Theory of Valuation: ASSIGNMENT - Spring 2020

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1 Exercise 1

In a Black and Scholes market, the riskless asset provides the instantaneous risk-free rate $r = 1\%$. The risky asset S_1 satisfies the stochastic differential equation

$$\frac{\mathrm{d}S_1(t)}{S_1(t)} = \mu_1 \mathrm{d}t + \sigma_{11}\mathrm{d}W_1^{\mathbb{P}}(t)$$

where $W_1^{\mathbb{P}}$ is a standard Brownian motion under the historical probability measure $\mathbb{P}$. The price of the risky asset today is $S_1(0) = 1$, $\mu_1 = 5\%$ and the volatility is $\sigma_{11} = 20\%$. Let $T = 2$.

1.1 1

Find the risk neutral probability $\mathbb{Q}$ that a European call option written on S_1 with strike price $K_1 = 1$ and maturity $T = 2$ closes at maturity out of the money.

A European call option expires out of the money if $S(T) < K$. Therefore, we need to compute the following risk neutral probability:

$$\mathbb{Q}(S_1(T) < K_1)$$

where we can replace the dynamics of $S_1(t)$ under the risk-neutral probability measure $\mathbb{Q}$. When moving from $\mathbb{P}$ to $\mathbb{Q}$, the distribution is still Lognormal but the drift is now riskless and equal to the risk-free rate. The diffusion is still the same.

$$\mathbb{Q}(S_1(T) < K_1) = \mathbb{Q}\left(S_1(0)e^{\left(r-\frac{\sigma_{11}^2}{2}\right)T+\sigma_{11}W_1^{\mathbb{Q}}(T)} < K_1\right)$$

where $W_1^{\mathbb{Q}}(T) \sim \mathcal{N}(0, T)$.

$$\mathbb{Q}(S_1(T) < K_1) = \mathbb{Q}\left(S_1(0)e^{\left(r-\frac{\sigma_{11}^2}{2}\right)T+\sigma_{11}z\sqrt{T}} < K_1\right)$$

where $z \sim \mathcal{N}(0, 1)$.

$$
\begin{aligned}
\mathbb{Q}(S_1(T) < K_1) &= \mathbb{Q}\left(\left(r - \frac{\sigma_{11}^2}{2}\right)T + \sigma_{11} z\sqrt{T} < \ln\left(\frac{K_1}{S_1(0)}\right)\right) = \\
&= \mathbb{Q}\left(z < \frac{\ln\left(\frac{K_1}{S_1(0)}\right) - \left(r - \frac{\sigma_{11}^2}{2}\right)T}{\sigma_{11}\sqrt{T}}\right) = \\
&= \mathbb{Q}\left(z < \frac{\ln\left(\frac{1}{1}\right) - \left(0.01 - \frac{0.2^2}{2}\right)2}{0.2\sqrt{2}}\right) = \\
&= \mathbb{Q}\left(z < 0.0707\right) = \mathcal{N}(0.0707) = 0.5282
\end{aligned}
$$

1.2 2

The final payoff of a long position in a forward contract on S_1 with maturity $T = 2$ is $S_1(T) - F$. The forward price F is settled at time $t = 0$ in such a way that the initial no-arbitrage price of the forward contract is zero. Determine F.

$$
S_x(0) = \mathbb{E}^{\mathbb{Q}}((S_1(T) - F)e^{-rT}) = 0
$$

Since $S_1(T)$ is Lognormal,

$$
F = \mathbb{E}^{\mathbb{Q}}(S_1(T)) = S_1(0)e^{rT} = e^{0.01(2)} = 1.0202
$$

Compute the historical probability that the final payoff of the long position in the forward contract is positive.

