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THE OPTIMUM STRATEGY IN BLACKJACK

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This article discusses the card game blackjack as played in the casinos of Las Vegas. The basic rules for the game are described in detail. The player's strategic problems are analyzed with the objective of finding the strategy maximizing his mathematical expectation.

A mathematical expression is derived giving a general solution to the player's problem of standing pat with a given hand versus drawing additional cards. No general solutions are possible for the other major strategic problems, however, and a detailed examination of individual situations is required. The formulas and methods for the case analysis are stated, but computational details are omitted. Similarly, the formula for the player's mathematical expectation is stated, but its numerical evaluation is not described. Detailed discussion is given to the problems arising in the combinatorial type of computations required by blackjack.

The "optimum strategy" determined by the above analysis differs substantially from the published strategies of card experts and the usual style of play in the casinos.

I. THE GAME OF BLACKJACK

BLACKJACK or twenty-one is one of the most widely played games in American homes and clubs and traditionally rivals poker for popularity in the Armed Forces. In Las Vegas, Reno, and other parts of the wide-open spaces blackjack ranks with poker, roulette, and craps as one of the four standard gambling games. Of these four, however, blackjack is by far the most neglected in the scientific literature of gambling and offers a relatively unexplored area for mathematical and statistical analysis.

It should be made clear at the outset that this paper deals exclusively with the "house" game of blackjack and not the "private" game. In the house game a representative of the gambling casino is permanent dealer, and his strategy is completely fixed by known house rules. The fixed and known nature of the dealer's strategy is vital in reducing the mathematical and computational problems in analyzing blackjack to manageable proportions.

Each gambling casino has a set of blackjack rules which agree with those of other casinos on the main points but which usually differ on details. Therefore, in selecting a variation of the game of blackjack for analysis, the best that could be done was to consider rules which are common but not universal. A presentation of these rules follows.

* The authors were ably assisted by Otto Dysktra, General Foods Laboratories, Hoboken, N. J.

1. *The Number of Players.* A dealer and from one to six players.
2. *The Pack.* An ordinary 52-card deck.
3. *Betting.* The players make their bets *before* any cards are dealt. The house establishes a minimum and a maximum bet.
4. *The Deal.* The players and dealer each receive two cards. Each player gets both cards face down. The dealer receives *one card face up* and one card face down. Cards received face down in the deal or draw are commonly known as "hole cards."
5. *The Numerical Value of the Cards.* The numerical value of an ace is 1 or 11 as the player chooses, the numerical value of a face card is 10, and the numerical value of all other cards is simply their face value. The numerical value or total of a hand is the sum of the numerical values of the cards in the hand.
6. *Object of the Player.* To obtain a total which is greater than the dealer's but does not exceed 21.
7. *Naturals.* An ace and a face card or ten dealt on the first two cards to either player or dealer constitutes a "natural" or "blackjack." If a player has a natural and the dealer does not, the player receives $1\frac{1}{2}$ times his original bet from the dealer. If a player does not have a natural and the dealer does, the player loses his original bet. If both player and dealer have naturals, no money changes hands.
8. *The Draw.* A player is not required to increase the number of cards in his hand and may look at his hole cards and elect to "stand." Otherwise, he may require that the dealer give him additional cards, face up, one at a time. If the player goes over 21 ("busts"), he immediately turns up his hole cards and pays his bet to the dealer. After each player has drawn his cards, starting with the player at the dealer's left and proceeding in a clockwise fashion, the dealer turns up his hole card. If his total is *16 or less*, he must draw a card and continue to draw cards until his total is *17 or more*, at which point he must stand. If the dealer has an ace, and counting it as 11 would bring his total to 17 or more without exceeding 21, he must count the ace as 11 and stand.
9. *The Settlement.* If the player does not go over 21 ("bust") and the dealer does, the player wins an amount equal to his original bet. If neither player nor dealer busts, the person with the higher total wins an amount equal to the player's original bet. If neither player nor dealer busts and both have the same total, no money changes hands.
10. *Splitting Pairs.* In the following rules a pair is defined as two cards which are identical except for suit, such as two jacks, two aces, or two tens. If the player's hole cards form a pair, he may choose to turn them face up and treat them as the initial cards in two separate "twin" hands. This strategy is known as "splitting pairs." The original bet goes on one of the split cards, and an equal amount is bet on the other card. The player automatically receives a second card face down on each of the split cards and may continue drawing cards face up to both twin hands as long as he desires. An exception to this rule is made in the case of split aces where the player may draw only *one* more card to each ace. Furthermore, if a face card or ten falls on one of the split aces, the hand is not counted as a natural but as ordinary 21. (Similarly, the player splitting a pair of face cards or tens who draws an ace holds an ordinary

21.) Finally, if a player splits a pair and receives a third card of the same type, he is not permitted to undertake further splitting.

