# CS101 Algorithms and Data Structures

Graph traversal

Textbook Ch 22.2/3/5

# Outline

- Graph traversal
  - Breadth-first
  - Depth-first
- Applications
  - Connectedness
  - Unweighted path length
  - Identifying bipartite graphs

# Outline

We will look at traversals of graphs

- Breadth-first or depth-first traversals
- Must avoid cycles
- Depth-first traversals can be recursive or iterative
- Problems that can be solved using traversals

# Graph Traversal

Traversals of a graph

- A means of visiting all the vertices in a graph
- Also called *searches*

Similar to tree traversal, we have breadth-first and depth-first traversals on graphs

- Breadth-first requires a queue
- Depth-first requires a stack

B q D s

# Graph Traversal

Different from tree traversal: there may be multiple paths between two vertices.

To avoid visiting a vertex for multiple times, we have to track which vertices have already been visited

- We may have an indicator variable in each vertex
- We may use a hash table or a bit array
- Requiring $\Theta(|V|)$ memory

The time complexity of graph traversal cannot be better than and should not be worse than $\Theta(|V| + |E|)$

- Connected graphs simplify this to $\Theta(|E|)$
- Worst case: $\Theta(|V|^2)$

# Breadth-first traversal

Breadth-first traversal on a graph:

- Choose any vertex, mark it as visited and push it onto queue
- While the queue is not empty:
  - Pop the top vertex *v* from the queue
  - For each vertex adjacent to *v* that has not been visited:
    - Mark it visited, and
    - Push it onto the queue

This continues until the queue is empty

- If there are no unvisited vertices, the graph is connected

The size of the queue is $O(|V|)$

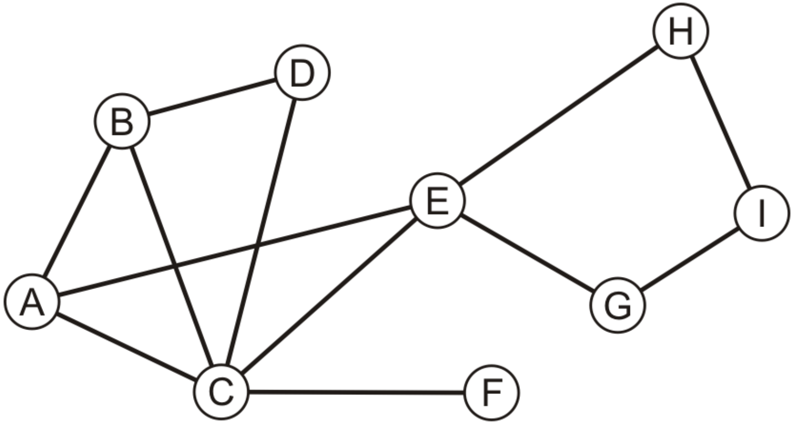
# Breadth-first traversal

The size of the queue is $O(|V|)$

– The actual size depends both on:
  - The number of edges, and
  - The out-degree of the vertices

# Example

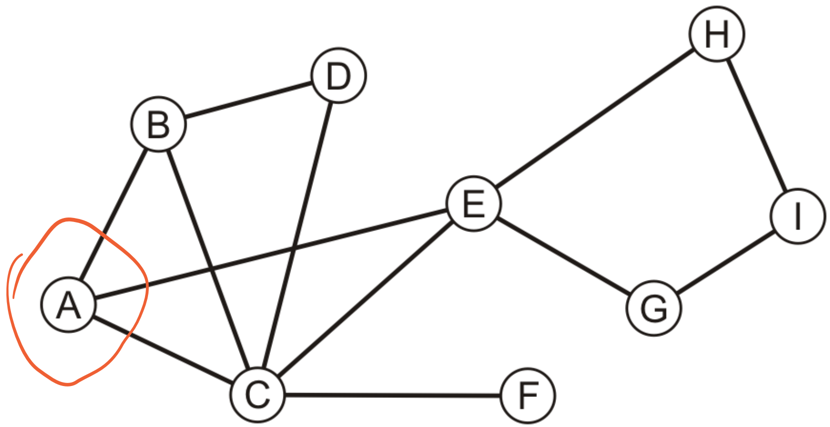
Consider this graph



# Example

Performing a breadth-first traversal

– Push the first vertex onto the queue



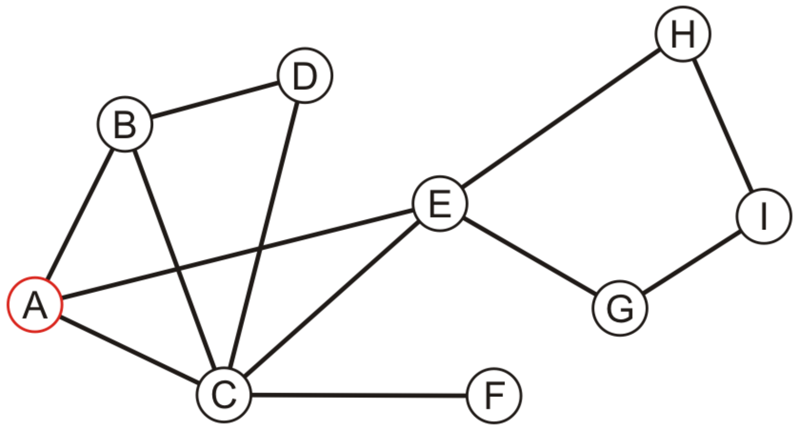

| | | | | | |
|---|---|---|---|---|---|
| A | | | | | |

# Example

Performing a breadth-first traversal

– Pop A and push B, C and E

A



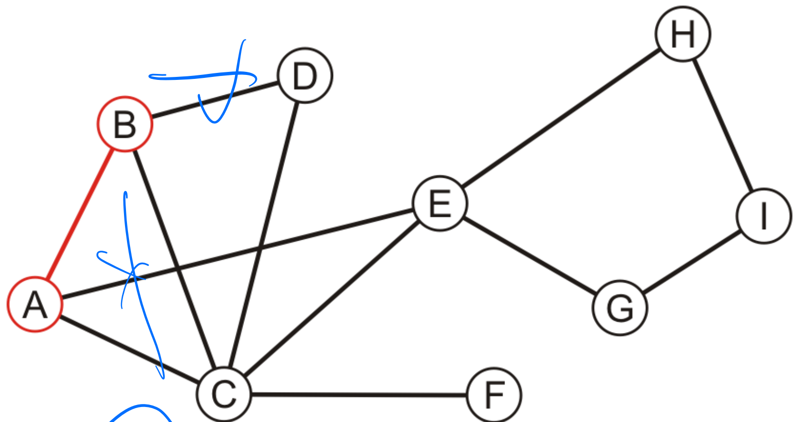

| | | | | | |
|---|---|---|---|---|---|
| B | C | E | | | |

# Example

Performing a breadth-first traversal:

– Pop B and push D

A, B



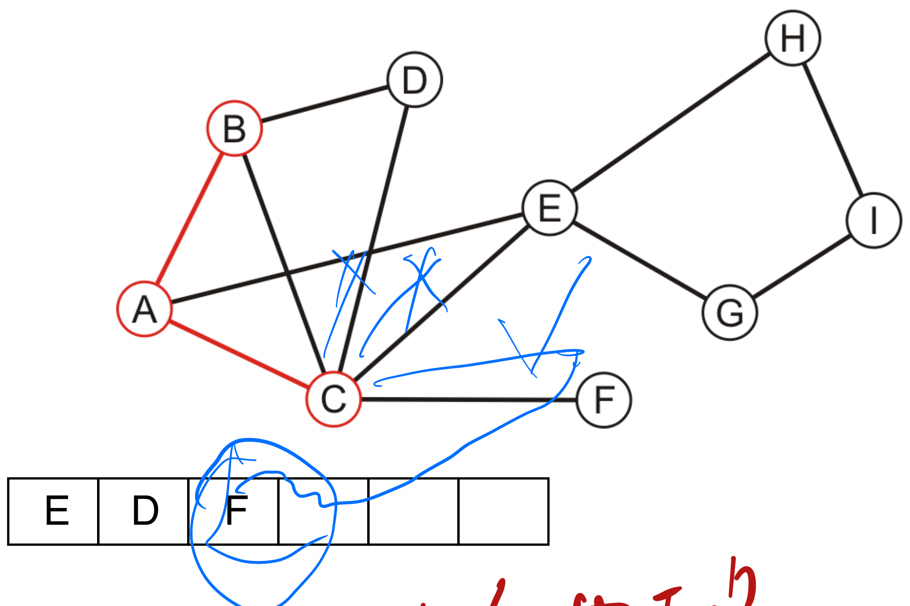

| | | | | | |
|---|---|---|---|---|---|
| C | E | D | | | |

# Example

Performing a breadth-first traversal:

– Pop C and push F

A, B, C




不能已历过的也不能 queue

的:

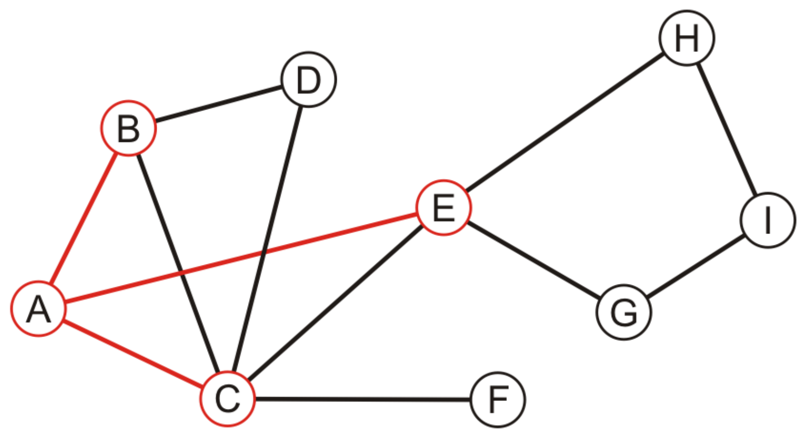
push时打印

# Example

Performing a breadth-first traversal:

– Pop E and push G and H

A, B, C, E



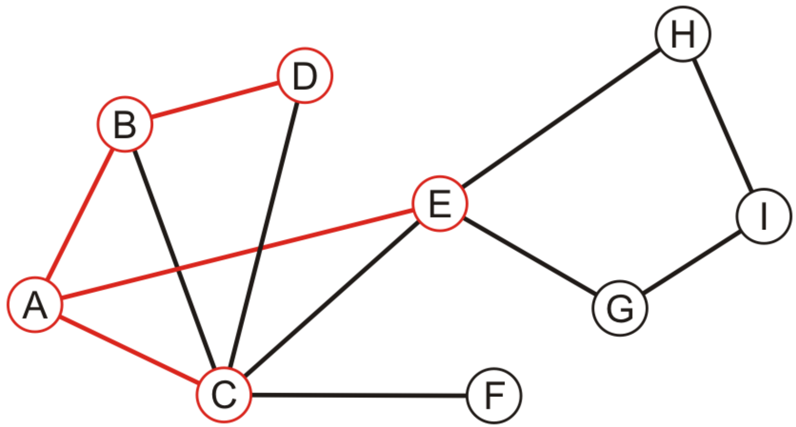

| | | | | | |
|---|---|---|---|---|---|
| D | F | G | H | | |

# Example

Performing a breadth-first traversal:

– Pop D

A, B, C, E, D



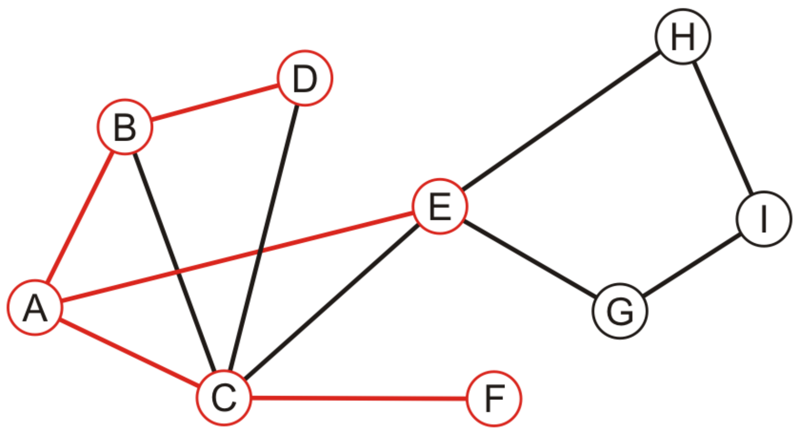

| | | | | | |
|---|---|---|---|---|---|
| F | G | H | | | |

# Example

Performing a breadth-first traversal:

– Pop F

A, B, C, E, D, F



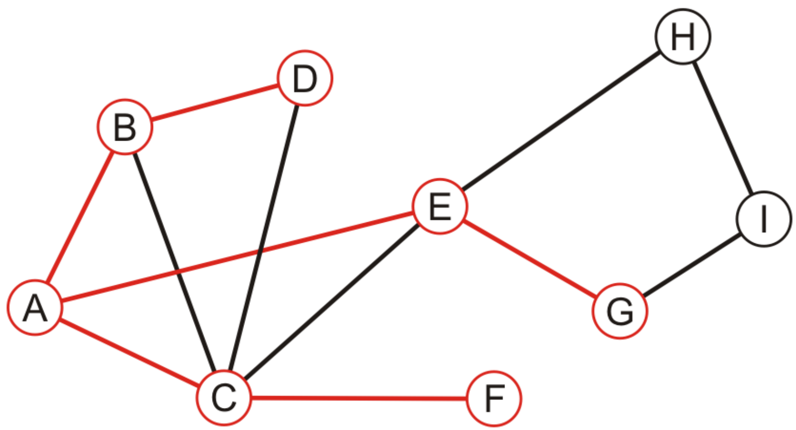

| | | | | | |
|---|---|---|---|---|---|
| G | H | | | | |

# Example

Performing a breadth-first traversal:

– Pop G and push I

A, B, C, E, D, F, G



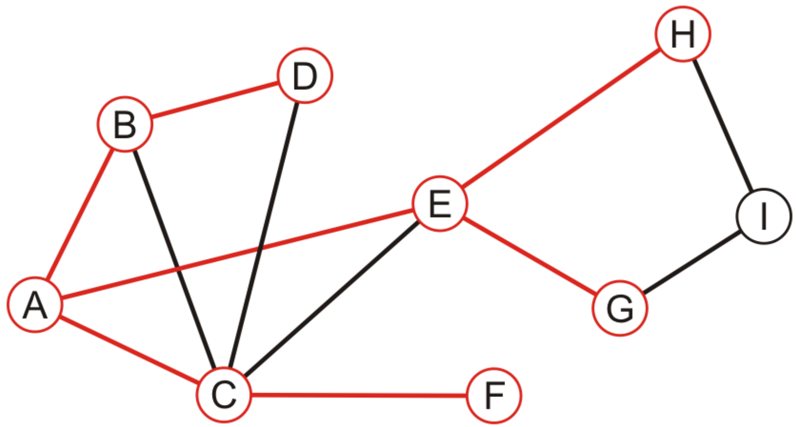

| | | | | | |
|---|---|---|---|---|---|
| H | I | | | | |

# Example

Performing a breadth-first traversal:

– Pop H

A, B, C, E, D, F, G, H



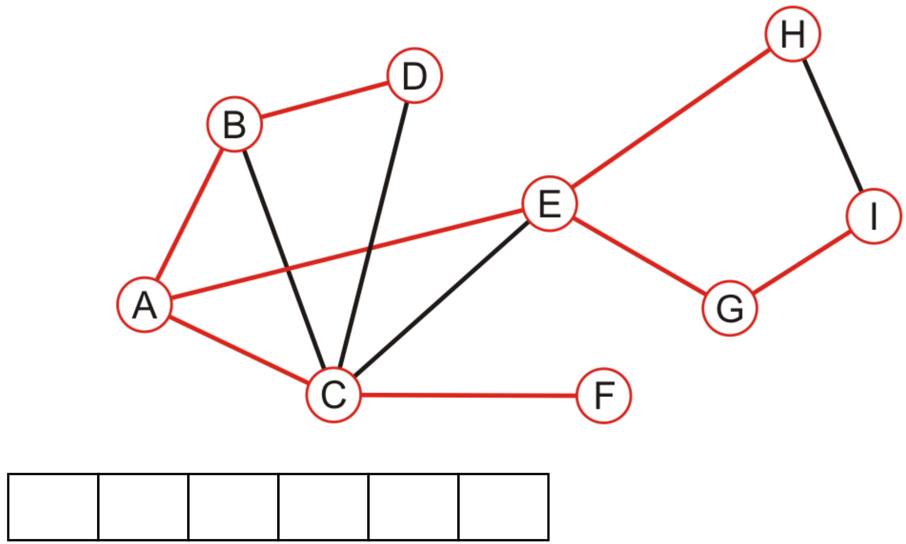

| I | | | | | |
|---|---|---|---|---|---|

# Example

Performing a breadth-first traversal:

– Pop I

A, B, C, E, D, F, G, H, I



empty-finish

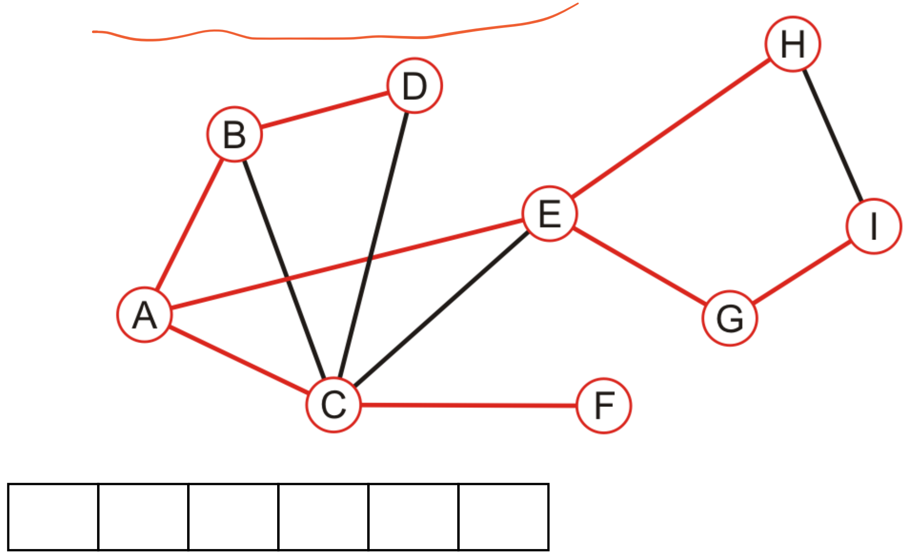
# Example

pop 序

Performing a breadth-first traversal:

– The queue is empty: we are finished

A, B, C, E, D, F, G, H, I



# Iterative depth-first traversal

An implementation can use a queue

```
void Graph::depth_first_traversal( Vertex *first ) const {
    unordered_map<Vertex *, int> hash;
    hash.insert( first );
    std::queue<Vertex *> queue;
    queue.push( first );

    while ( !queue.empty() ) {
        Vertex *v = queue.front();
        queue.pop();
        // Perform an operation on v

        for ( Vertex *w : v->adjacent_vertices() ) {
            if ( !hash.member( w ) ) {
                hash.insert( w );
                queue.push( w );
            }
        }
    }
}
```

D

# Depth-first traversal

Depth-first traversal on a graph:

- Choose any vertex, mark it as visited
- From that vertex:
  - If there is another adjacent vertex not yet visited, go to it
  - Otherwise, go back to the previous vertex
- Continue until no visited vertices have unvisited adjacent vertices

Two implementations:

- Recursive
- Use a stack

# Recursive depth-first traversal

A recursive implementation uses the call stack for memory:

```
void Graph::depth_first_traversal( Vertex *first ) const {
    std::unordered_map<Vertex *, int> hash;
    hash.insert( first );

    first->depth_first_traversal( hash );
}

void Vertex::depth_first_traversal( unordered_map<Vertex *, int> &hash ) const {
    // Perform an operation on this

    for ( Vertex *v : adjacent_vertices() ) {
        if ( !hash.member( v ) ) {
            hash.insert( v );
            v->depth_first_traversal( hash );
        }
    }
}
```

# Depth-first traversal

A recursive implementation:

```
void Vertex::depth_first_traversal() const {
    for ( Vertex *v : adjacent_vertices() ) {
        if ( !v->visited() ) {
            v->mark_visited();
            v->depth_first_traversal();
        }
    }
}
```

# Iterative depth-first traversal

for memory

An iterative implementation can use a stack

```
void Graph::depth_first_traversal( Vertex *first ) const {
    unordered_map<Vertex *, int> hash;
    hash.insert( first );
    std::stack<Vertex *> stack;
    stack.push( first );

    while ( !stack.empty() ) {
        Vertex *v = stack.top();
        stack.pop();
        // Perform an operation on v

        for ( Vertex *w : v->adjacent_vertices() ) {
            if ( !hash.member( w ) ) {
                hash.insert( w );
                stack.push( w );
            }
        }
    }
}
```

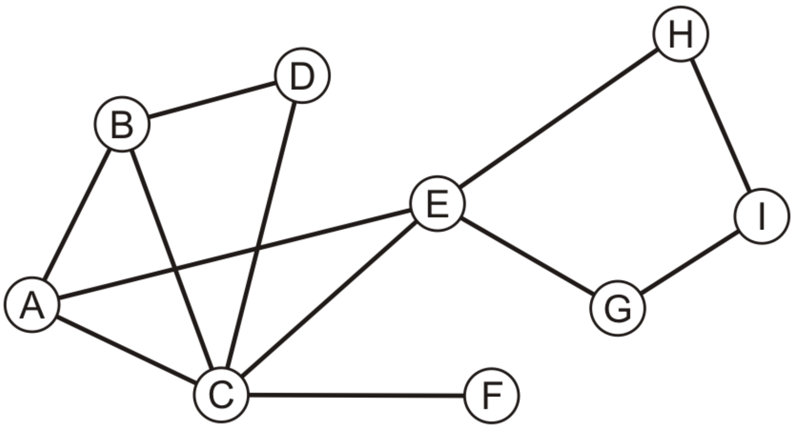
# Depth-first traversal

Use a stack:

- Choose any vertex
  - Mark it as visited
  - Place it onto an empty stack
- While the stack is not empty:
  - If the vertex on the top of the stack has an unvisited adjacent vertex *v*,
    - Mark *v* as visited
    - Place *v* onto the top of the stack (push)
  - Otherwise, pop the top of the stack

# Example

Perform a recursive depth-first traversal on this same graph

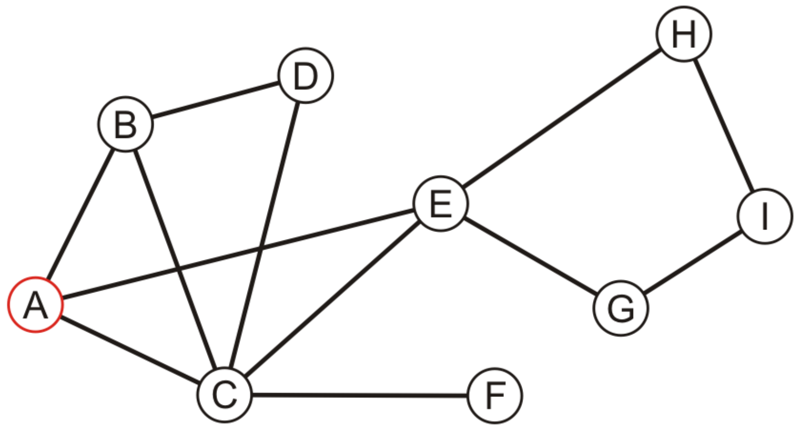


# Example

Performing a recursive depth-first traversal:

- Visit the first node

A

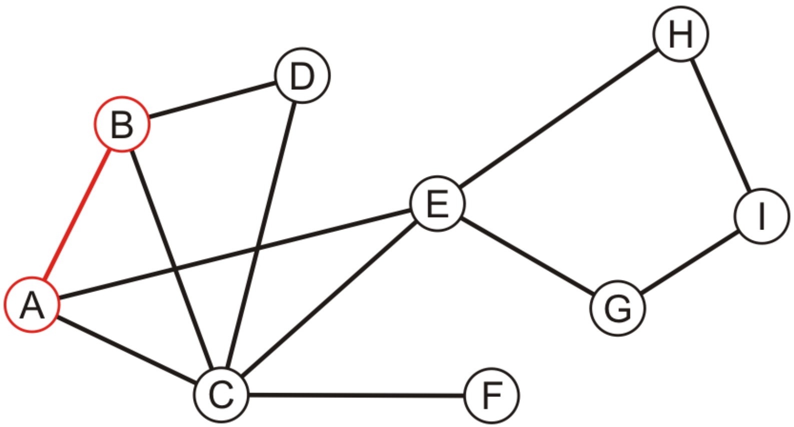


# Example

Performing a recursive depth-first traversal:

– A has an unvisited neighbor

A, B



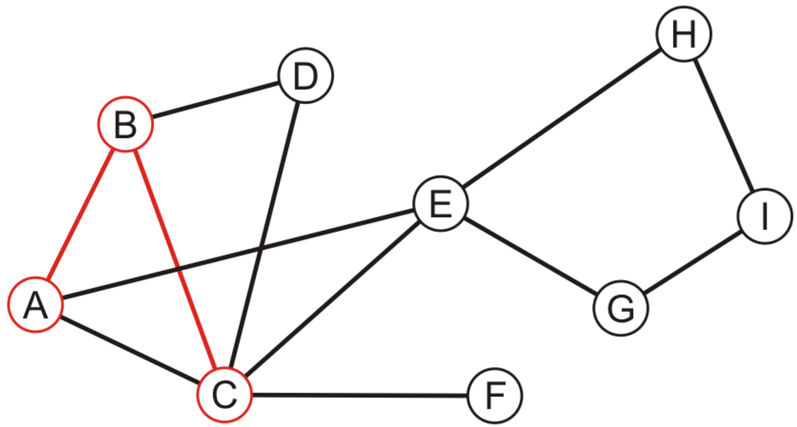
# Example

Performing a recursive depth-first traversal:

– B has an unvisited neighbor

A, B, C

没有D!

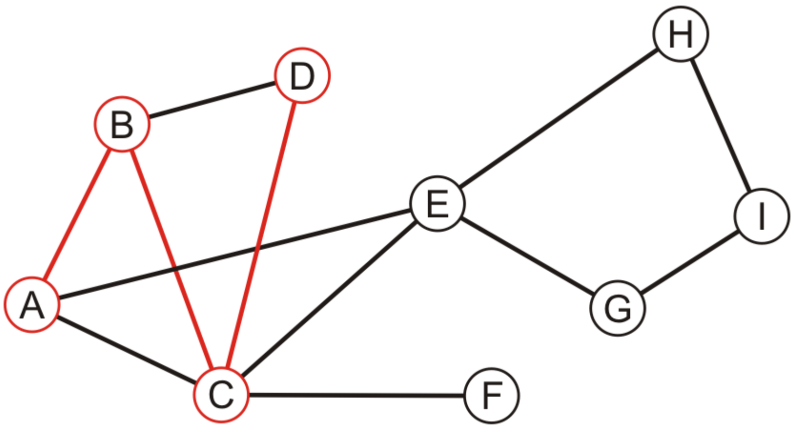


# Example

Performing a recursive depth-first traversal:

– C has an unvisited neighbor

A, B, C, D



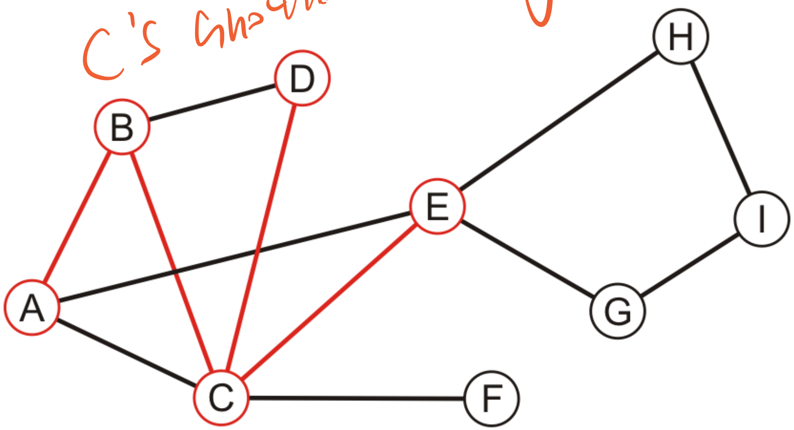
# Example

Performing a recursive depth-first traversal:

– D has no unvisited neighbors, so we return to C

A, B, C, D, E

C's another neighbor

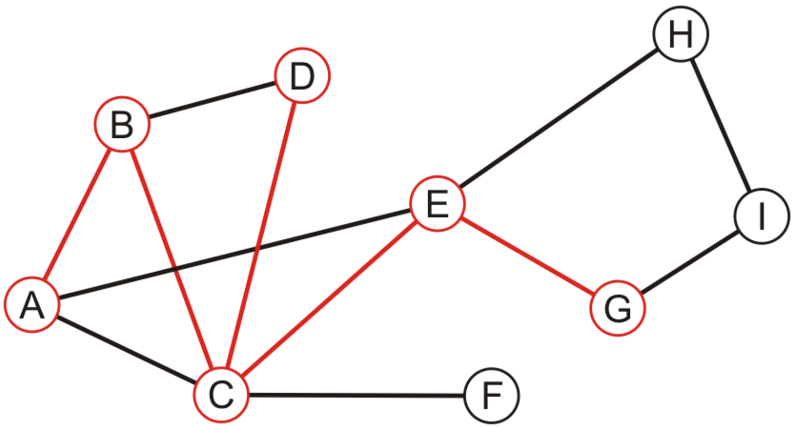


# Example

Performing a recursive depth-first traversal:

– E has an unvisited neighbor

A, B, C, D, E, G

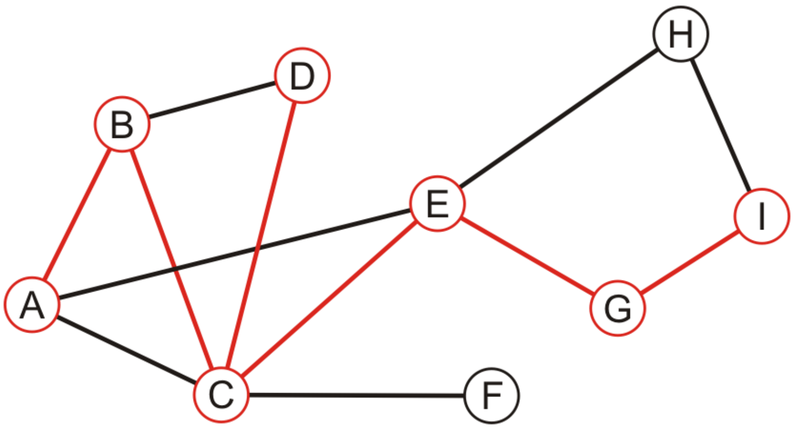


# Example

Performing a recursive depth-first traversal:

– F has an unvisited neighbor

A, B, C, D, E, G, I

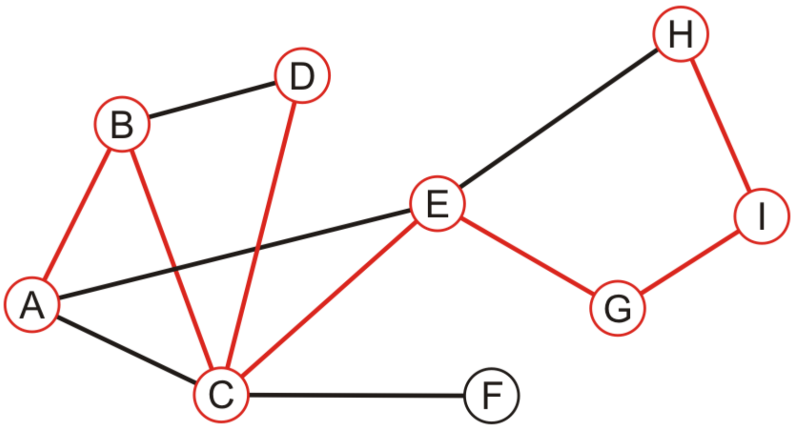


# Example

Performing a recursive depth-first traversal:

– H has an unvisited neighbor

A, B, C, D, E, G, I, H



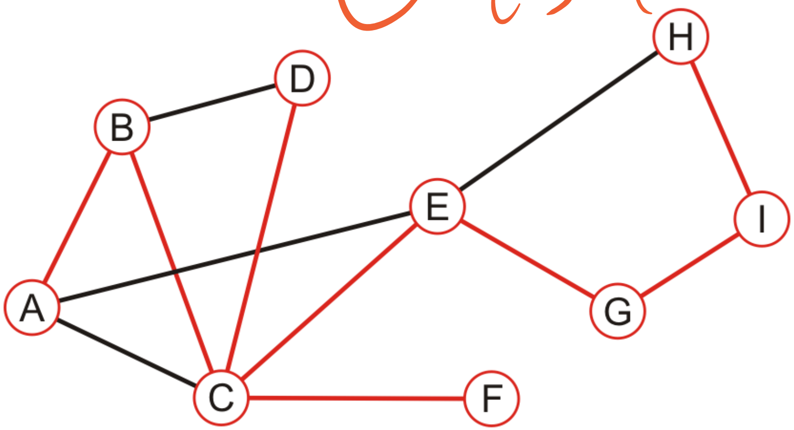
# Example

Performing a recursive depth-first traversal:

– We recurse back to C which has an unvisited neighbour

A, B, C, D, E, G, I, H, F

C's (一路回去找的)

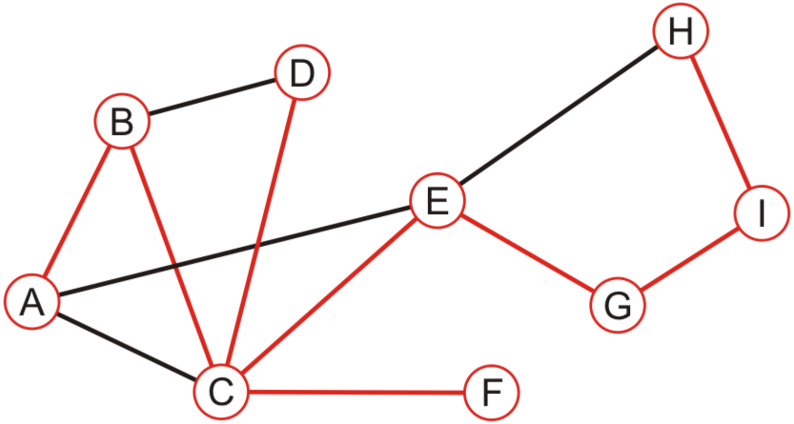


# Example

Performing a recursive depth-first traversal:

– We recurse finding that no other nodes have unvisited neighbours

A, B, C, D, E, G, I, H, F

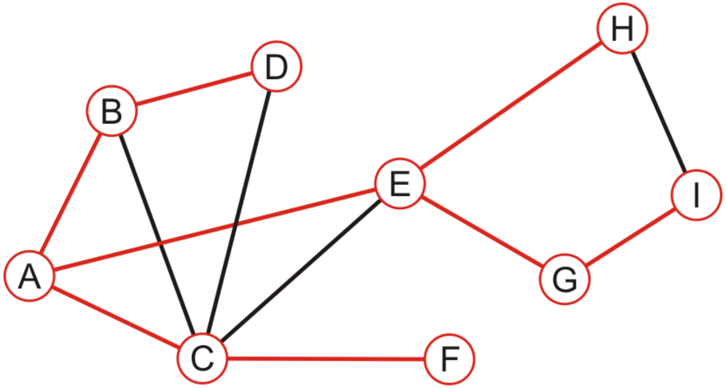


# Comparison

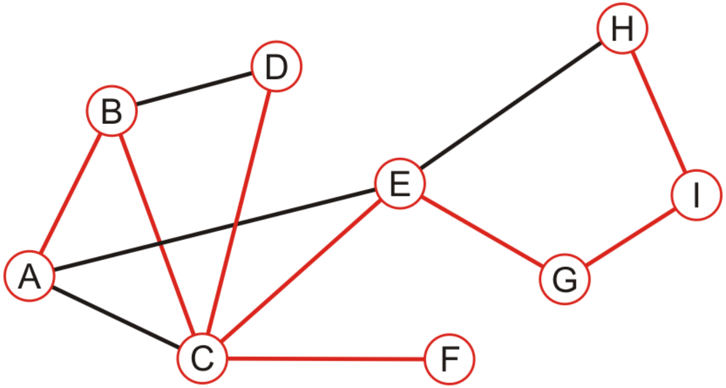
The order in which vertices can differ greatly

– An iterative depth-first traversal may also be different again

A, B, C, E, D, F, G, H, I




A, B, C, D, E, G, I, H, F



# Applications

Applications of tree traversals include:

- Determining connectiveness and finding connected sub-graphs
- Determining the path length from one vertex to all others
- Testing if a graph is bipartite
- Determining maximum flow
- Cheney's algorithm for garbage collection

# Summary

This topic covered graph traversals

- Considered breadth-first and depth-first traversals
- Depth-first traversals can recursive or iterative
- More overhead than traversals of rooted trees
- Considered a STL approach to the design
- Considered an example with both implementations
- They are also called *searches*

# References

Wikipedia, http://en.wikipedia.org/wiki/Graph_traversal
http://en.wikipedia.org/wiki/Depth-first_search
http://en.wikipedia.org/wiki/Breadth-first_search

# Outline

- Graph traversal
  - Breadth-first
  - Depth-first
- Applications
  - Connectedness
  - Unweighted path length
  - Identifying bipartite graphs

# Outline

We will use graph traversals to determine:

- Whether one vertex is connected to another
- The connected sub-graphs of a graph

# Connected

First, let us determine whether one vertex is connected to another

– $v_j$ is connected to $v_k$ if there is a path from the first to the second

Strategy:

– Perform a breadth-first traversal starting at $v_j$
– If the vertex $v_k$ is ever found during the traversal, return true
– Otherwise, return false

# Connected

B

Consider implementing a breadth-first traversal on an undirected graph:

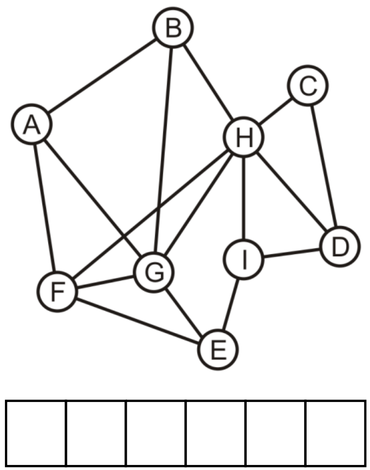
- Choose any vertex, mark it as visited and push it onto queue
- While the queue is not empty:
  - Pop to top vertex *v* from the queue
  - For each vertex adjacent to *v* that has not been visited:
    - Mark it visited, and
    - Push it onto the queue

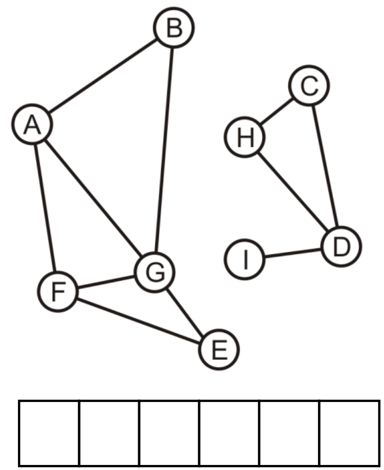
This continues until the queue is empty

- Note: if there are ~~no unvisited vertices,~~ the graph is connected,

# Determining Connections

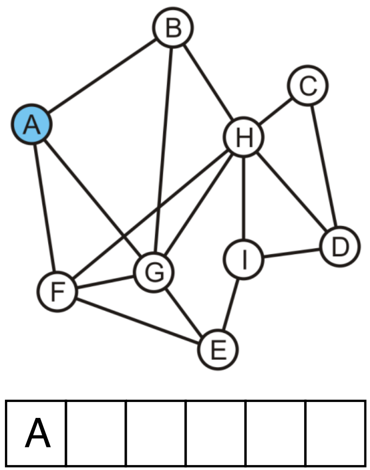
Is A connected to D?

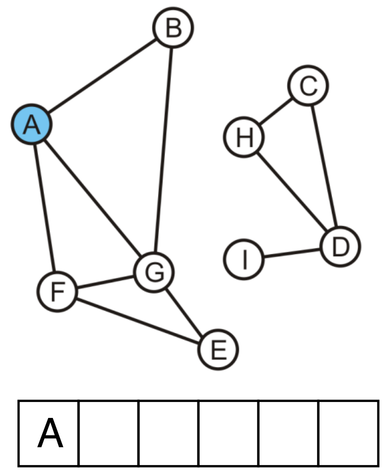


# Determining Connections

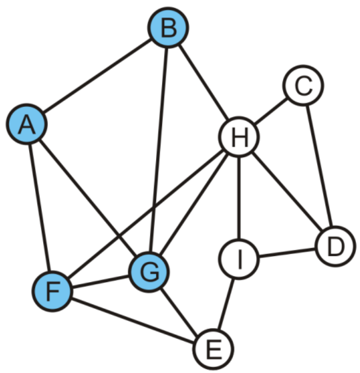
Vertex A is marked as visited and pushed onto the queue



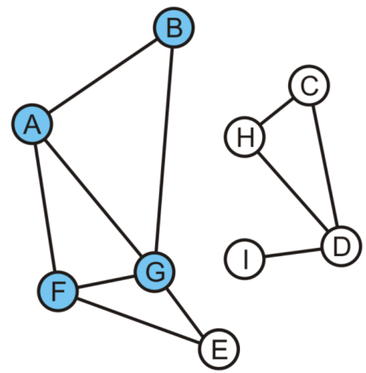
# Determining Connections

Pop the head, A, and mark and push B, F and G




| | | | | | |
|---|---|---|---|---|---|
| B | F | G | | | |

乱序



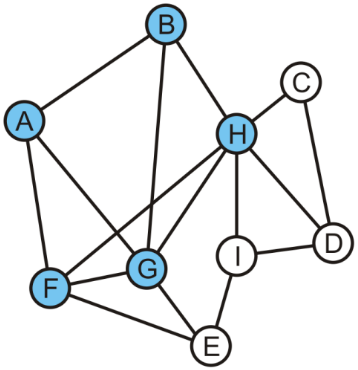

| | | | | | |
|---|---|---|---|---|---|
| B | F | G | | | |

序。

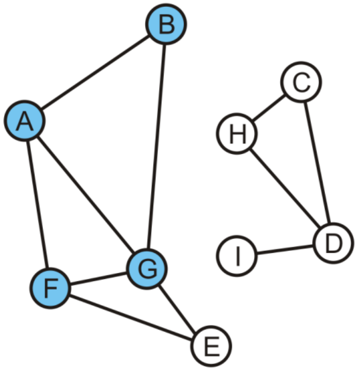
# Determining Connections

Pop B and mark and, in the left graph, mark and push H

– On the right graph, B has no unvisited adjacent vertices




| | | | | | |
|---|---|---|---|---|---|
| F | G | H | | | |

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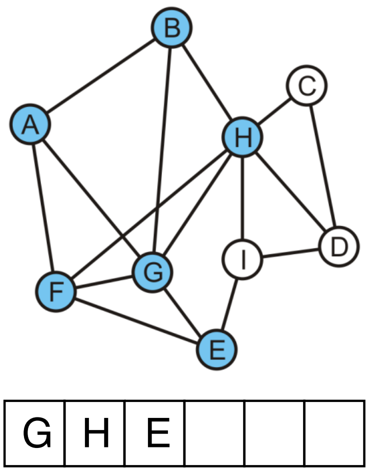
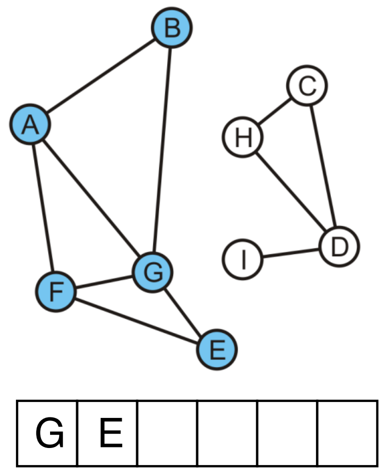
|---|---|---|---|---|---|
| F | G | | | | |

