CS101 Algorithms and Data Structures

Dynamic Programming

Textbook Ch 15

Fibonacci numbers

Consider this function:

```
double F( int n ) {
    return ( n <= 1 ) ? 1.0 : F(n - 1) + F(n - 2);
}
```

The run-time of this algorithm is

$$T(n) = \begin{cases} \Theta(1) & n \le 1 \\ T(n-1) + T(n-2) + \Theta(1) & n > 1 \end{cases}$$

Fibonacci numbers

Consider this function:

```
double F( int n ) {
    return ( n <= 1 ) ? 1.0 : F(n - 1) + F(n - 2);
}
```

The runtime is similar to the actual definition of Fibonacci numbers:

$$T(n) = \begin{cases} \Theta(1) & n \le 1 \\ T(n-1) + T(n-2) + \Theta(1) & n > 1 \end{cases} \qquad F(n) = \begin{cases} 1 & n \le 1 \\ F(n-1) + F(n-2) + 1 & n > 1 \end{cases}$$

$T(n) = O(2^n)$

2^n

Fibonacci numbers

Problem:

- To calculate $F(44)$, it is necessary to calculate $F(43)$ and $F(42)$
- However, to calculate $F(43)$, it is also necessary to calculate $F(42)$
- It gets worse, for example
  - $F(40)$ is called 5 times
  - $F(30)$ is called 620 times
  - $F(20)$ is called 75 025 times
  - $F(10)$ is called 9 227 465 times
  - $F(0)$ is called 433 494 437 times

Surely we don't have to recalculate $F(10)$ almost ten million times…

Fibonacci numbers

Here is a possible solution:

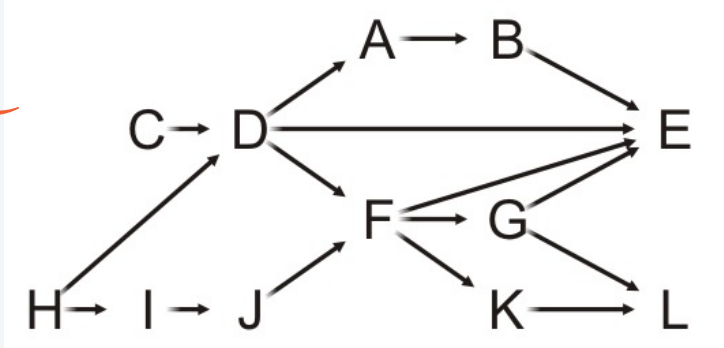
- To avoid calculating values multiple times, store intermediate calculations in a table
- When storing intermediate results, this process is called *memoization*
  - The root is *memo*
- We save (*memoize*) computed answers for possible later reuse, rather than re-computing the answer multiple times

Connected

Determining if two vertices are connected in a DAG, we could implement the following:

```
bool Weighted_graph::connected( int i, int j ) {
    if ( adjacent( i, j ) ) {
        return true;
    }

    for ( int v : neighbors( i ) ) {
        if ( connected( v, j ) ) {
            return true;
        }
    }
    return false;
}
```



- What are the issues with this implementation?

Dynamic programming

In solving optimization problems, the top-down approach may require repeatedly obtaining optimal solutions for the same sub-problem

- Mathematician Richard Bellman initially formulated the concept of dynamic programming in 1953 to solve such problems
- This isn't new, but Bellman formally defined this process

Dynamic programming

Dynamic programming is distinct from divide-and-conquer, as the divide-and-conquer approach works well if the sub-problems are essentially unique

- Storing intermediate results would only waste memory

If sub-problems re-occur, the problem is said to have *overlapping sub-problems*

Algorithmic paradigms

Greed. Process the input in some order, myopically making irrevocable decisions.

Divide-and-conquer. Break up a problem into independent subproblems; solve each subproblem; combine solutions to subproblems to form solution to original problem.

Dynamic programming. Break up a problem into a series of overlapping subproblems; combine solutions to smaller subproblems to form solution to large subproblem.

fancy name for
caching intermediate results
in a table for later reuse

memo

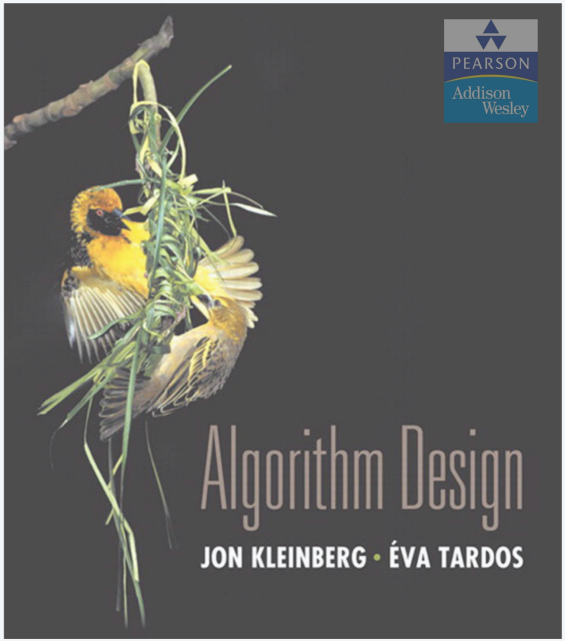
Dynamic programming applications

Application areas.

- Computer science: AI, compilers, systems, graphics, theory, ….
- Operations research.
- Information theory.
- Control theory.
- Bioinformatics.

Some famous dynamic programming algorithms.