11. *Doubling Down.* After looking at his hole cards a player may elect to double his bet and draw one and only one more card. This strategy is known as "doubling down." A player who elects to double down turns up his hole cards and receives his third card face down. A player splitting any pair except aces, after receiving an additional card on each of the split cards, may elect to double down on one or both of his twin hands.

The reader who is primarily interested in card playing results should skip to Section VI for a description of the optimum strategy. Section II gives the derivation of the basic mathematical formulas; Section III discusses the probability the dealer obtains various final totals; Section IV describes the methods of analysis for special situations—soft hands, doubling down, and splitting pairs; and Section V presents the method of calculating the player's mathematical expectation.

II. THE BASIC "DECISION EQUATION"

Consider two types of hands held by the player: the type of hand where the player's total x has one unique value not exceeding 21, and the type of hand consisting of one or more aces in such a way that the player's total has two values not exceeding 21. (In this situation x is defined to equal the larger total.) The second type of hand is known in gambling terminology as a "soft" hand or "soft" total and requires a separate strategy.

Some notation must be defined. Let D be the numerical value of the dealer's up card. $D=2, 3, \dots, 10, (1, 11)$. Let $M(D)$ be an integer such that if the dealer's up card is D and the player's total x is unique and less than $M(D)$, the player should draw; while if x (unique) $\geq M(D)$, the player should stand. The set of integers $M(D)$ are known as the minimum standing numbers for unique hands. Let us define $M^*(D)$ in the same way for soft hands.

The assumption is made that a good strategy for drawing may be defined by the set of minimum standing numbers $M(D)$ and $M^*(D)$. In other words, if it is good strategy for a player to stand on a given total, it is assumed to be good strategy for him to stand on all higher totals. This assumption is almost always correct and, in particular, holds true when player and dealer draw from a full deck. The assumption breaks down in certain "pathological" cases, however, when draws are made from a severely depleted deck with an unusual assortment of cards remaining.

The first step is to compare the mathematical expectation of player 1 using $M(D)=x$ with that of player 2 using $M(D)=x+1$ where x takes on integral values not exceeding 21. The comparison is $M^*(D)=x$ versus $M^*(D)=x+1$ in the case of soft hands. Players 1 and 2 employ the same strategy except when their total is x . Player 1 stands in this situation, while in the case of unique x player 2 draws exactly one more card. Thus a comparison of the mathematical expectation of players 1 and 2 for unique hands is equivalent to comparing $E_{s,x}$, the expectations of a player standing on a total of x , with $E_{d,x}$, the expectation of a player with a total of x who draws *exactly one* card. In the case of soft hands player 1 stands, while player 2 draws *one or more* cards. For example,

if player 2 draws a five to soft 17, obtaining a total of unique 12, in most cases he should draw again. Thus the comparison of $E_{s,x}$ and $E_{d,x}$ in the case of soft hands is *not* equivalent to a comparison of the mathematical expectations of players 1 and 2. We shall use the result, however, that if $E_{d,x} > E_{s,x}$, the player should draw to soft x .

When, for unique x , $E_{d,x} - E_{s,x}$ is a non-increasing function of x , $M(D)$ is easily obtained as the smallest integral value of x for which $E_{d,x} - E_{s,x} < 0$. The function is non-increasing in almost all cases, including the case of drawing from a full deck, and increases with x only in certain of those "pathological" situations previously discussed.

The following derivation of $E_{d,x} - E_{s,x}$ holds for both unique and soft hands. Let us define T , a random variable, as the final total obtained by the dealer. If $T > 21$ or if $T < x$, the player standing on x wins the bet, assumed here to be one unit. If $T = x$, no money changes hands, while if $x < T \leq 21$, the player loses one unit. Consequently,

$$\begin{aligned} E_{s,x} &= P(T > 21) + P(T < x) - P(x < T \leq 21) \\ &= 2P(T > 21) - 1 + 2P(T < x) + P(T = x). \end{aligned}$$

In discussing $E_{d,x}$ one must define a second random variable, J , as the total obtained by the player upon drawing one card. In cases where this total can take on two values not exceeding 21, J represents the larger total.

