

# CS101 Algorithms and Data Structures

Disjoint Sets  
Textbook Ch 21



# Disjoint Sets

- Definition: a set of elements partitioned into a number of disjoint subsets

For example, a partition of the 10 numerals

1, 2, 3, 4, 5, 6, 7, 8, 9, 0

into three disjoint subsets

$\{1, 2, 3, 5, 7\}, \{4, 6, 9, 0\}, \{8\}$

- Also called:
  - union–find data structure
  - merge–find set

# Operations on Disjoint Sets

There are two operations we would like to perform on disjoint sets:

- Determine if two elements are in the same disjoint set, and
- Take the union of two disjoint sets creating a single set

We will determine if two objects are in the same disjoint set by defining a **find** function

- **find(a)**: find the representative object of the disjoint set that **a** belongs to
- Given two elements **a** and **b**, they are in the same set if

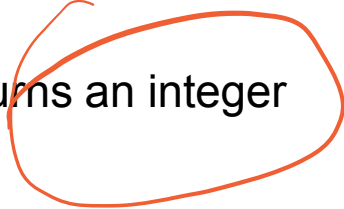
**find( a ) == find( b )**

# Implementation

What `find` returns is irrelevant so long as:

- If `a` and `b` are in the same set, `find( a ) == find( b )`
- If `a` and `b` are not in the same set, `find( a ) != find( b )`

Here we assume `find` returns an integer



# Implementation

Here is a poor implementation:

- Have two arrays and the second array stores the representative objects

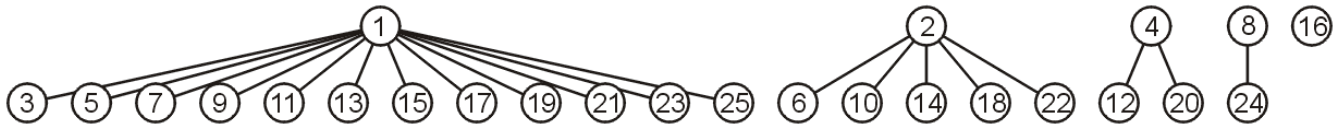
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	2	1	4	1	2	1	8	1	2	1	4	1	2	1	16	1	2	1	4	1	2	1	8	1

- Given the index of an element, finding the representative object is  $\Theta(1)$
- However, taking the union of two sets is  $\Theta(n)$ 
  - It would be necessary to check each array entry

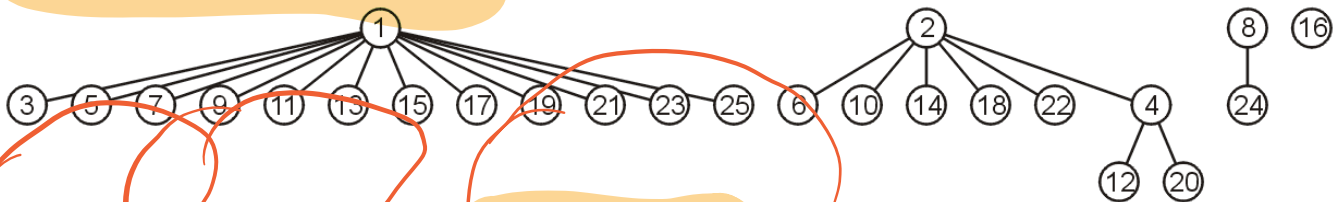
# Implementation

As an alternate implementation, let each disjoint set be represented by a general tree

- The root of the tree is the representative object



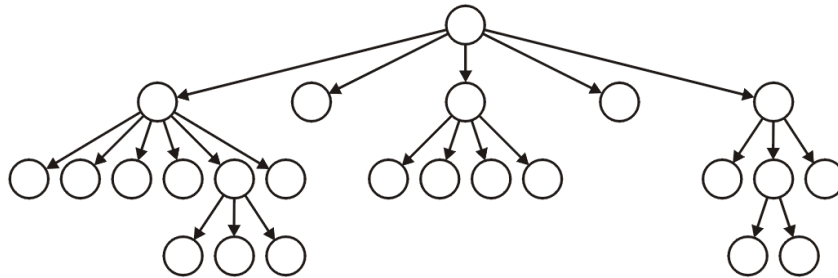
To take the union of two such sets, we will simply attach one tree to the root of the other



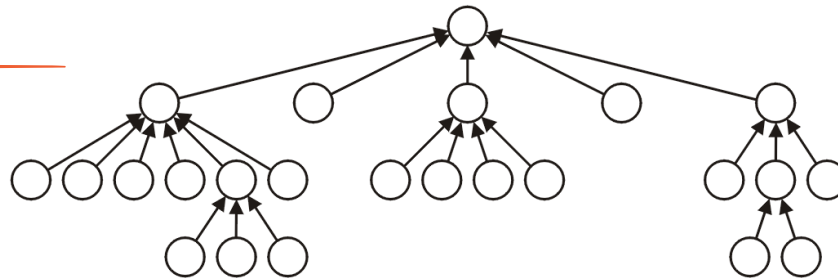
Find and union are now both  $O(h)$

# Implementation

Normally, a node points to its children:



We are only interested in the root; therefore, our interest is in storing the parent



# Implementation

For simplicity, assume we are creating disjoint sets for the  $n$  integers

$0, 1, 2, \dots, n-1$



We will define an array

```
parent = new int[n];
```

$n$

wood \$1

If `parent[i] == i`, then `i` is a root node

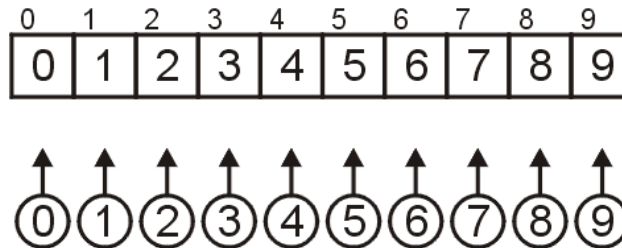
Initially, each integer is in its own set

```
for ( int i = 0; i < n; ++i ) {  
    parent[i] = i;  
}
```



# Example

Consider the following disjoint set on the ten decimal digits:



$\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}$

# Implementation

- `find( int i )`
  - Find the root element of the tree that contains `i`
- `set_union( int i, int j )`
  - Find the root elements of `i` and `j`
  - Update the parent of one root element to be the other root element

# Implementation

We will define the function

```
size_t Disjoint_set::find( size_t i ) const {  
    while( parent[i] != i ) {  
        i = parent[i];  
    }  
    return i;  
}
```