$$
\begin{aligned}
\mathbb{P}(S_1(T) > F) &= \mathbb{P}\left(S_1(0)e^{\left(\mu_1 - \frac{\sigma_{11}^2}{2}\right)T + \sigma_{11}W_1^{\mathbb{P}}(T)} > F\right) = \\
&= \mathbb{P}\left(\left(\mu_1 - \frac{\sigma_{11}^2}{2}\right)T + \sigma_{11}W_1^{\mathbb{P}}(T) > \ln\left(\frac{F}{S_1(0)}\right)\right) = \\
&= \mathbb{P}\left(\left(\mu_1 - \frac{\sigma_{11}^2}{2}\right)T + \sigma_{11} z\sqrt{T} > \ln\left(\frac{F}{S_1(0)}\right)\right) = \\
&= \mathbb{P}\left(z > \frac{\ln\left(\frac{F}{S_1(0)}\right) - \left(\mu_1 - \frac{\sigma_{11}^2}{2}\right)T}{\sigma_{11}\sqrt{T}}\right) = \\
&= \mathbb{P}\left(z > \frac{\ln\left(\frac{1.0202}{1}\right) - \left(0.05 - \frac{0.2^2}{2}\right)2}{0.2\sqrt{2}}\right) = \\
&= \mathbb{P}(z > -0.1414) = \mathcal{N}(0.1414) = 0.5562
\end{aligned}
$$

If the volatility σ_{11} increases, does this probability increase or decrease?

$$\begin{aligned}
\mathbb{P}(S_1(T) > F) &= \mathbb{P}\left(S_1(0)e^{\left(\mu_1 - \frac{\sigma_{11}^2}{2}\right)T + \sigma_{11}W_1^{\mathbb{P}}(T)} > S_1(0)e^{rT}\right) = \\
&= \mathbb{P}\left(\left(\mu_1 - \frac{\sigma_{11}^2}{2}\right)T + \sigma_{11}W_1^{\mathbb{P}}(T) > rT\right) = \\
&= \mathbb{P}\left(\left(\mu_1 - \frac{\sigma_{11}^2}{2}\right)T + \sigma_{11}z\sqrt{T} > rT\right) = \\
&= \mathbb{P}\left(z > \frac{rT - \left(\mu_1 - \frac{\sigma_{11}^2}{2}\right)T}{\sigma_{11}\sqrt{T}}\right) = \\
&= \mathbb{P}\left(z > \frac{\left(r - \mu_1 + \frac{\sigma_{11}^2}{2}\right)\sqrt{T}}{\sigma_{11}}\right) = \\
&= \mathcal{N}\left(-\frac{\left(r - \mu_1 + \frac{\sigma_{11}^2}{2}\right)\sqrt{T}}{\sigma_{11}}\right)
\end{aligned}$$

First, note that $\mathcal{N}(\cdot)$ is an increasing function of its argument. Then we can derive the argument:

$$\frac{\partial}{\partial\sigma_{11}}\left[-\frac{\left(r - \mu_1 + \frac{\sigma_{11}^2}{2}\right)\sqrt{T}}{\sigma_{11}}\right] = -\frac{\left(\sigma_{11}^2 - 2(r - \mu_1)\right)\sqrt{T}}{2\sigma_{11}^2} = -\frac{\left(\sigma_{11}^2 - 2(0.01 - 0.05)\right)\sqrt{2}}{2\sigma_{11}^2} = -\frac{\sqrt{2}\left(\sigma_{11}^2 + 0.08\right)}{2\sigma_{11}^2}$$

σ_{11}^2 is surely larger than zero, because it is always positive. Because of the negative sign before, the total quantity is negative. Since $\mathcal{N}(\cdot)$ is an increasing function of its argument, the probability decreases when the volatility increases.