- Avidan–Shamir for seam carving.
- Unix diff for comparing two files.
- Viterbi for hidden Markov models.
- De Boor for evaluating spline curves.
- Bellman–Ford–Moore for shortest path.
- Knuth–Plass for word wrapping text in TEX.
- Cocke–Kasami–Younger for parsing context-free grammars.
- Needleman–Wunsch/Smith–Waterman for sequence alignment.



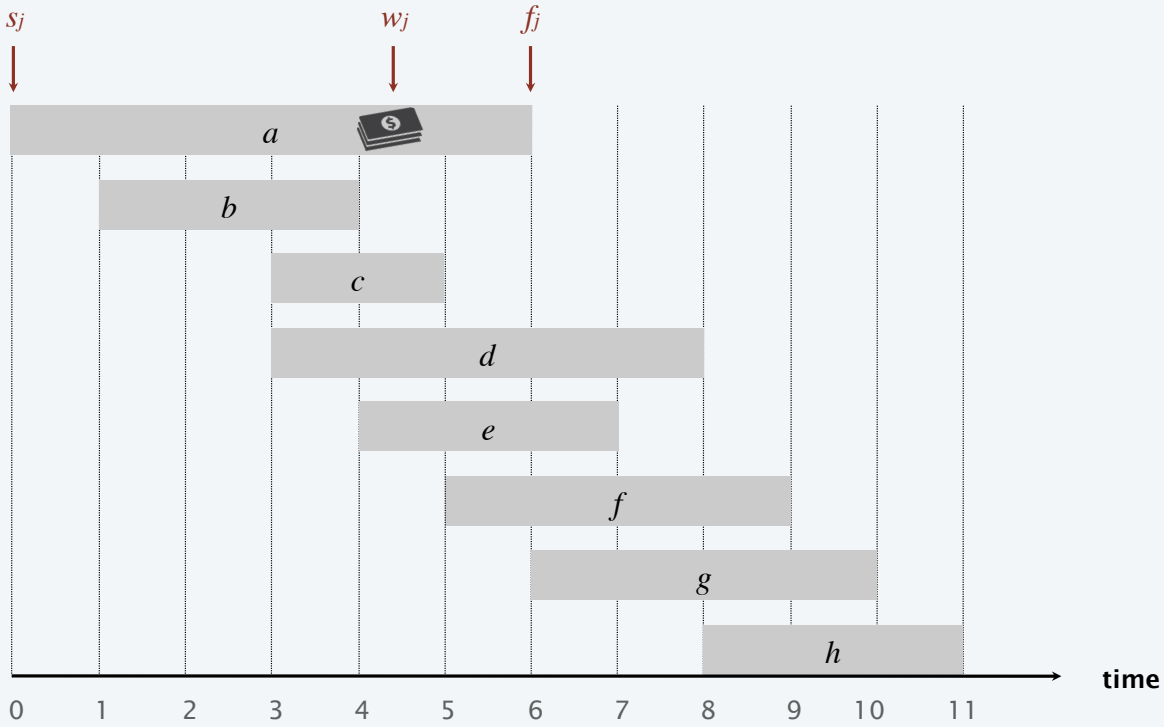
SECTIONS 6.1–6.2

DYNAMIC PROGRAMMING

- *weighted interval scheduling*
- *segmented least squares*
- *knapsack problem*

Weighted interval scheduling

- Job j starts at s_j, finishes at f_j, and has weight $w_j > 0$.
- Two jobs are compatible if they don't overlap.
- Goal: find max-weight subset of mutually compatible jobs.



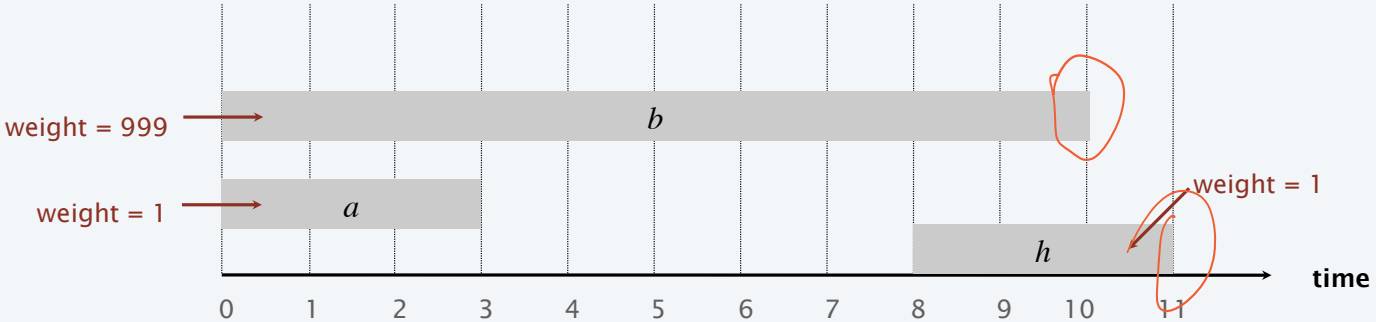
Earliest-finish-time first algorithm

Earliest finish-time first.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Recall. Greedy algorithm is correct if all weights are 1.

Observation. Greedy algorithm fails spectacularly for weighted version.



