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Given a family of topological spaces $(X_\alpha, \mathscr{T}_\alpha)$, we can define the **box topology** $\mathscr{T}_{\text{box}}$ on $\prod_\alpha X_\alpha$ using the base

$$\mathcal{B} = \left\{ \prod_\alpha U_\alpha : U_\alpha \in \mathscr{T}_\alpha \right\}.$$

Then

$$\mathscr{T}_{\text{box}} = \left\{ U \subset \prod_\alpha X_\alpha : \text{ for any } (x_\alpha) \in U \text{ there exists } U_\alpha \in \mathscr{T}_\alpha \text{ such that } (x_\alpha) \in \prod_\alpha U_\alpha \subset U \right\}.$$

Also recall that the **product topology** $\mathscr{T}_{\text{product}}$ on $\prod_\alpha X_\alpha$ is defined using the subbase

$$\mathcal{S} = \bigcup_\beta \left\{ \pi_\beta^{-1}(V_\beta) : V_\beta \in \mathscr{T}_\beta \right\},$$

so that the open sets are the unions of subsets $\prod_\alpha U_\alpha$, where $U_\alpha \in \mathscr{T}_\alpha$ with the additional condition that $U_\alpha = X_\alpha$ for all but finitely many indices α. The reason for this choice of open sets is that these are the *least* needed to make the projection onto the α-th factor π_α continuous for all indices α.

Any map $f\colon Y \to \prod_\alpha X_\alpha$ is continuous with respect to the product topology if and only if each of its component maps $\pi_\alpha \circ f\colon Y \to X_\alpha$ is continuous. However, this is not true for the box topology when there are infinitely many factors. For example, the map

$$f\colon \mathbb{R} \to (\mathbb{R}^{\mathbb{N}}, \mathscr{T}_{\text{box}}), \quad t \mapsto (t, t, t, \ldots),$$

is not continuous, since

$$(-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots \in \mathscr{T}_{\text{box}}$$

but

$$f^{-1}\left((-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots\right) = \{0\},$$

which is not open in $\mathbb{R}$.

Problem 1 Let X_α be topological spaces, and $A_\alpha \subset X_\alpha$. Consider the box topology $\mathscr{T}_{\text{box}}$ and the product topology $\mathscr{T}_{\text{product}}$ on $\prod_\alpha X_\alpha$.

(1) Show that in both topologies, we have $\overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}$.

(2) Prove that for $\mathscr{T}_{\text{box}}$, $\operatorname{Int}\left(\prod_\alpha A_\alpha\right) = \prod_\alpha \operatorname{Int}(A_\alpha)$, and give an example showing that this equality may fail for $\mathscr{T}_{\text{product}}$.

Solution (1) Let us show the inclusion in both directions for either topology.

$\boxed{\supset}$ Let (x_α) be a point of $\prod_\alpha \overline{A_\alpha}$ and $U = \prod_\alpha U_\alpha$ be a basis element for either $\mathscr{T}_{\text{box}}$ or $\mathscr{T}_{\text{product}}$ that contains (x_α). Since $x_\alpha \in \overline{A_\alpha}$ and U_α is an open neighborhood of x_α, we can choose a point

$y_\alpha \in U_\alpha \cap (A_\alpha \setminus \{x_\alpha\})$ for each α. Then $(y_\alpha) \in U \cap \left(\prod_\alpha A_\alpha \setminus \{(x_\alpha)\}\right)$. Since U is arbitrary, it follows that $(x_\alpha) \in \overline{\prod_\alpha A_\alpha}$.

$\boxed{\subset}$ Suppose $(x_\alpha) \in \overline{\prod_\alpha A_\alpha}$ in either topology. For each given index β, let V_β be an arbitrary open set of X_β containing x_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod_\alpha X_\alpha$ in either topology, if $x_\beta \notin A_\beta$, then $\pi_\beta^{-1}(V_\beta)$ contains a point $(y_\alpha) \in \prod_\alpha A_\alpha$. Then $y_\beta \in V_\beta \cap (A_\beta \setminus \{x_\beta\})$. Since V_β is arbitrary, it follows that $x_\beta \in \overline{A_\beta}$.

(2) We first show the inclusion in both directions for $\mathscr{T}_{\text{box}}$.

$\boxed{\supset}$ Since each $\text{Int}(A_\alpha)$ is open in X_α, the inclusion holds.