1.3 3

Suppose a new stock S_2 is introduced in the market constituted by B and the risky asset S_1. The new stock S_2 satisfies the stochastic differential equation

$$\frac{\mathrm{d}S_2(t)}{S_2(t)} = \mu_2\mathrm{d}t + \sigma_{21}\mathrm{d}W_1^{\mathbb{P}}(t) + \sigma_{22}\mathrm{d}W_2^{\mathbb{P}}(t)$$

where $W_2^{\mathbb{P}}$ is a standard Brownian motion under the historical probability measure $\mathbb{P}$-independent of $W_1^{\mathbb{P}}$. The initial price of the risky asset S_2 is $S_2(0) = 3$, $\mu_2 = 8\%$ and the the volatilities are $\sigma_{21} = 10\%$ and $\sigma_{22} = 30\%$.

Compute the market price of risk vector.

The market price of risk is defined as

$$\nu = \frac{\mu - r}{\sigma}$$

We set the following linear system:

$$\begin{cases}\sigma_{11}\nu_1 + \sigma_{12}\nu_2 = \mu_1 - r \\ \sigma_{21}\nu_1 + \sigma_{22}\nu_2 = \mu_2 - r\end{cases}$$

The solution is

$$\begin{cases} \nu_1 = 0.2 \\ \nu_2 = 0.1667 \end{cases}$$

Is the market constituted by the riskless bond B, and the two risky securities S_1 and S_2, free of arbitrage opportunities? Is it complete?

The market price of risk vector does exist and is unique. Therefore, by Girsanov theorem, the equivalent martingale measure exists and is unique. Then there is a unique risk-neutral probability $\mathbb{Q}$. Since there is a unique martingale measure, by the second Fundamental Theorem of Asset Pricing the market is arbitrage free and complete.

1.4 4

Let $n = 5$: compute the stochastic differential of

$$Y(t) = S_1(t)^n S_2(t)$$

under the risk-neutral measure $\mathbb{Q}$.

By Itô's Lemma,

$$f(x,y,t) = x^n y$$

$$\begin{cases} \dfrac{\partial f(x,y,t)}{\partial t} = 0 \\ \dfrac{\partial f(x,y,t)}{\partial x} = nx^{n-1}y \\ \dfrac{\partial f(x,y,t)}{\partial y} = x^n \\ \dfrac{\partial^2 f(x,y,t)}{\partial x \partial y} = nx^{n-1} \\ \dfrac{\partial^2 f(x,y,t)}{\partial x^2} = (n^2-n)x^{n-2}y \\ \dfrac{\partial^2 f(x,y,t)}{\partial y^2} = 0 \end{cases}$$

$$\mathrm{d}f(x,y,t) = nx^{n-1}y\mathrm{d}x + x^n\mathrm{d}y + nx^{n-1}\mathrm{d}x\mathrm{d}y + \frac{1}{2}(n^2-n)x^{n-2}y\mathrm{d}x^2$$

$$\mathrm{d}Y(t) = nS_1(t)^{n-1}S_2(t)\mathrm{d}S_1(t) + S_1(t)^n\mathrm{d}S_2(t) + nS_1(t)^{n-1}\mathrm{d}S_1(t)\mathrm{d}S_2(t) + \frac{1}{2}(n^2-n)S_1(t)^{n-2}S_2(t)\mathrm{d}S_1(t)^2$$

Replacing $\mathrm{d}S_1(t)$ and $\mathrm{d}S_2(t)$ with the risk-neutral dynamics yields

$$\begin{aligned} \mathrm{d}Y(t) = {} & nS_1(t)^{n-1}S_2(t)\left(rS_1(t)\mathrm{d}t + \sigma_{11}S_1(t)\mathrm{d}W_1^{\mathbb{Q}}(t)\right) + \\ & S_1(t)^n\left(rS_2(t)\mathrm{d}t + \sigma_{21}S_2(t)\mathrm{d}W_1^{\mathbb{Q}}(t) + \sigma_{22}S_2(t)\mathrm{d}W_2^{\mathbb{Q}}(t)\right) + \\ & nS_1(t)^{n-1}\left(rS_1(t)\mathrm{d}t + \sigma_{11}S_1(t)\mathrm{d}W_1^{\mathbb{Q}}(t)\right)\left(rS_2(t)\mathrm{d}t + \sigma_{21}S_2(t)\mathrm{d}W_1^{\mathbb{Q}}(t) + \sigma_{22}S_2(t)\mathrm{d}W_2^{\mathbb{Q}}(t)\right) + \\ & \frac{1}{2}(n^2-n)S_1(t)^{n-2}S_2(t)\left(rS_1(t)\mathrm{d}t + \sigma_{11}S_1(t)\mathrm{d}W_1^{\mathbb{Q}}(t)\right)^2 \end{aligned}$$