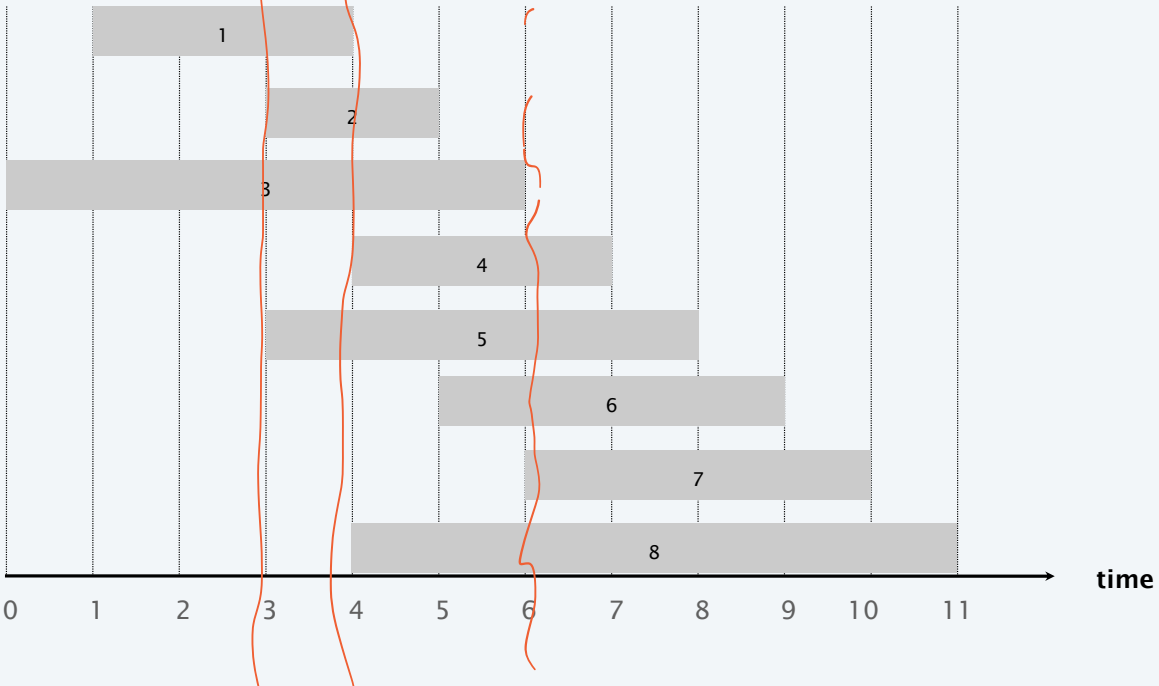
Weighted interval scheduling

Convention. Jobs are in ascending order of finish time: $f_1 \leq f_2 \leq \ldots \leq f_n$.

Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j.

Ex. $p(8) = 1, p(7) = 3, p(2) = 0$.

i is leftmost interval that ends before j begins



Dynamic programming: binary choice

Def. $OPT(j)$ = max weight of any subset of mutually compatible jobs for subproblem consisting only of jobs $1, 2, \ldots, j$.

Goal. $OPT(n)$ = max weight of any subset of mutually compatible jobs.

Case 1. $OPT(j)$ does not select job j.

- Must be an optimal solution to problem consisting of remaining jobs $1, 2, \ldots, j-1$.

Case 2. $OPT(j)$ selects job j.

- Collect profit w_j.
- Can't use incompatible jobs $\{ p(j)+1, p(j)+2, \ldots, j-1 \}$.
- Must include optimal solution to problem consisting of remaining compatible jobs $1, 2, \ldots, p(j)$.

optimal substructure property (proof via exchange argument)

Bellman equation.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ OPT(j-1),\ w_j + OPT(p(j)) \} & \text{if } j > 0 \end{cases}$$

Weighted interval scheduling: brute force

BRUTE-FORCE $(n, s_1, \ldots, s_n, f_1, \ldots, f_n, w_1, \ldots, w_n)$

Sort jobs by finish time and renumber so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

Compute $p[1], p[2], \ldots, p[n]$ via binary search.

RETURN COMPUTE-OPT(n).

COMPUTE-OPT(j)

IF ($j = 0$)

  RETURN 0.

ELSE

  RETURN max {COMPUTE-OPT($j-1$), w_j + COMPUTE-OPT($p[j]$) }.

Dynamic programming: quiz 1

What is running time of COMPUTE-OPT(n) in the worst case?

A. $\Theta(n \log n)$

B. $\Theta(n^2)$

C. $\Theta(1.618^n)$

D. $\Theta(2^n)$

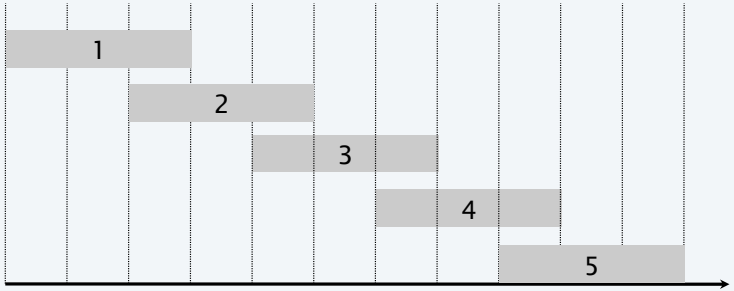
```
COMPUTE-OPT(j)

IF (j = 0)
    RETURN 0.
ELSE
    RETURN max {COMPUTE-OPT(j – 1), wj + COMPUTE-OPT(p[j]) }.
```

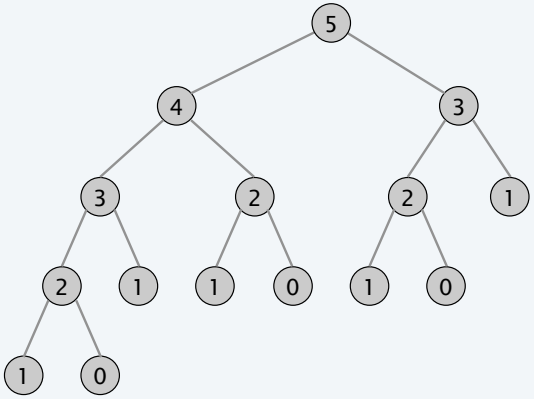
Weighted interval scheduling: brute force

Observation. Recursive algorithm is spectacularly slow because of overlapping subproblems ⇒ exponential-time algorithm.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



p(1) = 0, p(j) = j-2



recursion tree

Weighted interval scheduling: memoization

Top-down dynamic programming (memoization).