*return parent(i)*

$T_{find}(n) = O(h)$

# Implementation

Initially, you will note that

`find( i ) != find( j )`

for `i != j`, and therefore, we begin with each integer being in its own set

We must next look at the *union* operation

- how to join two disjoint sets into a single set

# Implementation

This function is also easy to define:

```
void set_union( size_t i, size_t j ) {  
    i = find( i );  
    j = find( j );  
  
    if ( i != j ) {  
        // slightly sub-optimal...  
        parent[j] = i;  
    }  
}
```

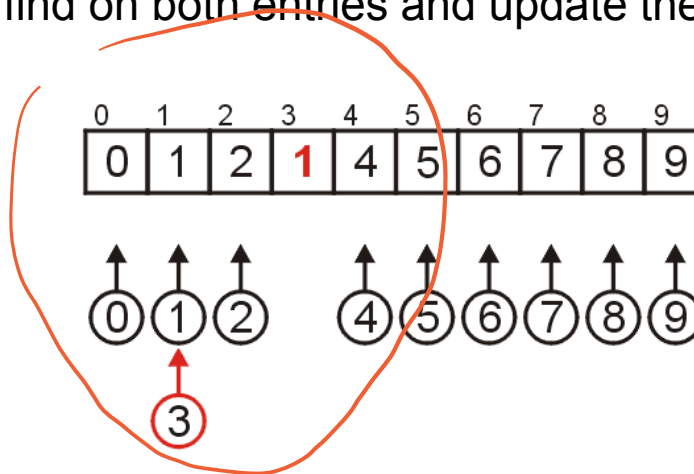
$$T_{set\_union}(n) = 2T_{find}(n) + \Theta(1)$$
$$= O(h)$$

# Example

If we take the union of the sets containing 1 and 3

```
set_union(1, 3);
```

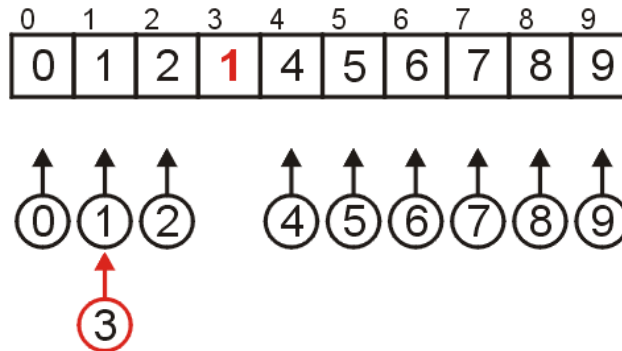
we perform a find on both entries and update the second



$\{0\}, \{1, 3\}, \{2\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}$

# Example

Now, `find(1)` and `find(3)` will both return the integer 1



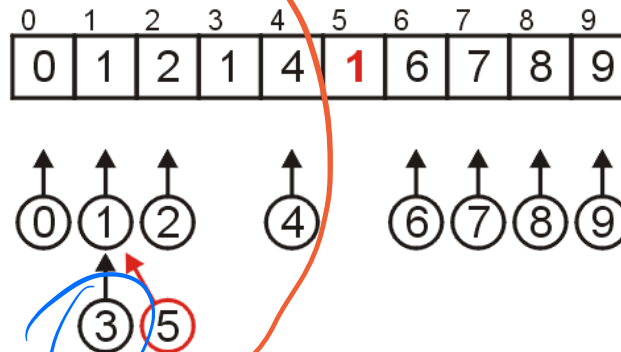
$\{0\}, \{1, 3\}, \{2\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}$

# Example

Next, take the union of the sets containing 3 and 5,

`set_union(3, 5);`

we perform a find on both entries and update the second

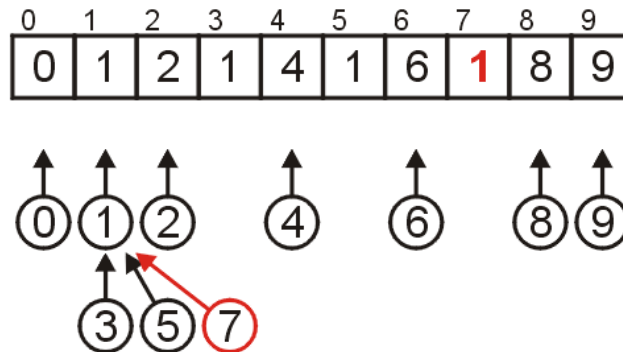


$\{0\}, \{1, 3, 5\}, \{2\}, \{4\}, \{6\}, \{7\}, \{8\}, \{9\}$



# Example

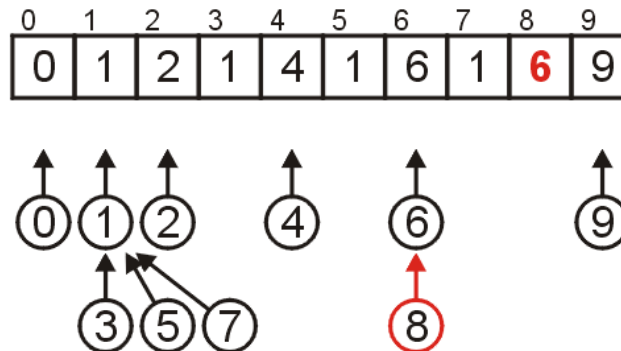
Now, if we take the union of the sets containing 5 and 7  
`set_union(5, 7);`



$\{0\}, \{1, 3, 5, 7\}, \{2\}, \{4\}, \{6\}, \{8\}, \{9\}$

# Example

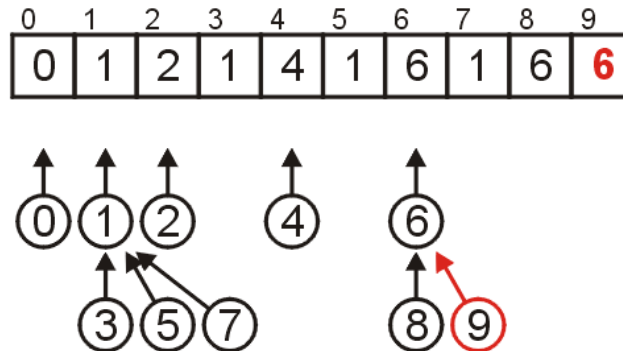
Taking the union of the sets containing 6 and 8  
`set_union(6, 8);`



