Harmonic Analysis

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Chapter 1

Introduction

1.1 Supplement knowledge

(Interpolation) Let f be a measurable function on the measure space (X, M, μ). The distribution function of f is defined as $d_f : [0, \infty) \to [0, \infty]$ given by

$$d_f(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\})$$

as a function of λ.

(Weak L^p norm) For $0 < p < \infty$, the weak L^p norm of f is defined as

$$\|f\|_{p,\infty} = \sup_{\lambda > 0} \left(\lambda^p \mu(\{x \in X : |f(x)| > \lambda\})\right)^{\frac{1}{p}}.$$

(Weak L^p space) The weak L^p space, denoted by $L^{p,\infty}$, is the set of all functions f such that $\|f\|_{p,\infty} < \infty$.

Remark: The weak L^∞ space is identical to the standard L^∞ space.

Proposition 1.1.1. If $0 < p < \infty$, then

$$\|f\|_p^p = \int |f|^p d\mu = p \int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha.$$

Proof. Assuming μ is a suitable measure, we have

$$\begin{aligned} p \int_0^\infty \alpha^{p-1} \mu(\{x \in X : |f(x)| > \alpha\}) d\alpha &= p \int_0^\infty \alpha^{p-1} \int_X \chi_{\{x : |f(x)| > \alpha\}} d\mu d\alpha \\ &\overset{\text{Fubini}}{=} \int_X \int_0^{|f(x)|} p\alpha^{p-1} d\alpha d\mu(x) \\ &= \int_X |f(x)|^p d\mu(x) = \|f\|_p^p. \end{aligned}$$

□

Theorem 1.1.2 (The Marcinkiewicz Interpolation Theorem)**.** Let (X, μ) and (Y, ν) be measure spaces. Suppose that p_0, p_1, q_0, q_1 are elements of $[1, \infty]$ satisfying $p_0 \leqslant q_0$, $p_1 \leqslant q_1$, and $q_0 \neq q_1$. Consider T to be a sublinear map ($|T(cf)| = c|T(f)|$ and $|T(f + g)| \leqslant |T(f)| + |T(g)|$) from $L^{p_0}(\mu) + L^{p_1}(\mu)$ to the space of measurable functions on Y. If T is of weak type (p_0, q_0) and (p_1, q_1), then for any $0 < t < 1$ and $\left(\frac{1}{p}, \frac{1}{q}\right) = (1-t)\left(\frac{1}{p_0}, \frac{1}{q_0}\right) + t\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$, the map T is of strong type (p, q).Then, B_p remains bounded as $p \to p_j$ if $p_j < \infty$ (or as $p \to \infty$ if $p_j = \infty$).

Remark. Weak type means that there exist constants C_0 and C_1 such that $\|Tf\|_{L^{q_0,\infty}} \leqslant C_0\|f\|_{L^{p_0}}$ and $\|Tf\|_{L^{q_1,\infty}} \leqslant C_1\|f\|_{L^{p_1}}$. Strong type means that there exists a constant B_p depending only on p, p_j, q_j, C_j (for $j = 0, 1$) such that $\|Tf\|_{L^q} \leqslant B_p\|f\|_{L^p}$.

Theorem 1.1.3 (The Riesz-Thorin Interpolation Theorem)**.** Let (X, μ) and (Y, ν) be two measure spaces, and let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$. If $q_0 = q_1 = \infty$, assume further that ν is semifinite. Let T be a linear operator from $L^{p_0}(X) + L^{p_1}(X)$ into $L^{q_0}(Y) + L^{q_1}(Y)$ such that:

1. $\|Tf\|_{L^{q_0}} \leqslant M_0\|f\|_{L^{p_0}}$ for all $f \in L^{p_0}$ (strong type (p_0, q_0)).
2. $\|Tf\|_{L^{q_1}} \leqslant M_1\|f\|_{L^{p_1}}$ for all $f \in L^{p_1}$ (strong type (p_1, q_1)).

Then, for any $0 < \theta < 1$ and $\left(\frac{1}{p}, \frac{1}{q}\right) = (1-\theta)\left(\frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$, we have $\|Tf\|_{L^q} \leqslant M_0^{1-\theta}M_1^{\theta}\|f\|_{L^p}$ for all $f \in L^p$ (strong type (p, q)).

1.2 Maximal functions

$f \in L^1_{loc}(\mathbb{R}^n)$ The Hardy-Littlewood maximal function of f is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)|\, dy \quad \text{where} \quad B_r = B(0, r) \subset \mathbb{R}^n$$

which can also be expressed as

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B(x,r)} |f(y)|\, dy$$

or equivalently as

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} (\chi_{B_r} * |f|)(x)$$

where χ_{B_r} is the characteristic function of the ball B_r.

An alternative definition is given by

$$\tilde{M}f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)|\, dy$$

where the supremum is taken over all balls B containing x. It can be shown that $Mf(x)$ and $\tilde{M}f(x)$ are equivalent in the sense that there exists a constant $C > 0$ such that

$$Mf(x) \leq \tilde{M}f(x) \leq CMf(x)$$

denoted as $\tilde{M}f(x) \lesssim Mf(x)$.

Lemma 1.2.1 (A finite version of the Vitali covering lemma)**.** Let E be a measurable subset of $\mathbb{R}^n$ that is the union of a finite collection of balls $\{B_j\}$. Then one can select a disjoint subcollection $B_1, \ldots, B_m$ of the $\{B_j\}$ so that

$$\sum_{k=1}^{m} |B_k| > C|E|$$

with $C = 3^{-n}$.

Theorem 1.2.2. Let f be a function defined on $\mathbb{R}^n$.
(a) The operator M is of weak type $(1,1)$, i.e., if $f \in L^1$, $\lambda > 0$, then

$$m(\{x : |Mf(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f\|_1$$

where m denotes the Lebesgue measure.
(b) M is of strong type (p,p) for $1 < p \leq \infty$, i.e.,

$$\|Mf\|_p \leq A_p \|f\|_p$$

where A_p is a constant depending only on p and n.

Remark. If f is in L^1 and is not identically zero, then $Mf \notin L^\infty$.

Since f is not identically zero, there exist $\varepsilon > 0$ and $R > 0$ such that

$$\int_{B_R} |f| \geq \varepsilon > 0$$

If $|x| > R$, then $B_R \subset B(x, 2|x|)$ and

$$Mf(x) \geq \frac{1}{|B(x,2|x|)|} \int_{B(x,2|x|)} |f|\, dv \geq \frac{C}{|x|^n} \int_{B_R} |f|\, dv \geq \frac{C\varepsilon}{|x|^n}$$

which is not in L^1. Therefore, $Mf \notin L^\infty$.

Proof. Obviously, $\|Mf\|_\infty \leq \|f\|_\infty$, so by the Marcinkiewicz interpolation theorem, it suffices to prove (a). Let $E_\lambda = \{x : |\tilde{M}f(x)| > \lambda\}$. For all $x \in E_\lambda$, there exists a ball B_x such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)|\, dy > \lambda$$

Fix a compact subset K of E_λ. Since K is covered by $\bigcup_{x \in E_\lambda} B_x$, we can select a finite subcover $K \subset \bigcup_{l=1}^N B_l$. By the covering lemma, there exist disjoint balls $B_{i1}, \ldots, B_{ik}$ such that

$$m\left(\bigcup_{l=1}^N B_l\right) \leq C \sum_{j=1}^k m(B_{ij})$$

Therefore,

$$m(K) \leq m\left(\bigcup_{l=1}^N B_l\right) \leq C \sum_{j=1}^k m(B_{ij}) \leq \frac{C}{\lambda} \sum_{j=1}^k \int_{B_{ij}} |f|\, dy = \frac{C}{\lambda} \int_{\bigcup B_{ij}} |f|\, dy \leq \frac{C}{\lambda} \|f\|_1$$

Letting $m(K) \to m(E_\lambda)$ completes the proof of (a). □

As a corollary, we obtain the Lebesgue differentiation theorem:

Theorem 1.2.3. If $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f(x) \quad \text{for a.e. } x$$

1.3 Approximation to the identity.

Definition 1.3.1. Let φ be a function belonging to $L'(\mathbb{R}^n)$, the dual space of Lebesgue integrable functions on $\mathbb{R}^n$, satisfying $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$. Define $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi\left(\frac{x}{\varepsilon}\right)$ for $\varepsilon > 0$. The family of functions $\{\varphi_\varepsilon\}_{\varepsilon > 0}$ is known as an approximation to the identity.

Theorem 1.3.2. Suppose $\{\varphi_\varepsilon\}_{\varepsilon > 0}$ is an approximation to the identity. Then for any function f in $L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, we have

$$\lim_{\varepsilon \to 0} \|\varphi_\varepsilon * f - f\|_p = 0$$

where $*$ denotes convolution.

Remark. There exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ converging to 0 as $k \to \infty$ such that $\varphi_{\varepsilon_k} * f(x) = f(x)$ almost everywhere.

Proof. Since $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$, we can write

$$\varphi_\varepsilon * f(x) - f(x) = \int_{\mathbb{R}^n} \varphi(y)[f(x - \varepsilon y) - f(x)]\, dy$$

Given $\tilde{\varepsilon} > 0$, choose $\delta > 0$ such that if $|h| < \delta$, then $\|f(t+h) - f(t)\| < \frac{\tilde{\varepsilon}}{2\|\varphi\|_1}$. For this fixed δ, if ε is sufficiently small, then

$$\int_{|y|\geq \frac{\delta}{\varepsilon}} |\varphi(y)|\, dy \leq \frac{\tilde{\varepsilon}}{4\|f\|_p}$$

Using Minkowski's integral inequality, we have

$$\begin{aligned}
\|\varphi_\varepsilon * f - f\|_p &\leq \int_{\mathbb{R}^n} |\varphi(y)| \|f(x - \varepsilon y) - f(x)\|_{L^p}\, dy \\
&< \int_{|y|<\frac{\delta}{\varepsilon}} |\varphi(y)| \|f(x - \varepsilon y) - f(x)\|_{L^p}\, dy + 2\|f\|_p \int_{|y|\geq\frac{\delta}{\varepsilon}} |\varphi(y)|\, dy \\
&\leq \frac{\tilde{\varepsilon}}{2\|\varphi\|_1} \cdot \|\varphi\|_1 + 2\|f\|_p \cdot \frac{\tilde{\varepsilon}}{4\|f\|_p} = \tilde{\varepsilon}
\end{aligned}$$

$\square$

Definition 1.3.3 (Schwartz Space $S(\mathbb{R}^n)$)**.** The Schwartz space $S(\mathbb{R}^n)$ is defined as the set of all infinitely differentiable functions f on $\mathbb{R}^n$ such that for any multi-indices α and β, the supremum $\sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)|$ is finite. Here, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $D^\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \cdots \partial_{x_n}^{\beta_n}$, where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}^n$.

As an example, the function $f(x) = e^{-|x|^2}$ belongs to $S(\mathbb{R}^n)$. If f belongs to the Schwartz space S, then for any multi-index α, the function $D^\alpha(x^\beta f)$ also belongs to S.