- Cache result of subproblem j in $M[j]$.
- Use $M[j]$ to avoid solving subproblem j more than once.

TOP-DOWN$(n, s_1, \ldots, s_n, f_1, \ldots, f_n, w_1, \ldots, w_n)$

Sort jobs by finish time and renumber so that $f_1 \le f_2 \le \ldots \le f_n$.

Compute $p[1], p[2], \ldots, p[n]$ via binary search.

$M[0] \leftarrow 0$. ← global array

RETURN M-COMPUTE-OPT(n).

nlogn

M-COMPUTE-OPT(j)

IF ($M[j]$ is uninitialized)

$M[j] \leftarrow \max \{$ M-COMPUTE-OPT $(j-1),\ w_j +$ M-COMPUTE-OPT$(p[j]) \}$.

RETURN $M[j]$.

All : nlogn .

Weighted interval scheduling: running time

Claim. Memoized version of algorithm takes $O(n \log n)$ time.

Pf.

- Sort by finish time: $O(n \log n)$ via mergesort.
- Compute $p[j]$ for each j : $O(n \log n)$ via binary search.

- M-COMPUTE-OPT(j): each invocation takes $O(1)$ time and either
  - (1) returns an initialized value $M[j]$
  - (2) initializes $M[j]$ and makes two recursive calls

- Progress measure $\Phi = \#$ initialized entries among $M[1..n]$.
  - initially $\Phi = 0$; throughout $\Phi \leq n$.
  - increases Φ by $1 \Rightarrow \leq 2n$ recursive calls.

- Overall running time of M-COMPUTE-OPT(n) is $O(n)$. ■

Weighted interval scheduling: finding a solution

Q. DP algorithm computes optimal value. How to find optimal solution?

A. Make a second pass by calling FIND-SOLUTION(n).

FIND-SOLUTION(j)

IF $(j = 0)$

  RETURN $\emptyset$.

ELSE IF $(w_j + M[p[j]] > M[j-1])$

  RETURN $\{ j \} \cup$ FIND-SOLUTION($p[j]$).

ELSE

  RETURN FIND-SOLUTION($j-1$).

$$M[j] = \max \{ M[j-1],\ w_j + M[p[j]] \}.$$

Analysis. # of recursive calls $\leq n \Rightarrow O(n)$.

Weighted interval scheduling: bottom-up dynamic programming

Bottom-up dynamic programming. Unwind recursion.

BOTTOM-UP$(n, s_1, \ldots, s_n, f_1, \ldots, f_n, w_1, \ldots, w_n)$

Sort jobs by finish time and renumber so that $f_1 \le f_2 \le \ldots \le f_n$.

Compute $p[1], p[2], \ldots, p[n]$.

$M[0] \leftarrow 0$.

previously computed values

FOR $j = 1$ TO n

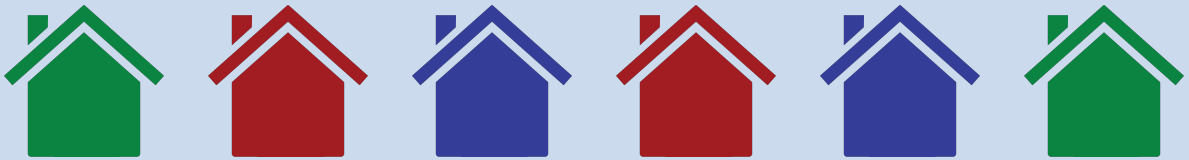
  $M[j] \leftarrow \max \{ M[j-1],\ w_j + M[p[j]] \}$.

Running time. The bottom-up version takes $O(n \log n)$ time.

HOUSE COLORING PROBLEM

Goal. Paint a row of n houses red, green, or blue so that

- No two adjacent houses have the same color.
- Minimize total cost, where $cost(i, color)$ is cost to paint i given color.



	A	B	C	D	E	F
	7	6	7	8	9	20
	3	8	9	22	12	8
	16	10	4	2	5	7

cost to paint house i the given color

HOUSE COLORING PROBLEM

Subproblems.

- $R[i]$ = min cost to paint houses $1, \ldots, i$ with i red.
- $G[i]$ = min cost to paint houses $1, \ldots, i$ with i green.
- $B[i]$ = min cost to paint houses $1, \ldots, i$ with i blue.
- Optimal cost = $\min \{ R[n], G[n], B[n] \}$.

Dynamic programming equation.

不能相同. 但要上一个最小.

- $R[i+1] = cost(i+1, red) + \min \{ B[i], G[i] \}$
- $G[i+1] = cost(i+1, green) + \min \{ R[i], B[i] \}$
- $B[i+1] = cost(i+1, blue) + \min \{ R[i], G[i] \}$

overlapping subproblems

Running time. $O(n)$. (不用 sort)

M_1, M_2, M_3.

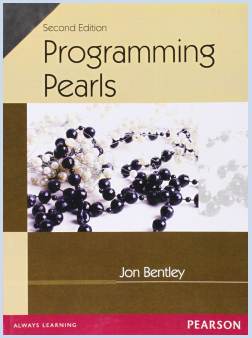
Maximum subarray problem

Goal. Given an array x of n integer (positive or negative), find a contiguous subarray whose sum is maximum.

12	5	−1	31	−61	59	26	−53	58	97	−93	−23	84	−15	6

187

Applications. Computer vision, data mining, genomic sequence analysis, technical job interviews, ...



