R_f. Then its inverse function f^{-1} has a domain $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$ and is defined by:
$(f^{-1})(y) = x \Leftrightarrow f(x) = y$
Example:
Let $g(x) = 3 + x + 2^x$. Is g invertible ? Yes, because its a strictly increasing function.
The inverse of $y = a^x$ is $y = \log_a x$ where $a > 1$.
The inverse of $y = a^x$ is $y = \log_a x$ where $0 < a < 1$.

5 Logarithms

The logarithm to the bare a is defined as the inverse function of the exponential function with bare a, that is $y = \log_a x$ means that $x = a^y$



5.1 Laws of logarithms

1. $\log_a(bc) = \log_a b + \log_a c$
2. $\log_a b^c = c * \log_a b$
3. $\log_a \frac{b}{c} = \log_a b - \log_a c$
4. $\log_a c = \log_a b * \log_b c$

6 Trigonometry

A standard position of an angle occurs when we place its vertex at the origin of a coordinate system and initial side on the positive x-axis.
A positive angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Negative angles are obtained by a clockwise rotation. Angles can be measured in radians. Mandatory for this class.
For an acute angle α the trigonometric functions are defined as ratios.
$\sin(\alpha) = \frac{opposite}{hypothenuse}$
$\cos(\alpha) = \frac{adjacent}{hypothenuse}$
$\tan(\alpha) = \frac{opposite}{adjacent}$
$\cot(\alpha) = \frac{adjacent}{opposite}$

The signs of the trig functions for angles in each of the quadrants can be remembered with: "**A**ll **S**tudents **T**ake **C**alculus"

6.1 Trigonometric identities

We have a angle α and a point $P(x, y)$

- $\forall\alpha \in \mathbb{R} \sin^2 \alpha + \cos^2 \alpha = 1$
- $\forall\alpha \in \mathbb{R} \sin(-\alpha) = -\sin \alpha$
- $\forall\alpha \in \mathbb{R}\ cos(-\alpha) = \cos \alpha$
- $\forall\alpha \in \mathbb{R} \sin(\alpha + 2\pi) = \sin(\alpha)$ and $\cos(\alpha + 2\pi) = \cos(\alpha)$
- $\sin(x + y) = \sin(x)\cos(y) + \cos(y)\sin(x)$
- $\sin(x - y) = \sin(x)\cos(y) - \cos(y)\sin(x)$
- $\sin(2x) = 2\sin(x)\cos(x)$
- If we denote $x + y = \alpha$ and $x - y = \beta$, then
  $\sin(\alpha) + \sin(\beta) = 2\sin(\frac{\alpha+\beta}{2}) * \cos(\frac{\alpha-\beta}{2})$
- $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
- $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
- $\cos(\alpha) + \cos(\beta) = 2\cos(\frac{\alpha+\beta}{2}) * \cos(\frac{\alpha-\beta}{2})$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$
- $\cos^2(x) = \frac{1+\cos(2x)}{2}$

6.2 Reduction formulas

- $\sin(\alpha + k\frac{\pi}{2}) = \sin(\alpha)\cos(k\frac{pi}{2}) + \cos(\alpha)\sin(k\frac{\pi}{2})$

  when k is even $\pm\sin(\alpha)$

  when k is odd $\pm\cos(\alpha)$

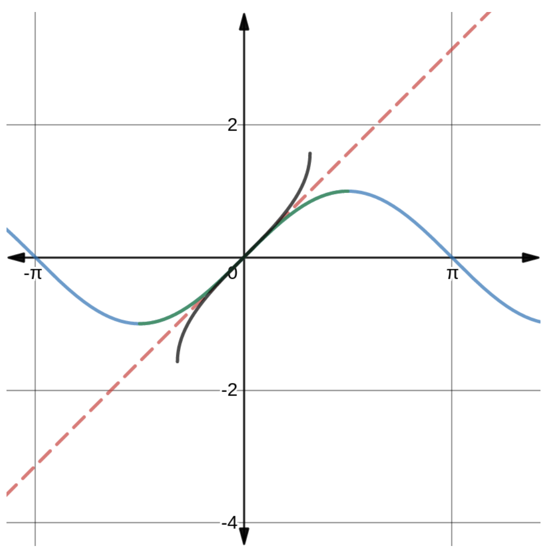
- $\cos(\alpha + l\frac{\pi}{2}) = \cos(\alpha)\cos(k\frac{\pi}{2}) - \sin(\alpha)\sin(k\frac{\pi}{2})$

