

Mathematical Analysis

Lecture

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1 Rules

No textbook, so take notes.

Classes are mandatory

2 Requirements

During the classes we will start with a quiz, every practice. To pass the course you need 50% of points from the quizzed. A Quizes is 15min every quiz is worth 5 points. You get points from your top 10 quizzes.

3 Notation

3.1 Number sets

1. Natural Numbers $N = \{1, 2, 3, \dots\}$
2. Integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
3. Rational $\mathbb{Q} = \{\frac{p}{q}; p, q \in \mathbb{Z}, q \neq 0\}$
4. Irrational *ex.* : $\sqrt{2}, \pi, \dots$
5. Real Numbers $\mathbb{R} = \text{Rational} + \text{Irrational}$

3.2 Sets notation

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$(a, \infty) = \{x \in \mathbb{R} : x > a\}$$

$$(a, b) - \text{open interval}$$

$$[a, b] - \text{closed interval}$$

$A \subset B$ A is a subset of B

$x \in A$ X is an element of A, x belongs to A

$x \notin A$ X is not an element of A, x does not belong to A

3.3 Cartesian Product

Given two sets A and B, we can form the set consisting of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$. This set is called the Cartesian product of A and B and is denoted by $A \times B$

$A \times B = \{(a, b) : a \in A, b \in B\}$

If $A = B$, then $A \times A$ is denoted by A^2

3.4 Quantifiers

1. Existential \exists "There exists x such that", "For at least one x"
2. Universal \forall "For all x", "For each x", "For every x"

Example:

$$\exists t > 0 \forall x \in \mathbb{R} x^2 + 4x + 4 > t$$

The statement above is false

The negation of the statement:

$$\forall t > 0 \exists x_0 \in \mathbb{R} x_0^2 + 4x_0 + 4 \leq t$$

4 Functions

A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

In our class $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ The set A is called the domain of the function f and will be denoted D_f .

The range of the function f is the set of all possible values of $f(x)$ as x varies throughout the domain. The range of f will be denoted by R_f .

The most common method for visualizing a function is its.

If f is a function with domain D_f then its graph is the set of ordered pairs.

$$\{(x, y) \in \mathbb{R}^2 : x \in D_f, y = f(x)\}$$

Example:

Min function

$$\begin{aligned}
f(x) &= \min\{x, x^2\} \\
f(2) &= \min\{2, 4\} = 2 \\
f\left(\frac{1}{2}\right) &= \min\left\{\frac{1}{2}, \frac{1}{4}\right\} = \frac{1}{4}
\end{aligned} \tag{1}$$

Absolute

$$\begin{aligned}
f(x) &= |x| = \{x, \text{if } x \geq 0 \text{ or } -x, \text{if } x < 0\} \\
f(x) &= |x - 2| = \{x - 2, \text{if } x \geq 2 \text{ or } -(x - 2), \text{if } x < 2\}
\end{aligned} \tag{2}$$

$|x - a|$ represents the distance between x and a

4.1 The Vertical Line Test

A curve in the XY plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

4.2 Classes of functions

1. Periodic functions

We say that f is a periodic function if

$$\exists T > 0 \forall x \in D_f (x \pm T \in D_f \text{ and } f(x + T) = f(x))$$

A periodic function is a function that repeats its values after some determined period has been added to its independent variable.

2. Symmetric functions

• Even

A function f is called even if:

$$\forall x \in D_f (-x \in D_f) \text{ and } f(-x) = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the Y axis.

If f is even D_f is symmetric about the Y Axis.

• Odd

A function f is called odd if:

$$\forall x \in D_f (-x \in D_f) \text{ and } f(-x) = -f(x)$$

The graph of an odd function is symmetric about the origin.
 If an odd function is defined at $x=0$ then $f(0)$ must be 0!!