# Determining Connections

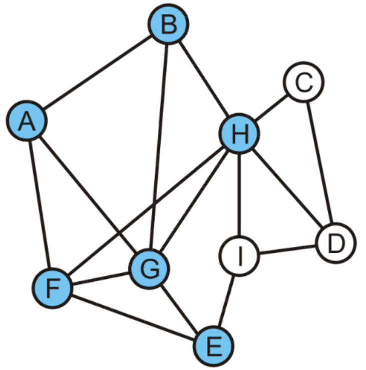
Popping F results in the pushing of E



# Determining Connections

In either graph, G has no adjacent vertices that are unvisited

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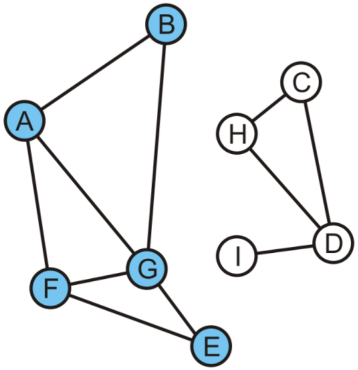
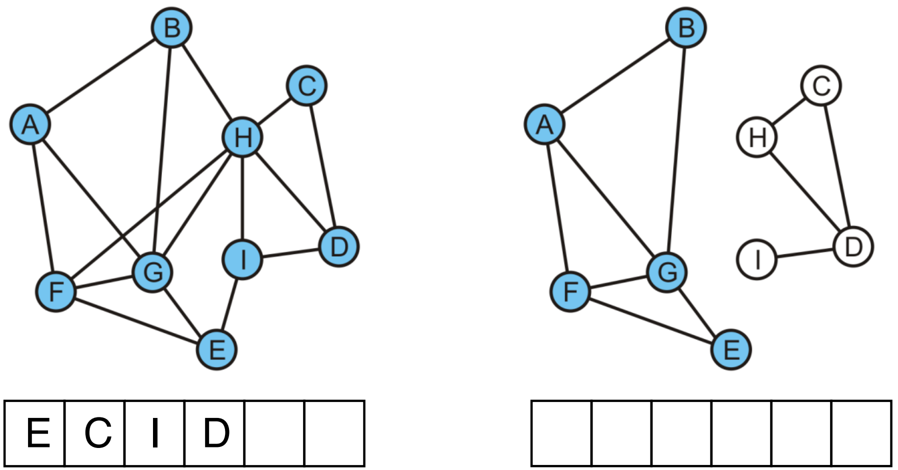
|---|---|---|---|---|---|
| H | E | | | | |




| | | | | | |
|---|---|---|---|---|---|
| E | | | | | |

# Determining Connections

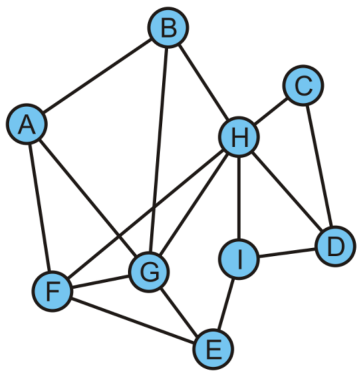
Popping H on the left graph results in C, I, D being pushed



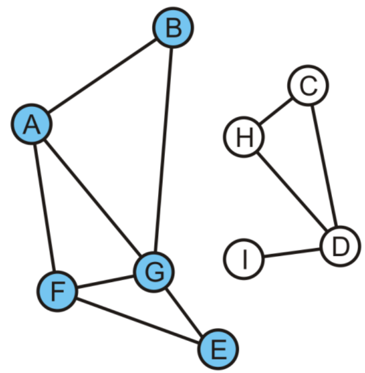
# Determining Connections

On the left, D is now visited

– We determine A is connected to D




| | | | | | |
|---|---|---|---|---|---|
| E | C | I | D | | |

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|---|---|---|---|---|---|
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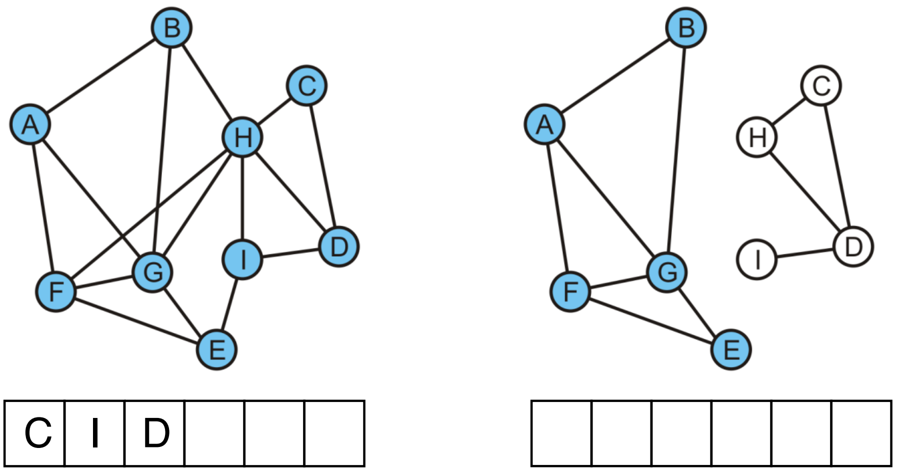
# Determining Connections

On the right, the queue is empty and D is not visited

– We determine A is not connected to D



# Connected Components

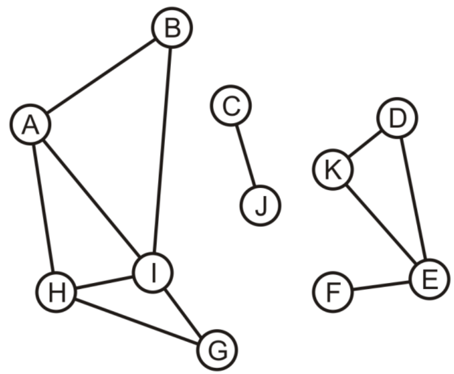
Suppose we want to partition the vertices into connected sub-graphs

- While there are unvisited vertices in the tree:
  - Select an unvisited vertex and perform a traversal on that vertex
  - Each vertex that is visited in that traversal is added to the set initially containing the initial unvisited vertex
- Continue until all vertices are visited

We would use a disjoint set data structure for maximum efficiency

# Connected Components

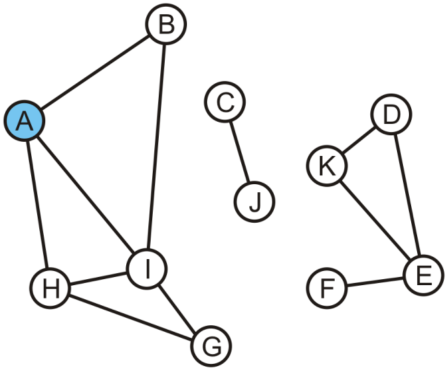
Here we start with a set of singletons




| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| **A** | **B** | **C** | **D** | **E** | **F** | **G** | **H** | **I** | **J** | **K** |

# Connected Components

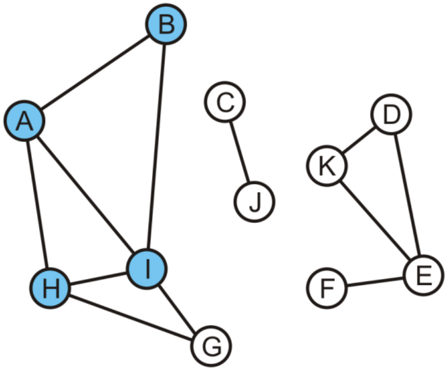
The vertex A is unvisited, so we start with it




| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| **A** | **B** | **C** | **D** | **E** | **F** | **G** | **H** | **I** | **J** | **K** |

# Connected Components

Take the union of with its adjacent vertices: $\{$A, B, H, I$\}$

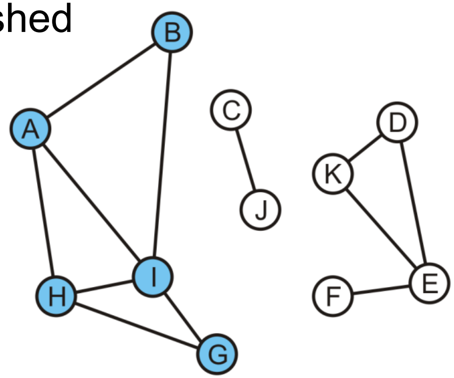



| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| **A** | **A** | **C** | **D** | **E** | **F** | **G** | **A** | **A** | **J** | **K** |

# Connected Components

As the traversal continues, we take the union of the set $\{G\}$ with the set containing H: $\{A, B, G, H, I\}$

– The traversal is finished

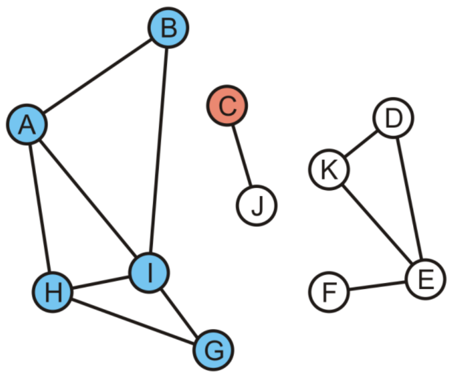



| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| **A** | **A** | **C** | **D** | **E** | **F** | **A** | **A** | **A** | **J** | **K** |

lz3

# Connected Components

Start another traversal with C: this defines a new set $\{C\}$



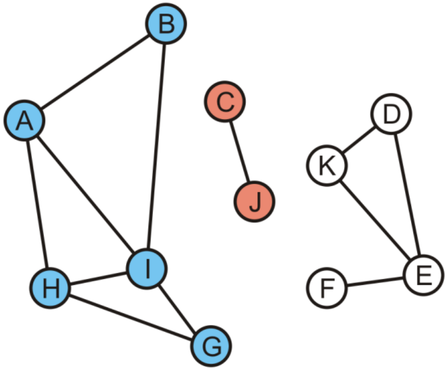

| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| **A** | **A** | **C** | **D** | **E** | **F** | **A** | **A** | **A** | **J** | **K** |