$\boxed{\subset}$ Let $(x_\alpha) \in \text{Int}\left(\prod_\alpha A_\alpha\right)$, then for each α there is an open set $U_\alpha \subset X_\alpha$ containing x_α with

$$\prod_\alpha U_\alpha \subset \text{Int}\left(\prod_\alpha A_\alpha\right) \subset \prod_\alpha A_\alpha.$$

This implies $U_\alpha \subset A_\alpha$ for each α, hence $U_\alpha \subset \text{Int}(A_\alpha)$ and $(x_\alpha) \in \prod_\alpha \text{Int}(A_\alpha)$.

This equality fails for $\mathscr{T}_{\text{product}}$. Take $X_n = \mathbb{R}$ and $A_n = (0,1)$ for each $n \in \mathbb{N}$. Then $\prod_{n=1}^\infty \text{Int}(A_n) = \prod_{n=1}^\infty (0,1)$. However, by the definition of product topology, this cannot be an open set in $\prod_{n=1}^\infty X_n$, and hence $\text{Int}\left(\prod_{n=1}^\infty A_n\right) \neq \prod_{n=1}^\infty \text{Int}(A_n)$. □

Problem 2 Let (X_n, d_n) be compact metric spaces. Define a metric on $X = \prod_{n=1}^\infty X_n$ by

$$d((x_n),(y_n)) := \sum_{n=1}^\infty \frac{d_n(x_n,y_n)}{[1+\text{diam}(X_n)]\cdot 2^n}.$$

Prove that on X the metric topology $\mathscr{T}_d$ coincides with the product topology $\mathscr{T}_{\text{product}}$.

Proof $\boxed{\mathscr{T}_d \subset \mathscr{T}_{\text{product}}}$ Suppose $U \in \mathscr{T}_d$. Then for any $(a_n) \in U$ there exists $r > 0$ such that $\mathbb{B}((a_n), r) \subset U$. Consider

$$V = \mathbb{B}_1\left(a_1, \tfrac{r}{2}\right) \times \cdots \times \mathbb{B}_N\left(a_N, \tfrac{r}{2}\right) \times X_{N+1} \times X_{N+2} \times \cdots \in \mathscr{T}_{\text{product}}.$$

Take $N \in \mathbb{N}$ such that $2^{-N} < \frac{r}{2}$, then for all $(b_n) \in V$ we have

$$\begin{aligned} d((a_n),(b_n)) &= \sum_{n=1}^\infty \frac{d_n(a_n,b_n)}{[1+\text{diam}(X_n)]\cdot 2^n} \leqslant \sum_{n=1}^N \frac{\frac{r}{2}}{2^n} + \sum_{n=N+1}^\infty \frac{1}{2^n} \\ &= \frac{r}{2}\left(1 - \frac{1}{2^N}\right) + \frac{1}{2^N} < \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Hence $(a_n) \in V \subset \mathbb{B}((a_n), r) \subset U$, which implies $U \in \mathscr{T}_{\text{product}}$.

$\boxed{\mathscr{T}_{\text{product}} \subset \mathscr{T}_d}$ Suppose $V \in \mathscr{T}_{\text{product}}$, which has the form

$$V = V_1 \times \cdots \times V_k \times X_{k+1} \times X_{k+2} \times \cdots ,$$

where each V_i is open in X_i. For any $(x_n) \in V$, there exists $r_1, \ldots, r_k > 0$ such that $\mathbb{B}(x_i, r_i) \subset V_i$ for $1 \leqslant i \leqslant k$. Let

$$r_0 = \min\left\{\frac{r_1}{[1+\operatorname{diam}(X_1)] \cdot 2^1}, \ldots, \frac{r_k}{[1+\operatorname{diam}(X_k)] \cdot 2^k}\right\} > 0,$$

then $(x_n) \in \mathbb{B}((x_n), r_0) \subset V$. Hence $V \in \mathscr{T}_d$. □

Problem 3 Show that $\mathbb{S}^n$ is path-connected for all $n \geqslant 1$.

Proof 1 Given distinct points $x, y \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$:

- If $y \neq -x$, then the path
$$\gamma\colon [0,1] \to \mathbb{S}^n, \quad t \mapsto \frac{(1-t)x + ty}{\|(1-t)x + ty\|}$$
is continuous and connects x to y.

