Theory of Valuation: ASSIGNMENT - Spring 2020

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Contents

1 Exercise 2

Under the objective probability measure $\mathbb{P}$, the underlying stock price dynamics is

$$\frac{\mathrm{d}S}{S} = (r - q + \sigma\nu)\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{P}}$$

where $r > 0$ is the risk-free rate, q is the dividend yield, $\sigma\nu$ is the risk premium ($\sigma > 0$ is the volatility and $\nu > 0$ is the market price of risk), and $\{W^{\mathbb{P}}\}$ is a Wiener process under $\mathbb{P}$. The current date is t and S is the current stock price. Also,

$$\beta_1 = -\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right) + \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}$$
$$\beta_2 = -\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right) - \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}$$

Assume $q > 0$, so that $\beta_1 > 1$. Your initial capital is $H = 2000$ euro. You cash in the additional amount of $1000\beta_1$ euro by selling short the underlying stock. The total amount of $1000(2 + \beta_1)$ euro is invested by you as follows: $1000(1 + \beta_1)$ euro are used to buy the perpetual American put with strike price K and the residual amount 1000 is employed to buy the perpetual American call with strike price K $\left(\frac{\beta_2}{\beta_2 - 1}K = L^* < S < U^* = \frac{\beta_1}{\beta_1 - 1}K\right)$. Work out the excess expected total return on your portfolio over the next instant.

Total return on the portfolio over the next instant under the physical measure $\mathbb{P}$ can be expressed as

$$\begin{aligned} \mathrm{d}H &= -1000\frac{\beta_1}{S}(\mathrm{d}S + qS\mathrm{d}t) + 1000\frac{1+\beta_1}{P}\mathrm{d}P + \frac{1000}{C}\mathrm{d}C \\ &= -\frac{H}{2}\beta_1\frac{\mathrm{d}S + qS\mathrm{d}t}{S} + \frac{H}{2}(1+\beta_1)\frac{\mathrm{d}P}{P} + \frac{H}{2}\frac{\mathrm{d}C}{C} \end{aligned}$$

The dynamics of each element of the portfolio are known. The underlying stock price, under the objective probability measure $\mathbb{P}$, has the following dynamics:

$$\begin{aligned} \frac{\mathrm{d}S + qS\mathrm{d}t}{S} &= (r + \sigma\nu)\mathrm{d}t + \sigma\mathrm{d}W^{\mathbb{P}} \\ \mathbb{E}_t^{\mathbb{P}}\left(\frac{\mathrm{d}S + qS\mathrm{d}t}{S}\right) &= (r + \nu\sigma)\mathrm{d}t \end{aligned}$$

The no-arbitrage value, $P(S, L^*)$, of the perpetual American put dynamics is

$$
\begin{aligned}
\frac{\mathrm{d}P}{P} &= (r+\beta_2\sigma\nu)\mathrm{d}t + \beta_2\sigma\mathrm{d}W^{\mathbb{P}} \\
\mathbb{E}_t^{\mathbb{P}}\left(\frac{\mathrm{d}P}{P}\right) &= (r+\beta_2\nu\sigma)\mathrm{d}t
\end{aligned}
$$

The no-arbitrage value, $C(S, U^*)$, of the perpetual American call dynamics is

$$
\begin{aligned}
\frac{\mathrm{d}C}{C} &= (r+\beta_1\sigma\nu)\mathrm{d}t + \beta_1\sigma\mathrm{d}W^{\mathbb{P}} \\
\mathbb{E}_t^{\mathbb{P}}\left(\frac{\mathrm{d}C}{C}\right) &= (r+\beta_1\nu\sigma)\mathrm{d}t
\end{aligned}
$$

After plugging the above results into the expression for $\mathrm{d}H$, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}H}{H} &= -\frac{1}{2}\beta_1\left((r+\sigma\nu)\mathrm{d}t + \sigma\mathrm{d}W^{\mathrm{P}}\right) + \frac{1}{2}(1+\beta_1)\left((r+\beta_2\sigma\nu)\mathrm{d}t + \beta_2\sigma\mathrm{d}W^{\mathrm{P}}\right) + \frac{1}{2}(1+\beta_1)\left((r+\beta_1\sigma\nu)\mathrm{d}t + \beta_1\sigma\mathrm{d}W^{\mathrm{P}}\right) = \\
&= \left(r + \frac{1}{2}\beta_2\sigma\nu(1+\beta_1)\right)\mathrm{d}t + \frac{1}{2}(1+\beta_1)\beta_2\sigma\mathrm{d}W^{\mathbb{P}}
\end{aligned}
$$

Taking expectations of both sides and exploiting the fact that $\mathbb{E}(\mathrm{d}W) = 0$,

$$
\mathbb{E}_t^{\mathbb{P}}\left(\frac{\mathrm{d}H}{H}\right) = r\mathrm{d}t + \left(\frac{\sigma\nu}{2}\left(1+\beta_1\right)\beta_2\right)\mathrm{d}t
$$

So the excess expected total return on the portfolio over the next instant is

$$
\mathbb{E}_t^{\mathbb{P}}\left(\frac{\mathrm{d}H}{H}\right) - r\mathrm{d}t = \left(\frac{\sigma\nu}{2}(1+\beta_1)\beta_2\right)\mathrm{d}t
$$