Proposition 1.3.4. If $f \in S(\mathbb{R}^n)$, then for any multi-index β and any natural number N, there exists a constant $C_{N,\beta}$ such that $|D^\beta f(x)| \leqslant \frac{C_{N,\beta}}{(1+|x|)^N}$. As a consequence, $D^\beta f$ belongs to L^p for all $p \geqslant 1$.

Proposition 1.3.5. Let $\varphi \in L'(\mathbb{R}^n)$ with $\int \varphi = 1$, and define $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$. Then, for all $f \in S(\mathbb{R}^n)$,

$$\lim_{\varepsilon\to 0}(f * \varphi_\varepsilon)(x) = f(x).$$

Proof. Consider the convolution $f * \varphi_\varepsilon(x) = \int f(x-\varepsilon y)\varphi(y)dy$. We have the estimate

$$|f(x-\varepsilon y)\varphi(y)| \leqslant \|f\|_\infty |\varphi(y)|,$$

which belongs to L' since $\varphi \in L'(\mathbb{R}^n)$ and f is bounded. Therefore, by the Dominated Convergence Theorem (DCT),

$$\lim_{\varepsilon\to 0}(f * \varphi_\varepsilon)(x) = \int f(x)\varphi(y)dy = f(x)\int \varphi(y)dy = f(x),$$

as desired. □

Theorem 1.3.6. Let $\varphi \in L^1$ with $\int \varphi = 1$, and define $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$ and $\psi(x) = \sup_{|y|\geqslant|x|}|\varphi(y)|$ (the least decreasing radial majorant of φ). Suppose that $\psi \in L^1$. Then,

(1) For all $f \in L^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant \infty$, we have $\sup_{\varepsilon>0}|f * \varphi_\varepsilon(x)| \leqslant AMf(x)$ almost everywhere, where $A = \int_{\mathbb{R}^n}\psi(x)dx$ and Mf is the Hardy-Littlewood maximal function of f.

(2) For all $f \in L^p$ with $1 \leqslant p < \infty$, we have $\lim_{\varepsilon\to 0} f * \varphi_\varepsilon(x) = f(x)$ almost everywhere. (This is known as differentiation of the approximation to identity.)

Proof. (1) It suffices to show that $\sup_{\varepsilon>0}|f_1 * \psi_\varepsilon(x)| \leqslant \|\psi\|_1 Mf(x)$ for any radial decreasing function $\psi \in L^1(\mathbb{R}^n)$ (i.e., $\psi(x) \leqslant \psi(y)$ if $|x| > |y|$).

Step 1: Assume first that ψ is a simple function, say $\psi(x) = \sum_{j=1}^m b_j \chi_{R_j}(x)$ where $\{R_j\}$ are annuli centered at the origin and $\{b_j\}$ are distinct positive numbers with $b_1 > \cdots > b_m$. We can write

$$\psi(x) = \sum_{j=1}^{m}(b_j - b_{j+1})\chi_{B_j}(x)$$

where B_j are concentric balls centered at the origin, $B_j = \{x : |x| \leqslant r_j\}$ for some r_j, and $b_{m+1} = 0$. Let $a_j = b_j - b_{j+1} > 0$ for $j = 1, \ldots, m-1$ and $a_m = b_m > 0$. Then,

$$\int \psi(x)dx = \sum_{j=1}^{m} a_j|B_j|.$$

Now,

$$|f| * \psi(x) = \int |f|(x-y)\psi(y)dy = \sum_{j=1}^{m} a_j \int_{B_j} |f|(x-y)dy \leqslant Mf(x) \sum_{j=1}^{m} a_j |B_j| = \|\psi\|_1 Mf(x).$$

Step 2: For a general radial decreasing function $\psi \in L^1(\mathbb{R}^n)$, there exists a sequence of simple functions $\{\psi_k\}$ as in Step 1 such that $0 \leqslant \psi_1 \leqslant \cdots \leqslant \psi_k \leqslant \cdots$ and $\psi_k \leqslant \psi$ for all $k \in \mathbb{N}$, with $\psi(x) = \lim_{k\to\infty} \psi_k(x)$. Clearly, $\|\psi_k\|_1 \leqslant \|\psi\|_1$ for all k. By Fatou's Lemma, we have

$$\begin{aligned} |f| * \psi(x) = \int |f|(x-y) \lim_{k\to\infty} \psi_k(y)dy &\leqslant \liminf_{k\to\infty} \int |f|(x-y)\psi_k(y)dy \\ &\leqslant \liminf_{k\to\infty} \|\psi_k\|_1 Mf(x) \\ &\leqslant \|\psi\|_1 Mf(x). \end{aligned}$$

Step 3: Replace ψ in Step 2 by ψ_ε to get

$$|f| * \psi_\varepsilon(x) \leqslant \|\psi_\varepsilon\|_1 Mf(x) = \|\psi\|_1 Mf(x).$$

Taking the supremum over $\varepsilon > 0$ completes the proof of part (1).

(2) For $1 \leqslant p < \infty$, it is known that $\lim_{\varepsilon\to 0} \|f * \varphi_\varepsilon - f\|_p = 0$, which implies that $\lim_{\varepsilon_k \to 0} f * \varphi_{\varepsilon_k}(x) = f(x)$ almost everywhere for any sequence $\{\varepsilon_k\}$ converging to 0. It remains to show that $\lim_{\varepsilon\to 0} f * \varphi_\varepsilon(x)$ exists almost everywhere.

Define $\Omega f(x) = |\limsup_{\varepsilon\to 0} f * \varphi_\varepsilon(x) - \liminf_{\varepsilon\to 0} f * \varphi_\varepsilon(x)|$. If $f \in S(\mathbb{R}^n)$ (the Schwartz space), then $\lim_{\varepsilon\to 0} f * \varphi_\varepsilon(x) = f(x)$ for all $x \in \mathbb{R}^n$, so $\Omega f(x) \equiv 0$. For a general $f \in L^p$ with $1 \leqslant p < \infty$, we can write $f = f_1 + f_2$ where $f_1 \in S(\mathbb{R}^n)$ and $\|f_2\|_p$ is arbitrarily small, since the Schwartz space is densely embedded in the L^p space. Then,

$$\Omega f(x) \leqslant \Omega f_1(x) + \Omega f_2(x) = \Omega f_2(x) \leqslant 2AMf_2(x).$$

Therefore,

$$|\{x : \Omega f(x) \geqslant \varepsilon\}| \leqslant |\{x : Mf_2(x) > \frac{\varepsilon}{2A}\}| \leqslant C \left(\frac{\|f_2\|_p}{\varepsilon/2A} \right)^p.$$

Since $\|f_2\|_p$ can be made arbitrarily small, we conclude that $|\{x : \Omega f(x) > \varepsilon\}| = 0$ for all $\varepsilon > 0$, and hence $|\{x : \Omega f(x) > 0\}| = 0$. This completes the proof of part (2) for $1 \leqslant p < \infty$.

For $p = \infty$, fix any ball B and let B_1 be a larger ball containing B with δ the distance from B to $\mathbb{R}^n \setminus B_1$. Write $f = f_1 + f_2$ where $f_1 = f\chi_{B_1}$ and $f_2 = f - f_1$. Then $f_1 \in L^1$ and for $x \in B$, we have

$$|f_2 * \varphi_\varepsilon(x)| = \left|\int f_2(x-y)\varphi_\varepsilon(y)dy\right| \leqslant \|f\|_\infty \int_{|y|\geqslant\delta} |\varphi_\varepsilon(y)|dy = \|f\|_\infty \int_{|y|\geqslant\delta/\varepsilon} |\varphi(y)|dy \to 0$$

as $\varepsilon \to 0$. Since $\lim_{\varepsilon\to 0} f_1 * \varphi_\varepsilon(x) = f_1(x)$ almost everywhere, we conclude that $\lim_{\varepsilon\to 0}(f_1 + f_2) * \varphi_\varepsilon(x) = f_1(x) = f(x)$ for almost every $x \in B$. This completes the proof of part (2) for $p = \infty$. □

Theorem 1.3.7. Let $\{T_\varepsilon\}$ be a family of linear operators mapping from $L^p(\mathbb{R}^n)$ into the space of measurable functions from $\mathbb{R}^n$ to $\mathbb{C}$. Define the operator T^* as $T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$. If T^* is of weak type (p,q), then the set $\{f \in L^p(\mathbb{R}^n) : \lim_{\varepsilon\to\varepsilon_0} T_\varepsilon f(x) = f(x) \text{ almost everywhere}\}$ is closed in $L^p(\mathbb{R}^n)$.

Remark. T^* is the maximal operator associated with the family $\{T_\varepsilon\}$.

Proof. Consider a sequence of functions $\{f_n\} \subset L^p$ converging to f in L^p and satisfying $T_\varepsilon f_n(x) \to f_n(x)$ almost everywhere. Then,

$$\begin{aligned}
&m\left(\left\{x : \limsup_{\varepsilon\to\varepsilon_0} |T_\varepsilon f(x) - f(x)| > \lambda\right\}\right)\\
&\leqslant m\left(\left\{x : \limsup_{\varepsilon\to\varepsilon_0} |T_\varepsilon (f-f_n)(x) - (f-f_n)(x)| > \lambda\right\}\right)\\
&\leqslant m\left(\left\{x : |T^*(f-f_n)(x)| > \frac{\lambda}{2}\right\}\right) + m\left(\left\{x : |(f-f_n)(x)| > \frac{\lambda}{2}\right\}\right)\\
&\leqslant \left(\frac{2C}{\lambda}\|f-f_n\|_p\right)^q + \left(\frac{2}{\lambda}\|f-f_n\|_p\right)^p \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore,

$$m\left(\left\{x : \limsup_{\varepsilon\to\varepsilon_0} |T_\varepsilon f(x) - f(x)| > \lambda\right\}\right) = 0.$$

Hence,

$$m\left(\left\{x : \limsup_{\varepsilon\to\varepsilon_0} |T_\varepsilon f(x) - f(x)| > 0\right\}\right) \leqslant \sum_{k=1}^{\infty} m\left(\left\{x : \limsup_{\varepsilon\to\varepsilon_0} |T_\varepsilon f(x) - f(x)| > \frac{1}{k}\right\}\right) = 0.$$

Thus, f belongs to the given set. □

Theorem 1.3.8. If $|\phi(x)| \leqslant \psi(x)$ almost everywhere, where ψ is a non-negative, radially decreasing, and integrable function, and $f \in L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, then

$$\lim_{\varepsilon\to 0} \phi_\varepsilon * f(x) = \left(\int \phi\right) f(x)$$

almost everywhere.