- If $y = -x$, then we can choose some $z \in \mathbb{S}^n$ such that $z \neq \pm x$. By the previous case, there exist paths $\gamma_1, \gamma_2 \in \mathcal{C}([0,1], \mathbb{S}^n)$ such that γ_1 connects x to z and γ_2 connects z to y. Then the concatenation of γ_1 and γ_2,
$$\gamma_1 * \gamma_2\colon [0,1] \to \mathbb{S}^n, \quad t \mapsto \begin{cases} \gamma_1(2t), & 0 \leqslant t \leqslant 1/2, \\ \gamma_2(2t-1), & 1/2 \leqslant t \leqslant 1, \end{cases}$$
is a path connecting x to y. □

Proof 2 Consider the spherical coordinates representation of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$:

$$\begin{aligned} x_1 &= \cos(\theta_1), \\ x_2 &= \sin(\theta_1)\cos(\theta_2), \\ x_3 &= \sin(\theta_1)\sin(\theta_2)\cos(\theta_3), \\ &\ \vdots \\ x_n &= \sin(\theta_1)\cdots\sin(\theta_{n-1})\cos(\theta_n), \\ x_{n+1} &= \sin(\theta_1)\cdots\sin(\theta_{n-1})\sin(\theta_n). \end{aligned}$$

For any two points
$$\boldsymbol{\theta}^{(i)} = \left(\theta_1^{(i)}, \ldots, \theta_n^{(i)}\right) \in \mathbb{S}^n, \quad i = 1, 2,$$
we can define a path $\gamma \in \mathcal{C}([0,1], \mathbb{S}^n)$ from $\boldsymbol{\theta}^{(1)}$ to $\boldsymbol{\theta}^{(2)}$ by

$$\gamma(t) = (1-t)\boldsymbol{\theta}^{(1)} + t\boldsymbol{\theta}^{(2)}.$$ □

Proof 3 Choose the north pole $N = (0, 0, \ldots, 0, 1) \in \mathbb{S}^n$, and define the stereographic projection

$$\sigma\colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$$

by projecting points from N onto the hyperplane $x_{n+1} = 0$.

Given distinct points $x, y \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$:

- ◇ If $x = -y = N$, then the path

$$\gamma\colon [0,1] \to \mathbb{S}^n, \quad t \mapsto (0, \ldots, 0, \sin(\pi t), \cos(\pi t))$$

  connects x to y.

- ◇ If $x, y \neq N$, then we can find a path $\gamma \in \mathcal{C}([0,1], \mathbb{R}^n)$ connecting $\sigma(x)$ to $\sigma(y)$ in $\mathbb{R}^n$. Then the composition $\sigma^{-1} \circ \gamma$ is a path in $\mathbb{S}^n$ connecting x to y.

- ◇ If $x = N$ and $y \neq -N$, then we can first reflect both points about the hyperplane $x_{n+1} = 0$ to get x' and y', then find a path from x' to y' as in the previous case, and finally reflect the path back to get a path from x to y. □

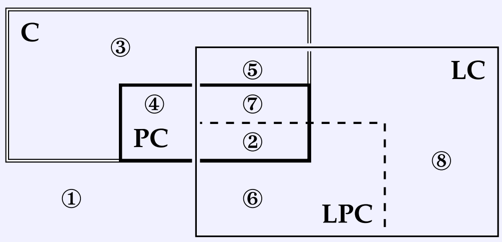
Problem 4 A topological space X is **locally (path-)connected** if for each $x \in X$, and each open neighborhood U of x, there is a (path-)connected open neighborhood V of x with $V \subset U$.

(1) Is $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ connected/locally connected/path-connected/locally path-connected?

(2) For simplicity, let us denote

C = connected, **LC** = locally connected,
PC = path-connected, **LPC** = locally path-connected.

Give examples for regions ① to ⑧ in the following diagram:



(3) Prove that if X is compact and locally connected, then X has finitely many connected components. Can we remove the local connectedness condition?

(4) Prove that X is locally connected if and only if for any open set U in X, any connected component of U is open. In particular, any connected component of a locally connected space is open.

(5) Suppose X is locally connected, $f\colon X \to Y$ is continuous. Prove that if f is either open or closed, then $f(X)$ is locally connected. Can we remove the assumption "f is either open or closed"?

Proof (1) ① Since $\mathbb{R}$ is uncountable, there are no disjoint non-empty open subsets in $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$, so it is connected.