Example: Check if function is even or odd.

$$f(x) = \frac{3^x - 3^{-x}}{x}$$

- (a) Check if domain is symmetric

$$D_f = \mathbb{R} \setminus \{0\}$$

- (b) Substitute $-x$ for x

- Monotonicity:

A function is monotonic if it is increasing, or decreasing, or non-decreasing, or non-increasing.

- Increasing:

A function f is called increasing on a set $I \subset D_f$,
 if $\forall x_1, x_2 \in I [(x_1 < x_2) \Rightarrow (f(x_1) < f(x_2))]$

- Non-decreasing:

A function f is called increasing on a set $I \subset D_f$,
 if $\forall x_1, x_2 \in I [(x_1 < x_2) \Rightarrow (f(x_1) \leq f(x_2))]$

- Decreasing:

A function f is called increasing on a set $I \subset D_f$,
 if $\forall x_1, x_2 \in I [(x_1 < x_2) \Rightarrow (f(x_1) > f(x_2))]$

- Non-increasing:

A function f is called increasing on a set $I \subset D_f$,
 if $\forall x_1, x_2 \in I [(x_1 < x_2) \Rightarrow (f(x_1) \geq f(x_2))]$

Algebraic way to check monotonicity:

let $f(x) = \frac{1}{1+x^2}$ and $I = (-\infty, 0]$

Take any 2 points $x_1, x_2 \in I$ with $x_1 < x_2$.

$$f(x_2) - f(x_1) = \frac{1}{1+x_2^2} - \frac{1}{1+x_1^2} =$$

$$\frac{1+x_1^2-1+x_2^2}{(1+x_1^2)(1+x_2^2)} =$$

$$\frac{(x_1-x_2)(x_1+x_2)}{(1+x_2^2)(1+x_1^2)}$$

$$x_1 - x_2 < 0$$

$$x_1 + x_2 < 0$$

4.3 New functions from Old functions

- Vertical and horizontal shifts:

Suppose $c > 0$.

To obtain the graph of $y = f(x) + c$ shift the graph of y a distance of c units upwards. If $y = f(x) - c$ shift downwards.

To obtain the graph of $y = f(x - c)$ shift the graph y a distance of c units to the right.

To obtain the graph of $y = f(x + c)$ shift the graph y a distance of c units to the left.

- Vertical and horizontal stretching and reflecting:

Suppose $c > 1$.

To obtain the graph $y = c * f(x)$ stretch y vertically by a factor of c .

To obtain the graph $y = f(c * x)$ compress the graph of y horizontally by a factor of c .

To obtain the graph $y = -f(x)$ reflect the graph of y about the x axis.

To obtain the graph $y = f(-x)$ reflect the graph of y about the y axis.

- Algebra of functions:

let f and g be functions with domains D_f and D_g . Then the functions $f + g$, $f - g$, fg and $\frac{f}{g}$ are as follows:

$$(f \pm g)(x) = f(x) \pm g(x); D_{f \pm g} = D_f \cap D_g$$

$$(f * g)(x) = f(x) * g(x); D_{f * g} = D_f \cap D_g$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}; D_{\frac{f}{g}} = D_f \cap D_g$$

4.4 Composite Functions

Given 2 functions f and g the composite function is denoted by $f \circ g$ and is defined as $(f \circ g)(x) = f(g(x))$.

Example:

If $f(x) = \sqrt{2 - x}$ and $g(x) = \sqrt{x}$, then:

$$(f \circ g)(x) = f(g(x)) = \sqrt{2 - \sqrt{x}}$$

$$(g \circ f)(x) = g(f(x)) = \sqrt{\sqrt{2 - x}}$$

For a \sqrt{x} to be defined, we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have a $2 - \sqrt{x} \geq 0$ that is $\sqrt{x} \leq 2$ or $x \leq 4$. One can see that $D_{f \circ g} = [0, 4]$ therefore $D_{g \circ f} = (-\infty, 2]$