$T \geq 17$, so if $J < 17$, the player wins when $T > 21$ and loses for all other values of T . His mathematical expectation in this situation is

$$P(T > 21) - [1 - P(T > 21)] = 2P(T > 21) - 1.$$

If $17 \leq J \leq 21$, the player's mathematical expectation is

$$P(T > 21) + P(T < J) - P(J < T \leq 21).$$

If $J > 21$, the player's mathematical expectation is -1 .

The value of J affects T only through eliminating the possibility that the dealer draws one particular card of value $J - x$. Consequently, little error is introduced in making the assumption that J and T are independent and writing

$$\begin{aligned} E_{d,x} &= P(J < 17)[2P(T > 21) - 1] - P(J > 21) \\ &\quad + \sum_{j=17}^{21} P(J = j)[P(T > 21) + P(T < j) - P(j < T \leq 21)]. \end{aligned}$$

Subtracting off $E_{s,x}$ and performing some straightforward algebraic manipulation

$$\begin{aligned} E_{d,x} - E_{s,x} &= -2P(T < x) - P(T = x) - 2P(T > 21)P(J > 21) \\ &\quad + 2P(T < J \leq 21) + P(T = J \leq 21). \end{aligned}$$

This is the most general form of the decision equation.

Since $T \geq 17$, the first two terms are zero for $x < 17$. $P(J > 21)$ is also zero for x unique and less than 12 and for all values of soft x . Consequently,

$E_{d,x} - E_{s,x} \geq 0$ for $x(\text{unique}) < 12$ and $x(\text{soft}) < 17$, and, therefore, $M(D) > 11$ and $M^*(D) > 16$ for all D .

Let us consider next the evaluation of the decision equation when $12 \leq x(\text{unique}) \leq 16$. The first two terms are zero, and the last term may be re-written using the independence of J and I .

$$E_{d,x} - E_{s,x} = -2P(T > 21)P(J > 21) + \sum_{t=17}^{21} P(T=t) [2P(t < J \leq 21) + P(J=t)].$$

An assumption is introduced at this point that the probability distribution of $J-x$, the single card drawn by the player, is given by $P(J-x=10)=4/13$ and $P(J-x=i)=1/13$ $i=2, 3, \dots, 9, (1, 11)$, i.e., each card in the deck has an equal chance of being drawn by the player. This assumption may be incorrect in individual hands but holds true in the sample space composed of all 52! permutations of the deck. With this assumption

$$P(J > 21) = 1/13(x-8) \text{ for } x(\text{unique}) \geq 12, \text{ and}$$

$$P(t < J \leq 21) = 1/13(21-t), \quad P(J=t) = 1/13 \text{ for } 17 \leq t \leq 21. \text{ Then}$$

$$E_{d,x} - E_{s,x} = -2/13(x-8)P(T > 21) + \sum_{t=17}^{21} 1/13(43-2t)P(T=t).$$

It is not necessary to compute $E_{d,x} - E_{s,x}$ for all values of $12 \leq x \leq 16$. One may set $E_{d,x} - E_{s,x} = 0$, and, since the function decreases linearly with x , obtain a single solution, $x = x_0$.

$$x_0 = 8 + \frac{\sum_{t=17}^{21} (21\frac{1}{2} - t)P(T=t)}{P(T > 21)}.$$

Then if $x_0 < 12$, $M(D) = 12$; if $x_0 > 16$, $M(D) > 16$; and if $12 \leq x_0 \leq 16$, $M(D) = [x_0] + 1$ where, in general, $[z]$ is defined to be the largest integer not greater than z . As one might expect, the greater the probability that the dealer busts, the lower the player's minimum standing number. Less obvious is the result that for a given value of $P(T > 21)$ the greater the dealer's chances for a good hand, the lower the player's minimum standing number. For example, if $P(T > 21) = 2/5$ and $P(T = 18) = 3/5$, $M(D) = 14$; while if $P(T > 21) = 2/5$ and $P(T = 19) = 3/5$, $M(D) = 12$.

In the case where $x(\text{unique}) = 17$

$$E_{d,17} - E_{s,17} = -18/13P(T > 21) - 5/13P(T = 17) + \sum_{t=18}^{21} 1/13(43-2t)P(T=t).$$

Since an evaluation of $P(T=t)$ will show that $E_{d,17} - E_{s,17} < 0$ for all D , and consequently $M(D) \leq 17$, it is not necessary to make any further evaluations of the decision equation for unique hands.