Proof. By some known results, $\sup_{\varepsilon>0} |\phi_\varepsilon * f(x)|$ is of weak type $(1,1)$ and strong type (p,p) for $1 < p < \infty$. From the previous theorem, the set

$$\left\{ f \in L^p(\mathbb{R}^n) : \lim_{\varepsilon \to 0} \phi_\varepsilon * f(x) = \left(\int \phi \right) f(x) \text{ a.e.} \right\}$$

is closed in $L^p(\mathbb{R}^n)$. Since $S \subset \left\{ f \in L^p(\mathbb{R}^n) : \lim_{\varepsilon \to 0} \phi_\varepsilon * f(x) = \left(\int \phi \right) f(x) \text{ a.e.} \right\} \subset L^p$, taking the closure, we have $\bar{S} = L^p$. Therefore,

$$\left\{ f \in L^p(\mathbb{R}^n) : \lim_{\varepsilon \to 0} \phi_\varepsilon * f(x) = \left(\int \phi \right) f(x) \text{ a.e.} \right\} = L^p.$$

$\square$

Example 1.3.9. Let $P(x) = \frac{C_n}{(1+|x|^2)^{\frac{n+1}{2}}}$ be the Poisson kernel, where $C_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}$. Define

$$P_t(x) = C_n \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}.$$

Now, let $u(x,t) = P_t(x) * f(x)$. Then, for $f \in L^p$, the function $u(x,t)$ solves the Dirichlet problem

$$\begin{cases} \left(\Delta_x + \frac{\partial^2}{\partial t^2} \right) u = 0 & \text{in } \mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\}, \\ u(x,0) = f(x) & \text{a.e. on } \mathbb{R}^n. \end{cases}$$

Chapter 2

Fourier Transform

2.1 The Fourier Transform on L^1 space

Definition 2.1.1. Let f be a function in $L^1(\mathbb{R}^n)$. The Fourier transform of f, denoted by $\hat{f}(\xi)$, is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx, \quad \text{for } \xi \in \mathbb{R}^n.$$

Proposition 2.1.2. Suppose $f \in L^1(\mathbb{R}^n)$. Then the following properties hold:

(a) The L^∞-norm of $\hat{f}$ is bounded by the L^1-norm of f, i.e., $\|\hat{f}\|_\infty \leqslant \|f\|_1$.

(b) The function $\hat{f}$ is uniformly continuous on $\mathbb{R}^n$.

(c) As $|\xi|$ approaches infinity, $\hat{f}(\xi)$ tends to zero. This is known as the Riemann-Lebesgue Lemma.

(d) If f and g are functions in $L^1(\mathbb{R}^n)$ such that their product $f\cdot g$ is also in $L^1(\mathbb{R}^n)$, then the Fourier transform of their sum is the product of their Fourier transforms, i.e., $\widehat{f+g} = \hat{f}\cdot\hat{g}$.

Proof. (c): Consider the following manipulation of the Fourier transform:

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx \\
&= \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\cdot(-1)e^{-2\pi i\xi\cdot\frac{\xi}{2|\xi|^2}}\,dx \\
&= -\int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi\cdot\left(x+\frac{\xi}{2|\xi|^2}\right)}\,dx \\
&= -\int_{\mathbb{R}^n} f\left(x-\frac{\xi}{2|\xi|^2}\right)e^{-2\pi i x\cdot\xi}\,dx.
\end{aligned}$$

Then, we have

$$\begin{aligned}
|\hat{f}(\xi)| &= \frac{1}{2}|\hat{f}(\xi) + \hat{f}(\xi)| \\
&= \frac{1}{2}\left|\int_{\mathbb{R}^n}\left[f(x) - f\left(x - \frac{\xi}{2|\xi|^2}\right)\right] e^{-2\pi i x\cdot\xi}\,dx\right| \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^n}\left|f(x) - f\left(x - \frac{\xi}{2|\xi|^2}\right)\right| dx \\
&\to 0 \quad \text{as } |\xi| \to \infty.
\end{aligned}$$

$\square$

Proposition 2.1.3. Let f be a function in $L^1(\mathbb{R})$. Then the following properties of the Fourier transform hold:

(1) The Fourier transform of $f(x-b)$ is $e^{-2\pi i\xi\cdot b}\hat{f}(\xi)$.

(2) The Fourier transform of $e^{2\pi i x\cdot h}f(x)$ is $\hat{f}(\xi - h)$.

(3) For any positive real number t, the Fourier transform of $t^{-n}f(t^{-1}x)$ is $\hat{f}(t\xi)$.

(4) Let ρ be an orthogonal transformation on $\mathbb{R}^n$, i.e., a linear transformation that preserves the inner product, satisfying $\rho(x)\cdot\rho(y) = x\cdot y$. Then the Fourier transform of $f\circ\rho$ is $\hat{f}\circ\rho(\xi)$.

(5) If f is a radial function, then $\hat{f}$ is also radial.

Proof. (4):

$$\begin{aligned}
(f\circ\rho)^\wedge(\xi) &= \int_{\mathbb{R}^n} f(\rho x)e^{-2\pi i x\cdot\xi}\,dx \\
&\overset{y=\rho x}{=} \int_{\mathbb{R}^n} f(y)e^{-2\pi i\rho^{-1}y\cdot\xi}\,dy \\
&= \int_{\mathbb{R}^n} f(y)e^{-2\pi i y\cdot\rho\xi}\,dy \\
&= \hat{f}(\rho\xi).
\end{aligned}$$

(5) To show that $\hat{f}(\xi_1) = \hat{f}(\xi_2)$ when $|\xi_1| = |\xi_2|$, we can use a rotation ρ such that $\rho\xi_1 = \xi_2$. Then, by property (4), we have

$$\hat{f}(\xi_2) = \hat{f}(\rho\xi_1) = (f\circ\rho)^\wedge(\xi_1) = \hat{f}(\xi_1).$$

$\square$

Theorem 2.1.4. Let $f \in L^1(\mathbb{R})$. Then,

(1) $\frac{\partial\hat{f}(\xi)}{\partial\xi_k} = (-2\pi i x_k f(x))^\wedge(\xi)$ provided that $x_k f \in L^1$.

(2) If $f \in C^1\cap C_0$ and $\frac{\partial f}{\partial x_k} \in L^1$, then $\left(\frac{\partial f}{\partial x_k}\right)^\wedge(\xi) = 2\pi i\xi_k\hat{f}(\xi)$.

Here, C_0 denotes the set of continuous functions that vanish at infinity, i.e., $C_0 = \{f \in C(x) : \forall \varepsilon > 0, \{x||f(x)| \geqslant \varepsilon\}$ is compact $\}$.

Proof. (1): Consider $h = (0, \dots, 0, h_k, 0, \dots, 0)$ where h_k is the k^{th} element. We have

$$\frac{\hat{f}(\xi + h) - \hat{f}(\xi)}{h_k} = \int_{\mathbb{R}^n} \frac{e^{-2\pi i x_k h_k} - 1}{h_k} f(x) e^{-2\pi i x \cdot \xi} \, dx.$$

Observe that $\left| \frac{e^{-2\pi i x_k h_k} - 1}{h_k} f(x) e^{-2\pi i x \cdot \xi} \right| \leqslant 2\pi |x_k f(x)|$. By the Dominated Convergence Theorem (D.C.T.),

$$\lim_{h_k \to 0} \frac{\hat{f}(\xi + h) - \hat{f}(\xi)}{h_k} = \int_{\mathbb{R}^n} -2\pi i x_k f(x) e^{-2\pi i x \cdot \xi} \, dx = (-2\pi i x_k f(x))^\wedge (\xi).$$

□

Corollary 2.1.5. For $\alpha \in \mathbb{Z}_+^n$ and $D^\alpha = (\partial_{x_1})^{\alpha_1} \cdots (\partial_{x_n})^{\alpha_n}$, let $P(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. Define the differential operator $P(D) = \sum_{|\alpha| \leqslant d} a_\alpha D^\alpha$. Then,

$$P(D)\hat{f}(\xi) = (P(-2\pi i x) f(x))^\wedge(\xi).$$

Moreover, for $f \in S(\mathbb{R}^n)$ (the Schwartz space),

$$(P(D)f)^\wedge(\xi) = P(2\pi i \xi)\hat{f}(\xi).$$

Definition 2.1.6 (Inverse Fourier Transform)**.** If $f \in L^1$, the inverse Fourier transform of f is defined as

$$\check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \xi \cdot x} \, d\xi \, (= \hat{f}(-x)).$$

Lemma 2.1.7. If $f, g \in L'$, then $\int \hat{f}(\xi) g(\xi) d\xi = \int f(x) \hat{g}(x) dx$ (the multiplication formula).

Proof.

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{f}(\xi)g(\xi)d\xi &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx g(\xi)d\xi \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} g(\xi)e^{-2\pi i x\cdot\xi}d\xi f(x)dx \\ &= \int_{\mathbb{R}^n} f(x)\hat{g}(x)dx.\end{aligned}$$

□

Lemma 2.1.8. $\left(e^{-\pi|x|^2}\right)^{\wedge}(\xi) = e^{-\pi|\xi|^2}$.

Proof.

$$\begin{aligned}\left(e^{-\pi|x|^2}\right)^{\wedge}(\xi) &= \int_{\mathbb{R}^n} e^{-\pi|x|^2}e^{-2\pi i x\cdot\xi}dx \\ &= \prod_{i=1}^{n}\int_{\mathbb{R}} e^{-\pi x_i^2}e^{-2\pi i x_i\xi_i}dx_i.\end{aligned}$$

It suffices to show $\left(e^{-\pi x^2}\right)^{\wedge}(\xi) = e^{-\pi\xi^2}$ for $x,\xi\in\mathbb{R}^1$.

The function $f(x) = e^{-\pi x^2}$ is the solution of the initial value problem (I.V.P):

$$\begin{cases} u' + 2\pi x u = 0, \\ u(0) = 1. \end{cases}$$

If $f\in S$ satisfies the initial value problem, then $\hat{f}$ also satisfies the same initial value problem. Indeed,

$$\begin{aligned}\hat{f}(0) &= \int_{\mathbb{R}^n} f(x)dx = 1, \\ 0 &= (f' + 2\pi x f)^{\wedge}(\xi) \\ &= 2\pi i\xi\hat{f}(\xi) + \frac{1}{-i}(\hat{f})'(\xi) \\ &= i(\hat{f}(\xi) + 2\pi\xi\hat{f}(\xi)).\end{aligned}$$

Therefore, by uniqueness, $\hat{f} = f$.

Remark. $\left(e^{-\pi a|x|^2}\right)^{\wedge}(\xi) = a^{-\frac{n}{2}}e^{-\pi\frac{|\xi|^2}{a}}$ for $a>0$.

□

Theorem 2.1.9 (The Fourier inversion theorem for L^1 functions)**.** If $f,\hat{f}\in L^1(\mathbb{R}^n)$, then $(\hat{f})^{\vee} = f$ almost everywhere (a.e.).

Remark. Since $\hat{f} \in L^1$, then $(\hat{f})^\vee \in C_0$. Modify f on a set of measure zero such that

$$(\hat{f})^\vee(x) = f(x) \quad \forall x.$$

Definition 2.1.10. $G_\varepsilon(f) = \int_{\mathbb{R}^n} f(x)e^{-\varepsilon|x|^2}dx$ is the Gauss means of $\int_{\mathbb{R}^n} f(x)dx$.

