

Mathematical Analysis

Lecture

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Contents

1	Rules	3
2	Requirements	3
3	Notation	3
3.1	Number sets	3
3.2	Sets notation	3
3.3	Cartesian Product	4
3.4	Quantifiers	4
4	Functions	4
4.1	The Vertical Line Test	5
4.2	Classes of functions	5
4.3	New functions from Old functions	6
4.4	Composite Functions	7
4.5	One-to-one functions	8
4.5.1	Horizontal Line Test	8
5	Logarithms	9
5.1	Laws of logarithms	9
6	Trigonometry	10
6.1	Trigonometric identities	11
6.2	Reduction formulas	12
6.3	Inverse of trigonometric functions / Cyclometric functions	12

7	Sequences	13
7.1	Limit Laws for convergent sequences	14
7.1.1	Squeeze Theorem for sequences	14
7.2	Monotonic sequence theorem	14
7.3	Infinite Limits	14
7.3.1	Infinity Theorem	14
8	Limits Involving Infinity	15
9	Asymptotes	15
10	Continuity	15
10.1	Theorem	16
10.2	Theorem	16
10.3	Theorem	16
10.4	Theorem	16
11	Derivatives	17
11.1	Theorem	17
11.2	Derivative rules	17
11.3	Derivatives of Inverse Functions	17

1 Rules

No textbook, so take notes.

Classes are mandatory

2 Requirements

During the classes we will start with a quiz, every practice. To pass the course you need 50% of points from the quizzed. A Quizes is 15min every quiz is worth 5 points. You get points from your top 10 quizzes.

3 Notation

3.1 Number sets

1. Natural Numbers $N = \{1, 2, 3, \dots\}$
2. Integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
3. Rational $\mathbb{Q} = \{\frac{p}{q}; p, q \in \mathbb{Z}, q \neq 0\}$
4. Irrational *ex.* : $\sqrt{2}, \pi, \dots$
5. Real Numbers $\mathbb{R} = \text{Rational} + \text{Irrational}$

3.2 Sets notation

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$(a, \infty) = \{x \in \mathbb{R} : x > a\}$$

$$(a, b) - \text{openinterval}$$

$$[a, b] - \text{closedinterval}$$

$A \subset B$ A is a subset of B

$x \in A$ X is an element of A, x belongs to A

$x \notin A$ X is not an element of A, x does not belong to A

3.3 Cartesian Product

Given two sets A and B, we can form the set consisting of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$. This set is called the Cartesian product of A and B and is denoted by $A \times B$

$A \times B = \{(a, b) : a \in A, b \in B\}$

If $A = B$, then $A \times A$ is denoted by A^2

3.4 Quantifiers

1. Existential \exists "There exists x such that", "For at least one x"
2. Universal \forall "For all x", "For each x", "For every x"

Example:

$$\exists t > 0 \forall x \in \mathbb{R} x^2 + 4x + 4 > t$$

The statement above is false

The negation of the statement:

$$\forall t > 0 \exists x_0 \in \mathbb{R} x_0^2 + 4x_0 + 4 \leq t$$

4 Functions

A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

In our class $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ The set A is called the domain of the function f and will be denoted D_f .

The range of the function f is the set of all possible values of $f(x)$ as x varies throughout the domain. The range of f will be denoted by R_f .

The most common method for visualizing a function is its.

If f is a function with domain D_f then its graph is the set of ordered pairs.

$$\{(x, y) \in \mathbb{R}^2 : x \in D_f, y = f(x)\}$$

Example:

Min function

$$\begin{aligned}
f(x) &= \min\{x, x^2\} \\
f(2) &= \min\{2, 4\} = 2 \\
f\left(\frac{1}{2}\right) &= \min\left\{\frac{1}{2}, \frac{1}{4}\right\} = \frac{1}{4}
\end{aligned} \tag{1}$$

Absolute

$$\begin{aligned}
f(x) &= |x| = \{x, \text{if } x \geq 0 \text{ or } -x, \text{if } x < 0\} \\
f(x) &= |x - 2| = \{x - 2, \text{if } x \geq 2 \text{ or } -(x - 2), \text{if } x < 2\}
\end{aligned} \tag{2}$$

$|x - a|$ represents the distance between x and a

4.1 The Vertical Line Test

A curve in the XY plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

4.2 Classes of functions

1. Periodic functions

We say that f is a periodic function if

$$\exists T > 0 \forall x \in D_f (x \pm T \in D_f \text{ and } f(x + T) = f(x))$$

A periodic function is a function that repeats its values after some determined period has been added to its independent variable.