In the case of soft hands the only remaining use for the decision equation arises where $x(\text{soft}) = 17$. In that situation

$$E_{d,17} - E_{s,17} = -1/13P(T=17) + \sum_{t=18}^{21} 1/13(43-2t)P(T=t)$$

which an evaluation of $P(T=t)$ will show is positive for all D , proving that $M^*(D) > 17$.

III. EVALUATION OF THE DEALER'S PROBABILITIES, $P(T=t)$.

Calculation of x_0 and the above expressions for $E_{d,17} - E_{s,17}$ for unique and soft 17 clearly requires only an evaluation of the dealer's probabilities, a task accomplished in three stages. In the first stage an exact evaluation was made of $P(T_3=v)$, the probability that the dealer obtains a total of v on his first three cards. These numbers, known as "three-card probabilities," were computed separately for each value of D , the dealer's first card. In cases where the rules required the dealer to stand on two cards, the probabilities for the totals thus obtained were included with the three-card probabilities.

In the second stage a table was developed which gives approximate values for $P(T=t/T_p=t_p)$, the conditional probability that the dealer obtains a final total t ($t \geq 17$) given a partial total t_p ($t_p < 17$). The table was developed under the following simplifying assumptions: (1) the probability of drawing any card in the deck is $1/52$ ("equiprobability"); and (2) no matter how many cards the player draws, the probability of receiving any particular card on the next draw is still $1/52$ ("sampling with replacement").

In the third stage the results from the previous stages are combined yielding the following approximation for $P(T=t)$ for $t \geq 17$.

$$P(T=t) = P(T_3=t) + \sum_{j < 17} P(T_3=j)P(T=t/T_p=j).$$

One may wonder about the accuracy of the approximate values for $P(T=t)$. Unfortunately, calculation of exact values is an exceedingly laborious task which has been completed in only two cases, $D=6$ and 10. The comparison of the exact and approximate probabilities for $D=6$ shows considerably greater error in the approximate probabilities than in the case $D=10$. Furthermore, some complicated heuristic arguments not presented here indicate that the error for any D will not be appreciably greater than for the case $D=6$ given below.

t	17	18	19	20	21	>21
$P(T=t)$ Exact	.166948	.106454	.107192	.100705	.097878	.420824
$P(T=t)$ Approx.	.167625	.107234	.108017	.101260	.098364	.417499

The error in each approximate probability in the above table is less than 1% of the corresponding exact probability. Assuming that the errors are less than 1% for all D , an examination of the expression for x_0 previously derived indicates that the maximum possible error in x_0 is 2% of the second term. A study of the results for x_0 shows that $D=10$ is the only case where such an error might affect $M(D)$. In this case $x_0=16.01$ when calculated with approximate probabilities and 15.97 with exact. Since these values bracket 16.00, a more detailed analysis was made which in considering the probability distribution of $J-x$ assumed that one ten-counting card was withdrawn from the full deck. With this

more realistic assumption $x_0 > 16$ and $M(10) = 17$. It is easily shown, furthermore, that a maximum error of 1% in each value of $P(T=t)$ cannot possibly affect the conclusions that $M(D) \leq 17$ and $M^*(D) > 17$.

A table of approximate dealer's probabilities, $P(T=t)$, is given below for all values of t and D .

		t					natural	
	17	18	19	20	21	21	>21	
D	2	.141781	.134885	.131432	.123829	.119581	.000000	.348492
	3	.133533	.133052	.126197	.122563	.114903	.000000	.369751
	4	.132206	.116037	.122553	.117930	.114292	.000000	.396983
	5	.121374	.124511	.117753	.105446	.107823	.000000	.423092
	6	.167625	.107233	.108018	.101260	.098364	.000000	.417499
	7	.372743	.139017	.077841	.079409	.073437	.000000	.257552
	8	.131202	.363359	.129634	.068457	.070026	.000000	.237322
	9	.122256	.104217	.357550	.122256	.061079	.000000	.232643
	10	.114756	.113186	.114756	.328873	.036324	.078431	.213674
	(1, 11)	.128147	.131284	.129716	.131284	.051284	.313725	.114560