Let $h(x) = 3^{\sqrt{x+3}}$, write it as $f \circ g$:

$$f(x) = 3^x$$

$$g(x) = \sqrt{x+3}$$

$$(f^{-1} \circ f)(x) = x, \forall x \in D_f$$

$$(f \circ f^{-1})(x) = x, \forall x \in R_f$$

4.5 One-to-one functions

A function f is called an one-to-one function on a set $I \subset D_f$

$$\forall x_1, x_2 \in I [(x_1 \neq x_2) \Rightarrow (f(x_1) \neq f(x_2))]$$

Example:

- (Strictly) increasing function are 1-1
- Exponential functions are 1-1

4.5.1 Horizontal Line Test

A function f is one-to-one if and only if no horizontal line intersects the graph at most once

Let f be a 1-1 function with a domain D_f and a range R_f . Then its inverse function f^{-1} has a domain $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$ and is defined by:

$$(f^{-1})(y) = x \Leftrightarrow f(x) = y$$

Example:

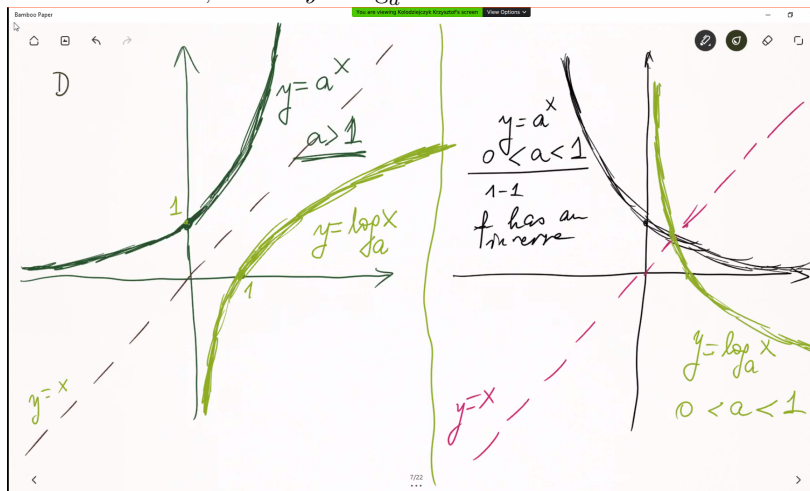
Let $g(x) = 3 + x + 2^x$. Is g invertible? Yes, because it's a strictly increasing function.

The inverse of $y = a^x$ is $y = \log_a x$ where $a > 1$.

The inverse of $y = a^x$ is $y = \log_a x$ where $0 < a < 1$.

5 Logarithms

The logarithm to the base a is defined as the inverse function of the exponential function with base a , that is $y = \log_a x$ means that $x = a^y$



5.1 Laws of logarithms

1. $\log_a(bc) = \log_a b + \log_a c$
2. $\log_a b^c = c * \log_a b$
3. $\log_a \frac{b}{c} = \log_a b - \log_a c$
4. $\log_a c = \log_a b * \log_b c$

6 Trigonometry

A standard position of an angle occurs when we place its vertex at the origin of a coordinate system and initial side on the positive x-axis.

A positive angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Negative angles are obtained by a clockwise rotation. Angles can be measured in radians. Mandatory for this class.

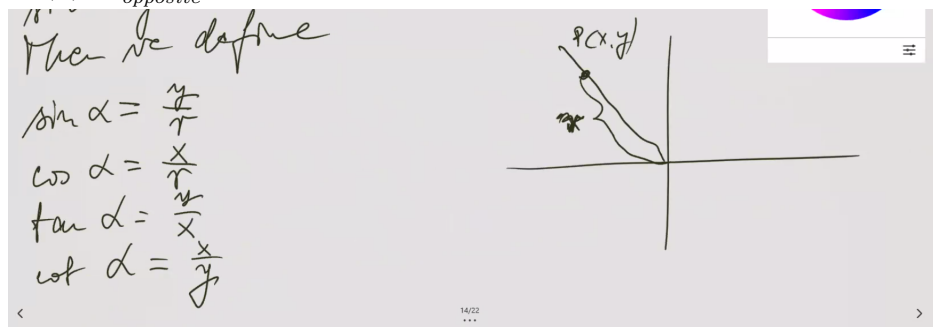
For an acute angle α the trigonometric functions are defined as ratios.