第一个被 traverse 到的

# Connected Components

We take the union of $\{C\}$ and its adjacent vertex J: $\{C, J\}$

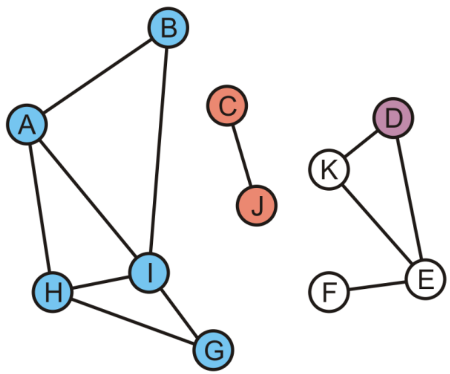
– This traversal is finished




| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| **A** | **A** | **C** | **D** | **E** | **F** | **A** | **A** | **A** | **C** | **K** |

# Connected Components

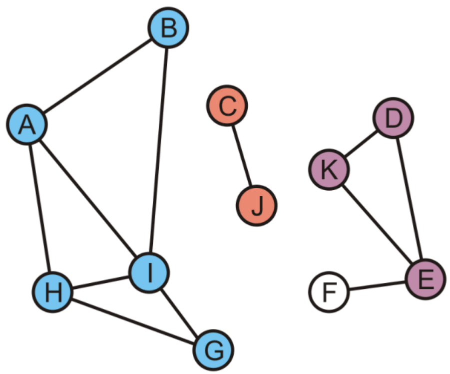
We start again with the set $\{D\}$




| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| **A** | **A** | **C** | **D** | **E** | **F** | **A** | **A** | **A** | **C** | **K** |

# Connected Components

K and E are adjacent to D, so take the unions creating $\{D, E, K\}$

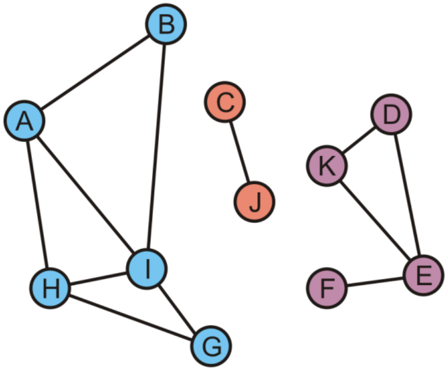



| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| **A** | **A** | **C** | **D** | **D** | **F** | **A** | **A** | **A** | **C** | **D** |

# Connected Components

Finally, during this last traversal we find that F is adjacent to E

– Take the union of {F} with the set containing E: {D, E, F, K}

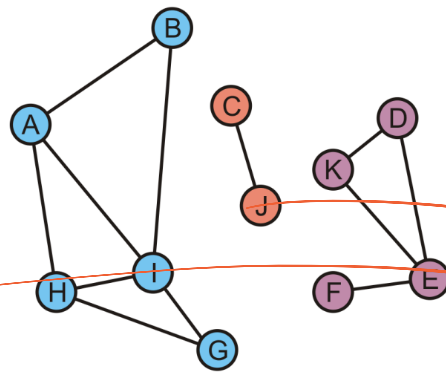



| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| A | A | C | D | D | D | A | A | A | C | D |

# Connected Components

All vertices are visited, so we are done

– There are three connected sub-graphs {A, B, G, H, I}, {C, J}, {D, E, F, K}




| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| A | A | C | D | D | D | A | A | A | C | D |

# Tracking Unvisited Vertices

How do you implement a set of unvisited vertices so as to:

- Find an unvisited vertex in $\Theta(1)$ time?
- Remove a vertex that has been visited from this list in $\Theta(1)$ time?

Bad solution

- We can simply flag vertices as visited, but this would require $O(|V|)$ time to find an unvisited vertex

Good solutions

- A hash table of unvisited vertices
- Or, an array of unvisited vertices, and we store for each vertex its position in the array

# Tracking Unvisited Vertices

Create two arrays:

- One array, unvisited, will contain the unvisited vertices
- The other, loc_in_unvisited, will contain the location of vertex $v_i$ in the first array

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G | H | I | J | K |

| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

- Or, instead of a second array, we may add a member variable in the vertex class

# Tracking Unvisited Vertices

Suppose we visit D

- D is in entry 3
- How shall we delete D in the first array?

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G | H | I | J | K |

| A | B | C | **D** | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | **3** | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

# Tracking Unvisited Vertices

Suppose we visit D

– D is in entry 3
– Copy the last unvisited vertex into this location and update the location array for this value

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | K | E | F | G | H | I | J | |

| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 3 |

# Tracking Unvisited Vertices

Suppose we visit G

– G is in entry 6

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | K | E | F | G | H | I | J | |

| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 3 |

# Tracking Unvisited Vertices

Suppose we visit G

- G is in entry 6
- Copy the last unvisited vertex into this location and update the location array for this value

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | K | E | F | J | H | I | | |

| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 6 | 3 |

# Tracking Unvisited Vertices

Suppose we now visit K

– K is in entry 3

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | K | E | F | J | H | I | | |

| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 6 | 3 |

# Tracking Unvisited Vertices

Suppose we now visit K

- K is in entry 3
- Copy the last unvisited vertex into this location and update the location array for this value

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | I | E | F | J | H | | | |

| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 3 | 6 | 3 |

# Tracking Unvisited Vertices

If we want to find an unvisited vertex, we simply return the last entry of the first array and return it

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | I | E | F | J | H | | | |

| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 3 | 6 | 3 |

# Tracking Unvisited Vertices

In this case, an unvisited vertex is H

– Removing it is trivial: just decrement the count of unvisited vertices

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | I | E | F | J | | | | |

| A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 3 | 6 | 3 |

# Tracking Unvisited Vertices

The actual algorithm is exceptionally fast:

– The initialization is $\Theta(|V|)$

```
int unvisited[nV];
int loc_in_unvisited[nV];

for ( int i = 0; i < nV; ++i ) {
      unvisited[i] = i;
      loc_in_unvisited[i] = i;
}
```

# Tracking Unvisited Vertices

The actual algorithm is exceptionally fast:

– Determining if the vertex $v_k$ is visited is fast: $\Theta(1)$

```
bool is_unvisited( int k ) const {
    return loc_in_unvisited[k] < count &&
           unvisited[ loc_in_unvisited[k] ] == k;
}
```

# Tracking Unvisited Vertices

The actual algorithm is exceptionally fast:

– Marking vertex $v_k$ as having been visited is also fast: $\Theta(1)$

```
void erase( int k ) {
    if ( !is_unvisited() ) {
        return;     // It has already been marked as visited
    }

    --count;
    int posn = loc_in_unvisited[k];
    unvisited[posn] = unvisited[count];
    loc_in_unvisited[unvisited[count]] = posn;
}
```

# Tracking Unvisited Vertices

The actual algorithm is exceptionally fast:

– Returning a vertex that is unvisited is also fast: (1)

```
int return unvisited() {
    if ( count == 0 ) {
        throw underflow();
    }

    --count;
    return unvisited[count];
}
```

# Summary

This topic covered connectedness

- Determining if two vertices are connected
- Determining the connected sub-graphs of a graph
- Tracking unvisited vertices

# References

Wikipedia, http://en.wikipedia.org/wiki/Connectivity_(graph_theory)

# Outline

- Graph traversal
  - Breadth-first
  - Depth-first
- Applications
  - Connectedness
  - Unweighted path length
  - Identifying bipartite graphs

# Determining Distances

Problem: in an unweighted graph, find the distances from one vertex $v$ to all the other vertices

- Distance: the length of the shortest path between two vertices

Method:

- Use a breadth-first traversal
- Vertices are added in *layers*
- The starting vertex $v$ is defined to be in the zeroth layer, $L_0$
- While the $k^{\text{th}}$ layer is not empty, all unvisited vertices adjacent to vertices in $L_k$ are added to the $(k + 1)^{\text{st}}$ layer

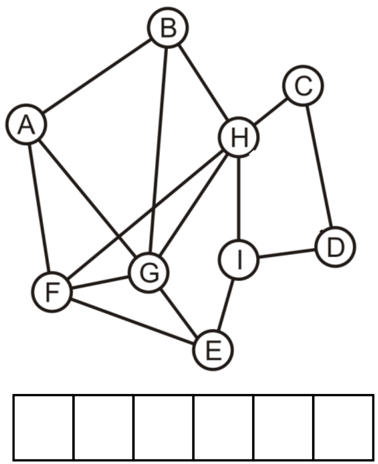
BFS,

The distance from $v$ to vertices in $L_k$ is $k$

Any unvisited vertices are said to have an infinite distance from $v$

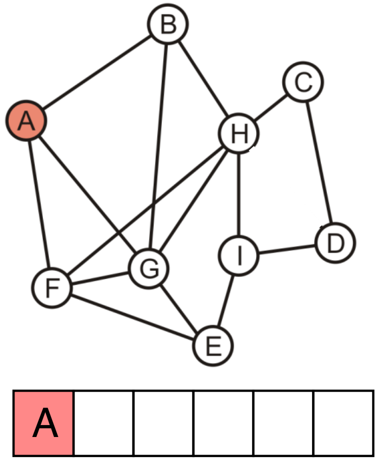
# Determining Distances

Consider this graph:  find the distance from A to each other vertex



# Determining Distances

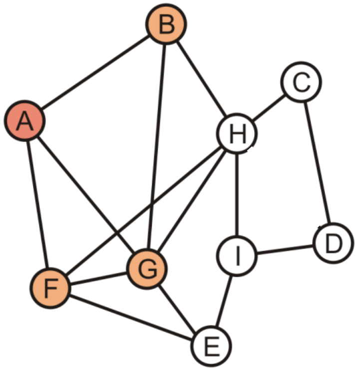
A forms the zeroeth layer, $L_0$



# Determining Distances

The unvisited vertices B, F and G are adjacent to A

– These form the first layer, $L_1$

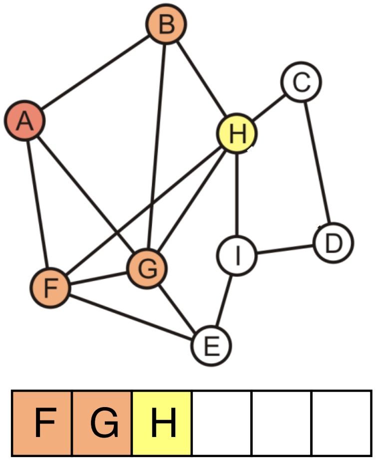



| | | | | | |
|---|---|---|---|---|---|
| B | F | G | | | |

# Determining Distances

We now begin popping $L_1$ vertices: pop B

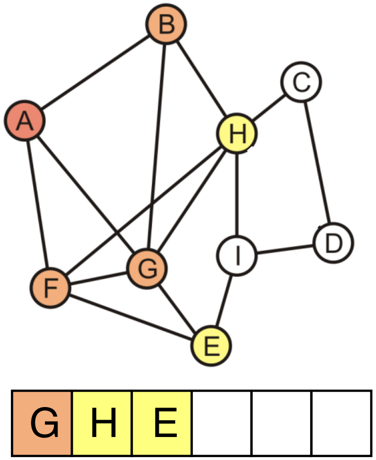
- H is adjacent to B
- It is tagged $L_2$