KADANE'S ALGORITHM

Def. $OPT(i)$ = max sum of any subarray of x whose rightmost index is i.

Goal. $\max_i OPT(i)$

Bellman equation.

$$OPT(i) = \begin{cases} x_1 & \text{if } i = 1 \\ \max\{ x_i,\ x_i + OPT(i-1) \} & \text{if } i > 1 \end{cases}$$

take only element i

take element i together with best subarray ending at index $i-1$

Running time. $O(n)$.

Maximum rectangle problem

Goal. Given an n-by-n matrix A, find a rectangle whose sum is maximum.

$$A = \begin{bmatrix} -2 & 5 & 0 & -5 & -2 & 2 & -3 \\ 4 & -3 & -1 & 3 & 2 & 1 & -1 \\ -5 & 6 & 3 & -5 & -1 & -4 & -2 \\ -1 & -1 & 3 & -1 & 4 & 1 & 1 \\ 3 & -3 & 2 & 0 & 3 & -3 & -2 \\ -2 & 1 & -2 & 1 & 1 & 3 & -1 \\ 2 & -4 & 0 & 1 & 0 & -3 & -1 \end{bmatrix}$$

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Applications. Databases, image processing, maximum likelihood estimation, technical job interviews, ...

BENTLEY'S ALGORITHM

Assumption. Suppose you knew the left and right column indices j and j'.

$$A = \begin{bmatrix} -2 & 5 & 0 & -5 & -2 & 2 & -3 \\ 4 & -3 & -1 & 3 & 2 & 1 & -1 \\ -5 & 6 & 3 & -5 & -1 & -4 & -2 \\ -1 & -1 & 3 & -1 & 4 & 1 & 1 \\ 3 & -3 & 2 & 0 & 3 & -3 & -2 \\ -2 & 1 & -2 & 1 & 1 & 3 & -1 \\ 2 & -4 & 0 & 1 & 0 & -3 & -1 \end{bmatrix} \quad x = \begin{bmatrix} -7 \\ 4 \\ -3 \\ 6 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

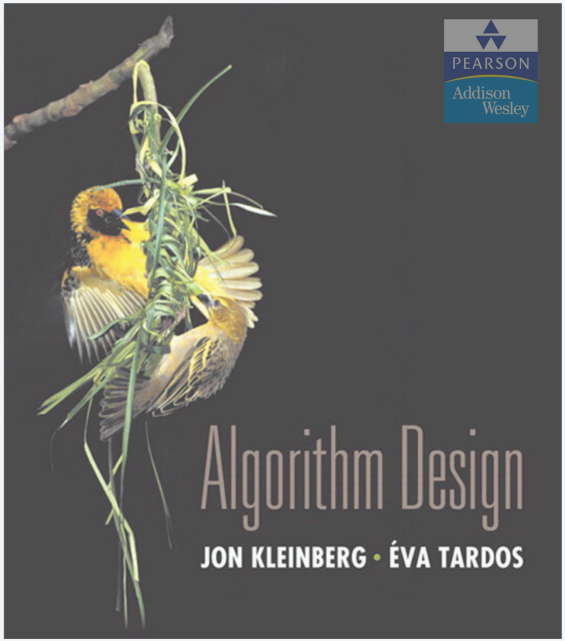
$0 - 5 - 2$

solve maximum subarray problem in this array

An $O(n^3)$ algorithm.

- Precompute cumulative row sums $S_{ij} = \sum_{k=1}^{j} A_{ik}$.
- For each $j < j'$:
  - define array x using row-sum differences: $x_i = S_{ij'} - S_{ij}$
  - run Kadane's algorithm in array x

Open problem. $O(n^{3-\varepsilon})$ for any constant $\varepsilon > 0$.



SECTION 6.3

DYNAMIC PROGRAMMING

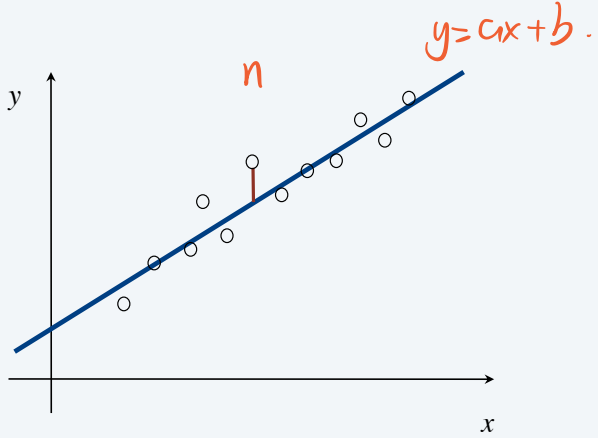
- *weighted interval scheduling*
- *segmented least squares*
- *knapsack problem*

Least squares

Least squares. Foundational problem in statistics.

- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.
- Find a line $y = ax + b$ that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$



Solution. Calculus $\Rightarrow$ min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented least squares

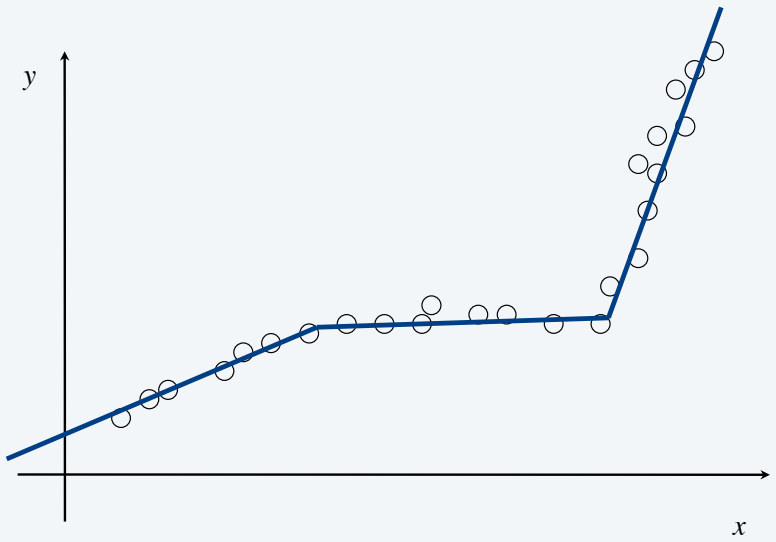
Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes $f(x)$.