2. Symmetric functions

• Even

A function f is called even if:

$$\forall x \in D_f (-x \in D_f) \text{ and } f(-x) = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the Y axis.

If f is even D_f is symmetric about the Y Axis.

• Odd

A function f is called odd if:

$$\forall x \in D_f (-x \in D_f) \text{ and } f(-x) = -f(x)$$

The graph of an odd function is symmetric about the origin.
 If an odd function is defined at $x=0$ then $f(0)$ must be 0!!

Example: Check if function is even or odd.

$$f(x) = \frac{3^x - 3^{-x}}{x}$$

- (a) Check if domain is symmetric

$$D_f = \mathbb{R} \setminus \{0\}$$

- (b) Substitute $-x$ for x

- Monotonicity:

A function is monotonic if it is increasing, or decreasing, or non-decreasing, or non-increasing.

- Increasing:

A function f is called increasing on a set $I \subset D_f$,

$$\text{if } \forall x_1, x_2 \in I [(x_1 < x_2) \Rightarrow (f(x_1) < f(x_2))]$$

- Non-decreasing:

A function f is called increasing on a set $I \subset D_f$,

$$\text{if } \forall x_1, x_2 \in I [(x_1 < x_2) \Rightarrow (f(x_1) \leq f(x_2))]$$

- Decreasing:

A function f is called increasing on a set $I \subset D_f$,

$$\text{if } \forall x_1, x_2 \in I [(x_1 < x_2) \Rightarrow (f(x_1) > f(x_2))]$$

- Non-increasing:

A function f is called increasing on a set $I \subset D_f$,

$$\text{if } \forall x_1, x_2 \in I [(x_1 < x_2) \Rightarrow (f(x_1) \geq f(x_2))]$$

Algebraic way to check monotonicity:

$$\text{let } f(x) = \frac{1}{1+x^2} \text{ and } I = (-\infty, 0]$$

Take any 2 points $x_1, x_2 \in I$ with $x_1 < x_2$.

$$f(x_2) - f(x_1) = \frac{1}{1+x_2^2} - \frac{1}{1+x_1^2} =$$

$$\frac{1+x_1^2-1+x_2^2}{(1+x_1^2)(1+x_2^2)} =$$

$$\frac{(x_1-x_2)(x_1+x_2)}{(1+x_2^2)(1+x_1^2)}$$

$$x_1 - x_2 < 0$$

$$x_1 + x_2 < 0$$

4.3 New functions from Old functions

- Vertical and horizontal shifts:

Suppose $c > 0$.

To obtain the graph of $y = f(x) + c$ shift the graph of y a distance of c units upwards. If $y = f(x) - c$ shift downwards.

To obtain the graph of $y = f(x - c)$ shift the graph y a distance of c units to the right.

To obtain the graph of $y = f(x + c)$ shift the graph y a distance of c units to the left.

- Vertical and horizontal stretching and reflecting:

Suppose $c > 1$.

To obtain the graph $y = c * f(x)$ stretch y vertically by a factor of c .

To obtain the graph $y = f(c * x)$ compress the graph of y horizontally by a factor of c .

To obtain the graph $y = -f(x)$ reflect the graph of y about the x axis.

To obtain the graph $y = f(-x)$ reflect the graph of y about the y axis.

- Algebra of functions:

let f and g be functions with domains D_f and D_g . Then the functions $f + g$, $f - g$, fg and $\frac{f}{g}$ are as follows:

$$(f \pm g)(x) = f(x) \pm g(x); D_{f \pm g} = D_f \cap D_g$$

$$(f * g)(x) = f(x) * g(x); D_{f * g} = D_f \cap D_g$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}; D_{\frac{f}{g}} = D_f \cap D_g$$

4.4 Composite Functions

Given 2 functions f and g the composite function is denoted by $f \circ g$ and is defined as $(f \circ g)(x) = f(g(x))$.