$\{0\}, \{1, 3, 5, 7\}, \{2\}, \{4\}, \{6, 8\}, \{9\}$

# Example

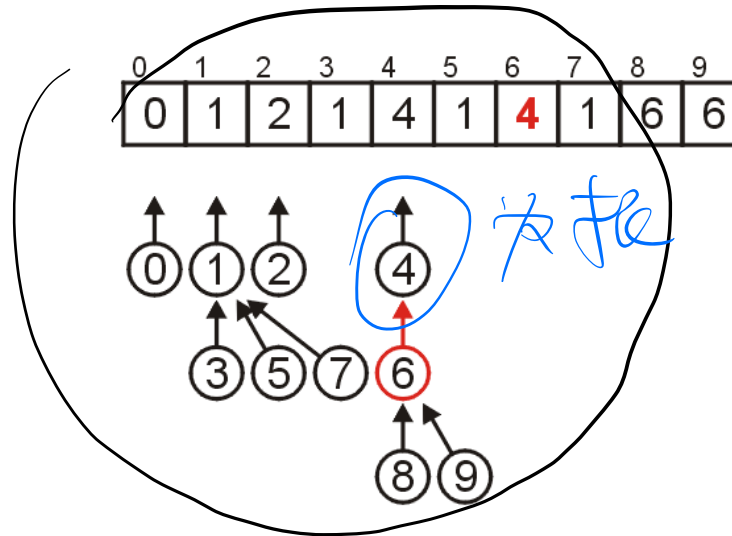
Taking the union of the sets containing 8 and 9  
`set_union(8, 9);`



$\{0\}, \{1, 3, 5, 7\}, \{2\}, \{4\}, \{6, 8, 9\}$

# Example

Taking the union of the sets containing 4 and 8  
`set_union(4, 8);`

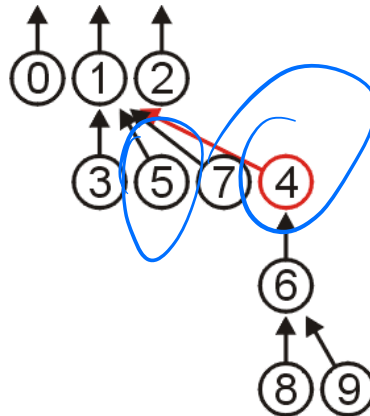


$\{0\}, \{1, 3, 5, 7\}, \{2\}, \{4, 6, 8, 9\}$

# Example

Finally, if we take the union of the sets containing 5 and 6  
`set_union(5, 6);`

0	1	2	3	4	5	6	7	8	9
0	1	2	1	1	1	4	1	6	6



$\{0\}, \{1, 3, 4, 5, 6, 7, 8, 9\}, \{2\}$

# Optimization 1

Problem:

- The height of the tree may grow very large

To optimize both `find` and `set_union`, we must minimize the height of the tree

- Therefore, point the root of the shorter tree to the root of the taller tree
- The height of the taller will increase if and only if the trees are equal in height

# Worst-Case Scenario

Let us consider creating the worst-case disjoint set

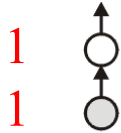
- The tallest tree with the least number of nodes

The worst case tree of height  $h$  must result from taking union of two worst case trees of height  $h-1$

# Worst-Case Scenario

Thus, building on this, we take the union of two sets with one element

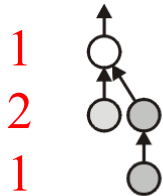
- We will keep track of the number of nodes at each depth





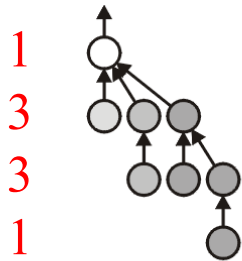
# Worst-Case Scenario

Next, we take the union of two sets, that is, we join two worst-case sets of height 1:



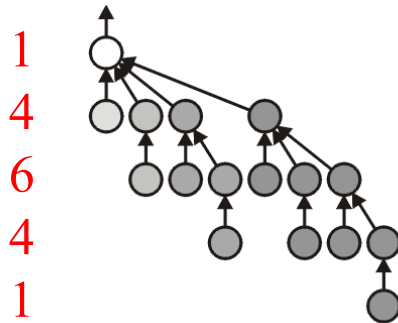
# Worst-Case Scenario

And continue, taking the union of two worst-case trees of height 2:



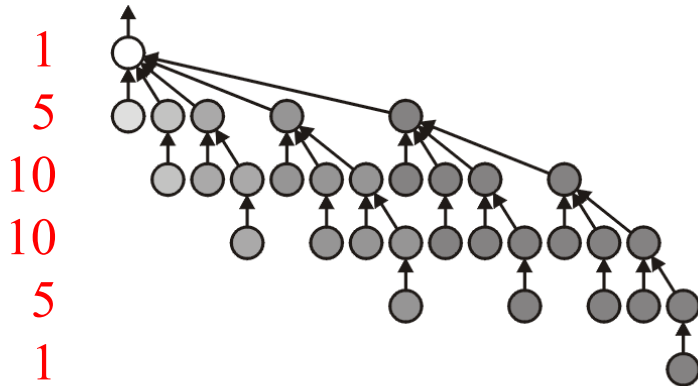
# Worst-Case Scenario

Taking the union of two worst-case trees of height 3:



# Worst-Case Scenario

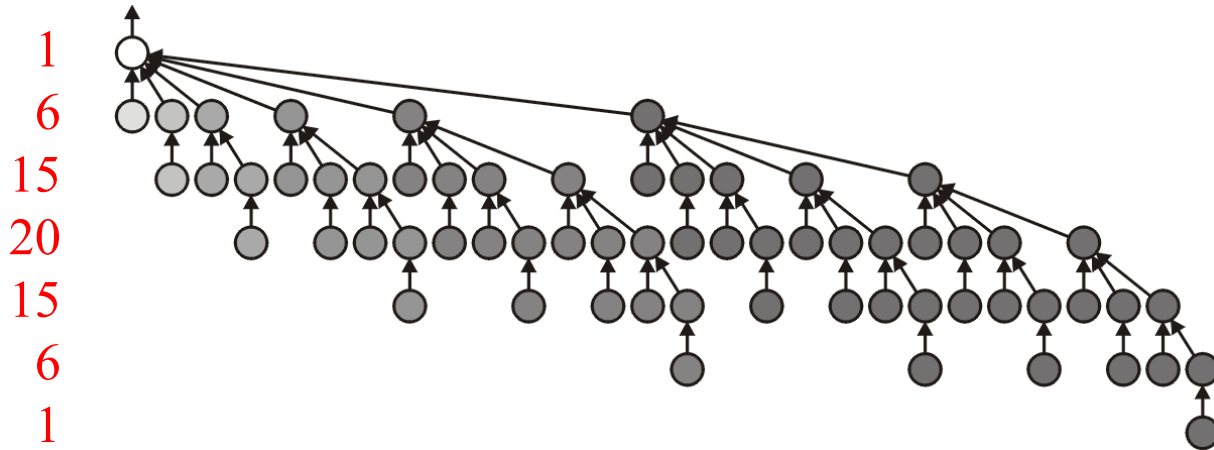
And of four:



# Worst-Case Scenario

And finally, take the union of two worst-case trees of height 5:

- These are *binomial trees*



# Worst-Case Scenario

From the construction, it should be clear that this would define Pascal's triangle

- The *binomial* coefficients

[illegible]

# Worst-Case Scenario

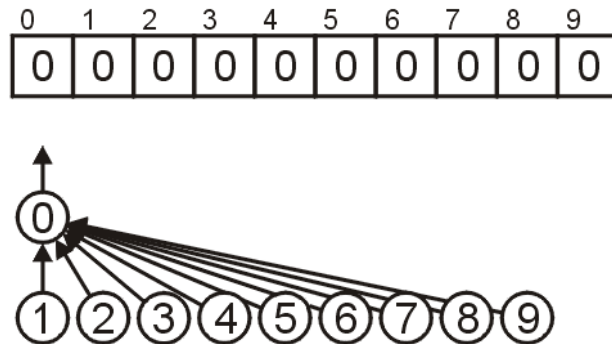


Thus, suppose we have a worst-case tree of height  $h$

- The number of nodes is  $\sum_{k=0}^h \binom{h}{k} = 2^h = n$
- The sum of node depth is  $\sum_{k=0}^h k \binom{h}{k} = h2^{h-1}$
- Therefore, the average depth is  $\frac{h2^{h-1}}{2^h} = \frac{h}{2} = \frac{\lg(n)}{2}$
- The height and average depth of the worst case are  $O(\ln(n))$

# Best-Case Scenario

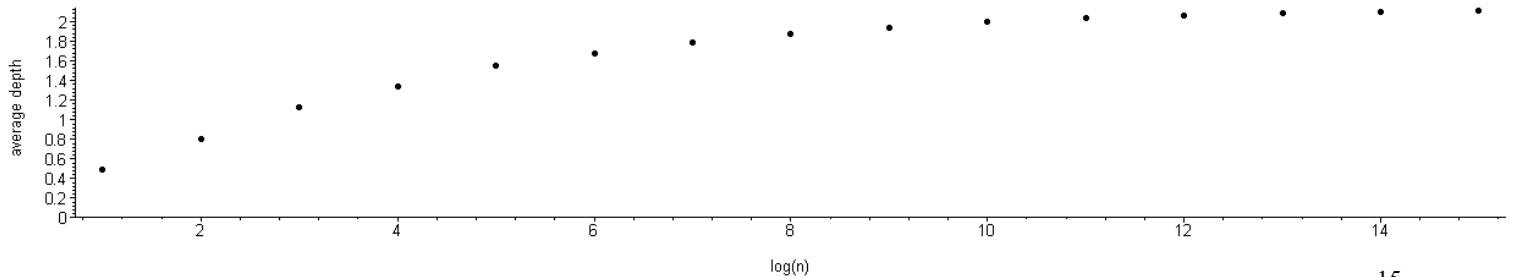
In the best case, all elements point to the same entry with a resulting height of  $\Theta(1)$ :





# Average-Case Scenario

The resulting graph shows the average height of a randomly generated disjoint set data structure with  $2^m$  elements



$$2^{15} = 32768$$

This suggests that the average height of such a tree is  $\mathcal{O}(\ln(n))$

## Optimization 2: Path Compression

Another optimization is that, whenever find is called, update the object to point to the root

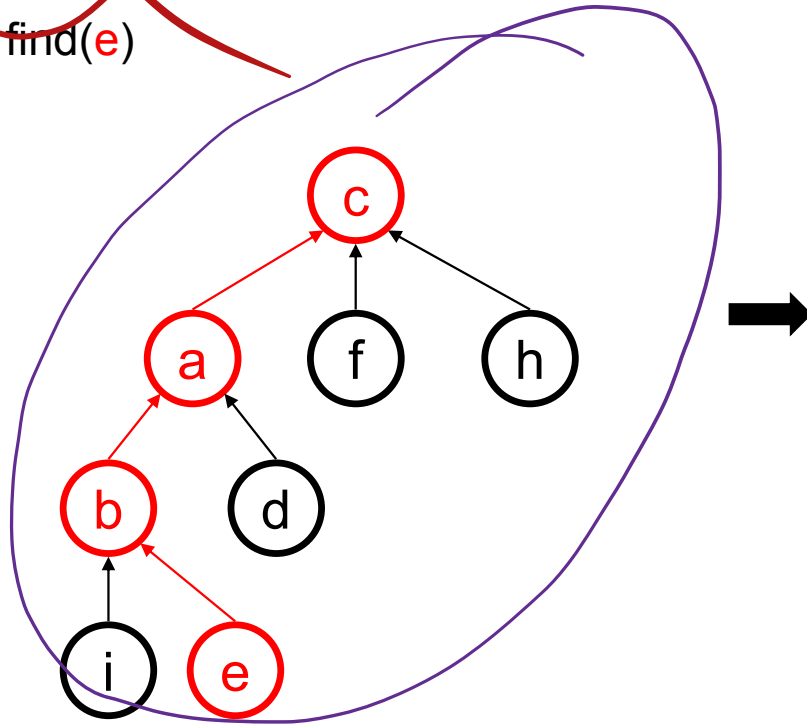
```
size_t Disjoint_set::find( size_t n ) {  
    if ( parent[n] == n ) {  
        return n;  
    } else {  
        parent[n] = find( parent[n] );  
        return parent[n];  
    }  
}
```

The next call to `find(n)` is  $\Theta(1)$

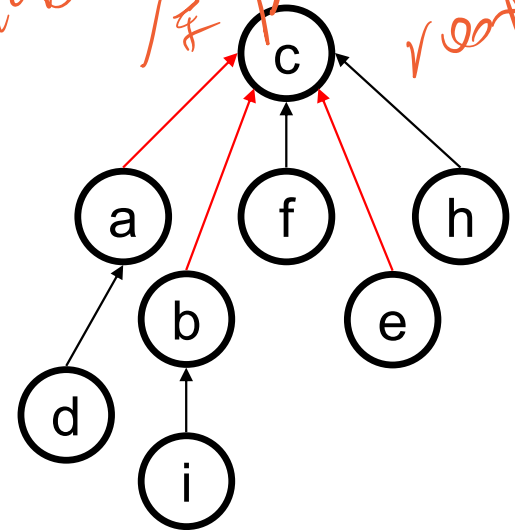
The cost is  $O(h)$  memory

## Optimization 2: Path Compression

find(e)



find for  
root point to  
root?