Computing both the quadratic and the cross-variations yields

$$\begin{aligned}
\mathrm{d}Y(t) = nS_1(t)^{n-1}S_2(t)&\left(rS_1(t)\mathrm{d}t + \sigma_{11}S_1(t)\mathrm{d}W_1^{\mathbb{Q}}(t)\right) + \\
&S_1(t)^n\left(rS_2(t)\mathrm{d}t + \sigma_{21}S_2(t)\mathrm{d}W_1^{\mathbb{Q}}(t) + \sigma_{22}S_2(t)\mathrm{d}W_2^{\mathbb{Q}}(t)\right) + \\
&nS_1(t)^{n-1}S_1(t)S_2(t)\sigma_{11}\sigma_{21}\mathrm{d}t + \\
&\frac{1}{2}(n^2-n)S_1(t)^{n-2}S_2(t)S_1(t)^2\sigma_{11}^2
\end{aligned}$$

Factorizing $Y(t)$ yields

$$\begin{aligned}
\mathrm{d}Y(t) = Y(t)&\left(nr\mathrm{d}t + n\sigma_{11}\mathrm{d}W_1^{\mathbb{Q}}(t)\right) + \\
&Y(t)\left(r\mathrm{d}t + \sigma_{21}\mathrm{d}W_1^{\mathbb{Q}}(t) + \sigma_{22}\mathrm{d}W_2^{\mathbb{Q}}(t)\right) + \\
&Y(t)\left(n\sigma_{11}\sigma_{21}\mathrm{d}t\right) + \\
&Y(t)\left(\frac{1}{2}(n^2-n)\sigma_{11}^2\mathrm{d}t\right)
\end{aligned}$$

The stochastic differential is then

$$\begin{aligned}
\mathrm{d}Y(t) = Y(t)&\left(\left(nr + r + n\sigma_{11}\sigma_{21} + \frac{1}{2}(n^2-n)\sigma_{11}^2\right)\mathrm{d}t + (n\sigma_{11}+\sigma_{21})\mathrm{d}W_1^{\mathbb{Q}}(t) + \sigma_{22}\mathrm{d}W_2^{\mathbb{Q}}(t)\right) = \\
= Y(t)&\left(\left(5(0.01) + 0.01 + 5(0.2)(0.1) + \frac{1}{2}(5^2-5)0.2^2\right)\mathrm{d}t + (5(0.2)+0.1)\,\mathrm{d}W_1^{\mathbb{Q}}(t) + 0.3\mathrm{d}W_2^{\mathbb{Q}}(t)\right) = \\
= Y(t)&\left(0.56\mathrm{d}t + 1.1\mathrm{d}W_1^{\mathbb{Q}}(t) + 0.3\mathrm{d}W_2^{\mathbb{Q}}(t)\right)
\end{aligned}$$

Let

$$\sigma_Y = \|[n\sigma_{11}+\sigma_{21},\ \sigma_{22}]\| = \sqrt{(n\sigma_{11}+\sigma_{21})^2 + \sigma_{22}^2} = \sqrt{(5(0.2)+0.1)^2 + 0.3^2} = 1.1402$$

Which is the distribution of Y under $\mathbb{Q}$? Compute the risk-neutral drift and volatility of Y.