Example:

If $f(x) = \sqrt{2 - x}$ and $g(x) = \sqrt{x}$, then:

$$(f \circ g)(x) = f(g(x)) = \sqrt{2 - \sqrt{x}}$$

$$(g \circ f)(x) = g(f(x)) = \sqrt{\sqrt{2 - x}}$$

For a \sqrt{x} to be defined, we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have a $2 - \sqrt{x} \geq 0$ that is $\sqrt{x} \leq 2$ or $x \leq 4$. One can see that $D_{f \circ g} = [0, 4]$ therefore $D_{g \circ f} = (-\infty, 2]$

Let $h(x) = 3^{\sqrt{x+3}}$, write it as $f \circ g$:

$$f(x) = 3^x$$

$$g(x) = \sqrt{x+3}$$

$$(f^{-1} \circ f)(x) = x, \forall x \in D_f$$

$$(f \circ f^{-1})(x) = x, \forall x \in R_f$$

4.5 One-to-one functions

A function f is called an one-to-one function on a set $I \subset D_f$

$$\forall x_1, x_2 \in I [(x_1 \neq x_2) \Rightarrow (f(x_1) \neq f(x_2))]$$

Example:

- (Strictly) increasing function are 1-1
- Exponential functions are 1-1

4.5.1 Horizontal Line Test

A function f is one-to-one if and only if no horizontal line intersects the graph at most once

Let f be a 1-1 function with a domain D_f and a range R_f . Then its inverse function f^{-1} has a domain $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$ and is defined by:

$$(f^{-1})(y) = x \Leftrightarrow f(x) = y$$

Example:

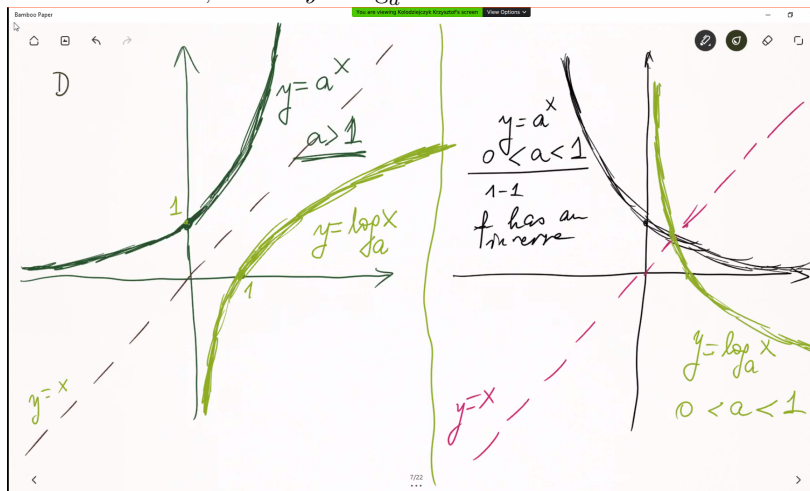
Let $g(x) = 3 + x + 2^x$. Is g invertible? Yes, because it's a strictly increasing function.

The inverse of $y = a^x$ is $y = \log_a x$ where $a > 1$.

The inverse of $y = a^x$ is $y = \log_a x$ where $0 < a < 1$.

5 Logarithms

The logarithm to the base a is defined as the inverse function of the exponential function with base a , that is $y = \log_a x$ means that $x = a^y$



5.1 Laws of logarithms

1. $\log_a(bc) = \log_a b + \log_a c$
2. $\log_a b^c = c * \log_a b$
3. $\log_a \frac{b}{c} = \log_a b - \log_a c$
4. $\log_a c = \log_a b * \log_b c$

6 Trigonometry

A standard position of an angle occurs when we place its vertex at the origin of a coordinate system and initial side on the positive x-axis.

A positive angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Negative angles are obtained by a clockwise rotation. Angles can be measured in radians. Mandatory for this class.