# Time complexity

With both optimization methods, could it be any better than  $\mathbf{O}(\ln(n))$ ?

- is there something better?

The amortized time complexity is  $\mathbf{O}(\alpha(n))$  where  $\alpha(n)$  is the inverse of the function  $A(n, n)$  where  $A(m, n)$  is the Ackermann function:

$$A(m, n) = \begin{cases} n + 1 & \text{if } m = 0 \\ A(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m - 1, A(m, n - 1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

The first values are:

$$A(0, 0) = 1, \quad A(1, 1) = 3, \quad A(2, 2) = 7, \quad A(3, 3) = 61$$

# Time complexity

$A(4, 4) = 2^{A(3, 4)} - 3$  where  $A(3, 4)$  is the 19729-decimal-digit number

$A(3, 4) = 200352993040684646497907251$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 115 116 117 118 119 120 121 122 123 124 125 126 127 128 129 130 131 132 133 134 135 136 137 138 139 140 141 142 143 144 145 146 147 148 149 150 151 152 153 154 155 156 157 158 159 160 161 162 163 164 165 166 167 168 169 170 171 172 173 174 175 176 177 178 179 180 181 182 183 184 185 186 187 188 189 190 191 192 193 194 195 196 197 198 199 200 201 202 203 204 205 206 207 208 209 210 211 212 213 214 215 216 217 218 219 220 221 222 223 224 225 226 227 228 229 230 231 232 233 234 235 236 237 238 239 240 241 242 243 244 245 246 247 248 249 250 251 252 253 254 255 256 257 258 259 260 261 262 263 264 265 266 267 268 269 270 271 272 273 274 275 276 277 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2022 2023 2024 2025 2026 2027 2028 2029 2030 2031 2032 2033 2034 2035 2036 2037 2038 2039 2040 2041 2042 2043 2044 2045 2046 2047 2048 2049 2050 2051 2052 2053 2054 2055 2056 2057 2058 2059 2060 2061 2062 2063 2064 2065 2066 2067 2068 2069 2070 2071 2072 2073 2074 2075 2076 2077 2078 2079 2080 2081 2082 2083 2084 2085 2086 2087 2088 2089 2090 2091 2092 2093 2094 2095 2096 2097 2098 2099 2100 2101 2102 2103 2104 2105 2106 2107 2108 2109 2110 2111 2112 2113 2114 2115 2116 2117 2118 2119 2120 2121 2122 2123 2124 2125 2126 2127 2128 2129 2130 2131 2132 2133 2134 2135 2136 2137 2138 2139 2140 2141 2142 2143 2144 2145 2146 2147 2148 2149 2150 2151 2152 2153 2154 2155 2156 2157 2158 2159 2160 2161 2162 2163 2164 2165 2166 2167 2168 2169 2170 2171 2172 2173 2174 2175 2176 2177 2178 2179 2180 2181 2182 2183 2184 2185 2186 2187 2188 2189 2190 2191 2192 2193 2194 2195 2196 2197 2198 2199 2200 2201 2202 2203 2204 2205 2206 2207 2208 2209 2210 2211 2212 2213 2214 2215 2216 2217 2218 2219 2220 2221 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2422 2423 2424 2425 2426 2427 2428 2429 2430 2431 2432 2433 2434 2435 2436 2437 2438 2439 2440 2441 2442 2443 2444 2445 2446 2447 2448 2449 2450 2451 2452 2453 2454 2455 2456 2457 2458 2459 2460 2461 2462 2463 2464 2465 2466 2467 2468 2469 2470 2471 2472 2473 2474 2475 2476 2477 2478 2479 2480 2481 2482 2483 2484 2485 2486 2487 2488 2489 2490 2491 2492 2493 2494 2495 2496 2497 2498 2499 2500 2501 2502 2503 2504 2505 2506 2507 2508 2509 2510 2511 2512 2513 2514 2515 2516 2517 2518 2519 2520 2521 2522 2523 2524 2525 2526 2527 2528 2529 2530 2531 2532 2533 2534 2535 2536 2537 2538 2539 2540 2541 2542 2543 2544 2545 2546 2547 2548 2549 2550 2551 2552 2553 2554 2555 2556 2557 2558 2559 2560 2561 2562 2563 2564 2565 2566 2567 2568 2569 2570 2571 2572 2573 2574 2575 2576 2577 2578 2579 2580 2581 2582 2583 2584 2585 2586 2587 2588 2589 2590 2591 2592 2593 2594 2595 2596 2597 2598 2599 2600 2601 2602 2603 2604 2605 2606 2607 2608 2609 2610 2611 2612 2613 2614 2615 2616 2617 2618 2619 2620 2621 2622 2623 2624 2625 2626 2627 2628 2629 2630 2631 2632 2633 2634 2635 2636 2637 2638 2639 2640 2641 2642 2643 2644 2645 2646 2647 2648 2649 2650 2651 2652 2653 2654 26

# Time complexity

Therefore, we (as engineers) can, in clear conscience, state that the time complexity is  $\Theta(1)$

- There are no physical circumstances where  $\alpha(n)$  could be anything more than 4