Theorem 2.1.11. If $f \in L^1(\mathbb{R}^n)$, then

$$\left\| \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x\cdot\xi}e^{-4\pi^2\varepsilon^2|\xi|^2}d\xi - f(x) \right\|_{L^1(dx)} \to 0$$

as $\varepsilon \to 0$.

Proof.

$$\begin{aligned}
\int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x\xi}e^{-4\pi^2\varepsilon^2|\xi|^2}d\xi &= \int f(y)\left(e^{2\pi i x\cdot\xi}e^{-4\pi^2\varepsilon^2|\xi|^2}\right)^\wedge(y)dy \\
&= \int_{\mathbb{R}^n} f(y)\left(e^{-4\pi^2\varepsilon^2|\xi|^2}\right)^\wedge(y-x)dy \\
&= \int_{\mathbb{R}^n} f(y)\varepsilon^{-n}(4\pi)^{-\frac{n}{2}}e^{-\frac{1}{4}\left|\frac{y-x}{\varepsilon}\right|^2}dy \\
&= \int_{\mathbb{R}^n} f(y)\varphi_\varepsilon(x-y)dy \\
&= f * \varphi_\varepsilon(x),
\end{aligned}$$

where $\varphi(x) = (4\pi)^{-\frac{n}{2}}e^{-\frac{1}{4}|x|^2} \in L^1$ and $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. ☐

Next we prove Theorem 2.1.9

Proof. By the previous theorem, there exists a sequence $\{\varepsilon_k\}_{k=1}^\infty$ converging to 0 such that

$$\lim_{\varepsilon_k\to 0} \int \hat{f}(\xi)e^{2\pi i x\cdot\xi}e^{-4\pi^2\varepsilon_k^2|\xi|^2}\,d\xi = f(x) \text{ for almost every } x.$$

Since $\hat{f} \in L'$, by the Dominated Convergence Theorem, we have

$$f(x) = \int \lim_{\varepsilon_k\to 0} \hat{f}(\xi)e^{2\pi i x\cdot\xi}e^{-4\pi^2\varepsilon_k^2|\xi|^2}\,d\xi = \int \hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi = (\hat{f})^\vee(x).$$

☐

Corollary 2.1.12. If $f_1, f_2 \in L'$ and $\hat{f}_1(\xi) = \hat{f}_2(\xi)$, then $f_1(x) = f_2(x)$ for almost every x.

Proof. Set $f = f_1 - f_2$. Then $f \in L^1$ and $\hat{f} = \widehat{f_1 - f_2} = \hat{f}_1 - \hat{f}_2 = 0 \in L^1$. By Fourier inversion, $f = (\hat{f})^\vee = 0$, thus $f_1 = f_2$ almost everywhere. ☐

Remark. Let $\mathcal{F}f(\xi) = \hat{f}(\xi)$ denote the Fourier transform of f evaluated at ξ.

2.2 The Fourier Transform on L^2 and $L^p, (1 < p < 2)$ space

Theorem 2.2.1. Suppose that $|\varphi(x)| \leq \frac{c}{(1+|x|)^{n+\varepsilon_0}}$ for some constants $c, \varepsilon_0 > 0$ and $\int_{\mathbb{R}^n} \varphi(x)\, dx = a$. If $f \in L^p$ for $1 \leq p \leq \infty$, then

$$\lim_{\varepsilon \to 0} f * \varphi_\varepsilon(x) = af(x)$$

holds for every x in the Lebesgue set of f.

Remark. The Lebesgue set L_f of f is defined as the set of points x where $f(x)$ is finite and

$$\lim_{r \to 0} \frac{1}{r^n} \int_{|y|<r} |f(x-y) - f(x)|\, dy = 0.$$

Proof. Let $x \in L_f$. For any $\delta > 0$, there exists $\eta > 0$ such that

$$\frac{1}{r^n} \int_{|y|<r} |f(x-y) - f(x)|\, dy \leq \delta$$

for all $r \leq \eta$. Now consider the expression

$$|f * \varphi_\varepsilon(x) - af(x)| = \left| \int_{\mathbb{R}^n} f(x-y)\varphi_\varepsilon(y)\, dy - \int_{\mathbb{R}^n} f(x)\varphi_\varepsilon(y)\, dy \right|.$$

This can be rewritten as

$$\left| \int_{\mathbb{R}^n} (f(x-y) - f(x))\varphi_\varepsilon(y)\, dy \right|.$$

We split the integral into two parts:

$$\underbrace{\int_{|y|\leq\eta} |f(x-y) - f(x)||\varphi_\varepsilon(y)|\, dy}_{I_1} + \underbrace{\int_{|y|>\eta} |f(x-y) - f(x)||\varphi_\varepsilon(y)|\, dy}_{I_2}.$$

We claim that $I_1 \leq A\delta$, where A is independent of ε, and $I_2 \to 0$ as $\varepsilon \to 0$. Since $|f * \varphi_\varepsilon(x) - af(x)| \leqslant I_1 + I_2 \leqslant A\delta + I_2$, we have

$$\limsup_{\varepsilon \to 0} |f * \varphi_\varepsilon(x) - af(x)| \leq A\delta.$$

As δ is arbitrary, we conclude that $\lim_{\varepsilon\to 0} f * \varphi_\varepsilon(x) = af(x)$. To estimate I_1, let $K \in \mathbb{N}$ be fixed such that $2^K \leqslant \eta/\varepsilon < 2^{K+1}$ when $\eta/\varepsilon \geqslant 2$. We define the set $B(0,\eta)$ as follows:

$$B(0,\eta) = \begin{cases} B(0,2^{-k}\eta) \cup \left(\bigcup_{i=1}^{K} \{y \mid 2^{-i}\eta \leq |y| < 2 \cdot 2^{-i}\eta\} \right), & \text{if } \eta/\varepsilon \geqslant 2, \\ B(0,\eta), & \text{if } \eta/\varepsilon < 2. \end{cases}$$

Case 1: $\eta/\varepsilon < 2$. In this case, we have

$$I_1 \leqslant c\varepsilon^{-n} \int_{B(0,\eta)} |f(x-y) - f(x)|\, dy \leqslant c\varepsilon^{-n}\delta\eta^n \leqslant c\delta.$$

Case 2: $\eta/\varepsilon \geqslant 2$. On the k-th annulus, we estimate

$$|\varphi_\varepsilon(y)| = \varepsilon^{-n} \left|\varphi\left(\varepsilon^{-1}y\right)\right| \leqslant C\varepsilon^{-n} \frac{1}{|\varepsilon^{-1}y|^{n+\varepsilon_0}} \leqslant C\varepsilon^{\varepsilon_0} \frac{1}{(2^k\eta)^{n+\varepsilon_0}}.$$

On the ball $B\left(0,2^{-k}\eta\right)$, we use the estimate $|\varphi_\varepsilon(y)| \leqslant C\varepsilon^{-n}$. Thus,

$$\begin{aligned} I_1 &\leqslant \sum_{k=1}^{K} c\varepsilon^{\varepsilon_0} \frac{1}{(2^k\eta)^{n+\varepsilon_0}} \delta \left(2 \cdot 2^{-k}\eta\right)^n + c\varepsilon^{-n}\delta\left(2^{-K}\eta\right)^n \\ &= c\delta \frac{\varepsilon^{\varepsilon_0}}{\eta^{\varepsilon_0}} \sum_{k=1}^{K} 2^{-k(n+\varepsilon_0-n)} + c\delta\left(2^{-K}\frac{\eta}{\varepsilon}\right)^n \\ &= c\delta \frac{\varepsilon^{\varepsilon_0}}{\eta^{\varepsilon_0}} \sum_{k=1}^{K} 2^{-k\varepsilon_0} + c\delta\left(2^{-K}\frac{\eta}{\varepsilon}\right)^n \\ &\leq c\delta \frac{\varepsilon^{\varepsilon_0}}{\eta^{\varepsilon_0}} \frac{1-2^{-K\varepsilon_0}}{1-2^{-\varepsilon_0}} + c\delta \\ &\leq c\delta \frac{1}{1-2^{-\varepsilon_0}} + c\delta = A\delta \end{aligned}$$

As for I_2, if p' is the conjugate exponent to p and $x = \mathbf{1}_{\{|y|:|y|>\eta\}}$, by Hölder's inequality we have:

$$\begin{aligned} I_2 &\leqslant \int (|f(x-y)| + |f(x)|) \left|\mathbf{1}_{\{|y|:|y|>\eta\}}\varphi_\varepsilon(y)\right| dy \\ &\leqslant \|f\|_p \left\|\mathbf{1}_{\{|y|:|y|>\eta\}}\varphi_\varepsilon\right\|_{p'} + |f(x)| \left\|\mathbf{1}_{\{|y|:|y|>\eta\}}\varphi_\varepsilon\right\|_1 \end{aligned}$$

It suffices to show $\left\|\mathbf{1}_{\{|y|>\eta\}}\varphi_\varepsilon\right\|_q \to 0$ as $\varepsilon \to 0$. If $q = \infty$, then

$$\left\|1_{\{|y|>\eta\}}\varphi_\varepsilon\right\|_\infty \leqslant \varepsilon^{-n} \frac{c}{(1+\varepsilon^{-1}\eta)^{n+\varepsilon_0}} \leq C\eta^{-n-\varepsilon_0}\varepsilon^{\varepsilon_0} \to 0$$

as $\varepsilon \to 0$.

If $q < \infty$, then

$$
\begin{aligned}
\left\|1_{\{|y|>\eta\}}\varphi_\varepsilon\right\|_q^q &= \int_{|y|>\eta} \left|\varepsilon^{-n}\varphi\left(\varepsilon^{-1}y\right)\right|^q dy \\
&= \varepsilon^{-nq}\int_{|z|\geqslant\frac{\eta}{\varepsilon}} |\varphi(z)|^q\cdot\varepsilon^n dz \\
&\leq C\varepsilon^{n(1-q)}\int_{|z|\geqslant\frac{\eta}{\varepsilon}} \frac{1}{(1+|z|)^{(n+\varepsilon_0)q}}dz \\
&\leq C\varepsilon^{n(1-q)}\int_{r=\frac{\eta}{\varepsilon}}^{\infty} \frac{r^{n-1}}{(1+r)^{(n+\varepsilon_0)q}}dr \\
&\leq C\eta^{n-(n+\varepsilon_0)q}\varepsilon^{\varepsilon_0 q} \longrightarrow 0
\end{aligned}
$$

as $\varepsilon \to 0$. □

Corollary 2.2.2. Suppose $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \geq 0$. If f is continuous at 0, then $\hat{f} \in L^1$ and $f(x) = \int \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$ almost everywhere. In particular, $f(0) = \int \hat{f}(\xi)d\xi$.