Q. What is a reasonable choice for $f(x)$ to balance accuracy and parsimony?

goodness of fit

number of lines



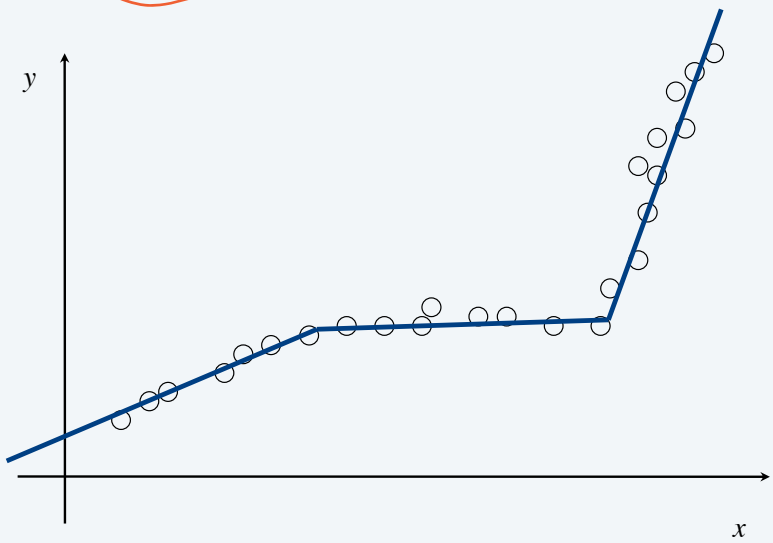
Segmented least squares

Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes $f(x)$.

Goal. Minimize $f(x) = E + c\,L$ for some constant $c > 0$, where

- E = sum of the sums of the squared errors in each segment.
- L = number of lines.



lines.

Dynamic programming: multiway choice

Notation.

- $OPT(j)$ = minimum cost for points $p_1, p_2, \ldots, p_j$.
- e_{ij} = SSE for for points $p_i, p_{i+1}, \ldots, p_j$.

To compute $OPT(j)$:

- Last segment uses points $p_i, p_{i+1}, \ldots, p_j$ for some $i \le j$.
- Cost = $e_{ij} + c + OPT(i-1)$.

optimal substructure property
(proof via exchange argument)

Bellman equation.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min\limits_{1 \le i \le j} \{ e_{ij} + c + OPT(i-1) \} & \text{if } j > 0 \end{cases}$$

Segmented least squares algorithm

SEGMENTED-LEAST-SQUARES($n, p_1, \ldots, p_n, c$)

FOR j = 1 TO n

  FOR i = 1 TO j

    Compute the SSE e_{ij} for the points $p_i, p_{i+1}, \ldots, p_j$.

$M[0] \leftarrow 0$

FOR $j = 1$ TO n

  $M[j] \leftarrow \min_{1 \le i \le j} \{ e_{ij} + c + M[i-1] \}$.

previously computed value

RETURN $M[n]$.

Segmented least squares analysis

Theorem. [Bellman 1961] DP algorithm solves the segmented least squares problem in $O(n^3)$ time and $O(n^2)$ space.

Pf.

- Bottleneck = computing SSE e_{ij} for each i and j.

$$a_{ij} = \frac{n \sum_k x_k y_k - (\sum_k x_k)(\sum_k y_k)}{n \sum_k x_k^2 - (\sum_k x_k)^2}, \quad b_{ij} = \frac{\sum_k y_k - a_{ij} \sum_k x_k}{n}$$

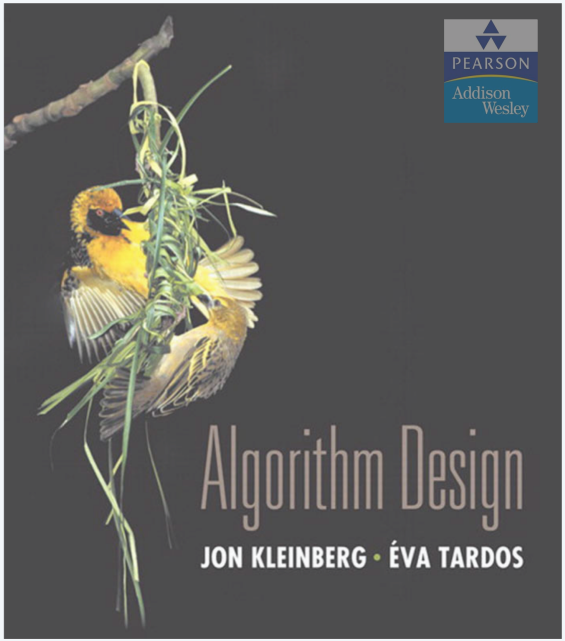
- $O(n)$ to compute e_{ij}. ▪

Remark. Can be improved to $O(n^2)$ time.