The distribution of Y under $\mathbb{Q}$ is Lognormal with drift $\mu_Y^{\mathbb{Q}} = 0.56$ and volatility $\sigma_Y = 1.1402$:

$$Y(t) = Y(0)e^{\left(\left(\mu_Y^{\mathbb{Q}} - \frac{1}{2}\sigma_Y^2\right)t + \sigma_Y W^{\mathbb{Q}}(t)\right)} = 3e^{\left(\left(0.56 - \frac{1}{2}1.1402^2\right)t + 1.1402W^{\mathbb{Q}}(t)\right)} = 3e^{\left(-0.09t + 1.1402W^{\mathbb{Q}}(t)\right)}$$

1.5 5

The terminal payoff of a European derivative with maturity $T = 2$ is

$$X(T) = \begin{cases} e^{rT} & \text{if} \quad Y(T) < Y(0) \\ 0 & \text{if} \quad Y(T) \geq Y(0) \end{cases}$$

Determine its no-arbitrage price at $t = 0$.

$$
\begin{aligned}
S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left(X(T)e^{-rT}\right) = \\
&= \mathbb{E}^{\mathbb{Q}}\left(\mathbb{I}_{Y(T)<Y(0)}e^{rT}e^{-rT}\right) = \\
&= \mathbb{E}^{\mathbb{Q}}\left(\mathbb{I}_{Y(T)<Y(0)}\right) = \\
&= \mathbb{Q}\left(Y(T) < Y(0)\right) = \\
&= \mathbb{Q}\left(Y(0)e^{\left(\left(\mu_Y^{\mathbb{Q}}-\frac{1}{2}\sigma_Y^2\right)T+\sigma_Y W^{\mathbb{Q}}(T)\right)} < Y(0)\right) = \\
&= \mathbb{Q}\left(\left(\mu_Y^{\mathbb{Q}}-\frac{1}{2}\sigma_Y^2\right)T+\sigma_Y W^{\mathbb{Q}}(T) < 0\right) = \\
&= \mathbb{Q}\left(\left(\mu_Y^{\mathbb{Q}}-\frac{1}{2}\sigma_Y^2\right)T+\sigma_Y z\sqrt{T} < 0\right) = \\
&= \mathbb{Q}\left(z < -\frac{\left(\mu_Y^{\mathbb{Q}}-\frac{1}{2}\sigma_Y^2\right)T}{\sigma_Y\sqrt{T}}\right) = \\
&= \mathbb{Q}\left(z < -\frac{\left(0.56-\frac{1}{2}1.1402^2\right)2}{1.1402\sqrt{2}}\right) = \\
&= \mathbb{Q}\left(z < 0.1116\right) = \mathcal{N}\left(0.1116\right) = 0.5444
\end{aligned}
$$

1.6 6

Determine the no-arbitrage price at of the derivative of the previous point for any $t \in (0, T)$.

$$
\begin{aligned}
S_X(t) &= \mathbb{E}^{\mathbb{Q}}\left(X(T)e^{-r(T-t)}|\mathcal{F}_t\right) = \\
&= \mathbb{E}^{\mathbb{Q}}{}_t\left(X(T)\right)e^{-r(T-t)} = \\
&= \mathbb{E}^{\mathbb{Q}}{}_t\left(\mathbb{I}_{Y(T)<Y(0)}e^{rT}e^{-r(T-t)}\right) = \\
&= \mathbb{E}^{\mathbb{Q}}{}_t\left(\mathbb{I}_{Y(T)<Y(0)}e^{rt}\right)
\end{aligned}
$$