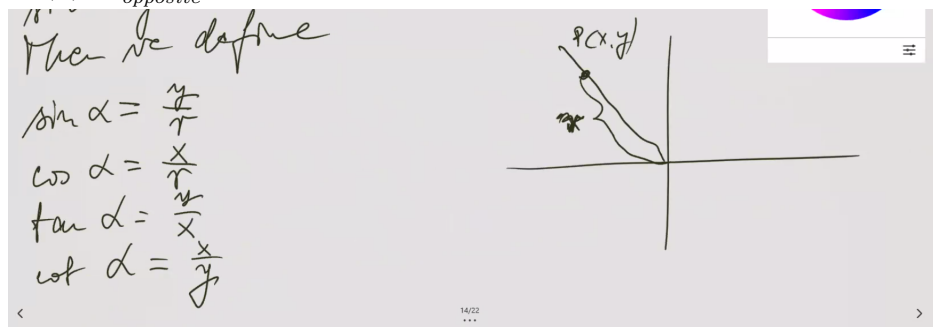
For an acute angle α the trigonometric functions are defined as ratios.

$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot(\alpha) = \frac{\text{adjacent}}{\text{opposite}}$$



The signs of the trig functions for angles in each of the quadrants can be remembered with: "All Students Take Calculus"

6.1 Trigonometric identities

We have a angle α and a point $P(x, y)$

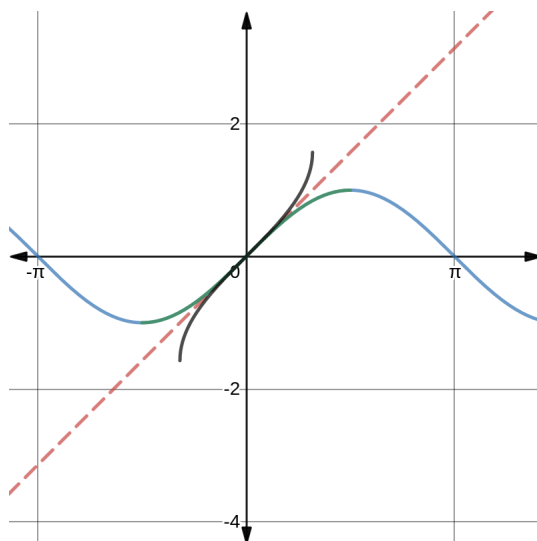
- $\forall \alpha \in \mathbb{R} \sin^2 \alpha + \cos^2 \alpha = 1$
- $\forall \alpha \in \mathbb{R} \sin(-\alpha) = -\sin \alpha$
- $\forall \alpha \in \mathbb{R} \cos(-\alpha) = \cos \alpha$
- $\forall \alpha \in \mathbb{R} \sin(\alpha + 2\pi) = \sin(\alpha)$ and $\cos(\alpha + 2\pi) = \cos(\alpha)$
- $\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y)$
- $\sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y)$
- $\sin(2x) = 2 \sin(x) \cos(x)$
- If we denote $x + y = \alpha$ and $x - y = \beta$, then
 $\sin(\alpha) + \sin(\beta) = 2 \sin(\frac{\alpha+\beta}{2}) * \cos(\frac{\alpha-\beta}{2})$
- $\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$
- $\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y)$
- $\cos(\alpha) + \cos(\beta) = 2 \cos(\frac{\alpha+\beta}{2}) * \cos(\frac{\alpha-\beta}{2})$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$
- $\cos^2(x) = \frac{1+\cos(2x)}{2}$