$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot(\alpha) = \frac{\text{adjacent}}{\text{opposite}}$$



The signs of the trig functions for angles in each of the quadrants can be remembered with: "All Students Take Calculus"

6.1 Trigonometric identities

We have a angle α and a point $P(x, y)$

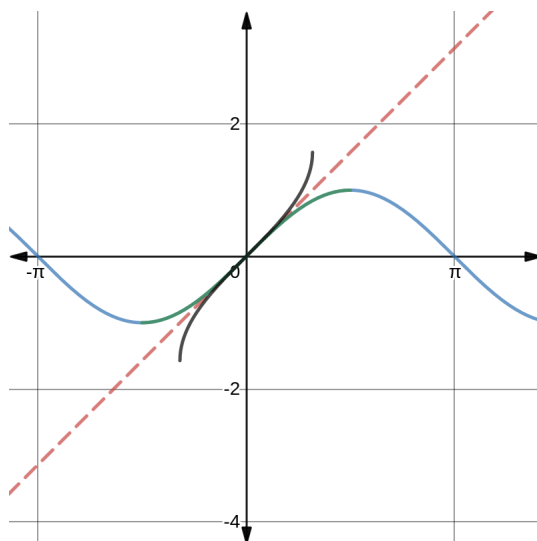
- $\forall \alpha \in \mathbb{R} \sin^2 \alpha + \cos^2 \alpha = 1$
- $\forall \alpha \in \mathbb{R} \sin(-\alpha) = -\sin \alpha$
- $\forall \alpha \in \mathbb{R} \cos(-\alpha) = \cos \alpha$
- $\forall \alpha \in \mathbb{R} \sin(\alpha + 2\pi) = \sin(\alpha)$ and $\cos(\alpha + 2\pi) = \cos(\alpha)$
- $\sin(x + y) = \sin(x) \cos(y) + \cos(y) \sin(x)$
- $\sin(x - y) = \sin(x) \cos(y) - \cos(y) \sin(x)$
- $\sin(2x) = 2 \sin(x) \cos(x)$
- If we denote $x + y = \alpha$ and $x - y = \beta$, then
 $\sin(\alpha) + \sin(\beta) = 2 \sin(\frac{\alpha+\beta}{2}) * \cos(\frac{\alpha-\beta}{2})$
- $\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$
- $\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y)$
- $\cos(\alpha) + \cos(\beta) = 2 \cos(\frac{\alpha+\beta}{2}) * \cos(\frac{\alpha-\beta}{2})$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$
- $\cos^2(x) = \frac{1+\cos(2x)}{2}$

6.2 Reduction formulas

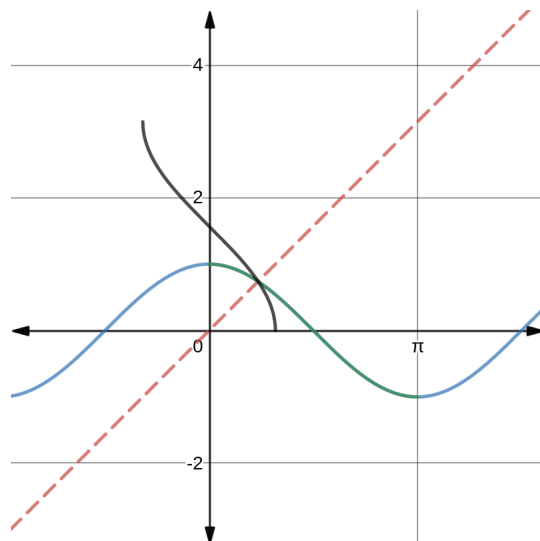
- $\sin(\alpha + k\frac{\pi}{2}) = \sin(\alpha)\cos(k\frac{\pi}{2}) + \cos(\alpha)\sin(k\frac{\pi}{2})$
when k is even $\pm \sin(\alpha)$
when k is odd $\pm \cos(\alpha)$
- $\cos(\alpha + l\frac{\pi}{2}) = \cos(\alpha)\cos(l\frac{\pi}{2}) - \sin(\alpha)\sin(l\frac{\pi}{2})$
when k is even $\cos(\alpha)$
when k is odd $\pm \sin(\alpha)$