Since the function is $\mathcal{F}_t$-measurable,

$$
S_X(t) = \mathbb{Q}_t\left(Y(T) < Y(0)\right)e^{rt}
$$

Note that

$$
\begin{aligned}
\mathbb{Q}_t\left(Y(T) < Y(0)\right) &= \mathbb{Q}_t\left(\frac{Y(T)}{Y(t)} < \frac{Y(0)}{Y(t)}\right) = \\
&= \mathbb{Q}_t\left(\frac{Y(0)e^{\left(\left(\mu_Y^{\mathbb{Q}}-\frac{1}{2}\sigma_Y^2\right)T+\sigma_Y W^{\mathbb{Q}}(T)\right)}}{Y(0)e^{\left(\left(\mu_Y^{\mathbb{Q}}-\frac{1}{2}\sigma_Y^2\right)t+\sigma_Y W^{\mathbb{Q}}(t)\right)}} < \frac{Y(0)}{Y(t)}\right) = \\
&= \mathbb{Q}_t\left(e^{\left(\left(\mu_Y^{\mathbb{Q}}-\frac{1}{2}\sigma_Y^2\right)(T-t)+\sigma_Y\left(W^{\mathbb{Q}}(T)-W^{\mathbb{Q}}(t)\right)\right)} < \frac{Y(0)}{Y(t)}\right)
\end{aligned}
$$

Since $W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)$ is $\mathcal{F}_t$-independent and $\frac{Y(0)}{Y(t)}$ is $\mathcal{F}_t$-measurable, we have that

$$\begin{aligned}\mathbb{Q}_t\left(Y(T) < Y(0)\right) &= \mathbb{Q}_t\left(e^{\left(\left(\mu_Y^{\mathbb{Q}} - \frac{1}{2}\sigma_Y^2\right)(T-t) + \sigma_Y\left(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)\right)\right)} < \frac{Y(0)}{Y(t)}\right) = \\ &= \mathbb{Q}_t\left(e^{\left(\left(\mu_Y^{\mathbb{Q}} - \frac{1}{2}\sigma_Y^2\right)(T-t) + \sigma_Y z\sqrt{T-t}\right)} < \frac{Y(0)}{Y(t)}\right) = \\ &= \mathbb{Q}_t\left(z < \frac{\ln\left(\frac{Y(0)}{Y(t)}\right) - \left(\mu_Y^{\mathbb{Q}} - \frac{1}{2}\sigma_Y^2\right)(T-t)}{\sigma_Y\sqrt{T-t}}\right) = \\ &= \mathcal{N}\left(\frac{\ln\left(\frac{Y(0)}{Y(t)}\right) - \left(\mu_Y^{\mathbb{Q}} - \frac{1}{2}\sigma_Y^2\right)(T-t)}{\sigma_Y\sqrt{T-t}}\right)\end{aligned}$$

Then

$$S_X(t) = \mathcal{N}\left(\frac{\ln\left(\frac{Y(0)}{Y(t)}\right) - \left(\mu_Y^{\mathbb{Q}} - \frac{1}{2}\sigma_Y^2\right)(T-t)}{\sigma_Y\sqrt{T-t}}\right) e^{rt}$$

Is the replicating portfolio of this derivative long or short on S_1 and S_2?

Let

$$\xi = \frac{\ln\left(\frac{Y(0)}{Y(t)}\right) - \left(\mu_Y^{\mathbb{Q}} - \frac{1}{2}\sigma_Y^2\right)(T-t)}{\sigma_Y\sqrt{T-t}}$$

The replicating portfolio is made up by:

- θ_1 units of S_1;
- θ_2 units of S_2.

