Algebra and Analytic Geometry lecture

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Contents

Chapter 1

Complex Numbers

1.1 Sets of numbers

- Natural Numbers $N = \{1, 2, 3, 4, ...\}$
- Integers $Z = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
- Rational Numbers $Q = \{\frac{p}{q}; q, p \subset Z, q \neq 0\}$
- Real Numbers R*Any distance from 0 on the number line*

1.2 Operation law on numbers

1. Commutative law
   $a + b = b + a$
   $ab = ba$
2. Associative law
   $(a + b) + c = a + (b + c)$ $(ab)c = a(cb)$ $a(b + c) = ab + bc$

For $\{N, Z, Q, R\}$ all the laws listed above are true

1.3 Divisibility

$A|B$ if there is a $c \subset N$

1.4 Prime Numbers

A natural number $\neq 1$ is called a prime if it has only two divisors, namely 1 and itself.
$Examples: 2, 3, 5, 7, ...$

1.4.1 Theorem

Every natural number can be uniquely (up to orders of factors) described as a product of prime numbers.
$Example: 24 = 2 * 2 * 2 * 3$

1.5 Principle of Mathematical Induction

Law for proving statements

If $p_1, p_2, ..., p_k$ are statements and:

1. p_1 is true
2. if p_k is true, then p_{k+1} is also true

Example:

1.6 Binomial formula

Newtons formula for expanding powers of sums
$(x+y)^1 = x + y$
$(x+y)^2 = x^2 + 2xy + y^2$
$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + x^2$
$(x+y)^4 = \textit{To difficult to remember or to expand}$

1.6.1 Theorem

$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}a^{n-n}b^n$

This can be simplified to:
$\sum\limits_{k=0}^{n} \binom{n}{k}a^{n-k}b^k$

1.6.2 Find a given coefficient of a given equation

Question: Find the coefficient of x^4 in $(2x - \frac{1}{x})^6$

$(2x - \frac{1}{x})^6 = (2x + \frac{-1}{x})^6$

$\sum_{k=0}^{6} \binom{6}{k} (2x)^{6-k} (\frac{-1}{x})^k$

To calculate 'k' to substitute it into the equation we have to have only one x so:

$2^{6-k} x^{6-k} - x^{-k} -> x^{6-k-k} -> x^{6-2k}$

Based on that and the fact that we want to get the coefficient of x^4 we do the following:

$6 - 2k = 4$

$k = 1$

So we substitute and we get:

$\binom{6}{1} 2x^{6-1} * (\frac{-1}{x})^1 = \frac{6!}{5!} 2x^5 * \frac{-1}{x} = 6 * 2x^5 * \frac{-1}{x} = -12x^4$

Therefore the answer is -12

1.7 Complex numbers

Def: A complex number is a pair of real numbers.

Example: (2, 3) they are traditionally denoted by z, u, w .. .

The set of all complex numbers will be denoted by $\mathbb{C}$ so:

$\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\}$

if z = (x, y) $(x, y) \in \mathbb{R}$, then x is called the real part of z and denoted as $R_e z$ and y is the imaginary part.

Geometric interpretation:

A complex number z=(x, y) can be viewed as a point on a Cartesian plane. Such plane will be called the complex plane.

As Vectors on the plane with a initial point at the origin of the plane

1.7.1 Operations on complex numbers

1. Addition:
   f $z_1 = (x_1, y_1),\ z_2 = (x_2, y_2) \in \mathbb{C}$, then their sum is defined to be $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$
   $(1, 0) + (b, 0) = (a + b, 0)$

2. Multiplication:
   if $z_1 = (x_1, y_1), z_2 = (x_1, y_1)$ then their product is defined to be the number $z_1 z_2 = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$
   Example: $(1, 2) * (3, -1) = (1 * 3 - 2(-1),\ 1(-1), 3 * 2) = (5, 5)$
   Properties: Commutative and Associative
   $(a, 0)(b, 0) = (ab, 0)$
   $(0, 1)(0, 1) = (-1, 0)$
   $z = (x, y) = (x, 0) + (0, y) = (x, 0) + (0, 1)(y, 0) = x + iy$

