CMSC 451: Reductions & NP-completeness

Slides By: Carl Kingsford

Department of Computer Science
University of Maryland, College Park

Based on Section 8.1 of *Algorithm Design* by Kleinberg & Tardos.

Reductions as tool for hardness

We want prove some problems are computationally difficult.

As a first step, we settle for relative judgements:

Problem X is at least as hard as problem Y

To prove such a statement, we reduce problem Y to problem X:

If you had a black box that can solve instances of problem X, how can you solve any instance of Y using polynomial number of steps, plus a polynomial number of calls to the black box that solves X?

Polynomial Reductions

- If problem Y can be reduced to problem X, we denote this by $Y \leq_P X$.
- This means "Y is polynomal-time reducible to X."
- It also means that X is at least as hard as Y because if you can solve X, you can solve Y.
- Note: We reduce *to* the problem we want to show is the harder problem.

Polynomial Problems

Suppose:

- $Y \leq_P X$, and
- there is an polynomial time algorithm for X.

Then, there is a polynomial time algorithm for Y.

Why?

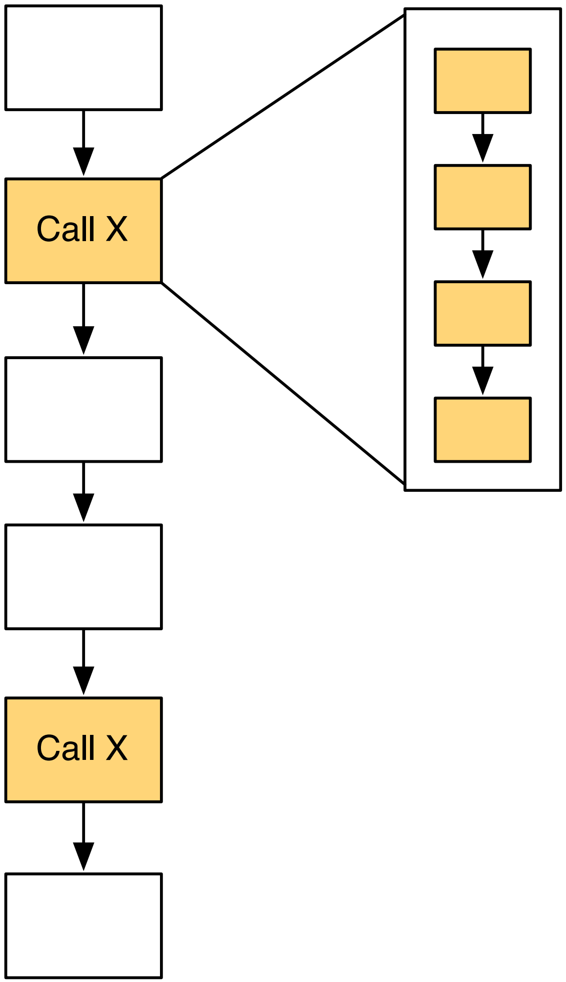
Polynomial Problems

Suppose:

- $Y \leq_P X$, and
- there is an polynomial time algorithm for X.

Then, there is a polynomial time algorithm for Y.

Why? Because polynomials compose.



We've Seen Reductions Before

Examples of Reductions:

- Max Bipartite Matching $\leq_P$ Max Network Flow.
- Image Segmentation $\leq_P$ Min-Cut.
- Survey Design $\leq_P$ Max Network Flow.
- Disjoint Paths $\leq_P$ Max Network Flow.

Reductions for Hardness

Theorem

If $Y \leq_P X$ and Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

Why? If we *could* solve X in polynomial time, then we'd be able to solve Y in polynomial time using the reduction, contradicting the assumption.

So: If we could find one hard problem Y, we could prove that another problem X is hard by reducing Y to X.

Vertex Cover

Def. A vertex cover of a graph is a set S of nodes such that every edge has at least one endpoint in S.

In other words, we try to "cover" each of the edges by choosing at least one of its vertices.

Vertex Cover

Given a graph G and a number k, does G contain a vertex cover of size at most k.

Independent Set to Vertex Cover

Independent Set

Given graph G and a number k, does G contain a set of at least k independent vertices?

Can we reduce independent set to vertex cover?

Vertex Cover

Given a graph G and a number k, does G contain a vertex cover of size at most k.

Relation btw Vertex Cover and Indep. Set

Theorem

If $G = (V, E)$ is a graph, then S is an independent set $\iff$ $V - S$ is a vertex cover.

Proof. $\implies$ Suppose S is an independent set, and let $e = (u, v)$ be some edge. Only one of u, v can be in S. Hence, at least one of u, v is in $V - S$. So, $V - S$ is a vertex cover.

