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Functions Homework

Problem 4

Everything that goes through f gets mapped to c, thus we get

$$g \circ f = a$$

$$f \circ g = c$$

Problem 6

We first calculate $g \circ f$. We do this by plugging in the result of f into g.

$$g \circ f = 3(\frac{1}{x^2+1}) + 2 = \frac{3}{x^2+1} + 2$$

To calculate $f \circ g$, we do the opposite, plugging the result of g into f.

$$f \circ g = \frac{1}{(3x+2)^2+1} = \frac{1}{9x^2+12x+5}$$

Problem 8

We do essentially the same thing, just pluggin in the results of f to g and vice versa.

$$\begin{aligned} g \circ f &= \{5(3m-4n)+2m+n, 3m-4n\} \\ &= \{15m-20n+2m+n, 3m-4n\} \\ &= \{17m-19n, 3m-4n\} \\ f \circ g &= \{3(5m+n)-4m, 2(5m+n)+m\} \\ &= \{11m+3n, 11m+2n\} \end{aligned}$$

Problem 4

For sake of visual clarity, we can find the inverse by substituting x for y and $f(x)$ for x.

$$x = e^{y^3+1}$$

$$\ln x = y^3 + 1$$

$$y = (\ln x - 1)^{\frac{1}{3}}$$

We get that the inverse is:

$$f^{-1}(x) = (\ln x - 1)^{\frac{1}{3}}.$$

Problem 8

The function is bijective, because for every result, there is always an X to get it, and if two results are the same, that must mean that the initial sets are the same. Furthermore

$\theta^{-1} = \theta$ as in that $\theta^{-1}(X) = \overline{X}$.

Problem 2

The images are:

$$\begin{aligned} f(\{1,2,3\}) &= \{3,8\} \\ f(\{4,5,6,7\}) &= \{1,2,4,6\} \\ f(\emptyset) &= \emptyset \\ f^{-1}(\{0,5,9\}) &= \emptyset \\ f^{-1}(\{0,3,5,9\}) &= \{1,3\} \end{aligned}$$

Problem 8

One such counterexample would be the function $f(x) = x$.

Problem 10

To prove this, we have to split the problem into two parts:

$$f^{-1}(Y \cap Z) \subseteq f^{-1}(Y) \cap f^{-1}(Z)$$

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To do the first part, we split the problem into four cases.

$$x \in Y, x \notin Z \tag{1}$$
$$x \notin Y, x \in Z \tag{2}$$
$$x \in Y, x \in Z \tag{3}$$
$$x \notin Y, x \notin Z \tag{4}$$

In the first and second and fourth case, we get that $x \notin Y \cap Z$, and thus we're done.
In the third case, we get that $x \in Y \cap Z$, thus $f^{-1}(x)$ is on the left side, however, we also get that $f^{-1}(x)$ is in $f^{-1}(Y) \cap f^{-1}(Z)$, so we're good, as $f^{-1}(x)$ belongs in both sides.
For the second part, we again use the split cases.
In the first, second, and fourth case, we get that $f^{-1}(x) \notin f^{-1}(Y) \cap f^{-1}(Z)$, and we're done
In the third case, we get that $f^{-1}(x) \in f^{-1}(Y) \cap f^{-1}(Z)$, but we also get that $f^{-1}(x) \in f^{-1}(Y \cap Z)$. We're fine, as$f^{-1}(x)$ belongs on both sides.
Thus we're done, as we've proved both sides.

Problem 12

If f is not injective, then $X \neq f^{-1}(f(X))$ for all $X \subseteq A$. This is because, if f is not injective, that must mean there are some $a, b \in A$ such that $f(a) = f(b)$. Here we can choose $X = \{a\}$. This means that $f^{-1} = \{a, b, \cdots\}$, which is not equal to the original X. On the other hand, if f is injective, we get that for every $x \in X$, $f(x)$ is unique.

This is important because this means that $f^{-1}(f(x))$ will result in x. Thus we get that $X = f^{-1}(f(X))$. Thus we proved the if and only if, as if f is not injective, it results in $X = f^{-1}(f(X))$ being false, while if f is injective, it results in $X = f^{-1}(f(X))$ being true.
For the second problem we do it the same way. If f is not surjective, this means that for some $y \in Y$, $f^{-1}(y)$ does not exist. This means that we can have $Y = \{y\}$, and then $f(f^{-1}(Y)) = \emptyset$, which is not equal to Y. Then to prove the other way, we look at if f is surjective, that would mean that for every $y \in Y$, $f^{-1}(y)$ exists, which means that $f(f^{-1}(y)) = y$. Thus we have proved both ways.

Problem 14

The statement is true, consider $x \in Y$, we can have X be the set of all points such that for $\forall a \in X$, $f(a) = x$. In other words, X is pre-image of just a singular point $x \in Y$. Now we have to prove equivalence. For the left side, we get that $X \in f^{-1}(Y)$, we then get that $f(X) = x$, then finally we get that $f^{-1}(x) = X$. Thus we get X on the left side, for the right side, we also get X, thus we're done. I like latex on vim.