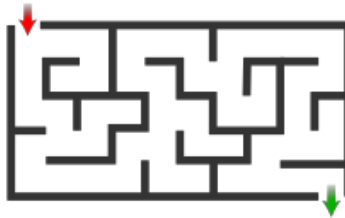
$\Theta(1)$

# Application: Maze Generation

A fun application is in the generation of mazes

Impress your (non-engineering) friends

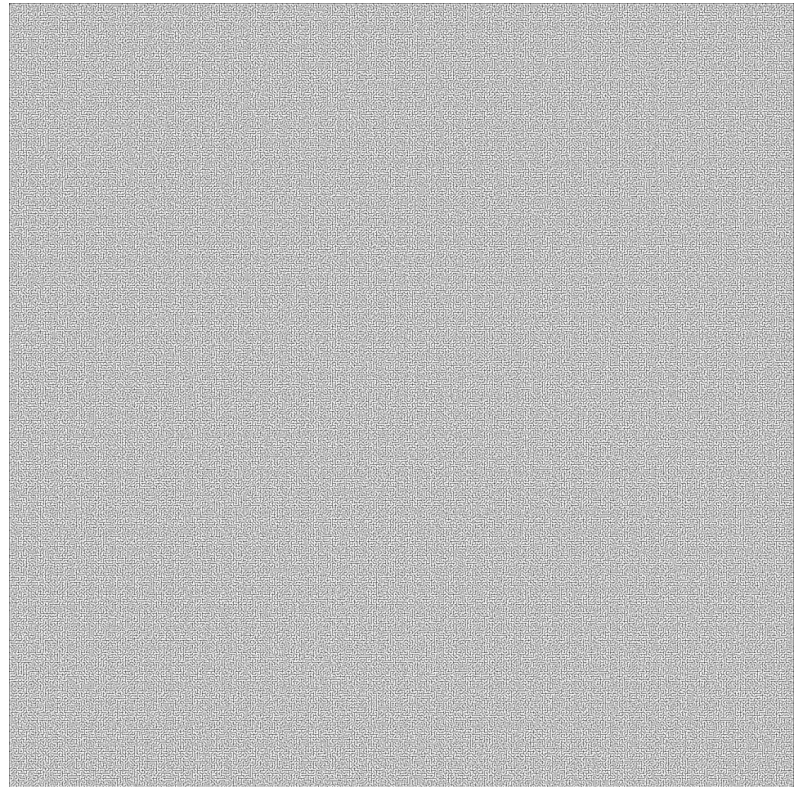
- They'll never guess how easy this is...



# Application: Maze Generation

Here we have a maze which spans a  $500 \times 500$  grid of squares where:

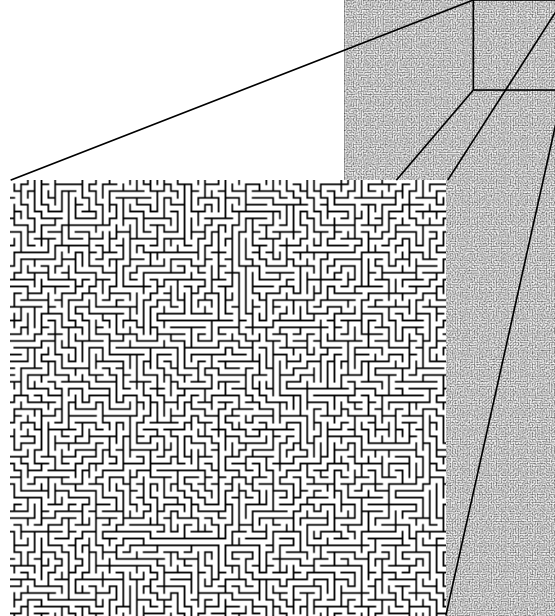
- There is one unique solution
- Each point can be reached by one unique path from the start





# Application: Maze Generation

Zooming in on the maze, you will note that it is rather complex and seemingly random

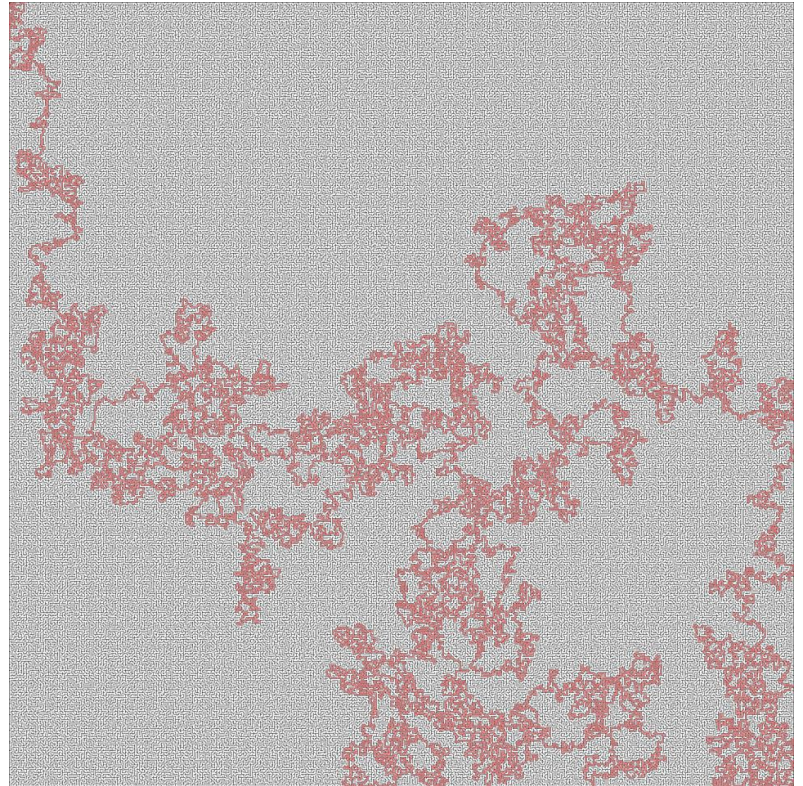


# Application: Maze Generation

Finding the solution is a problem for a different lecture

- Backtracking algorithms

We will look at creating the maze using disjoint sets



# Application: Maze Generation



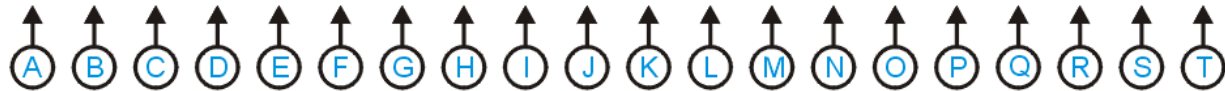
What we will do is the following:

- Start with the entire grid subdivided into squares
- Represent each square as a separate disjoint set
- Repeat the following algorithm:
  - Randomly choose a wall
  - If that wall connects two disjoint sets of cells, then remove the wall and union the two sets
- To ensure that you do not randomly remove the same wall twice, we can have an array of unchecked walls

# Application: Maze Generation

Let us begin with an entrance, an exit, and a disjoint set of 20 squares and 31 interior walls

A	1	B	2	C	3	D	4	E
5	6	7	8	9				
F	10	G	11	H	12	I	13	J
14	15	16	17	18				
K	19	L	20	M	21	N	22	O
23	24	25	26	27				
P	28	Q	29	R	30	S	31	T