- For each i : precompute cumulative sums

$$\sum_{k=1}^{i} x_k, \quad \sum_{k=1}^{i} y_k, \quad \sum_{k=1}^{i} x_k^2, \quad \sum_{k=1}^{i} x_k y_k$$

- Using cumulative sums, can compute e_{ij} in $O(1)$ time.



SECTION 6.4

DYNAMIC PROGRAMMING

- *weighted interval scheduling*
- *segmented least squares*
- *knapsack problem*

Knapsack problem

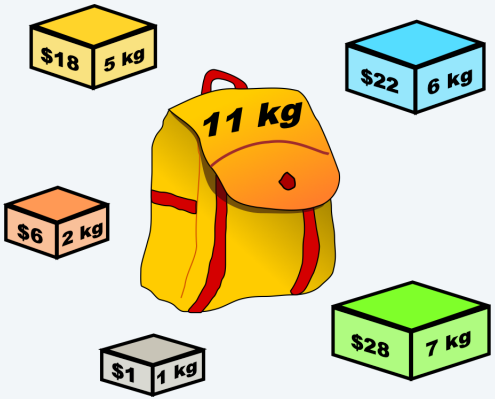
Goal. Pack knapsack so as to maximize total value.

- There are n items: item i provides value $v_i > 0$ and weighs $w_i > 0$.
- Knapsack has weight capacity of W.

Assumption. All input values are integral.

Ex. $\{ 1, 2, 5 \}$ has value \$35 (and weight 10).

Ex. $\{ 3, 4 \}$ has value \$40 (and weight 11).



i	v_i	w_i
1	US\$1	1 kg
2	US\$6	2 kg
3	US\$18	5 kg
4	US\$22	6 kg
5	US\$28	7 kg

knapsack instance (weight limit W = 11)

Dynamic programming: quiz 2

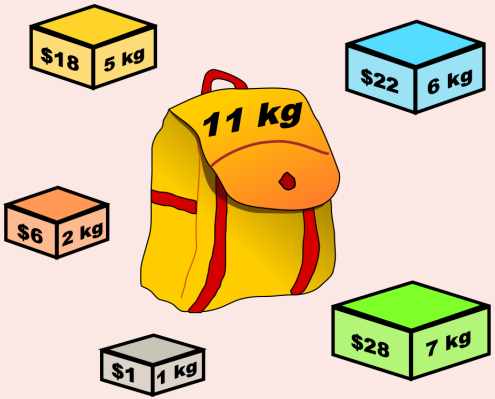
Which algorithm solves knapsack problem?

A. Greedy by value: repeatedly add item with maximum v_i.

B. Greedy by weight: repeatedly add item with minimum w_i.

C. Greedy by ratio: repeatedly add item with maximum ratio v_i / w_i.

D. Dynamic programming.



i	v_i	w_i
1	US$1	1 kg
2	US$6	2 kg
3	US$18	5 kg
4	US$22	6 kg
5	US$28	7 kg

knapsack instance
(weight limit W = 11)

Dynamic programming: quiz 3

Which subproblems?

A. $OPT(w)$ = max-profit with weight limit w.

B. $OPT(i)$ = max-profit subset of items $1, \ldots, i$.

C. $OPT(i, w)$ = max-profit subset of items $1, \ldots, i$ with weight limit w.

D. Any of the above.

Dynamic programming: false start

Def. $OPT(i)$ = max-profit subset of items $1, \ldots, i$.

Goal. $OPT(n)$.

Case 1. $OPT(i)$ does not select item i.

- OPT selects best of $\{ 1, 2, \ldots, i-1 \}$.

optimal substructure property
(proof via exchange argument)

Case 2. $OPT(i)$ selects item i.

- Selecting item i does not immediately imply that we will have to reject other items.
- Without knowing which other items were selected before i, we don't even know if we have enough room for i.

Conclusion. Need more subproblems!

Dynamic programming: adding a new variable

Def. $OPT(i, w)$ = max-profit subset of items $1, \ldots, i$ with weight limit w.

Goal. $OPT(n, W)$.

$w_i > w$

Case 1. $OPT(i, w)$ does not select item i.

possibly because $w_i > w$

- $OPT(i, w)$ selects best of $\{ 1, 2, \ldots, i-1 \}$ using weight limit w.

$w_i < w$

Case 2. $OPT(i, w)$ selects item i.

optimal substructure property
(proof via exchange argument)

- Collect value v_i.
- New weight limit $= w - w_i$.
- $OPT(i, w)$ selects best of $\{ 1, 2, \ldots, i-1 \}$ using this new weight limit.

Bellman equation.

Sub problems: exact same structure

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max \{ OPT(i-1, w),\ v_i + OPT(i-1, w - w_i) \} & \text{otherwise} \end{cases}$$

不选 i 选了 i

Knapsack problem: bottom-up dynamic programming

KNAPSACK$(n, W, w_1, \ldots, w_n, v_1, \ldots, v_n)$

FOR $w = 0$ TO W

  $M[0, w] \leftarrow 0$.

FOR $i = 1$ TO n

  FOR $w = 0$ TO W

    IF $(w_i > w)$ $M[i, w] \leftarrow M[i-1, w]$.

    ELSE $M[i, w] \leftarrow \max \{ M[i-1, w],\ v_i + M[i-1, w - w_i] \}$.

RETURN $M[n, W]$.

previously computed values

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max \{ OPT(i-1, w),\ v_i + OPT(i-1, w - w_i) \} & \text{otherwise} \end{cases}$$

Knapsack problem: bottom-up dynamic programming demo

i	v_i	w_i
1	US$1	1 kg
2	US$6	2 kg
3	US$18	5 kg
4	US$22	6 kg
5	US$28	7 kg

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{OPT(i-1,w),\ v_i + OPT(i-1,w-w_i)\} & \text{otherwise} \end{cases}$$

weight limit w

subset of items 1, …, i	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
{ }	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	35	40

OPT(i, w) = max-profit subset of items 1, …, i with weight limit w.