② For any open neighbourhood U of x and any two open sets $A, B \subset \mathbb{R}$ such that $U \cap A \neq \varnothing$ and $U \cap B \neq \varnothing$, we have $U \cap A, U \cap B \in \mathscr{T}_{\text{cocountable}}$. Hence $A \cap B \cap U = (U \cap A) \cap (U \cap B)$ is the intersection of two nonempty open sets, which is nonempty for the same reason as in ①. Therefore any open neighbourhood of x is connected, so $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is locally connected.

③ For any $f \in \mathcal{C}([0,1], (\mathbb{R}, \mathscr{T}_{\text{cocountable}}))$, the image $f([0,1])$ is compact. However, compact sets in $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ must be finite (any infinite subset A contains some countable subset $\{x_n\}_{n=1}^{\infty}$, and the open cover $\left\{\left(\bigcup_{k \neq n}\{x_k\}\right)^c\right\}_{n=1}^{\infty}$ of A has no finite subcover). Since the topology on any finite set in $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is discrete, $f([0,1])$ would be totally disconnected *(i.e., a topological space that has only singletons as connected subsets)* if it has more than one point. But now it is the continuous image of $[0,1]$, so $f([0,1])$ must be a singleton. Therefore f must be constant, which means $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is not path-connected.

④ Since $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is connected but not path-connected, it is not locally path-connected by the implication

$$\mathbf{C} + \mathbf{LPC} \implies \mathbf{PC}.$$

Hint〉 For any fixed $x \in X$, the set of points path-connected to x is a clopen subset of X.

(2) ① $\mathbb{Q}$ is neither **C** nor **LC**.

② $\mathbb{R}$ is both **PC** and **LPC**.

③ The topologist's sine curve is **C** but neither **PC** nor **LC**.

④ The topologist's sine curve with an additional path from $(0,0)$ to $(1,0)$ is **PC** but not **LC**.

⑤ $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is **C** and **LC** but not **PC**, as shown in (1).

⑥ $(0,1) \cup (1,2)$ is **LPC** but not **PC**.

⑦ Let L^* be the space $\Omega \times [0,1) \cup \{(\omega_1, 0)\}$ with the lexicographic order topology, where $\Omega = [0, \omega_1)$ is the minimal uncountable well-ordered set. Denote by C the quotient of L^* identifying its initial point $(0,0)$ and final point $(\omega_1, 0)$. Then C is **PC**, **LC**, but not **LPC**. (It is not locally path-connected at $[(\omega_1, 0)]$.)

⑧ The disjoint union of two copies of ⑤-type space is **LC** but neither **C** nor **LPC**.

(3) Since X is compact and locally connected, it can be covered by finitely many connected open sets. Therefore X has finitely many connected components. The local connectedness condition is necessary. For example, the Cantor set is compact, but it has uncountably many connected components.

(4) $(\Rightarrow)$ Let C be a connected component of U. Since any point in C has a connected open neighbourhood, which must lie in C, C is open.

$(\Leftarrow)$ For any $x \in X$ and any neighbourhood V of x, V contains an open neighbourhood U of x. Then the connected component of U containing x is the desired connected open neighbourhood of x which is contained in V.

(5) We shall use the characterization of local connectedness in (4). By our assumptions, $f\colon X \to f(X)$ is an identification map. Let V be an open set in $f(X)$ and C be a connected component of V. We want to show that C is open, i.e., $f^{-1}(C)$ is open in X. For any $x \in f^{-1}(C)$, we have $x \in f^{-1}(V)$, which is open in X. Since X is locally connected, there exists a connected open neighbourhood U of x such that $U \subset f^{-1}(V)$. Then $f(U)$ is also connected and $f(U) \cap C \ni f(x)$, so $f(U) \cup C$ is connected and contained in V. Since C is a connected component of V, we must have $f(U) \cup C = C$,

i.e., $f(U) \subset C$. Therefore $x \in U \subset f^{-1}(C)$, which implies $f^{-1}(C)$ is open. It follows that C is connected and by (4) we see that $f(X)$ is locally connected.

The assumption "f is either open or closed" is necessary. Let $(X, \mathscr{T}_X)$ be any non-locally connected space. Then $(X, \mathscr{T}_{\text{discrete}})$ is locally connected and the identity map $\mathrm{Id}_X : (X, \mathscr{T}_{\text{discrete}}) \to (X, \mathscr{T}_X)$ is continuous. However, the image is not locally connected. $\square$