The replicating portfolio must be self-financing and locally riskless.

$$\begin{cases}\theta_1(t) = \dfrac{\partial\mathcal{N}(\xi)}{\partial S_1(t)} e^{rT} \\ \theta_2(t) = \dfrac{\partial\mathcal{N}(\xi)}{\partial S_2(t)} e^{rT}\end{cases}$$

By the chain rule,

$$\frac{\partial\mathcal{N}(\xi)}{\partial S_i} = f_{\mathcal{N}(0,1)}(\xi)\frac{\partial\xi}{\partial S_i}$$

where $f_{\mathcal{N}(0,1)}(\cdot)$ denotes the standard Normal probability density function.

$$\begin{cases}\dfrac{\partial\xi}{\partial S_1(t)} = -\dfrac{n}{\sigma_Y\sqrt{T-t}S_1(t)} \\ \dfrac{\partial\xi}{\partial S_2(t)} = -\dfrac{1}{\sigma_Y\sqrt{T-t}S_2(t)}\end{cases}$$

Since $\theta_1(t) < 0$ and $\theta_2(t) < 0$, the replicating portfolio of this derivative is short on both S_1 and S_2.

Proof of the partial derivatives:

$$\frac{\partial}{\partial S_1(t)}\left[\frac{\ln\left(\frac{Y(0)}{S_1(t)^n S_2(t)}\right) - \left(\mu_Y^{\mathbb{Q}} - \frac{1}{2}\sigma_Y^2\right)(T-t)}{\sigma_Y\sqrt{T-t}}\right] =$$

$$= \frac{1}{\sigma_Y\sqrt{T-t}}\left(\frac{\partial}{\partial S_1(t)}\left[\ln\left(\frac{Y(0)}{S_1(t)^n S_2(t)}\right)\right] + \frac{\partial}{\partial S_1(t)}\left[-\left(\mu_Y^{\mathbb{Q}} - \frac{1}{2}\sigma_Y^2\right)(T-t)\right]\right) =$$

$$= \frac{\frac{S_1(t)^n S_2(t)}{Y(0)}\frac{\partial}{\partial S_1(t)}\left[\frac{Y(0)}{S_1(t)^n S_2(t)}\right]}{\sigma_Y\sqrt{T-t}} =$$

$$= \frac{\frac{Y(0)}{S_2(t)}\frac{\partial}{\partial S_1(t)}\left[\frac{1}{S_1(t)^n}\right] S_1(t)^n S_2(t)}{\sigma_Y\sqrt{T-t}Y(0)} =$$

$$= -\frac{(-n)S_1(t)^{-n-1}S_1(t)^n}{\sigma_Y\sqrt{T-t}} =$$

$$= -\frac{n}{\sigma_Y\sqrt{T-t}S_1(t)}$$

$$\frac{\partial}{\partial S_2(t)}\left[\frac{\ln\left(\frac{Y(0)}{S_1(t)^n S_2(t)}\right) - \left(\mu_Y^{\mathbb{Q}} - \frac{1}{2}\sigma_Y^2\right)(T-t)}{\sigma_Y\sqrt{T-t}}\right] =$$

$$= \frac{1}{\sigma_Y\sqrt{T-t}}\left(\frac{\partial}{\partial S_2(t)}\left[\ln\left(\frac{Y(0)}{S_1(t)^n S_2(t)}\right)\right] + \frac{\partial}{\partial S_2(t)}\left[-\left(\mu_Y^{\mathbb{Q}} - \frac{1}{2}\sigma_Y^2\right)(T-t)\right]\right) =$$

$$= \frac{\frac{S_1(t)^n S_2(t)}{Y(0)}\frac{\partial}{\partial S_2(t)}\left[\frac{Y(0)}{S_1(t)^n S_2(t)}\right]}{\sigma_Y\sqrt{T-t}} =$$

$$= \frac{\frac{Y(0)}{S_1(t)^n}\frac{\partial}{\partial S_2(t)}\left[\frac{1}{S_2(t)}\right] S_1(t)^n S_2(t)}{\sigma_Y\sqrt{T-t}Y(0)} =$$

$$= -\frac{\frac{\frac{\partial S_2(t)}{\partial S_2(t)}}{S_2(t)^2}S_2(t)}{\sigma_Y\sqrt{T-t}} =$$

$$= -\frac{1}{\sigma_Y\sqrt{T-t}S_2(t)}$$