Proof. Since f is continuous at 0, we have that $0 \in L_f$. Recall that

$$\int \hat{f}(\xi)e^{2\pi i x\cdot\xi}e^{-4\pi^2\varepsilon^2|\xi|^2}d\xi = \int f * \varphi_\varepsilon(x)dx$$

with $\varphi(x) = (4\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4}}$ and $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. By a previous result , we have at $x = 0$

$$f(0) = \lim_{\varepsilon\to 0}\int f * \varphi_\varepsilon(0)dx = \lim_{\varepsilon\to 0}\int \hat{f}(\xi)e^{-4\pi^2\varepsilon^2|\xi|^2}d\xi$$

Hence,

$$\|\hat{f}\|_1 = \int \hat{f}(\xi)d\xi = \int \lim_{\varepsilon\to 0}\hat{f}(\xi)e^{-4\pi^2\varepsilon^2|\xi|^2}d\xi \leq \liminf_{\varepsilon\to 0}\int \hat{f}(\xi)e^{-4\pi^2\varepsilon^2|\xi|^2}d\xi = f(0) < \infty$$

thus $\hat{f} \in L^1$. By the Dominated Convergence Theorem (D.C.T.),

$$f(0) = \int \lim_{\varepsilon\to 0}\hat{f}(\xi)e^{-4\pi^2\varepsilon^2|\xi|^2}d\xi = \int \hat{f}(\xi)d\xi$$

□

Now, let's proceed to define the Fourier Transform on L^2.

Theorem 2.2.3. If $f \in L^1 \cap L^2$, then $\|\hat{f}\|_2 = \|f\|_2 < \infty$ ($\hat{f} \in L^2$).

Proof. Let $g(x) = \overline{f(-x)}$. Then $h = f * g \in L^1$ ($\|h\|_1 = \|f * g\|_1 \leq \|f\|_1\|g\|_1 = \|f\|_1^2$). h

is bounded ($\|h\|_\infty \leq \|f\|_2\|g\|_2 = \|f\|_2^2$) and uniformly continuous. Thus $\hat{h} = \hat{f} * \hat{g} = \hat{f}\hat{g} = \hat{f}\overline{\hat{f}} = |\hat{f}|^2 \geq 0$. By the previous corollary, $\hat{h} \in L^1$ and $h(0) = \int \hat{h}(\xi)d\xi$. We thus have $\int |\hat{f}|^2 d\xi = h(0) = f * g(0) = \int f(x)g(0-x)dx = \int f(x)\overline{f(x)}dx = \int |f|^2 dx$.

□

Theorem 2.2.4. Let $f \in L^2(\mathbb{R}^n)$ and $\{g_n\} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $g_n \to f$ in $L^2(\mathbb{R}^n)$. Then $\hat{g}_n$ converges to a function $\mathcal{F}f$ in $L^2(\mathbb{R}^n)$. $\mathcal{F}f$ is independent of the particular sequence $\{g_n\}$ and $\mathcal{F}f$ is called the L^2 Fourier transform of f on $L^2(\mathbb{R}^n)$. The Fourier transform on $L^2(\mathbb{R}^n)$ will be denoted by $\mathcal{F}$, and we shall use the notation $\hat{f} = \mathcal{F}f$ whenever $f \in L^2(\mathbb{R}^n)$.

Proof. We have $\|\hat{g}_n - \hat{g}_m\|_2 = \|(g_n - g_m)^\wedge\|_2 = \|g_n - g_m\|_2 \to 0$as $m, n \to \infty$. Thus $\{\hat{g}_n\}$ is an Cauchy sequence in $L^2(\mathbb{R}^n)$. Then there exists an $L^2(\mathbb{R}^n)$ function, denoted by $\mathcal{F}f$, such that $\hat{g}_n \xrightarrow{L^2} \mathcal{F}f$. Assume $\{g_n\}$ and $\{\widetilde{g}_n\}$ both converge to f in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with respect to the L^2-norm. Consider the sequence $\{g_1, \tilde{g}_1, g_2, \tilde{g}_2, \ldots, g_n, \tilde{g}_n, \ldots\}$ in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ converging to f in $L^2(\mathbb{R}^n)$. Hence, there exists $h \in L^2(\mathbb{R}^n)$ such that $\{\hat{g}_1, \hat{\tilde{g}}_1, \ldots, \hat{g}_n, \widehat{\tilde{g}}_n, \ldots\} \xrightarrow{L^2} h$. Both $\hat{g}_n \xrightarrow{L^2} h$ and $\widehat{\tilde{g}}_n \xrightarrow{L^2} h$ converge to the same function in $L^2(\mathbb{R}^n)$. □

Theorem 2.2.5 (The Plancherel Theorem)**.** For any function f in the space $L^2(\mathbb{R}^n)$, the Fourier transform $\mathcal{F}f$ satisfies the equality $\|\mathcal{F}f\|_2 = \|f\|_2$.

Proof. Assume we have a sequence of functions g_n that converge to f in the L^2 norm, and each g_n belongs to the intersection of $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. Then, the Fourier transforms $\hat{g}_n$ converge to $\mathcal{F}f$ in the L^2 norm.
Since the Fourier transform preserves the L^2 norm for each g_n, i.e., $\|\hat{g}_n\|_2 = \|g_n\|_2$, we can take the limit as n approaches infinity to obtain $\|\mathcal{F}f\|_2 = \|f\|_2$. □

Corollary 2.2.6. The Fourier transform $\mathcal{F}$ is injective.

Then we will show $\mathcal{F}$ be a linear operator. For all $f, g \in L^2(\mathbb{R}^n)$, there exist sequences f_n and g_n such that f_n converges to f in L^2, g_n converges to g in L^2, and $f_n \cdot g_n$ belongs to both $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. We have that:

$$\mathcal{F}f = \lim_{n\to\infty} \hat{f}_n \quad \mathcal{F}g = \lim_{n\to\infty} \hat{g}_n$$

Consequently, the linearity of $\mathcal{F}$ is established as follows:

$$\mathcal{F}(f+g) = \lim_{\substack{n\to\infty \\ \text{in } L^2}} \widehat{f_n + g_n} = \lim_{\substack{n\to\infty \\ \text{in } L^2}} (\hat{f}_n + \hat{g}_n) = \lim_{\substack{n\to\infty \\ \text{in } L^2}} \hat{f}_n + \lim_{\substack{n\to\infty \\ \text{in } L^2}} \hat{g}_n = \mathcal{F}f + \mathcal{F}g$$

Thereby, we conclude that $\mathcal{F}$ is a unique bounded linear operator mapping from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Additionally, for any function f that belongs to both $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, $\mathcal{F}f$ is equivalent to its Fourier transform $\hat{f}$.

To demonstrate uniqueness, let us assume the existence of two distinct bounded linear operators, F_1 and F_2, both defined on $L^2(\mathbb{R}^n)$ with values in $L^2(\mathbb{R}^n)$. Furthermore, let these operators satisfy the property that for all functions f in the intersection of $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, $F_i f = \hat{f}$ for $i = 1, 2$.

For any function g in $L^2(\mathbb{R}^n)$, we can construct a sequence g_n belonging to the intersection of $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ such that g_n approaches g in the L^2 norm. Now, considering the differences between $F_1 g$ and $F_2 g$ in the L^2 norm, we have:

$$\|F_1 g - F_2 g\|_2 \leqslant \|F_1 g - F_1 g_n\|_2 + \|F_1 g_n - F_2 g_n\|_2 + \|F_2 g_n - F_2 g\|_2$$

The middle term, $\|F_1 g_n - F_2 g_n\|_2$, vanishes since both operators agree on g_n (by our assumption). Thus,

$$\|F_1 g - F_2 g\|_2 \leqslant c\|g - g_n\|_2 + c'\|g_n - g\|_2$$

for some constants c and c' depending on the boundedness of the operators. As n approaches infinity, this expression tends to zero, implying the equality of F_1 and F_2. This establishes the uniqueness of the operator $\mathcal{F}$.

Proposition 2.2.7. The Fourier transform $\mathcal{F} : L^2 \to L^2$ is surjective.

Proof. Since $\mathcal{F}$ is an isometry, its range $R(\mathcal{F})$ constitutes a closed subspace of L^2 (as established in Claim 1). Furthermore, the Schwartz space S is contained within $R(\mathcal{F})$ (as shown in Claim 2). Therefore, the image of $\mathcal{F}$ is indeed the entirety of L^2.

Claim 1:

Consider a sequence $\{\mathcal{F}f_n\}$ in $R(\mathcal{F})$ where $f_n \in L^2$ and $\mathcal{F}f_n$ converges to g in the L^2 norm. We can deduce that $\{\mathcal{F}f_n\}$ is a Cauchy sequence in L^2. Consequently, $\{f_n\}$ is also a Cauchy sequence in L^2 and converges to some $f \in L^2$. By continuity of $\mathcal{F}$, we have $\mathcal{F}f_n \to \mathcal{F}f$ in L^2 and thus $g = \mathcal{F}f$. This establishes that g belongs to $R(\mathcal{F})$.

Claim 2:

It is a known fact that the **Fourier transform maps the Schwartz space S onto itself bijectively.** To elaborate, if f belongs to S, then it is both integrable and bounded. For any multi-indices α and β, the function $\widehat{D^\alpha(x^\beta f)}$ is bounded since $D^\alpha(x^\beta f)$ also belongs to S. We have the identity $\widehat{D^\alpha(x^\beta f)} = C_{\alpha,\beta}\xi^\alpha D^\beta \hat{f}$ which implies that $\xi^\alpha D^\beta \hat{f}$ is bounded. This, in turn, means that $\hat{f}$ belongs to S.

The injectivity of $\mathcal{F}$ on S follows from the fact that if $f_1, f_2 \in S$ and $\mathcal{F}f_1 = \mathcal{F}f_2$, then f_1

and f_2 must be equal almost everywhere. Hence, they are equivalent.

To show the surjectivity of $\mathcal{F}$ on S, consider any $f \in S$. Define $F(x) = f(-x)$ and let $g = \hat{F}$. It can be shown that g belongs to S and that $\hat{g}(\xi) = f(\xi)$. This demonstrates that $\mathcal{F}$ is surjective onto S. $\square$

Theorem 2.2.8. The Fourier transform is a unitary operator on L^2.
Unitary: A linear operator on L^2 that is an isometry and maps onto L^2.

Corollary 2.2.9. The Fourier transform on L^2 preserves inner products: $\langle \mathcal{F}f, \mathcal{F}g\rangle = \langle f, g\rangle$ for all $f, g \in L^2$.

Proof. By the polarization identity, we have

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f\bar{g} = \frac{1}{4}\left(\|f+g\|_2^2 - \|f-g\|_2^2 + i\|f+ig\|_2^2 - i\|f-ig\|_2^2\right).$$

Since $\mathcal{F}$ is an isometry, it follows that $\|\mathcal{F}f\|_2 = \|f\|_2$ for all $f \in L^2$. Therefore, applying the polarization identity to $\mathcal{F}f$ and $\mathcal{F}g$, we obtain

$$\langle \mathcal{F}f, \mathcal{F}g\rangle = \frac{1}{4}\left(\|\mathcal{F}f+\mathcal{F}g\|_2^2 - \|\mathcal{F}f-\mathcal{F}g\|_2^2 + i\|\mathcal{F}f+i\mathcal{F}g\|_2^2 - i\|\mathcal{F}f-i\mathcal{F}g\|_2^2\right) = \langle f, g\rangle.$$

This completes the proof that the Fourier transform preserves inner products on L^2. $\square$

Theorem 2.2.10. The inverse of the Fourier transform, denoted by $\mathcal{F}^{-1}$, can be obtained by letting $(\mathcal{F}^{-1}g)(x) = (\mathcal{F}g)(-x)$ for all $g \in L^2(\mathbb{R}^n)$.