When $D=10$ or $(1, 11)$, the dealer immediately looks at his hole card. If he holds a natural, he announces it at once and proceeds with the settlement. Consequently, if the dealer holds a natural, the player has no opportunity to draw, double down, or split pairs. As a result the player examines these decisions secure in the knowledge that the dealer does not have a natural. Therefore, the important dealer's probabilities for $D=10$ and $(1, 11)$ are $P(T=t/T \neq \text{natural } 21)$, the conditional probabilities for various outcomes t given that the dealer does not obtain a natural. They are given below.

		t					natural	
	17	18	19	20	21	21	>21	
D	10	.124522	.122819	.124522	.356862	.039415	.000000	.231859
	(1, 11)	.186728	.191299	.189015	.191299	.074728	.000000	.166930

The entries in the tables of approximate values for $P(T=t)$ have been carried to six places to facilitate a comparison with the exact probabilities in the two cases mentioned and to improve certain numerical checks. Actually, the digits in the fourth and higher places are relatively meaningless and have been dropped in the final results for the mathematical expectation.

IV. METHODS OF ANALYSIS FOR SPECIAL SITUATIONS

A. Soft Hands

The method of analysis for all special situations requires tables of $P(H=h/H_p=h_p)$, the conditional probability the player obtains a final total of $h \{h \geq M(D)\}$ given that he has a partial total of $h_p \{h_p(\text{unique}) < M(D), h_p(\text{soft}) < M^*(D)\}$ and draws or stands following $M(D)$ and $M^*(D)$. The tables were worked out with the same assumptions of equiprobability and sampling with replacement as in the corresponding tables of conditional probabilities for the dealer. Separate tables were required for each of the following four cases: $D=2, 3$ where $M(D)=13$ and $M^*(D)=18$; $D=4, 5, 6$ where $M(D)=12$ and $M^*(D)=18$; $D=7, 8, (1, 11)$ where $M(D)=17$ and $M^*(D)=19$; and $D=9, 10$ where $M(D)=17$ and $M^*(D)=19$.

In the case of soft hands, the decision equation has been used to show that

$M^*(D) > 17$, and subsequent analysis need only compare the mathematical expectation of player 1 with $M^* = x$ versus that of player 2 with $M^* = x + 1$ for $x \geq 18$. The two players will use the same strategy except where their total is soft x . In this situation player 1 stands and player 2 draws one card and continues to draw so long as his total is less than $M(D)$. A comparison of the mathematical expectations of player 1 and 2 for soft hands is equivalent to comparing $E_{s,x}$, the expectation of a player standing on a total of x , with $E_{d^*,x}$, the expectation of a player with a soft total of x who draws one card and then draws or stands according to $M(D)$. $E_{s,x}$ has already been calculated. Adopting the convention that $E_{s,k} = -1$ for $k > 21$, $E_{d^*,x}$ is given by

$$E_{d^*,x} = \sum_{j \geq M} P(J = j) E_{s,j} + \sum_{j < M} P(J = j) \sum_{h \geq M} P(H = h/H_p = j) E_{s,h}.$$

While x does not appear explicitly in the right hand side above, its value affects $P(J = j)$. Since $E_{s,19} > E_{d^*,19}$ for all D , it was not necessary to evaluate $E_{s,x}$ and $E_{d^*,x}$ for $x \geq 20$.

B. Doubling Down

The player who elects to double down on a total of x turns up his hole cards, doubles his bet, and receives one and only one card face down. His expectation then is $2 E_{d,x}$. To determine whether he should double down the player must compare this expectation with $E_{M,M^*;x}$, the mathematical expectation of the player with a total of x who follows the drawing strategy given by M and M^* . Thus the player will double down if and only if $2 E_{d,x} - E_{M,M^*;x} > 0$. But $E_{M,M^*;x} \geq E_{d,x}$, therefore

$$2E_{d,x} - E_{M,M^*;x} \geq 2E_{d,x} - E_{d,x} = E_{d,x}.$$

This result and the equation for $E_{d,x}$ can be used to show immediately that doubling down is poor strategy where $x(\text{unique}) > 11$ or < 8 for all D and $x(\text{soft}) < 17$ for $D > 6$ and $D = (1, 11)$. Many combinations of D and x need to be examined individually, however, using the following formula for $E_{M,M^*;x}$.

$$E_{M,M^*;x} = \begin{cases} E_{s,x} & x(\text{unique}) \geq M \text{ or } x(\text{soft}) \geq M^* \\ \sum_{h \geq M} P(H = h/H_p = x) E_{s,h} & \text{otherwise.} \end{cases}$$

Define $X = X(D)$ as the set of values of x for which the player should double down when the dealer's up card is D .