3. Better multiplication:
   $(3+2i)(1-2i) = (3+2i)1+(3+2i)(2i) = 3+2i+6i-4 = -1+8i$

4. Inverse:
   $z = x + iy$
   $w = \frac{x}{x^2+y^2} = i\frac{-y}{x^2+y^2}$
   $w = \frac{1}{z}$

5. Conjugate:
   Properties:
   $\overline{z} = x + i(-y) = x - iy$

   (a) $\overline{z + w} = \overline{z} + \overline{w}$
   (b) $\overline{z * w} = \overline{z} * \overline{w}$
   (c) $\overline{z^n} = \overline{z}^n$

6. Modulus:
   if $z = x + iy$ the modulus is $|z| = \sqrt{x^2 + y^2}$ Properties:

   - $|z| \geq 0$ and $|z| = 0 \equiv z = 0$
   - $|\bar{z}| = |z|$
   - $|zw| = |z| * |w|$
   - $|z^n| = |z|^n \; n \in \mathbb{N}$
   - $|z + w| \leq |z| + |w|$
   - $z\bar{z} = |z|^2$

1.7.2 Polar form

A polar form of a complex number $z \neq 0$ is the form $z = r(\cos(\alpha) + i\sin(\alpha))$, where $r > 0$ Example:
$z = 3(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}))$
If $z = r(\cos\alpha + i\sin\alpha),\; w = rR(\cos\alpha + i\sin\alpha)$

- $zw = rR[\cos(\alpha + \beta) + i\sin(\alpha + \beta)]$
- $\frac{z}{w} = \frac{r}{R}[\cos(\alpha - \beta) + i\sin(\alpha - \beta)]$
- $z^n = r^n[\cos(n\alpha) + i\sin(n\alpha)]$

1.7.3 Geometry of algebra of complex numbers

Geometry of algebra of complex numbers.

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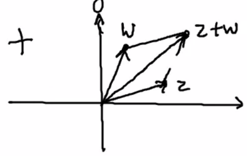
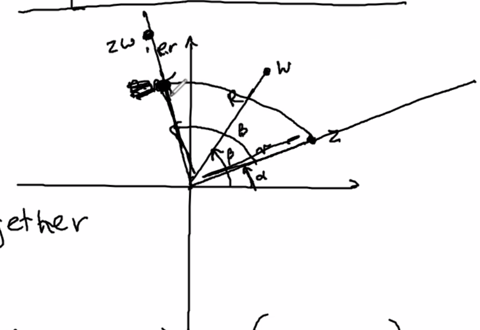


• $zw = \underbrace{r(\cos\alpha + i\sin\alpha)}_{z} \cdot \underbrace{R(\cos\beta + i\sin\beta)}_{w} =$
$= rR\left[\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right]$



So, multiplication of z by w is the rotation of z by β (about the origin) together with multiplication by R

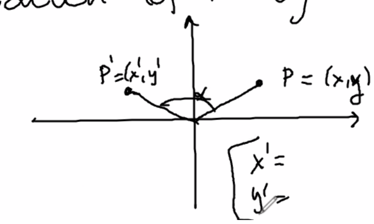
In particular, if $w = (\cos\beta + i\sin\beta)$ $\quad (|w| = 1)$
the product zw is simply the rotation of z by β.

Examples:

1) iz is the rotation of z by $\frac{\pi}{2}$ (counterclockwise)
(since $i = 1(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$)

Notice that we knew before that $|zw| = |z|\cdot|w|$

2) Consider the following problem:
Let $P = (x,y) \in \mathbb{R}^2$. Find coordinates of $P' = (x', y')$, which is the result of rotation of P by angle α about the origin.



$x' =$
$y' =$

1.7.4 Roots

Consider equation $z^n = W$.
The n^{th} root of a complex number w is the set $\{z : z^n = w\}$. We denote it by $\sqrt[n]{w}$. Properties:

- If $w = 0$, then $\sqrt[n]{0} = \{0\}$
- if $w \neq 0$, then $\sqrt[n]{0}$ has exactly n elements

Examples:

- $\sqrt{-1} = \{i, -i\}$
- $\sqrt{4}^{(\mathbb{C})} = \{2, -2\}$ it may be a real or complex root

If $w = r(\cos\alpha + i\sin\alpha)$, then $z^n = w$ has exactly n solutions given by $z_k = \sqrt[n]{r}[\cos(\frac{\alpha+2k\pi}{n}) + i\sin(\frac{\alpha+2k\pi}{k})]$.