Proof. Suppose first that $g \in S(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing functions. Then there exists $f \in S(\mathbb{R}^n)$ such that $g = \hat{f}$, where $\hat{f}$ denotes the Fourier transform of f. By the Fourier inversion formula for functions in the Schwartz space, we have

$$f = \mathcal{F}^{-1}g = (\hat{f})^{\vee} = \hat{g}(-x) = (\mathcal{F}g)(-x).$$

This shows that $(\mathcal{F}^{-1}g)(x) = (\mathcal{F}g)(-x)$ holds for all $g \in S(\mathbb{R}^n)$.

Now let $g \in L^2(\mathbb{R}^n)$ be arbitrary. By density of the Schwartz space in $L^2(\mathbb{R}^n)$, there exists a sequence $\{g_k\}$ in $S(\mathbb{R}^n)$ such that $g_k \to g$ in L^2 as $k \to \infty$. Using the triangle inequality and the fact that $\mathcal{F}$ is an isometry on L^2, we have

$$\begin{aligned}
&\|(\mathcal{F}g)(-x) - \mathcal{F}^{-1}g(x)\|_{L^2} \\
&\leq \|(\mathcal{F}g)(-x) - (\mathcal{F}g_k)(-x)\|_2 + \|(\mathcal{F}g_k)(-x) - \mathcal{F}^{-1}g_k(x)\|_2 + \|\mathcal{F}^{-1}g_k(x) - \mathcal{F}^{-1}g(x)\|_2 \\
&= \|g - g_k\|_2 + 0 + \|g_k - g\|_2 \to 0 \text{ as } k \to \infty.
\end{aligned}$$

This shows that $(\mathcal{F}g)(-x) = \mathcal{F}^{-1}g(x)$ almost everywhere, completing the proof. □

If a function f can be expressed as the sum of two functions f_1 and f_2 where f_1 belongs to L^1 and f_2 belongs to L^2, then we write $f = f_1 + f_2$. In this case, we define the Fourier transform of f as $\hat{f} = \hat{f}_1 + \hat{f}_2$.

Well-definedness: Suppose we have another decomposition of f as $g_1 + g_2$ where $g_1 \in L^1$ and $g_2 \in L^2$. Then, it follows that $f_1 + f_2 = g_1 + g_2$. Rearranging this equation, we obtain $f_1 - g_1 = g_2 - f_2$. Since both L^1 and L^2 are linear spaces, the difference $f_1 - g_1$ belongs to L^1 and the difference $g_2 - f_2$ belongs to L^2. Therefore, the Fourier transform of $f_1 - g_1$ exists and is equal to the Fourier transform of $g_2 - f_2$. This implies that $\hat{f}_1 + \hat{f}_2 = \hat{g}_1 + \hat{g}_2$, showing that the definition of the Fourier transform for functions in $L^1 + L^2$ is well-defined.

Definition 2.2.11. For $1 \leq p \leq 2$, since $L^p \subset L^1 + L^2$, we can apply the above definition to functions in L^p.

2.3 The Fourier Transform on $\mathcal{S}'$

At the begining, we define the **seminorm** $\|f\|_{\alpha,\beta} = \|x^\alpha D^\beta f\|_\infty$. Then we define a topology on $\mathcal{S}$ as follows: a sequence $\{f_k\} \subset \mathcal{S}$ converges in $\mathcal{S}$ to f if and only if

$$\forall \alpha, \beta \in \mathbb{N}_0^n, \quad \lim_{k\to\infty} \|f_k - f\|_{\alpha,\beta} = 0.$$

The space of bounded linear functionals on $\mathcal{S}$, denoted by $\mathcal{S}'$, is called the space of tempered distributions. A linear map $T : \mathcal{S} \to \mathbb{C}$ belongs to $\mathcal{S}'$ if $\lim_{k\to\infty} T(\phi_k) = 0$ whenever $\lim_{k\to\infty} \phi_k = 0$ in $\mathcal{S}$.

Theorem 2.3.1. The Fourier transform is a continuous map from $\mathcal{S}$ to $\mathcal{S}$.

Proof. We have

$$\|\hat{f}\|_{\alpha,\beta} = \|\xi^\alpha D^\beta \hat{f}(\xi)\|_\infty \leq C\|\widehat{D^\alpha(x^\beta f)}(\xi)\|_\infty \leq C\|D^\alpha(x^\beta f)\|_1.$$

The L^1 norm can be bounded by a finite linear combination of seminorms of f, which implies that the Fourier transform is a continuous map. $\square$

Remark. Using Leibniz's rule, we can write

$$D^\alpha(x^\beta f) = \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2} D^{\alpha_1}(x^\beta) D^{\alpha_2} f.$$

Then,

$$\begin{aligned}\|D^{\alpha_1}(x^\beta)D^{\alpha_2}f\|_1 &= \|(1+|x|^2)^{-N}(1+|x|^2)^N D^{\alpha_1}(x^\beta)D^{\alpha_2}f\|_1 \\ &\le \|(1+|x|^2)^N D^{\alpha_1}(x^\beta)D^{\alpha_2}f\|_\infty \|(1+|x|^2)^{-N}\|_1 \\ &< \infty,\end{aligned}$$

since the first factor is a finite linear combination of seminorms of f.

Definition 2.3.2. The Fourier transform (F.T.) of $T \in S'$ is the tempered distribution given by

$$\hat{T}(f) = T(\hat{f}) \quad \text{for } f \in S.$$

Remark. $\hat{T}$ is a tempered distribution since $f_k \to 0$ in S implies $\hat{f}_k \to 0$ in S and hence

$$T(\hat{f}_k) \to 0 \Rightarrow \hat{T} \in S'.$$

Examples of tempered distributions

(1) Let $f \in L^p$ with $1 \leqslant p \leqslant \infty$. Define

$$L(\varphi) = L_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx \quad \text{for } \varphi \in S.$$

$L_f \in S'$ since $\|L_f(\varphi)\| \le \|f\|_p \|\varphi\|_q \to 0$ as $\varphi \to 0$ in S (where q is the conjugate exponent of p). Then,

$$\hat{L}_f(\varphi) = L_f(\hat{\varphi}) = \int_{\mathbb{R}^n} f(x)\hat{\varphi}(x)\,dx.$$

If $1 \leqslant p \leqslant 2$ and $f \in L^p$, then for $\varphi \in S$,

$$\hat{L}_f(\varphi) = \int_{\mathbb{R}^n} f(x)\hat{\varphi}(x)\,dx = \int_{\mathbb{R}^n} \hat{f}(x)\varphi(x)\,dx,$$

where $\hat{f}$ is the Fourier transform of f in the sense of $L^{p'}$ norm (with p' being the conjugate exponent of p). A distribution $u \in S'$ **coincides with** a function h if

$$u(\varphi) = \int_{\mathbb{R}^n} h(x)\varphi(x)\,dx \quad \text{for all } \varphi \in S.$$

In this case, $\hat{L}_f$ coincides with $\hat{f}$.

Note: For any $p > 2$, there exists an $f \in L^p$ whose Fourier transform as a tempered distribution dose not **coincides with** a function.

(2) If μ is a finite Borel measure, the linear functional $L = L_\mu$ defined by

$$L(\varphi) = L_\mu(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\, d\mu(x) \quad \text{for } \varphi \in S$$

is a tempered distribution. Then,

$$\begin{aligned}\hat{L}_\mu(\varphi) = L_\mu(\hat{\varphi}) &= \int_{\mathbb{R}^n} \hat{\varphi}(\xi)\, d\mu(\xi)\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i x\cdot\xi}\, dx\, d\mu(\xi)\\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\, d\mu(\xi)\right)\varphi(x)\, dx.\end{aligned}$$

$\hat{L}_\mu(\varphi)$ coincides with $\hat{\mu}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\, d\mu(\xi)$.

(3) A measurable function f satisfying $\frac{f(x)}{(1+|x|^2)^k} \in L^p$, where $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}$, is called a tempered function. (When $p = \infty$, such a function is called a slowly increasing function.)

Theorem 2.3.3. A linear functional L on S is a tempered distribution if and only if there exist constants $C > 0$ and integers m and l such that

$$|L(\varphi)| \leqslant C \sum_{\substack{|\alpha|\leqslant l\\ |\beta|\leqslant m}} \|\varphi\|_{\alpha,\beta} \quad \forall \varphi \in S.$$

Convolution of a distribution with a function in S

For a function g on $\mathbb{R}^n$, its reflection $\tilde{g}$ is defined by $\tilde{g}(x) = g(-x)$. If $u, \varphi, \psi \in S$, then

$$\int_{\mathbb{R}^n} (u * \varphi)(x)\psi(x)\, dx = \int_{\mathbb{R}^n} u(x)(\tilde{\varphi} * \psi)(x)\, dx.$$

The mappings $\psi \longmapsto \int_{\mathbb{R}^n}(u * \varphi)(x)\psi(x)\, dx$ and $\theta \longmapsto \int_{\mathbb{R}^n} u(x)\theta(x)\, dx$ are linear functionals on S. Denote these functionals by $u * \varphi$ and u, respectively. Then, $(*)$ is given by

$$(u * \varphi)(\psi) = u(\tilde{\varphi} * \psi).$$

Definition 2.3.4. Let $u \in S'$ and $\varphi \in S$. Define the convolution $u * \varphi$ by $(u * \varphi)(\psi) = u(\tilde{\varphi} * \psi)$. Then, for all $u \in S'$ and $\varphi \in S$, we have $u * \varphi \in S'$ and the convolution is associative: $(u * \varphi) * \psi = u * (\varphi * \psi)$ whenever $u \in S'$ and $\varphi, \psi \in S$.