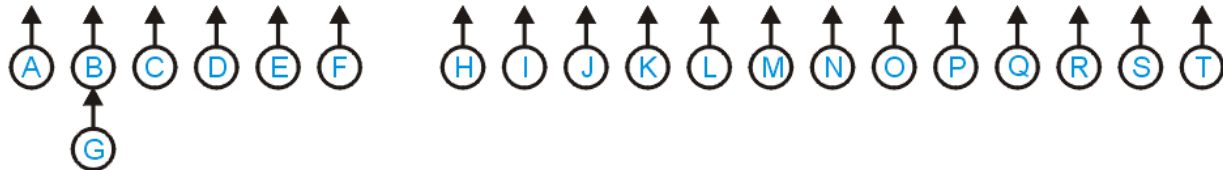
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
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# Application: Maze Generation

First, we select 6 which joins cells B and G

- Both have height 0

A	1	B	2	C	3	D	4	E
5				7	8			9
F	10	G	11	H	12	I	13	J
14	15		16		17		18	
K	19	L	20	M	21	N	22	O
23	24		25	26		27		
P	28	Q	29	R	30	S	31	T

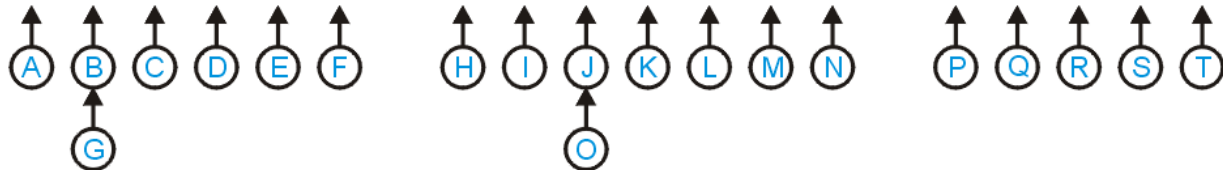


0	1	2	3	4	5	1	7	8	9	10	11	12	13	14	15	16	17	18	19
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# Application: Maze Generation

Next we select wall 18 which joins regions J and O

A	1	B	2	C	3	D	4	E
5				7	8			9
F	10	G	11	H	12	I	13	J
14	15	16	17					
K	19	L	20	M	21	N	22	O
23	24	25	26	27				
P	28	Q	29	R	30	S	31	T

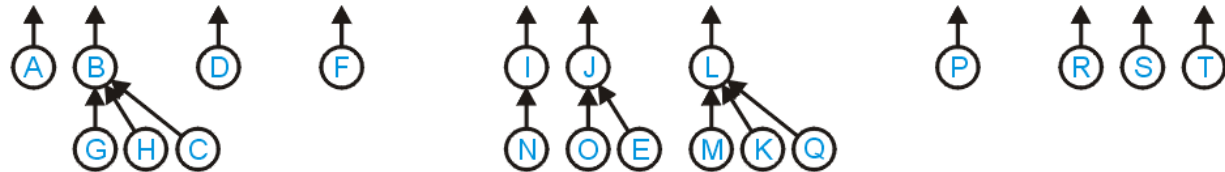
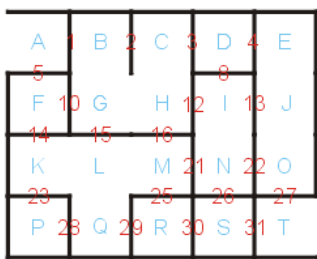


0	1	2	3	4	5	1	7	8	9	10	11	12	13	9	15	16	17	18	19
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# Application: Maze Generation

Next we select wall 23 and join the disjoint set Q with the set identified by L

- Again, Q has height 0 so we attach it to L

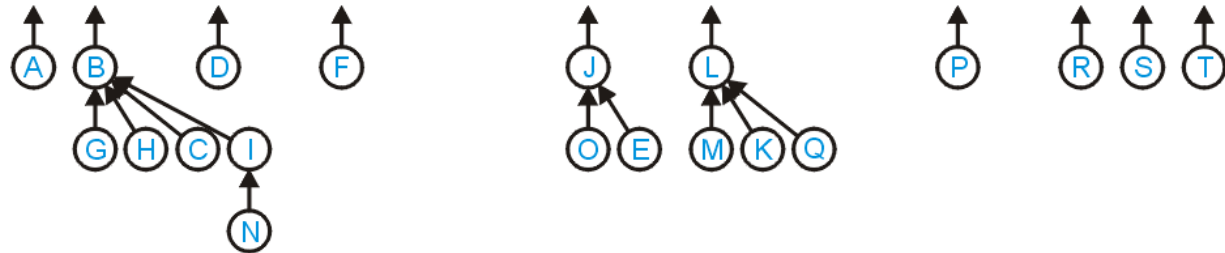
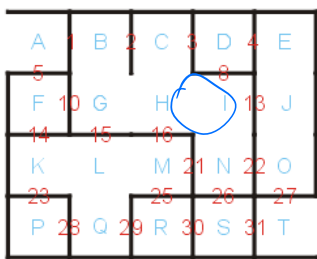


0	1	1	3	9	5	1	1	8	9	11	11	11	8	9	15	11	17	18	19
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# Application: Maze Generation

Next we select wall 12 which joints the disjoint sets identified by B and I

- They both have the same height, but B has more nodes, so we add I to the node B



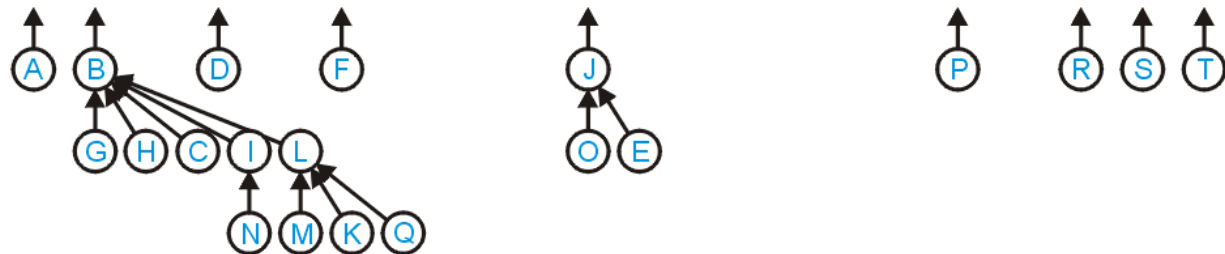
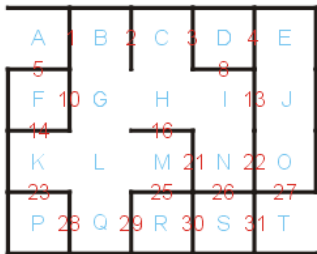
0	1	1	3	9	5	1	1	1	9	11	11	11	11	8	9	15	11	17	18	19
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# Application: Maze Generation