  when k is even $\cos(\alpha)$

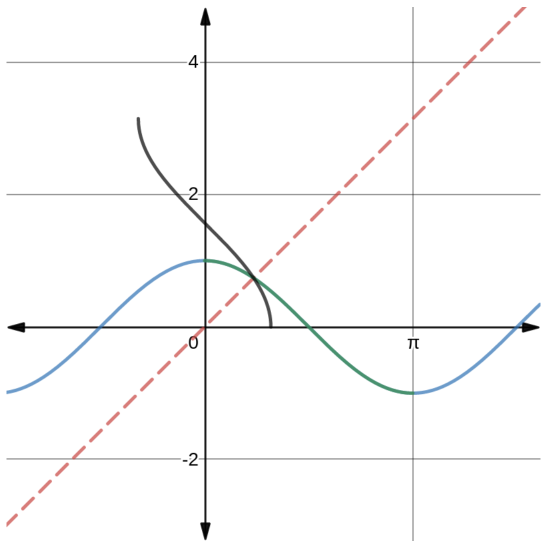
  when k is odd $\pm\sin(\alpha)$

6.3 Inverse of trigonometric functions / Cyclometric functions

The function $\sin(x)$ is not 1-1 but if we consider $f(x) = \sin(x)$ for $x \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ as a 1-1 function The existing inverse is called arcsine $\arcsin(x) = y \Leftrightarrow \sin(y) = x$ and $\frac{-\pi}{2} \leq \frac{\pi}{2}$



The function $\cos(x)$ is not 1-1 but if we consider $f(x) = \cos(x)$ for $x \in [0, \pi]$ as a 1-1 function The existing inverse is called arc-cosine $\arccos(x) = y \Leftrightarrow \sin(y) = x$ and $0 \leq \pi$



Example:
Show that $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$
$\arcsin(x) = \alpha \Leftrightarrow \sin(\alpha) = x$ and $\frac{-\pi}{2} \leq \alpha \leq \frac{\pi}{2}$
$\arccos(x) = \beta \Leftrightarrow \cos(\beta) = x$ and $0 \leq \beta \leq \pi$

7 Sequences

A sequence can be thought of as a list of numbers written in a definite order ex. $a_1, a_2, ..., a_n$. The numbers have special names
a_1 is called the first term of the sequence. (a_n) infinite sequence of numbers, $\{a_n\} = \{a_n : n \in \mathbb{N}\}$.
A sequence can be defined as a function whose domain is is the set $\mathbb{N}$. But we usually write a_n instead of the function notation f(n).
Sequences can be defined by:

1. By giving the formula for the nth term: $a_n = \frac{(-1)^{n+1}(n+\sqrt[n]{n})}{2^n}$
2. By a description: Let a_n be the digit in the n-th decimal place of the number $\sqrt{2}$
3. By a recursive relation: $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2} \; \forall \; n \geq 3$

7.1 Limit Laws for convergent sequences

If (a_n) and (b_n) are convergent sequences and c is a constant then:

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(c * a_n) = c \lim_{n\to\infty}(a_n)$
3. $\lim_{n\to\infty}(a_n * b_n) = \lim_{n\to\infty}(a_n) * \lim_{n\to\infty}(b_n)$
4. $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$ if $\lim_{n\to\infty}(b_n) \neq 0$
5. $\lim_{n\to\infty}(a_n)^p = (\lim_{n\to\infty} a_n)^p$
6. $\lim_{n\to\infty}(\sqrt[n]{a_n}) = \sqrt[k]{\lim_{n\to\infty}(a_n)}$
7. $\lim_{n\to\infty}(\sqrt[n]{a}) = 1$
8. $\lim_{n\to\infty}(\sqrt[n]{n}) = 1$

7.1.1 Squeeze Theorem for sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$.
and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$