Theorem 2.3.5. If $u \in S'$ and $\varphi \in S$, then the convolution $u * \varphi$ coincides with the function f defined by $f(x) = u(\tau_x \tilde{\varphi})$ for $x \in \mathbb{R}^n$, where τ_x denotes the translation by x (i.e., $\tau_x(g(y)) = g(y - x)$). Moreover, $f \in C^\infty$ and it as well as all its derivatives are slowly increasing, i.e., for all α there exist constants $C_\alpha, k_\alpha > 0$ such that

$$|(\partial^\alpha f)(x)| \leqslant C_\alpha (1 + |x|)^{k_\alpha}.$$

Proof. By the continuity of u and the fact that

$$\frac{\tau_{he_j}(\tau_x(\tilde{\varphi})) - \tau_x(\tilde{\varphi})}{h} \to -\tau_x(\partial_j \tilde{\varphi})$$

in S as $h \to 0$, we have

$$\frac{f(x + he_j) - f(x)}{h} = \frac{u(\tau_{x+he_j}\tilde{\phi}) - u(\tau_x \tilde{\phi})}{h} = u(\frac{\tau_{he_j}(\tau_{x+he_j}\tilde{\phi}) - \tau_x \tilde{\phi}}{h}) \to -u(T_x(\partial_j \tilde{\varphi})).$$

Considering higher-order derivatives, we find that the function f belongs to the class C^∞ and satisfies the relation:

$$\partial^\alpha f(x) = (-1)^{|\alpha|} u(\tau_x D^\alpha \tilde{\varphi}).$$

Moreover, we can estimate the magnitude of $\partial^\alpha f(x)$ as:

$$\begin{aligned}
|\partial^\alpha f(x)| &\leq c \sum_{|\gamma| \leq l} \sup_{y \in \mathbb{R}^n} |y^\gamma \tau_x (\partial^{\alpha+\beta} \tilde{\varphi})(y)| \\
&\leq c \sum_{\substack{|\gamma| \leq l \\ |\beta| \leq m}} \sup_{y \in \mathbb{R}^n} |(x + y)^\gamma (\partial^{\alpha+\beta} \tilde{\varphi})(y)| \\
&\leq C_l \sum_{|\beta| \leq m} \sup_{y \in \mathbb{R}^n} (1 + |x|^l + |y|^l) |(\partial^{\alpha+\beta} \tilde{\varphi})(y)|.
\end{aligned}$$

This estimation reveals that $|\partial^\alpha f(x)|$ is bounded by a polynomial of x.

Now, let's demonstrate that for any ψ belonging to the Schwartz space S, the following equality holds:

$$(u * \varphi)(\psi) = \int_{\mathbb{R}^n} f(x) \psi(x) \, dx.$$

To this end, we observe that:

$$\begin{aligned}
(u * \varphi)(\psi) &= u(\tilde{\varphi} * \psi) \\
&= u\left(\int_{\mathbb{R}^n} \tilde{\varphi}(x-y)\psi(y)\,dy\right) \\
&= u\left(\int_{\mathbb{R}^n} (\tau_y\tilde{\varphi})(x)\psi(y)\,dy\right) \\
&= \int_{\mathbb{R}^n} u(\tau_y\tilde{\varphi})\psi(y)\,dy.
\end{aligned}$$

Here, the transition from the Riemann sum of $\int_{\mathbb{R}^n}(\tau_y\tilde{\varphi})(x)\psi(y)\,dy$ to the integral is justified by the linearity and continuity of the functional u in the Schwartz space S. $\square$

Chapter 3

The Theory of Singular Integrals

3.1 The Hilbert transform: A model

Definition 3.1.1. The principal value of the function $\frac{1}{x}$ is denoted as $p.v.\frac{1}{x}$. We define $\omega_0 = p.v.\frac{1}{x}$ as follows: For $\varphi \in S(\mathbb{R})$ (functions in the Schwartz space), we have

$$\omega_0(\varphi) = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx$$

Proposition 3.1.2. $\omega_0 \in S'$, which means ω_0 is a tempered distribution.

Proof.

$$\begin{aligned}\omega_0(\varphi) &= \lim_{\varepsilon \to 0} \left(\int_{\varepsilon<|x|\le 1} \frac{\varphi(x)}{x}\, dx + \int_{|x|\ge 1} \frac{\varphi(x)}{x}\, dx \right) \\ &= \lim_{\varepsilon \to 0} \left(\int_{\varepsilon<|x|\le 1} \frac{\varphi(x)-\varphi(0)}{x}\, dx + \int_{|x|\ge 1} \frac{\varphi(x)}{x}\, dx \right)\end{aligned}$$

Since $\left|\frac{\varphi(x)-\varphi(0)}{x}\right| \le \|\varphi'\|_\infty$, by the Dominated Convergence Theorem (DCT), we obtain

$$\begin{aligned}|\omega_0(\varphi)| &\le \int_{\mathbb{R}} \left|\frac{\varphi(x)-\varphi(0)}{x}\right| dx + \int_{|x|\ge 1} \left|\frac{\varphi(x)}{x}\right| dx \\ &\le 2\|\varphi'\|_\infty + \int_{|x|\ge 1} \left|\frac{x\varphi(x)}{x^2}\right| dx \\ &\le 2\|\varphi'\|_\infty + 2\|x\varphi(x)\|_\infty\end{aligned}$$

Therefore, $\omega_0 \in S'$. $\square$

Definition 3.1.3. For $f \in S(\mathbb{R})$, the Truncated Hilbert Transform (at height ϵ) is defined as

$$H^{(\varepsilon)}(f)(x) = \frac{1}{\pi}\int_{|y|\geq\varepsilon} \frac{f(x-y)}{y}\,dy = \frac{1}{\pi}\int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,dy$$

The Hilbert Transform is then defined as

$$Hf(x) = \frac{1}{\pi}(w_0 * f)(x) = \lim_{\varepsilon\to 0} H^{(\varepsilon)}(f)(x)$$

Note. The convolution $\omega_0 * f(x)$ can be expressed as $\omega_0(\tau_x\tilde{f})$, where $\tau_x\tilde{f}(y) = \tilde{f}(y-x)$, i.e.,

$$\omega_0 * f(x) = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \frac{\tau_x\tilde{f}(y)}{y}\,dy = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy$$

Alternatively, the Hilbert Transform can also be represented as

$$Hf(x) = \frac{1}{\pi}p.v.\int_{-\infty}^{+\infty} \frac{f(x-y)}{y}\,dy = \frac{1}{\pi}p.v.\int_{-\infty}^{+\infty} \frac{f(y)}{x-y}\,dy$$

Remark. If we use $\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}\,dy$ to define the Hilbert Transform , its definition can be naturally extended to a broader class of functions. Given $x \in \mathbb{R}$, $Hf(x)$ is defined for all integrable functions f on $\mathbb{R}$ that satisfy the Hölder condition near x, i.e., there exist $C_x, \varepsilon_x > 0$ such that $|f(x) - f(y)| \leq C_x|x-y|^{\varepsilon_x}$ whenever $|y-x| < \delta_x$.

For piecewise smooth integrable functions, the Hilbert Transform is well-defined at the Hölder-Lipschitz continuous points of the function.

Proposition 3.1.4. $\left(\frac{1}{\pi}p.v.\frac{1}{x}\right)^{\wedge}(\xi)$ is coincide with $-i\,\mathrm{sgn}(\xi)$ i.e,

$$\left(\frac{1}{\pi}p.v.\frac{1}{x}\right)^{\wedge}(\varphi) = \int(-i\,\mathrm{sgn}(\xi))\varphi(\xi)\,d\xi$$

.

Proposition 3.1.5. $\widehat{Hf}(\xi)$ is coincide with $-i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$ for $f \in S(\mathbb{R})$ i.e, for all $\varphi \in S(\mathbb{R})$, we have

$$\widehat{Hf}(\varphi) = \int_{\mathbb{R}} -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)\varphi(\xi)\,d\xi.$$

Proof. In fact, $\widehat{Hf}(\varphi) = Hf(\hat{\varphi}) = \frac{1}{\pi}(\omega_0 * f)(\hat{\varphi})$ can be further derived as:

$$
\begin{aligned}
&= \frac{1}{\pi}\omega_0(\tilde{f} * \hat{\varphi}) = \frac{1}{\pi}\omega_0(\widehat{\tilde{f}}) \\
&= \frac{1}{\pi}\omega_0(\hat{f}\varphi) \\
&= \int_{\mathbb{R}} -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)\varphi(\xi)\,d\xi.
\end{aligned}
$$

this is what we desired. □

We consider $\widehat{Hf}(\xi)$ as a function and identify $\hat{H} \in S'$ with the function $-i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$. Therefore, for $f \in S$, we have $\|\hat{H}f\|_2 = \|\hat{f}\|_2$.

Using this isometric property, we can extend the definition of the Hilbert Transform to $L^2(\mathbb{R})$. If $f \in L^2(\mathbb{R})$, then $-i\,\mathrm{sgn}(\xi)\hat{f}(\xi) \in L^2$. We define $Hf(x) = (-i\,\mathrm{sgn}(\xi)\hat{f}(\xi))^\vee(x)$. If $f \in L^2$ and there exists a sequence $\{f_n\} \subset S$ converging to f in L^2, then $\{H\hat{f}_m\}$ is a Cauchy sequence in L^2 and thus converges to a function in L^2. For f in L^2, we can define its Hilbert transform via $g \in L^2$ where $\hat{g} = \lim_{m\to\infty} \widehat{Hf_m}$ in L^2.

Theorem 3.1.6. For $f \in S(\mathbb{R})$,
(1) H is of weak type $(1,1)$, i.e., $m\{x \in \mathbb{R} : |Hf(x)| > \lambda\} \leqslant \frac{c}{\lambda}\|f\|_1$;
(2) H is of strong type (p,p) for $1 < p < \infty$, i.e., $\|Hf\|_p \leqslant C_p\|f\|_p$.

Remark. (1) As $p \to \infty$, $C_p = O(p)$; as $p \to 1$, $C_p = O\left(\frac{1}{p-1}\right)$;
(2) If $f = \chi_{[0,1]}$, then $Hf(x) = \frac{1}{\pi}\log\left|\frac{x}{x-1}\right|$. Note that while $f \in L^1$, $Hf \notin L^1$ and similarly, while $f \in L^\infty$, $Hf \notin L^\infty$.

Lemma 3.1.7 (The Calderon-Zygmund Decomposition in $L^1(\mathbb{R}^n)$)**.** Let $f \in L'(\mathbb{R}^n)$ and $\lambda > 0$. Then f can be decomposed as $f = g + b$ where $|g| \leqslant \lambda$ a.e. and $b = \sum_Q x_Q f$. The summation is over a collection $B = \{Q\}$ of disjoint cubes, and for each Q, $\lambda < \frac{1}{|Q|}\int_Q |f(x)|dx \leqslant 2^n\lambda$ (①). Furthermore, $m\left(\cup_{Q\in B} Q\right) < \frac{1}{\lambda}\|f\|_1$ (②).