Selecting wall 15 joints the sets identified by B and L

- The tree B has height 2 while L has height 1 and therefore we attach L to B

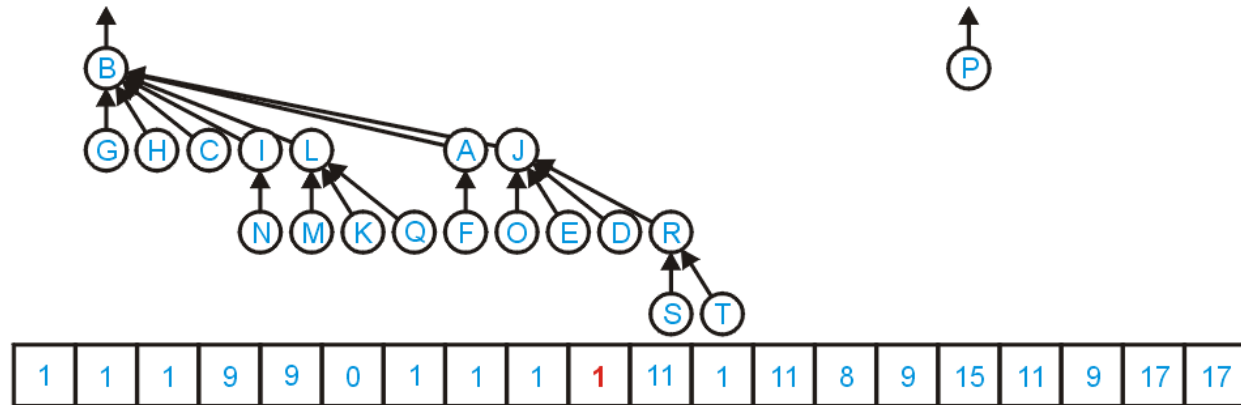
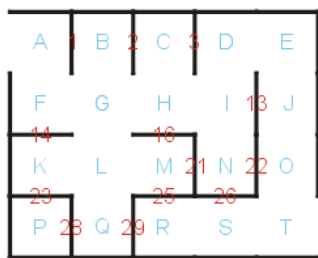


0	1	1	3	9	5	1	1	1	9	11	1	11	8	9	15	11	17	18	19
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# Application: Maze Generation

Selecting wall 8 joins sets identified by B and J

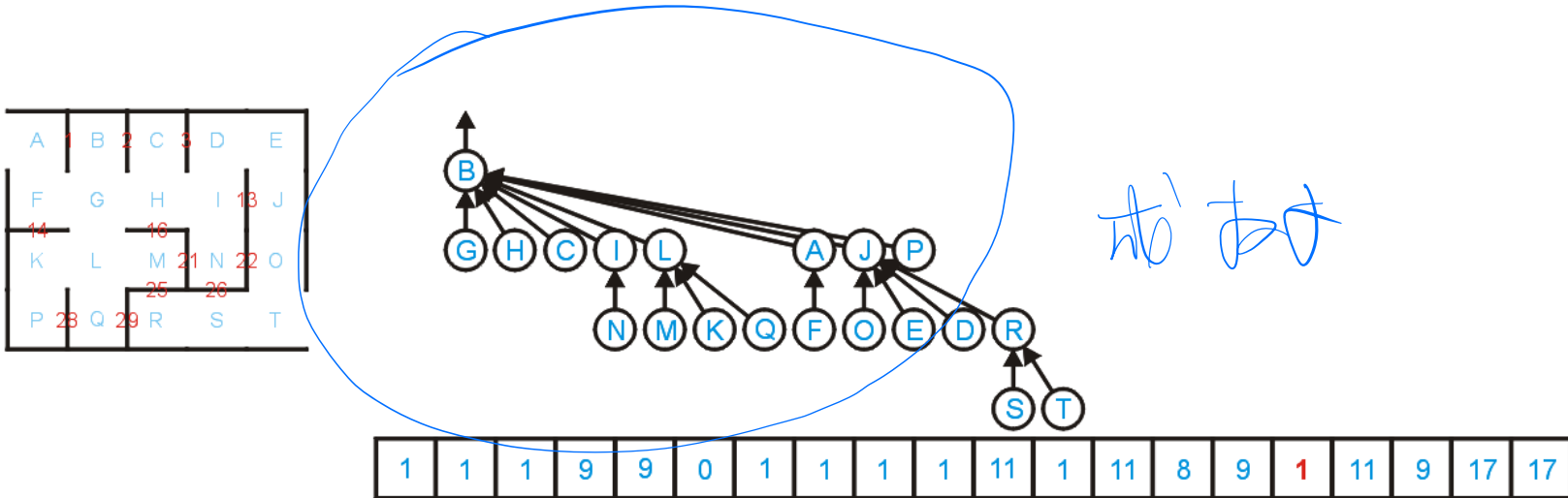
- They both have height 2 so we note that J has fewer nodes than B, so we add J to B



# Application: Maze Generation

Finally we select wall 23 which joins the disjoint set P and the disjoint set identified by B

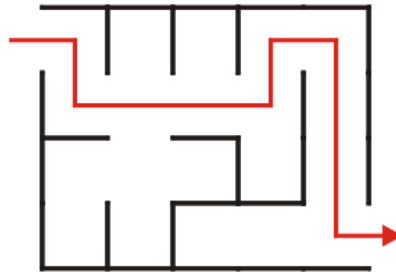
- P has height 0, so we attach it to B



# Application: Maze Generation

Thus we have a (rather trivial) maze where:

- there is one unique solution, and
- you can reach any square by a unique path from the starting point



How can we prove these two properties?

# Application: Maze Generation

A	1	B	2	C	3	D	4	E
5	6		7		8		9	
F	10	G	11	H	12	I	13	J
14	15	16	17	18				
K	19	L	20	M	21	N	22	O
23	24	25	26	27				
P	28	Q	29	R	30	S	31	T

●	1	●	2	●	3	●	4	●
5	6	7	8	9	10	11	12	13
●	14	●	15	●	16	●	17	●
18	19	20	21	22	23	24	25	26
●	27	●	28	●	29	●	30	●