C. Splitting Pairs

In the case of splitting a pair of y 's a separate analysis is required for every combination of y and D . The results for doubling down must be used inasmuch as a player may split a pair and subsequently double down. The player will split his y 's if and only if $E_{\text{split},y} > E_{\text{no-split},y}$ where

$$E_{\text{no-split},y} = \begin{cases} 2E_{d,2y} & 2y \in X \\ E_{M,M^*;2y} & 2y \notin X \end{cases}$$

$$\frac{1}{2} E_{\text{split},y} = \sum_{j \in X} P(J = j) 2E_{d,j} + \sum_{j \notin X} P(J = j) E_{M,M^*;j}.$$

In the special case where $y = (1, 11)$, the player splitting is allowed to draw only one more card and cannot double down. Thus $\frac{1}{2}E_{\text{split}, (1, 11)} = \sum_{\text{all } j} P(J=j)E_{s, j}$. Let $Y=Y(D)$ denote the values of y for which a pair of y 's should be split when the dealer's up card is D .

V. THE PLAYER'S MATHEMATICAL EXPECTATION

The mathematical expectation of the player is given by

$$E(W) = 1/13 \sum_{D \neq 10} E(W_D) + 4/13 E(W_{10}),$$

where W_D is the amount won by the player on a single hand when the dealer's up card is D . When $D=10$ and $(1, 11)$ one must calculate $E(W_D)$ under two conditions: (1) given the dealer does not obtain a natural, and (2) given that he does. These conditional expectations are then multiplied by the probabilities for events (1) and (2) to obtain $E(W_D)$.

The first step in obtaining $E(W_D)$ is to calculate the probabilities for the various hands formed by the player's two hole cards. It is assumed that the hole cards are drawn from a deck which is complete except for one D -counting card. In the case where $D=10$ or $(1, 11)$ and the dealer has a natural

$$E(W_D) = -1[1 - P(\text{the hole cards form a natural})].$$

In all other cases $E(W_D)$ is the sum of the following four terms:

- (1) $\frac{1}{2} P(\text{the hole cards form a natural}),$
- (2) $\sum_{y \in Y} P(\text{the hole cards are a pair of } y\text{'s}) E_{\text{split}, y},$
- (3) $\sum_{j \in X} P(\text{the hole cards total } j) 2E_{d, j},$
- (4) $\sum_{j \notin X} P(\text{the hole cards total } j) E_{M, M^*, j}.$

In the last two sums it is understood that the hole cards do not form a natural or a pair of y 's with $y \in Y$.

VI. DESCRIPTION OF THE OPTIMUM STRATEGY

The player's basic problems of strategy are (a) when to draw and stand, (b) when to double down, and (c) when to split pairs. These problems will be treated in order. A fourth subsection on the player's mathematical expectation is also included.

(a) *Drawing Strategy.* Ordinarily, the player's hand has one unique total not exceeding 21. However, sometimes the player will hold a hand with two possible totals not exceeding 21, e.g., an ace and a five is either 6 or 16. In gambling terminology this ambiguous type of hand is known as "soft" and requires a separate strategy.

Some notation will facilitate the description of the optimum strategy for drawing. Let D be the numerical value of the dealer's up card. $D=2, 3, \dots, 10, (1, 11)$. Let $M(D)$ be an integer such that if the dealer's up card is D and the player's total is unique and less than $M(D)$, the player should draw; while if

the player's total is unique and greater or equal to $M(D)$, the player should stand. The ten integers $M(D)$ are known as the minimum standing numbers for unique hands. Let us define $M^*(D)$ in the same way for soft hands with the understanding that "player's total" means the larger of the two possible totals. The optimum strategy may now be described as follows:

$$M(D) = \begin{cases} 13 & D = 2, 3 \\ 12 & D = 4, 5, 6 \\ 17 & D \geq 7, D = (1, 11) \end{cases} \quad M^*(D) = \begin{cases} 18 & D \leq 8, D = (1, 11) \\ 19 & D = 9, 10 \end{cases}$$

The most surprising aspect of the optimum strategy for unique hands is the low values of $M(D)$ for $D \leq 6$. Few experienced players recommend standing on a total of 13 under any circumstances, and standing on 12 would be completely out of the question. Culbertson, Morehead, and Mott-Smith, [1] for example, propose a strategy of $M(D) = 14$ for $D \leq 6$ and $M(D) = 16$ for $D \geq 7$ and $D = (1, 11)$. Also unexpected is the great "discontinuity" in $M(D)$ as D goes from 6 to 7. The optimum strategy for soft hands is not particularly surprising except perhaps for $D = 9, 10$. Most experts, including Culbertson et al., suggest $M^*(D) = 18$ for all D .