6.2 Reduction formulas

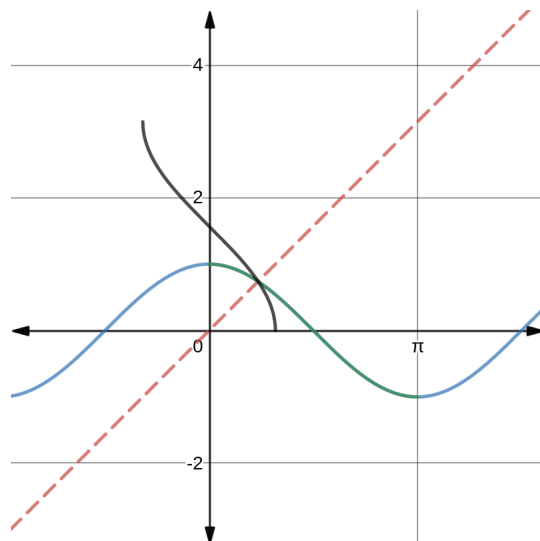
- $\sin(\alpha + k\frac{\pi}{2}) = \sin(\alpha) \cos(k\frac{\pi}{2}) + \cos(\alpha) \sin(k\frac{\pi}{2})$
when k is even $\pm \sin(\alpha)$
when k is odd $\pm \cos(\alpha)$
- $\cos(\alpha + l\frac{\pi}{2}) = \cos(\alpha) \cos(l\frac{\pi}{2}) - \sin(\alpha) \sin(l\frac{\pi}{2})$
when k is even $\cos(\alpha)$
when k is odd $\pm \sin(\alpha)$

6.3 Inverse of trigonometric functions / Cyclometric functions

The function $\sin(x)$ is not 1-1 but if we consider $f(x) = \sin(x)$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ as a 1-1 function The existing inverse is called arcsine $\arcsin(x) = y \Leftrightarrow \sin(y) = x$ and $-\frac{\pi}{2} \leq \frac{\pi}{2}$



The function $\cos(x)$ is not 1-1 but if we consider $f(x) = \cos(x)$ for $x \in [0, \pi]$ as a 1-1 function The existing inverse is called arc-cosine $\arccos(x) = y \Leftrightarrow \cos(y) = x$ and $0 \leq \pi$



Example:

Show that $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$

$\arcsin(x) = \alpha \Leftrightarrow \sin(\alpha) = x$ and $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$\arccos(x) = \beta \Leftrightarrow \cos(\beta) = x$ and $0 \leq \beta \leq \pi$

7 Sequences

A sequence can be thought of as a list of numbers written in a definite order ex.

a_1, a_2, \dots, a_n . The numbers have special names

a_1 is called the first term of the sequence. (a_n) infinite sequence of numbers, $\{a_n\} = \{a_n : n \in \mathbb{N}\}$.

A sequence can be defined as a function whose domain is the set \mathbb{N} . But we usually write a_n instead of the function notation $f(n)$.

Sequences can be defined by:

1. By giving the formula for the nth term: $a_n = \frac{(-1)^{n+1}(n + \sqrt[3]{n})}{2^n}$
2. By a description: Let a_n be the digit in the n-th decimal place of the number $\sqrt{2}$
3. By a recursive relation: $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2} \forall n \geq 3$

7.1 Limit Laws for convergent sequences

If (a_n) and (b_n) are convergent sequences and c is a constant then:

1. $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$
2. $\lim_{n \rightarrow \infty} (c * a_n) = c \lim_{n \rightarrow \infty} (a_n)$
3. $\lim_{n \rightarrow \infty} (a_n * b_n) = \lim_{n \rightarrow \infty} (a_n) * \lim_{n \rightarrow \infty} (b_n)$
4. $\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \frac{\lim_{n \rightarrow \infty} (a_n)}{\lim_{n \rightarrow \infty} (b_n)}$ if $\lim_{n \rightarrow \infty} (b_n) \neq 0$
5. $\lim_{n \rightarrow \infty} (a_n)^p = (\lim_{n \rightarrow \infty} a_n)^p$
6. $\lim_{n \rightarrow \infty} (\sqrt[p]{a_n}) = \sqrt[p]{\lim_{n \rightarrow \infty} (a_n)}$
7. $\lim_{n \rightarrow \infty} (\sqrt[p]{a}) = 1$
8. $\lim_{n \rightarrow \infty} (\sqrt[p]{n}) = 1$

7.1.1 Squeeze Theorem for sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$.

and

$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$

A sequence is bounded above if there is a number such that $a_n \leq M$

7.2 Monotonic sequence theorem

Every bounded and monotonic sequence is convergent (has a limit).