$\impliedby$ Suppose $V - S$ is a vertex cover, and let $u, v \in S$. There can't be an edge between u and v (otherwise, that edge wouldn't be covered in $V - S$). So, S is an independent set. $\square$

Independent Set $\leq_P$ Vertex Cover

Independent Set $\leq_P$ Vertex Cover

To show this, we change any instance of Independent Set into an instance of Vertex Cover:

- Given an instance of Independent Set $\langle G, k \rangle$,
- We ask our Vertex Cover black box if there is a vertex cover $V - S$ of size $\leq |V| - k$.

By our previous theorem, S is an independent set iff $V - S$ is a vertex cover. If the Vertex Cover black box said:

> `yes`: *then S must be an independent set of size $\geq k$.*
> `no`: *then there is no vertex cover $V - S$ of size $\leq |V| - k$, hence there is no independent set of size $\geq k$.*

Vertex Cover $\leq_P$ Independent Set

Actually, we also have:

Vertex Cover $\leq_P$ Independent Set

Proof. To decide if G has an vertex cover of size k, we ask if it has an independent set of size $n - k$. $\square$

So: Vertex Cover and Independent Set are equivalently difficult.

NP-completeness

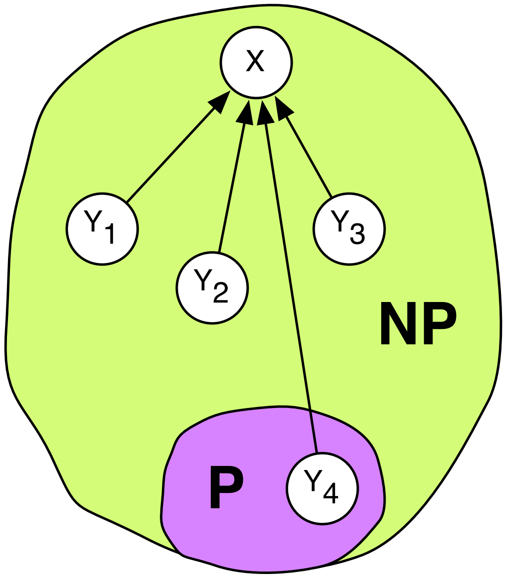
NPC

Def. We say X is NP-complete if:

- $X \in \mathbf{NP}$
- for all $Y \in \mathbf{NP}$, $Y \leq_P X$.

If these hold, then X can be used to solve every problem in **NP**.

Therefore, X is definitely at least as hard as every problem in **NP**.



X is NPC : solve all NP

NP-completeness and P=NP

Theorem

If X is NP-complete, then X is solvable in polynomial time if and only if $\mathbf{P} = \mathbf{NP}$.

Proof. If $\mathbf{P} = \mathbf{NP}$, then X can be solved in polytime.

Suppose X is solvable in polytime, and let Y be any problem in $\mathbf{NP}$. We can solve Y in polynomial time: reduce it to X.

Therefore, every problem in $\mathbf{NP}$ has a polytime algorithm and $\mathbf{P} = \mathbf{NP}$.

Reductions and NP-completeness

NPC → NPC

Theorem

If Y is NP-complete, and

1. *X is in NP*
2. *$Y \leq_P X$*

then X is NP-complete.

Y reduced to X !

Y NPC.

In other words, we can prove a new problem is NP-complete by reducing some other NP-complete problem to it.

Proof. Let Z be any problem in **NP**. Since Y is NP-complete, $Z \leq_P Y$. By assumption, $Y \leq_P X$. Therefore: $Z \leq_P Y \leq_P X$. $\square$

Some First NP-complete problem

We need to find some first NP-complete problem.

Finding the first NP-complete problem was the result of the Cook-Levin theorem.

We'll deal with this later. For now, trust me that:

- Independent Set is a *packing problem* and is NP-complete.
- Vertex Cover is a *covering problem* and is NP-complete.

Set Cover

Another very general and useful covering problem:

Set Cover

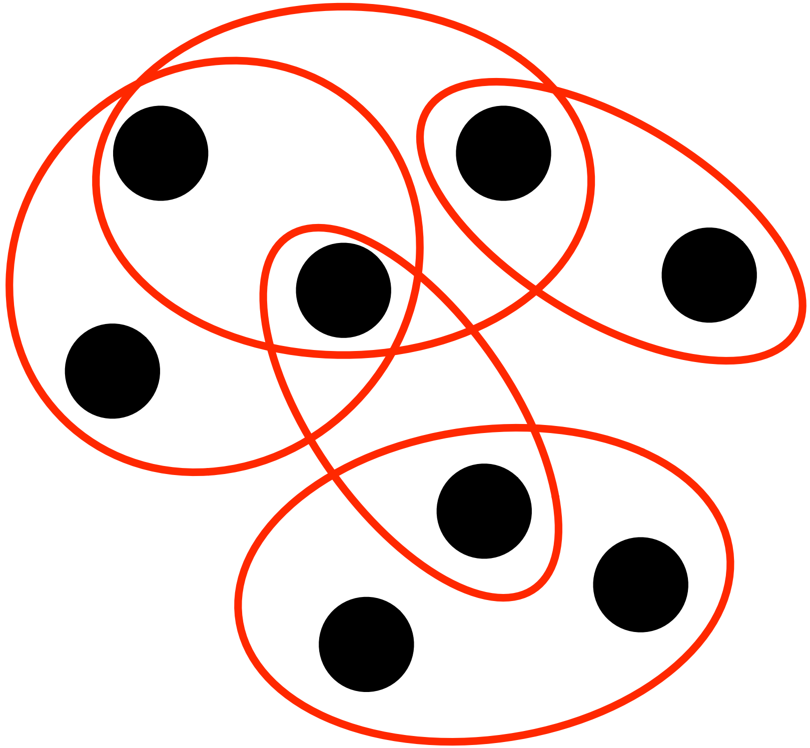
Given a set U of elements and a collection $S_1, \ldots, S_m$ of subsets of U, is there a collection of at most k of these sets whose union equals U?

We will show that

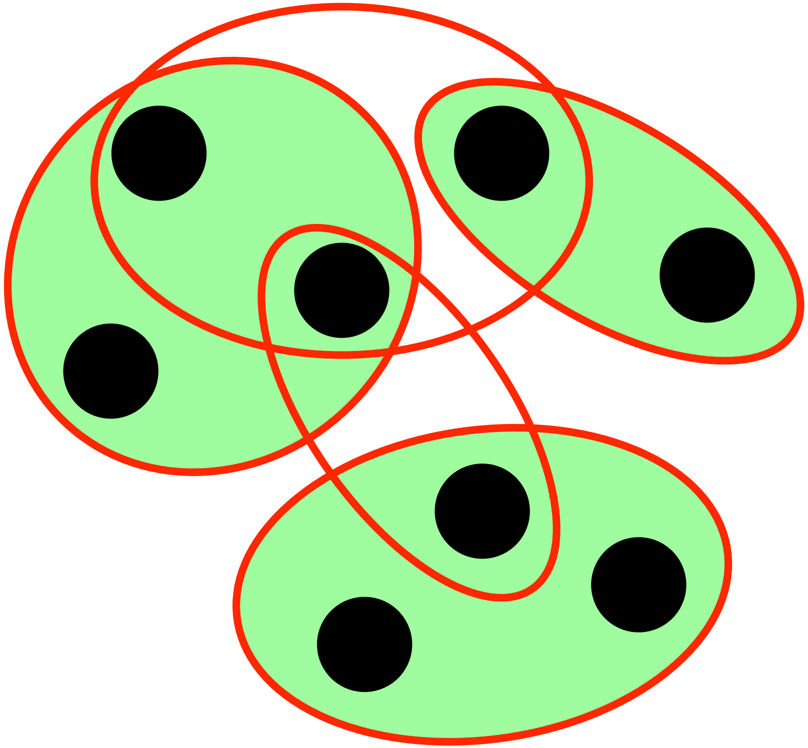
$$\textsc{Set Cover} \in NP$$
$$\textsc{Vertex Cover} \leq_P \textsc{Set Cover}$$

And therefore that Set Cover is NP-complete.

Set Cover, Figure



Set Cover, Figure



Vertex Cover $\leq_P$ Set Cover

Thm. Vertex Cover $\leq_P$ Set Cover

Proof. Let $G = (V, E)$ and k be an instance of VERTEX COVER. Create an instance of SET COVER:

- $U = E$
- Create a S_u for for each $u \in V$, where S_u contains the edges adjacent to u.

U can be covered by $\leq k$ sets iff G has a vertex cover of size $\leq k$.

Why? If k sets $S_{u_1}, \ldots, S_{u_k}$ cover U then every edge is adjacent to at least one of the vertices $u_1, \ldots, u_k$, yielding a vertex cover of size k.

If $u_1, \ldots, u_k$ is a vertex cover, then sets $S_{u_1}, \ldots, S_{u_k}$ cover U. $\square$

Last Step:

We still have to show that Set Cover is in **NP**!

The certificate is a list of k sets from the given collection.

We can check in polytime whether they cover all of U.

Since we have a certificate that can be checked in polynomial time, Set Cover is in **NP**.

Summary

You can prove a problem is NP-complete by reducing a known NP-complete problem to it.

We know the following problems are NP-complete:

- Vertex Cover
- Independent Set
- Set Cover

known $\xrightarrow{\text{reduce}}$ unknown

Warning: You should reduce the *known* NP-complete problem to the problem you are interested in. (You *will* mistakenly do this backwards sometimes.)