1.8 Polynomials

A (complex) polynomial is a function of the form: $W(z) = a_0 + a_1 z + ... + a_n z^n$, where $a_0, a_1, ..., a_n \in \mathbb{C}$ and are called coefficients. A degree of a nonzero polynomial is the largest n such that $a_n \neq 0$. For zero polynomial $W(z) = 0$ we agree that its degree is $-\infty$.
Examples:

1. $W(z) = (i+2)z^2 + z - i$
2. $W(z) = i + z^3$

If $a_0, a_1, ..., a_n \in \mathbb{R}$, then W(z) is called a real polynomial. We can add and multiply polynomials as functions. Moreover if W_1, W_2, W_3 are polynomials, then

- $W_1 + W_2 = W_2 + W_1$
- $(W_1 + W_2) + W_3 = W_1 + (W_2 + W_3)$
- $W_1 W_2 = W_2 W_1$

- $W_1(W_2W_3) = (W_1W_2)W_3 = W_1W_2W_3$
- $W_1(W_2 + W_3) = W_1W_2 + W_1W_3$

We can always divide two polynomials: $W(Z) : P(Z) = Q(z)$. If $W(Z), P(Z)$ are polynomials, then there are polynomials $Q(Z), R(Z)$ such that $W(Z) = P(Z)Q(Z) + R(Z)$

A complex number z_0 is a root of a polynomial $W(Z)$ if $W(z_0) = 0$.

1.8.1 Bezout Theorem

z_0 is a root of $W(z) \Leftrightarrow W(z)$ is divisible by z_0

1.8.2 Fundamental Theorem of Algebra

Every polynomial of degree ≥ 1 has at least one complex root. If $W(z)$ is a polynomial of degree $n \geq 1$, then there are complex numbers $z_1, z_2, ..., z_n$ such that $W(z) = A(z - z_1)(z - z_2)...(z - z_n)$

1.8.3 Rational Functions

A rational function is a function of the form $f(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials.
$f(x) = \frac{4x^2-x+2}{x-3ix^3}$.
If both P(x), Q(x) are real polynomials, then $f(x) = \frac{P(x)}{Q(x)}$ is called a <u>real rational function</u>.
A rational function $f(x) = \frac{P(x)}{Q(x)}$ is called a <u>proper rational function</u> if the degP < degQ.

Theorem

If $f(x) = \frac{P(x)}{Q(x)}$ is a rational function, then it can be written as a sum of a polynomial and a proper rational function.
Every real rational function is a sum of a real polynomial and a real proper rational function.

Partial Fraction

A partial fraction is a (proper) rational function of the form:

- $\frac{A}{(x-a)^n}$, $A \in \mathbb{R}$, $n \in \mathbb{N}$ 1st type
- $\frac{Ax+B}{(ax^2+bx+c)^m}$, $A, B, a, b, c \in \mathbb{R}$, $m \in \mathbb{N}$ 2nd Type

Partial fractions decomposition

Every proper rational function can be written as a sum of partial fractions. Moreover if $\frac{P(x)}{Q(x)}$ is a proper rational function then

- If $(x-a)^n$ is a factor of $Q(x)$ and $(x-a)^{n+1}$ is not, then in partial fractions decomposition we will get terms, $\frac{A_1}{(x-a)^1}$, $\frac{A_2}{(x-a)^2}$, ..., $\frac{A_n}{(x-a)^n}$
- If $ax^2 + bx + c$ with $\Delta < 0$ appears in decomposition of $Q(x)$ exactly n times
- No other terms will appear in the decomposition of $\frac{P(x)}{Q(x)}$

If $W_1(x) = a_0, a_1x + ... + a_nx^n$, $W_2(x) = b_0, b_1x + ... + b_nx^n$, then $W_1 = W_2 \leftrightarrow$
$\forall_{x \in \mathbb{R}} W_1(x) = W_2(x) \leftrightarrow$
$\forall_{k=0,1,2,...,n} a_k = b_k \leftrightarrow$
$\forall_{z \in \mathbb{C}} W_1(z) = W_2(x)$

Chapter 2

Matrices