6.3 Inverse of trigonometric functions / Cyclometric functions

The function $\sin(x)$ is not 1-1 but if we consider $f(x) = \sin(x)$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ as a 1-1 function The existing inverse is called arcsine $\arcsin(x) = y \Leftrightarrow \sin(y) = x$ and $-\frac{\pi}{2} \leq \frac{\pi}{2}$



The function $\cos(x)$ is not 1-1 but if we consider $f(x) = \cos(x)$ for $x \in [0, \pi]$ as a 1-1 function The existing inverse is called arc-cosine $\arccos(x) = y \Leftrightarrow \sin(y) = x$ and $0 \leq \pi$



Example:

Show that $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$

$\arcsin(x) = \alpha \Leftrightarrow \sin(\alpha) = x$ and $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$\arccos(x) = \beta \Leftrightarrow \cos(\beta) = x$ and $0 \leq \beta \leq \pi$

7 Sequences

A sequence can be thought of as a list of numbers written in a definite order ex.

a_1, a_2, \dots, a_n . The numbers have special names

a_1 is called the first term of the sequence. (a_n) infinite sequence of numbers,

$\{a_n\} = \{a_n : n \in \mathbb{N}\}$.

A sequence can be defined as a function whose domain is the set \mathbb{N} . But we usually write a_n instead of the function notation $f(n)$.

Sequences can be defined by:

1. By giving the formula for the nth term: $a_n = \frac{(-1)^{n+1}(n + \sqrt[3]{n})}{2^n}$
2. By a description: Let a_n be the digit in the n-th decimal place of the number $\sqrt{2}$
3. By a recursive relation: $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2} \forall n \geq 3$

7.1 Limit Laws for convergent sequences

If (a_n) and (b_n) are convergent sequences and c is a constant then:

1. $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$
2. $\lim_{n \rightarrow \infty} (c * a_n) = c \lim_{n \rightarrow \infty} (a_n)$
3. $\lim_{n \rightarrow \infty} (a_n * b_n) = \lim_{n \rightarrow \infty} (a_n) * \lim_{n \rightarrow \infty} (b_n)$
4. $\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \frac{\lim_{n \rightarrow \infty} (a_n)}{\lim_{n \rightarrow \infty} (b_n)}$ if $\lim_{n \rightarrow \infty} (b_n) \neq 0$
5. $\lim_{n \rightarrow \infty} (a_n)^p = (\lim_{n \rightarrow \infty} a_n)^p$
6. $\lim_{n \rightarrow \infty} (\sqrt[p]{a_n}) = \sqrt[p]{\lim_{n \rightarrow \infty} (a_n)}$
7. $\lim_{n \rightarrow \infty} (\sqrt[p]{a}) = 1$
8. $\lim_{n \rightarrow \infty} (\sqrt[p]{n}) = 1$

7.1.1 Squeeze Theorem for sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$.

and

$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$

A sequence is bounded above if there is a number such that $a_n \leq M$

7.2 Monotonic sequence theorem

Every bounded and monotonic sequence is convergent (has a limit).

7.3 Infinite Limits

$\lim_{n \rightarrow \infty} a_n = \infty$. This means that for every positive number M there is an integer n_0 such that $a_n > M$ whenever $n > n_0$.

$$\lim_{n \rightarrow \infty} a_n = \infty \Leftrightarrow \forall_{M>0} \exists_{n_0 \in \mathbb{N}} \sqrt{n} > M$$

7.3.1 Infinity Theorem

1. $a + \infty = \infty$
2. $a\infty =$