Knapsack problem: running time

Theorem. The DP algorithm solves the knapsack problem with n items and maximum weight W in $\Theta(n\,W)$ time and $\Theta(n\,W)$ space.

Pf.

weights are integers between 1 and W

- Takes $O(1)$ time per table entry.
- There are $\Theta(n\,W)$ table entries.
- After computing optimal values, can trace back to find solution: $OPT(i, w)$ takes item i iff $M[i, w] > M[i-1, w]$. ■

COIN CHANGING

Problem. Given n coin denominations $\{ c_1, c_2, \ldots, c_n \}$ and a target value V, find the fewest coins needed to make change for V (or report impossible).

Recall. Greedy cashier's algorithm is optimal for U.S. coin denominations, but not for arbitrary coin denominations.

Ex. { 1, 10, 21, 34, 70, 100, 350, 1295, 1500 }.

Optimal. 140¢ = 70 + 70.

COIN CHANGING

Def. $OPT(v)$ = min number of coins to make change for v.

Goal. $OPT(V)$.

Multiway choice. To compute $OPT(v)$,

- Select a coin of denomination c_i for some i.
- Select fewest coins to make change for $v - c_i$.

optimal substructure property
(proof via exchange argument)

Bellman equation.

$$OPT(v) = \begin{cases} \infty & \text{if } v < 0 \\ 0 & \text{if } v = 0 \\ \min\limits_{1 \le i \le n} \{ 1 + OPT(v - c_i) \} & \text{otherwise} \end{cases}$$

Running time. $O(n\,V)$.

Dynamic programming summary

Outline.

typically, only a polynomial number of subproblems

- Define a collection of subproblems.
- Solution to original problem can be computed from subproblems.
- Natural ordering of subproblems from "smallest" to "largest" that enables determining a solution to a subproblem from solutions to smaller subproblems.

Techniques.

- Binary choice: weighted interval scheduling.
- Multiway choice: segmented least squares.
- Adding a new variable: knapsack problem.

Top-down vs. bottom-up dynamic programming. Opinions differ.

$N! = N \cdot (N-1)!$

$1! = 1$

find $5!$?

Top down:

$5 \to 4 \to 3 \to 2 \to 1$

recursively break problem into sub problem. until to the state we known the value.

逆推

bottom up:

$1 \to 2 \to 3 \to 4 \to 5$.

knew

→ don't knew.

迭代

(有优势)

Both **Memo** 记忆

Memoization

Tabulation

动态规划至少有两种主要技术，它们并不相互排斥：

- 记忆 - 这是一种自由放任的方法：您假设您已经计算了所有子问题，并且您不知道最佳评估顺序是什么。通常，您会从根执行递归调用（或某种等效的迭代），并且要么希望您将接近最佳评估顺序，要么获得证明您将帮助您达到最佳评估顺序的证据。您将确保递归调用永远不会重新计算子问题，因为您*缓存*了结果，因此不会重新计算重复的子树。
  - *示例：* 如果您正在计算斐波那契数列`fib(100)`，您只需调用它，它就会调用`fib(100)=fib(99)+fib(98)`，它会调用`fib(99)=fib(98)+fib(97)`，...等.....，它会调用`fib(2)=fib(1)+fib(0)=1+0=1`. 然后它最终会解析`fib(3)=fib(2)+fib(1)`，但不需要重新计算`fib(2)`，因为我们缓存了它。
  - 这从树的顶部开始，并评估从叶子/子树返回到根的子问题。
- 制表 - 您也可以将动态规划视为“表格填充”算法（尽管通常是多维的，但在极少数情况下，此“表格”可能具有非欧几何*）。这类似于记忆，但更主动，并且涉及一个额外的步骤：您必须提前选择进行计算的确切顺序。这并不意味着顺序必须是静态的，而是意味着您比记忆具有更大的灵活性。
  - *示例：* 如果您正在执行斐波那契，您可以选择按以下顺序计算数字：`fib(2)`，`fib(3)`，`fib(4)`... 缓存每个值，以便您可以更轻松地计算下一个值。您也可以将其视为填充表（另一种形式的缓存）。
  - 我个人很少听到“制表”这个词，但这是一个非常体面的术语。有些人认为这是“动态规划”。
  - 在运行算法之前，程序员会考虑整棵树，然后编写一个算法来按照特定的顺序向根计算子问题，通常填写一个表格。
  - *脚注：有时，“桌子”本身并不是具有类似网格连接的矩形桌子。相反，它可能具有更复杂的结构，例如树，或特定于问题域的结构（例如地图上飞行距离内的城市），甚至格状图，虽然类似网格，但没有一个up-down-left-right连通性结构等。例如user3290797链接了一个动态规划的例子，在树中寻找最大独立集，对应于在树中填空。

（在最普遍的情况下，在“动态编程”范式中，我会说程序员考虑整个树，*然后*编写了一个算法来实现评估子问题的策略，该策略可以优化您想要的任何属性（通常是时间复杂度和空间复杂度的组合）。您的策略必须从某个特定的子问题开始，并且可能会根据这些评估的结果进行调整。在“动态编程”的一般意义上，您可能会尝试缓存这些子问题，更一般地说，尝试避免重新访问子问题，这些子问题可能有细微的区别，可能是各种数据结构中的图的情况。很多时候，这些数据结构的核心是数组或表格。如果我们不再需要子问题的解决方案，它们可以被丢弃。）