7.3 Infinite Limits

$\lim_{n \rightarrow \infty} a_n = \infty$. This means that for every positive number M there is an integer n_0 such that $a_n > M$ whenever $n > n_0$.

$$\lim_{n \rightarrow \infty} a_n = \infty \Leftrightarrow \forall_{M>0} \exists_{n_0 \in \mathbb{N}} \sqrt{n} > M$$

7.3.1 Infinity Theorem

1. $a + \infty = \infty$
2. $a\infty =$

8 Limits Involving Infinity

Let f be a function defined on both sides of a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that the value of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a (but not equal to a)

$$\forall M > 0 \exists \delta > 0 \forall x ((a < |x - a| < \delta) \Rightarrow (f(x) > M))$$

1. $p + \infty = \infty$

2. $p * \infty = \pm\infty$

3. $\frac{p}{\infty} = 0$

4. $p^\infty = \infty || 0$

9 Asymptotes

The line $x=a$ is called a vertical asymptote if the curve $y=f(x)$ if at least one of the following statements is true:

1. $\lim_{x \rightarrow a} f(x) = \infty$

2. $\lim_{x \rightarrow a} f(x) = -\infty$

10 Continuity

A function f is continuous at a if $\lim_{x \rightarrow a} f(x) = f(a)$.

A function f is continuous at a if and only if it is continuous from the right and continuous from the left at a .

A function f is continuous on an open interval (a, b) if it is continuous at every number in the interval.

We say that f is continuous on $[a, b]$ if it is continuous on (a, b) and if f is continuous from the right at a and continuous from the left at b . Example:

- $f(x) = x$ is continuous on \mathbb{R}
- $f(x) = \text{sign}(x)$ is continuous on $(0, \infty)$ and $(-\infty, 0)$, is discontinuous at 0
- $f(x) = \tan(x)$ is continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$

- $f(x) = \arcsin(x)$ is continuous on $[-1, 1]$
- $f(x) = \arctan(x)$ is continuous on \mathbb{R}

10.1 Theorem

If f and g are continuous at a and $c \in \mathbb{R}$, then the following functions are also continuous functions at a .

- $f + g$
- $f - g$
- $c \cdot f$
- $f \cdot g$
- $\frac{f}{g}$ if $g(a) \neq 0$

Corollary

- Any polynomial is continuous everywhere
- Any rational function is continuous whenever it is defined, that is, it is continuous on its domain

10.2 Theorem

If f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then $\lim_{x \rightarrow a} f(g(x)) = f(b)$
 In other words $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$.

10.3 Theorem

If g is continuous at a and f is continuous at $g(a)$ then $(f \circ g)$ is continuous at a .

10.4 Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let α be any number between $f(a)$ and $f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = \alpha$

11 Derivatives

The derivative of a function f at a number a , denoted $f'(a)$ is $f'(a) = \lim_{s \rightarrow 0} \frac{f(a+s) - f(a)}{s}$

or $f'(a) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}$.

A derivative is the slope of a tangent line.

A function f is differentiable at x_0 if $f'(x_0)$ exists.

11.1 Theorem

If f is differentiable at x_0 , then f is continuous at x_0

How can a function fail to be differentiable?

1. Has a 'corner'
2. Is discontinuous at x_0
3. Has a vertical tangent

11.2 Derivative rules

- $(cf)'(x) = c * f'(x)$
- $(f \pm g)'(x) = f'(x) \pm g'(x)$
- $f(x) = \sin x \quad f'(x) = \cos x$
- $f(x) = \cos x \quad f'(x) = -\sin x$
- $(f * g)'(x) = f'(x) * g(x) + f(x) * g'(x)$
- $f(x) = c^x \quad f'(x) = c^x \ln c$
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x) * g(x) - f(x) * g'(x)}{(g(x))^2}$
- $(f \circ g)'(x) = f'(g(x)) * g'(x)$
- $(f \circ g \circ h)'(x) = f'(g(h(x))) * g'(h(x)) * h'(x)$
- $(\log_a x)' = \frac{1}{x \ln a}$

11.3 Derivatives of Inverse Functions

Let f be a strictly monotonic function such that:

1. f is continuous near x_0
2. f is differentiable at x_0 and $f'(x_0) \neq 0$

Then $f^{-1}(y_0) = \frac{1}{f'(x_0)}$