# Determining Distances

Popping F pushes E onto the queue

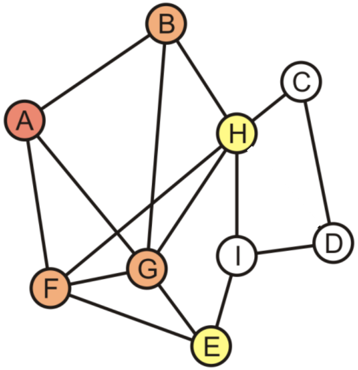
– It is also tagged $L_2$



# Determining Distances

We pop G which has no other unvisited neighbours

– G is the last $L_1$ vertex; thus H and E form the second layer, $L_2$

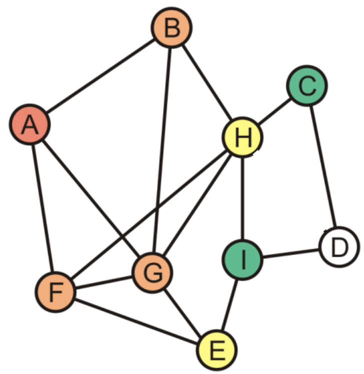



| | | | | | |
|---|---|---|---|---|---|
| H | E | | | | |

# Determining Distances

pop的层数+1

Popping H in $L_2$ adds C and I to the third layer $L_3$

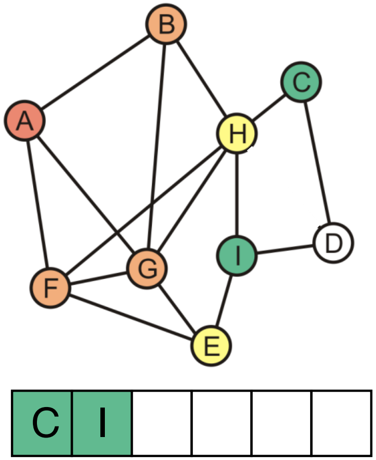



| | | | | | |
|---|---|---|---|---|---|
| E | C | I | | | |

# Determining Distances

E has no more adjacent unvisited vertices

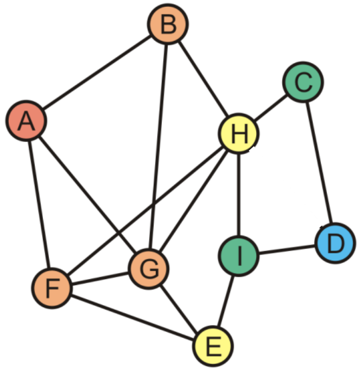
– Thus C and I form the third layer, $L_3$



# Determining Distances

The unvisited vertex D is adjacent to vertices in $L_3$

– This vertex forms the fourth layer, $L_4$




– Distance 1: B, F, G

– Distance 2: H, E

– Distance 3: C, I

– Distance 4: D

# Determining Distances

Theorem:

– If, in a breadth-first traversal of a graph, two vertices $v$ and $w$ appear in layers $L_i$ and $L_j$, respectively and $\{v, w\}$ is an edge in the graph, then $i$ and $j$ differ by at most one

BFS中层差最多1

Proof:

If $i = j$, we are done

If $i \neq j$, without loss of generality, assume $i < j$

Because $v \in L_i$, $w$ does not appear in any previous layer, and $\{v, w\}$ is an edge in the graph, it follows that $w \in L_{i+1}$

Thus, $j = i + 1$

Therefore, $i$ and $j$ differ by at most one

# Sumary

This topic found the unweighted path length from a single vertex to all other vertices

- A breadth-first traversal was used
- The first vertex is marked as layer $0$
- Vertices added to the queue by one in layer $k$ are marked as layer $k + 1$
- Later, we will see different algorithms for finding the shortest path length in weighted graphs

# References

Wikipedia, http://en.wikipedia.org/wiki/Shortest_path
http://en.wikipedia.org/wiki/Breadth-first_search

[1] Jon Kleinberg and Éva Tardos, *Algorithm Design,* Addison Wesley, 2006, §§3.2-5, pp.78-99.

# Outline

This topic looks at another problem solved by breadth-first traversals

- Determining if a graph is bipartite
- Definition of a bipartite graph
- The algorithm
- An example

# Outline

- Graph traversal
  - Breadth-first
  - Depth-first
- Applications
  - Connectedness
  - Unweighted path length
  - Identifying bipartite graphs

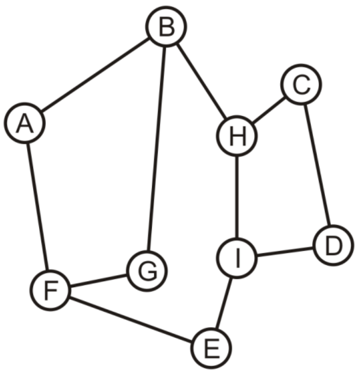
# Definition

Definition

– A *bipartite graph* is a graph where the vertices $V$ can be divided into two disjoint sets $V_1$ and $V_2$ such that **every** edge has one vertex in $V_1$ and the other in $V_2$

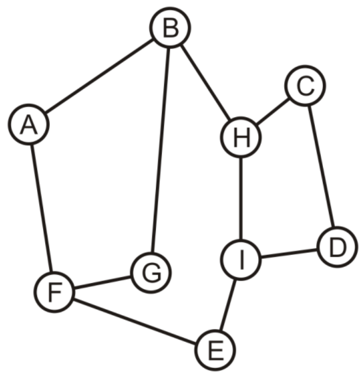
# Bipartite Graphs

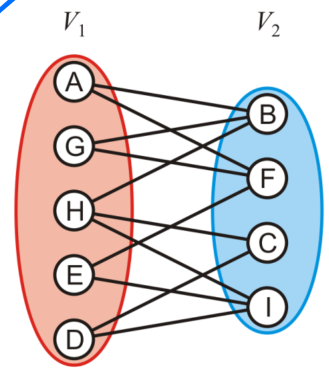
Consider this graph:  is it bipartite?



# Bipartite Graphs

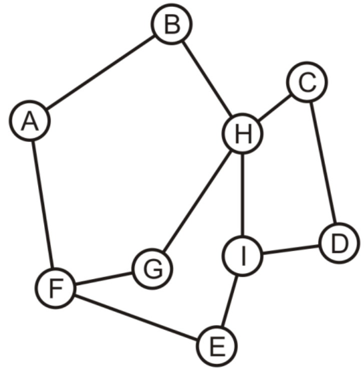
Yes: With a little work, it is possible to determine that we can decompose the vertices into two disjoint sets



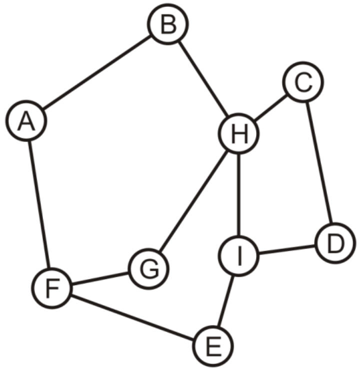
# Bipartite Graphs

Is this graph bipartite?



# Bipartite Graphs

In this case, it is not a bipartite graph




How can we determine if a graph is bipartite?

# Bipartite Graphs

Use a breadth-first traversal for a connected graph:

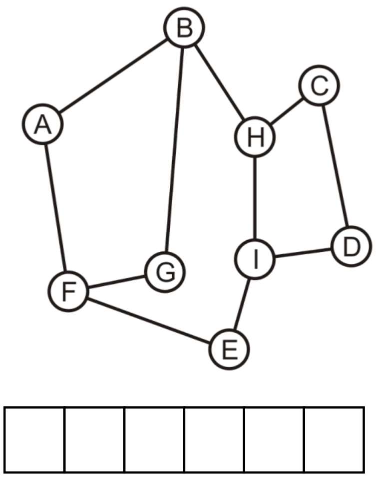
- Choose a vertex, mark it belonging to $V_1$ and push it onto a queue
- While the queue is not empty, pop the front vertex $v$ and
  - Any adjacent vertices that are already marked must belong to the set not containing $v$, otherwise, the graph is not bipartite (we are done);
  - Any unmarked adjacent vertices are marked as belonging to the other set and they are pushed onto the queue
- If the queue is empty, the graph is bipartite

BFS

# Bipartite Graphs

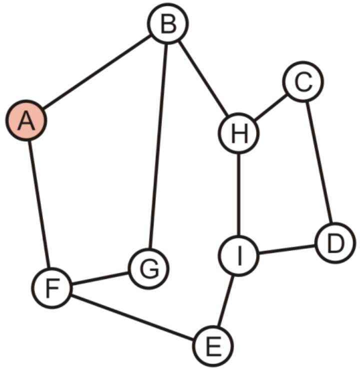
With the first graph, we can start with any vertex

– We will use colours to distinguish the two sets



# Bipartite Graphs

Push A onto the queue and colour it red

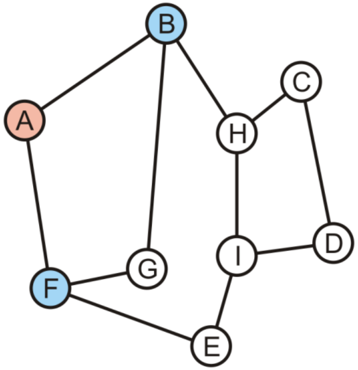



| | | | | | |
|---|---|---|---|---|---|
| A | | | | | |

# Bipartite Graphs

Pop A and its two neighbours are not marked:

– Mark them as blue and push them onto the queue

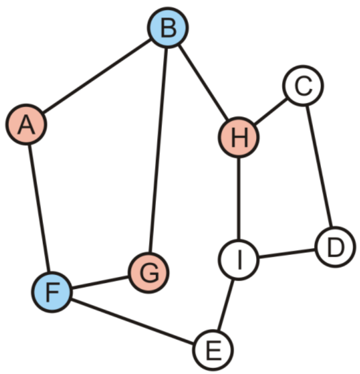



| | | | | | |
|---|---|---|---|---|---|
| B | F | | | | |

# Bipartite Graphs

Pop B—it is blue:

- Its one marked neighbour, A, is red (ok)
- Its other neighbours G and H are not marked: mark them red and push them onto the queue