(b) *Doubling Down*. The optimum strategy is given in the following table.

Player's <i>Unique</i> Two Card Total	≥ 12	11	10	9	≤ 8
Values of D Where the Player Should Double Down	none	$2 \leq D \leq 10$	$2 \leq D \leq 9$	$2 \leq D \leq 6$	none
Player's <i>Soft</i> Two Card Total	≥ 19	18	17	13-16	12
Values of D Where the Player Should Double Down	none	$D = 4, 5, 6$	$D = 3, 4, 5, 6$	$D = 5, 6$	$D = 5$

Note that two card soft 12 always consists of two aces. While the above table shows it is good strategy to double down on soft 12 when the dealer shows a five, more detailed analysis shows that it is even better strategy to split the aces. Similarly, while it is good strategy to stand rather than draw on two card soft 18 when $D = 4, 5, 6$, it is even better strategy to double down.

The fact that the player should double down so frequently may surprise many people. Doubling down is not common in the casinos, and, in particular, the idea of doubling down on soft hands probably does not occur to most players. Doubling down is not recommended by Culbertson et al. nor by any other writers on Blackjack encountered by the authors [2, 3, 4]. One may even go so far as to say that doubling down is the most neglected, under-rated aspect of Blackjack strategy.

(c) *Splitting Pairs*. The optimum strategy is given in the following table.

Type of Pair	ace's, 8's	9's	7's	6's, 3's, 2's	4's	10's, 5's, face cards no values
Values of D Where all values The Pair is Split		$2 \leq D \leq 6$ $D = 8, 9$	$2 \leq D \leq 8$	$2 \leq D \leq 7$	$D = 5$	

The fact that the optimum strategy calls for splitting aces will evoke no surprise as this is common strategy in the casinos, and is the only split always

recommended by the experts. (Culbertson et al. believe that aces are the only pair which should be split.) In other respects, however, the optimum strategy is quite surprising. Splitting eights, for example, is seldom seen in the casinos, while splitting tens and face cards is not uncommon. Furthermore, the detailed strategy for splitting nines, sevens, sixes, fours, threes and twos probably defies the intuition of even the most experienced players.

(d) *The Player's Mathematical Expectation.* Let W , a random variable, be the amount won by the player on a single hand under the following conditions: the player bets one unit of capital, the game of blackjack is defined by the rules given in Section I, and the player utilizes the optimum strategy described in (a), (b), (c) above. W has mathematical expectation $-.006$ and variance 1.1 .

While the player's overall expectation is $-.006$, $E(W_D)$, the player's conditional expectation given that the dealer's up card is D , shows considerable variability and is positive for seven out of the ten values of D .

D	2	3	4	5	6	7	8	9	10	(1, 11)
$E(W_D)$.090	.123	.167	.218	.230	.148	.056	-.043	-.176	-.363

The player's disadvantage in blackjack stems entirely from the rule that if both the player and dealer bust, the dealer wins. The player's disadvantage is smaller than in other popular house games such as craps and roulette where his mathematical expectation is, at best, $-.014$. Furthermore, the optimum strategy is considerably superior to many common strategies. The player, for example, who follows the strategy recommended by Culbertson et al. has an expectation of $-.036$. The player who mimics the dealer, drawing to 16 or less, standing on 17 or more, never doubling down or splitting pairs, has an expectation of $-.056$.

The optimum strategy was developed under the assumption that the player does not have the time or inclination to utilize the information available in the hands of the players preceding him in the draw. This information is non-existent when the player sits on the dealer's left and is greatest when the player is on the dealer's right. There are tremendous difficulties, however, in using this information except in an intuitive, non-scientific manner.

Practical considerations required that the optimum strategy be developed under the above major assumption and several less important assumptions discussed in Sections II-V. Because of the need for these simplifying assumptions, the strategy presented in this section could be more precisely described as a "practical" optimum strategy.

REFERENCES

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