Proof. For each $l \in \mathbb{Z}$, define a collection of dyadic cubes D_l as follows:

$$
D_l = \left\{ \prod_{i=1}^{n} \left[2^l m_i, 2^l(m_i+1)\right) : m_1, \ldots, m_n \in \mathbb{Z} \right\}
$$

Observe that if $Q \in D_l$ and $Q' \in D_{l'}$, then either $Q \cap Q' = \emptyset$, $Q \subset Q'$, or $Q' \subset Q$. Choose l_0 large enough so that for each $Q \in D_{l_0}$ satisfies $\frac{1}{|Q|}\int_Q |f(x)|dx \leq \lambda$. For each such cube, consider its 2^n "children" (or subcube) with side length 2^{l_0-1}. Each subcube Q' will have one of the following properties:

$$
\frac{1}{|Q'|}\int_{Q'} |f(x)|dx \leqslant \lambda \quad \text{or} \quad \frac{1}{|Q'|}\int_{Q'} |f(x)|dx > \lambda \tag{3.1}
$$

In the latter case, we stop and include Q' in the collection B. Observe that in this case,

$$\frac{1}{|Q'|}\int_{Q'}|f(x)|dx \leqslant \frac{2^n}{|Q|}\int_{Q}|f(x)|dx \leqslant 2^n\lambda$$

Let Q denote the parent cube of Q'. Therefore, (①) holds. If the first inequality in (3.1) is satisfied, then further subdivide Q' into its child cubes, each with half the side length of Q'. Continuing this process yields a collection of disjoint dyadic cubes B that satisfy (①). Consequently, (②) also holds because

$$\left|\bigcup_{Q\in B} Q\right| \leq \sum_{Q\in B}|Q| < \sum_{Q\in B}\frac{1}{\lambda}\int_{Q}|f(x)|dx = \frac{1}{\lambda}\int_{\bigcup Q}|f(x)|dx \leq \frac{1}{\lambda}\|f\|_1.$$

Now, consider a point $x_0 \in \mathbb{R}^n \setminus \bigcup_{Q\in B} Q$. Such an x_0 is contained in a decreasing sequence of dyadic cubes $\{Q_i\}$, each satisfying $\frac{1}{|Q_j|}\int_{Q_j}|f| \leq \lambda$. By Lebesgue's theorem, for such an x_0, we have $|f(x_0)| \leq \lambda$ almost everywhere.

Define $g = f - b = f - \sum_{Q\in B} x_Q f = 1_{\mathbb{R}^n\setminus\bigcup_{Q\in B}} f$. Since $\mathbb{R}^n \setminus \bigcup_{Q\in B} Q$ and $\mathbb{R}^n \setminus \bigcup_{Q\in\bar{Q}} Q$ differ only by a set of measure zero, it follows that $|g| \leq \lambda$ almost everywhere, as desired. □

Proof of boundedness of H (Theorem 3.1.6):
(1) Fix $\lambda > 0$. Using the $C - Z$ decomposition, there exist disjoint intervals $\{I_j\}$ such that

$$|f| \leq \lambda \text{ a.e. } x \notin \Omega = \bigcup_j I_j, \quad \lambda < \frac{1}{|I_j|}\int_{I_j}|f(x)|dx \leq 2\lambda, \quad |\Omega| \leq \frac{1}{\lambda}\|f\|_1.$$

Decompose f as $f = g + b$, where

$$g(x) = \begin{cases} f(x) & x \notin \Omega \\ \frac{1}{|I_j|}\int_{I_j} f(x)dx & x \in I_j, j \in \mathbb{N} \end{cases}$$

and $b(x) = \sum_j b_j(x)$ with $b_j(x) = \left(f(x) - \frac{1}{|I_j|}\int_{I_j} f(x)dx\right)\mathbf{1}_{I_j}(x)$. Then $|g(x)| \leq 2\lambda$ almost everywhere, and each b_j is supported on I_j with $\int_{I_j} b_j(x)dx = 0$.

Since $f = g + b$, we have $Hf = Hg + Hb$. Consequently,

$$|\{x : |Hf(x)| > \lambda\}| \leq \left|\left\{x : |Hg(x)| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x : |Hb(x)| > \frac{\lambda}{2}\right\}\right|.$$

For the first term, we have

$$
\begin{aligned}
\left|\left\{x : |Hg(x)| > \frac{\lambda}{2}\right\}\right| &\leqslant \frac{1}{(\lambda/2)^2}\int |Hg(x)|^2 dx = \frac{4}{\lambda^2}\int |g(x)|^2 dx \\
&\leqslant \frac{4}{\lambda^2}\left(\int |g(x)|dx\right)^2 \leqslant \frac{4}{\lambda^2}\left(\int |f(x)|dx\right)^2 \\
&\leqslant \frac{4}{\lambda^2}\|f\|_1^2 < \infty \quad (\text{Since } f \in L^1 \subset L^2)
\end{aligned}
$$

Let $2I_j$ be the interval with the same center as I_j and twice the length, and let $\Omega^* = \bigcup_j 2I_j$. Then $|\Omega^*| \le 2|\Omega|$ and

$$
\begin{aligned}
\left|\left\{x : |Hb(x)| > \frac{\lambda}{2}\right\}\right| &\le |\Omega^*| + \left|\left\{x \notin \Omega^* : |Hb(x)| > \frac{\lambda}{2}\right\}\right| \\
&\le 2|\Omega| + \frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega^*} |Hb(x)|dx \\
&\le \frac{2}{\lambda}\|f\|_1 + \frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega^*} |Hb(x)|dx
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_{\mathbb{R}\setminus\Omega^*} |Hb(x)|dx &\le \int_{\mathbb{R}\setminus\Omega^*}\left|\sum_j Hb_j(x)\right| dx \le \sum_j \int_{\mathbb{R}\setminus\Omega^*} |Hb_j(x)|dx \\
&\le \sum_j \int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|dx
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|dx &= \int_{\mathbb{R}\setminus 2I_j} \lim_{\varepsilon\to 0}\left|\int_{\substack{|x-y|>\varepsilon \\ y\in I_j}} \frac{b_j(y)}{x-y}dy\right| dx \\
&= \int_{\mathbb{R}\setminus 2I_j}\left|\int_{I_j} b_j(y)\left(\frac{1}{x-y} - \frac{1}{x-c_j}\right)dy\right| dx
\end{aligned}
$$

where c_j is the center of I_j and $\int_{\mathbb{R}} b_j(x)dx = 0$. Since $|y - c_j| \le \frac{1}{2}|I_j|$ and $|x-y| \ge \frac{1}{2}|x - c_j|$, then,

$$
\begin{aligned}
\int_{\mathbb{R}\setminus 2I_j}\left|\int_{I_j} b_j(y)\left(\frac{1}{x-y} - \frac{1}{x-c_j}\right)dy\right| dx &\le \int_{I_j} |b_j(y)| \int_{\mathbb{R}\setminus 2I_j} \frac{|y-c_j|}{|x-y||x-c_j|}dxdy \\
&\le \int_{I_j} |b_j(y)| \int_{\mathbb{R}\setminus 2I_j} \frac{|I_j|}{|x-c_j|^2}dxdy \\
&= 2\int_{I_j} |b_j(y)|dy
\end{aligned}
$$

Thus,

$$
\sum_j \int_{\mathbb{R}^{2I_j}} |Hb_j(x)|\, dx \leqslant 2\sum_j \int_{I_j} |b_j(x)|\, dx \leqslant 4\sum_j \int_{I_j} |f(x)|\, dx \leqslant 4\|f\|_1.
$$

Then, we can show that

$$\left|\left\{x : |Hb(x)| > \frac{\lambda}{2}\right\}\right| \leq \frac{10}{\lambda}\|f\|_1,$$

which proves the weak type $(1,1)$ estimate.

(2) H is of weak type $(1,1)$ and strong type $(2,2)$ (since $\|Hf\|_2 = \|f\|_2$). Hence, by interpolation, it is also of strong type (p,p) for $1 < p < 2$. If $2 < p < \infty$, then $p' < 2$. Consider the following estimate:

$$\begin{aligned}\|Hf\|_p &= \sup\left\{\left|\int_{\mathbb{R}^n} Hf \cdot g\,dx\right| : g \in C_c^\infty, \|g\|_{p'} \leq 1\right\}\\ &= \sup\left\{\left|-\int_{\mathbb{R}^n} f \cdot Hg\right| : g \in C_c^\infty, \|g\|_{p'} \leq 1\right\}.\end{aligned}$$

For $f, g \in S(\mathbb{R})$, we have

$$\int_{\mathbb{R}^n} Hf \cdot g = \int_{\mathbb{R}^n} Hf(\tilde{g})^\wedge = \int_{\mathbb{R}^n} \hat{H}_f \tilde{g} = \int_{\mathbb{R}^n} -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)\hat{g}(\xi)\,d\xi.$$

Furthermore, by a change of variables ($\eta = -\xi$), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)\hat{g}(\xi)\,d\xi\\ &= -\int_{\mathbb{R}^n} i\,\mathrm{sgn}(\eta)\hat{f}(\eta)\hat{g}(\eta)\,d\eta\\ &= -\int_{\mathbb{R}^n} \hat{f}(\eta)H\hat{g}(\eta)\,d\eta\\ &= -\int_{\mathbb{R}^n} (\hat{f})^\wedge Hg\\ &= -\int_{\mathbb{R}^n} f \cdot Hg.\end{aligned}$$

Using Holder's Inequality, we have

$$\|Hf\|_p \leqslant C\|f\|_p \sup\left\{\|g\|_{p'} : g \in C_c^\infty, \|g\|_{p'} \leq 1\right\} \leqslant C\|f\|_p.$$

Remark. We can extend the Hilbert transform to functions in L^p space. For $1 \leq p < \infty$ and any $f \in L^p$, there exists a sequence $\{f_k\} \subset S$ such that $f_k \to f$ in L^p. Since

$$\|Hf_m - Hf_n\|_p \leq C\|f_m - f_n\|_p \to 0$$

as $m, n \to \infty$, the sequence $\{Hf_m\}$ is Cauchy in L^p and converges to some $g = Hf$.

Another similar example is the **Riesz transform**.

Definition 3.1.8. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $w_j \in \mathcal{S}'$, let

$$\langle w_j, \varphi \rangle = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon \to 0} \int_{|y| \geqslant \varepsilon} \frac{y_j}{|y|^{n+1}} \varphi(y)\, dy.$$

and for $1 \leq j \leq n$, the **j-th Riesz transform** of f is given by

$$R_j(f)(x) = (w_j * f)(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy, \quad \forall f \in \mathcal{S}(\mathbb{R}^n),$$

where p.v. denotes the principal value integral.

Remark. The definition makes sense for any integrable function that satisfies the following property: for all x, there exist constants $C_x > 0$, $\varepsilon_x > 0$, and $\delta_x > 0$ such that

$$|f(x) - f(y)| \leqslant C_x |x-y|^{\varepsilon_x}$$

whenever $|x-y| < \delta_x$.

3.2 Singular Integrals

Definition 3.2.1. Let $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ satisfy, for some constant B,

1. $|K(x)| \leqslant B|x|^{-n}$ for $x \in \mathbb{R}^n \setminus \{0\}$ (size condition)
2. $\int_{|x|>2|y|} |K(x) - K(x-y)|\, dx \leq B$ for all y (smoothness condition)
3. $\int_{r<|x|<S} K(x)\, dx = 0$ for all $0 < r < S < \infty$ (cancellation condition)

Then K is called a Calderón-Zygmund kernel. The singular integral operator (or Calderón-Zygmund operator) with kernel K is defined as

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy, \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Lemma 3.2.2. Suppose that $|\nabla K(x)| \leq B|x|^{-n-1}$ for all $x \neq 0$ and some $B > 0$. Then

$$\int_{|x|>2|y|} |K(x) - K(x-y)|\, dx \leq CB$$

with $C = C(n)$ being a constant depending only on n.