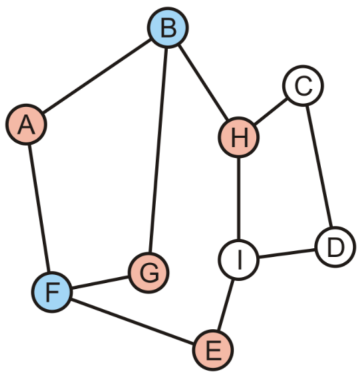



| | | | | | |
|---|---|---|---|---|---|
| F | G | H | | | |

# Bipartite Graphs

Pop F—it is blue:

- Its two marked neighbours, A and G, are red
- Its neighbour E is not marked: mark it red and pus it onto the queue

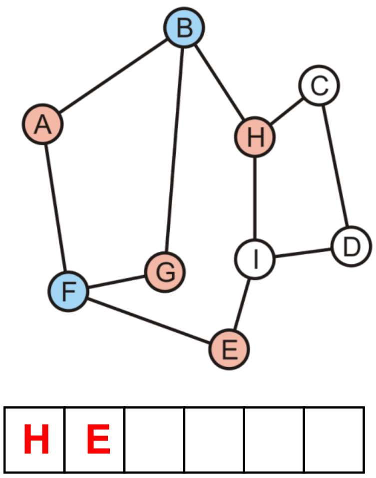



| G | H | E | | | |
|---|---|---|---|---|---|

# Bipartite Graphs

Pop G—it is red:

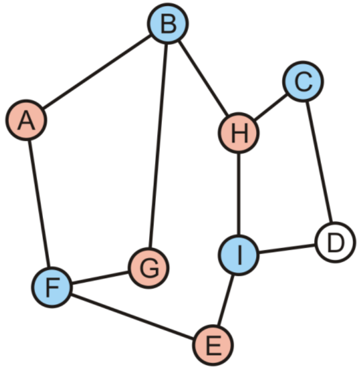
- Its two marked neighbours, B and F, are blue



# Bipartite Graphs

Pop H—it is red:

- Its marked neighbours, B, is blue
- It has two unmarked neighbours, C and I; mark them blue and push them onto the queue

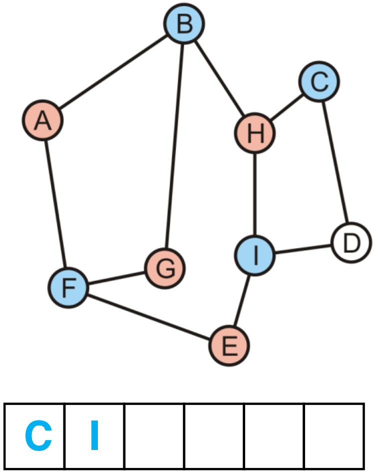



| | | | | | |
|---|---|---|---|---|---|
| E | C | I | | | |

# Bipartite Graphs

Pop E—it is red:

- Its marked neighbours, F and I, are blue

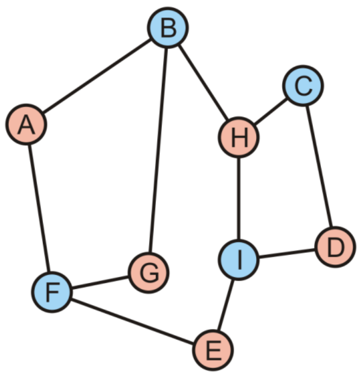


# Bipartite Graphs

Pop C—it is blue:

– Its marked neighbour, H, is red

– Mark D as red and push it onto the queue

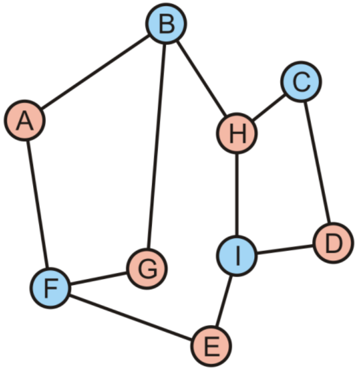


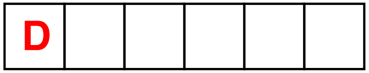

| | | | | | |
|---|---|---|---|---|---|
| I | D | | | | |

# Bipartite Graphs

Pop I—it is blue:

– Its marked neighbours, H, D and E, are all red

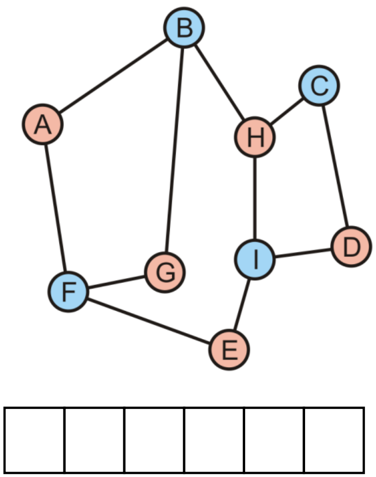


# Bipartite Graphs

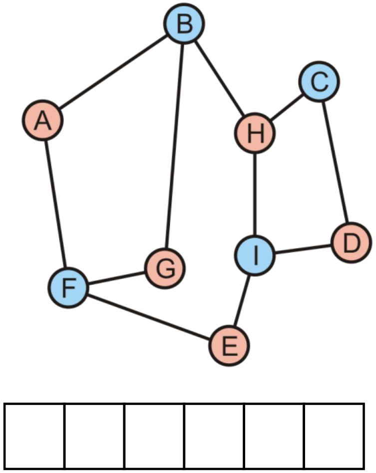
Pop D—it is red:

– Its marked neighbours, C and I, are both blue



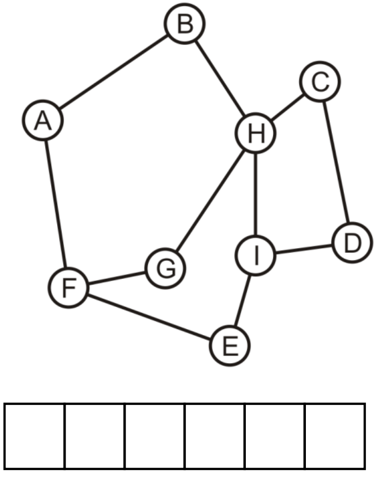
# Bipartite Graphs

The queue is empty, the graph is bipartite



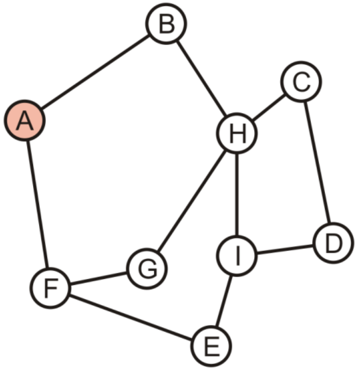
# Bipartite Graphs

Consider the other graph which was claimed to be not bipartite



# Bipartite Graphs

Push A onto the queue and colour it red

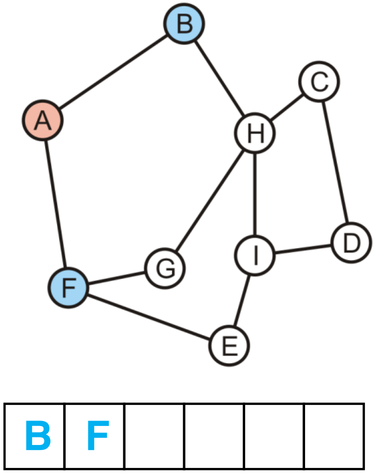



| A | | | | | |
|---|---|---|---|---|---|

# Bipartite Graphs

Pop A off the queue:

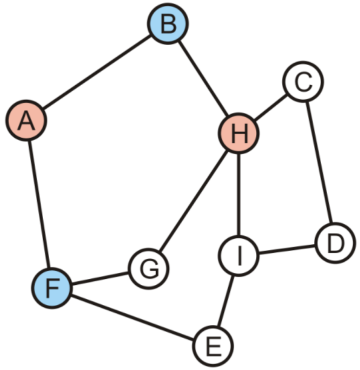
– Its neighbours are unmarked: colour them blue and push them onto the queue

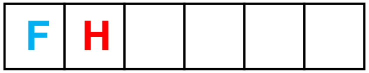


# Bipartite Graphs

Pop B off the queue:

- Its one neighbour, A, is red
- The other neighbour, H, is unmarked: colour it red and push it onto the queue



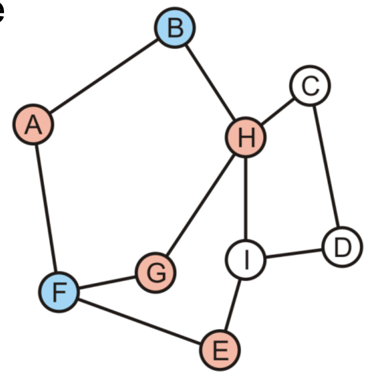



# Bipartite Graphs

Pop F off the queue:

– Its one neighbour, A, is red

– The other neighbours, E and G, are unmarked: colour them red and push it onto the queue

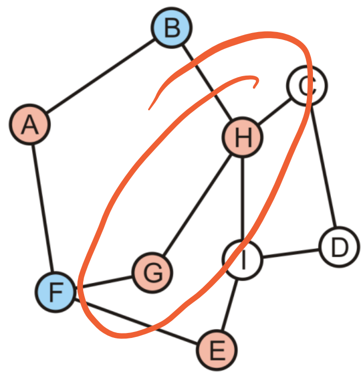



| | | | | | |
|---|---|---|---|---|---|
| H | E | G | | | |

[illegible]

# Bipartite Graphs

Pop H off the queue—it is red:
- Its one neighbour, G, is already red
- The graph is not bipartite




| | | | | | |
|---|---|---|---|---|---|
| E | G | | | | |

[illegible]

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# Bipartite Graphs

## Definition

Cycles that contains either an even number or an odd number of vertices are said to be *even cycles* and *odd cycles*, respectively

## Theorem

A graph is bipartite if and only if it does not contain any odd cycles

# Sumary

This topic looked at identifying bipartite graphs

- Perform a breadth-first traversal
- Each vertex is given one of two identifiers (we used color)
- The first vertex is identified as one color and pushed onto the queue
- When a vertex is popped:
  - Each unvisited neighbor is pushed onto the tree with the opposite color
  - Each visited neighbor must be the opposite color
    - If one is not, the graph is not bipartite

# References


Wikipedia, http://en.wikipedia.org/wiki/Breadth-first_search#Testing_bipartiteness
http://en.wikipedia.org/wiki/Breadth-first_search
http://en.wikipedia.org/wiki/Bipartite_graph

[1] Jon Kleinberg and Éva Tardos, *Algorithm Design,* Addison Wesley, 2006, §§3.2-5, pp.78-99.

# Outline

- Graph traversal
  - Breadth-first: use a queue
  - Depth-first: use recursion or stack
- Applications
  - Connectedness
  - Unweighted path length
  - Identifying bipartite graphs