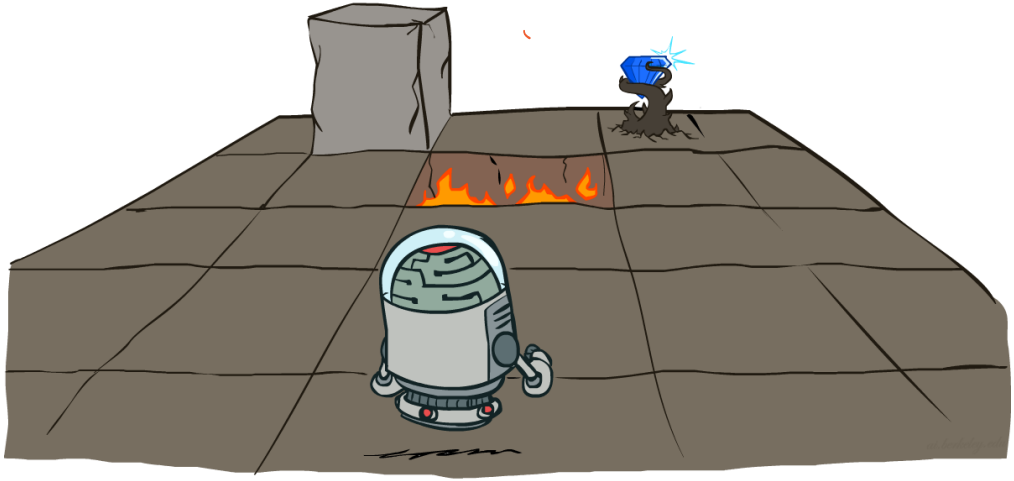
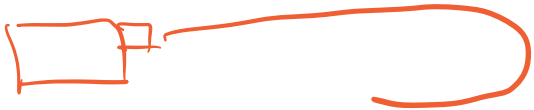
# Announcement

- Homework 4
  - Due: May 3, 11:59pm

# Markov Decision Processes

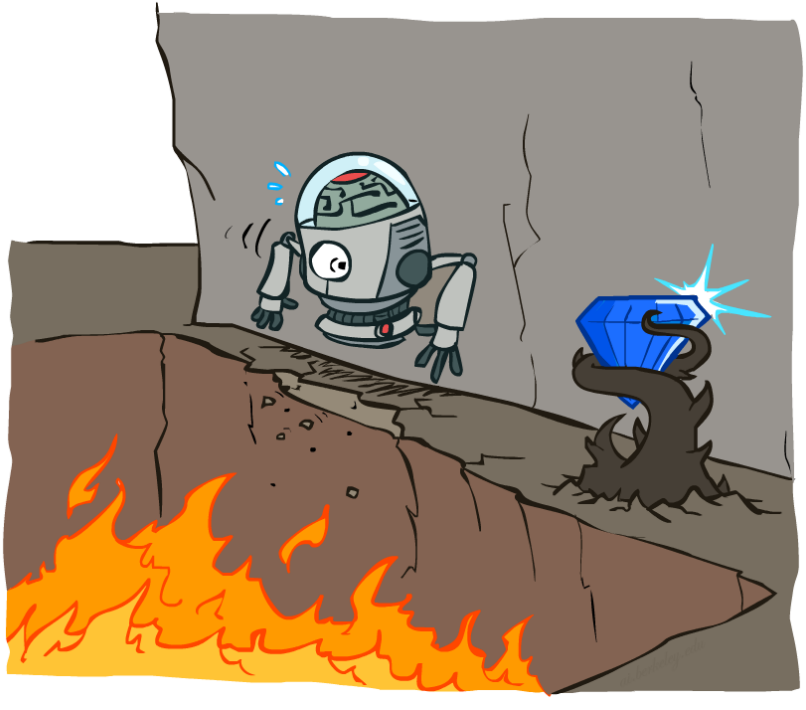


AIMA Chapter 17



[Adapted from slides by Dan Klein and Pieter Abbeel at UC Berkeley]

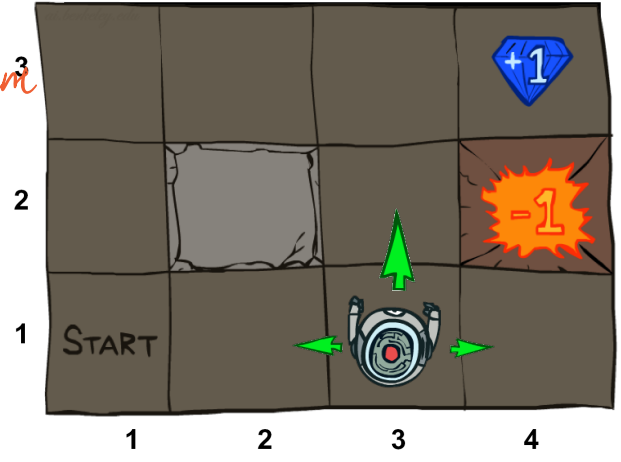
# Non-Deterministic Search



# Example: Grid World

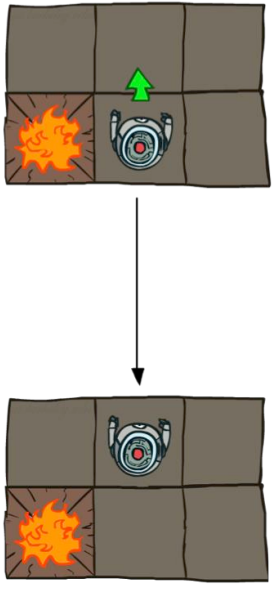
1. Not as planned
2. reward / punishment

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
  - Small "living" reward each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards

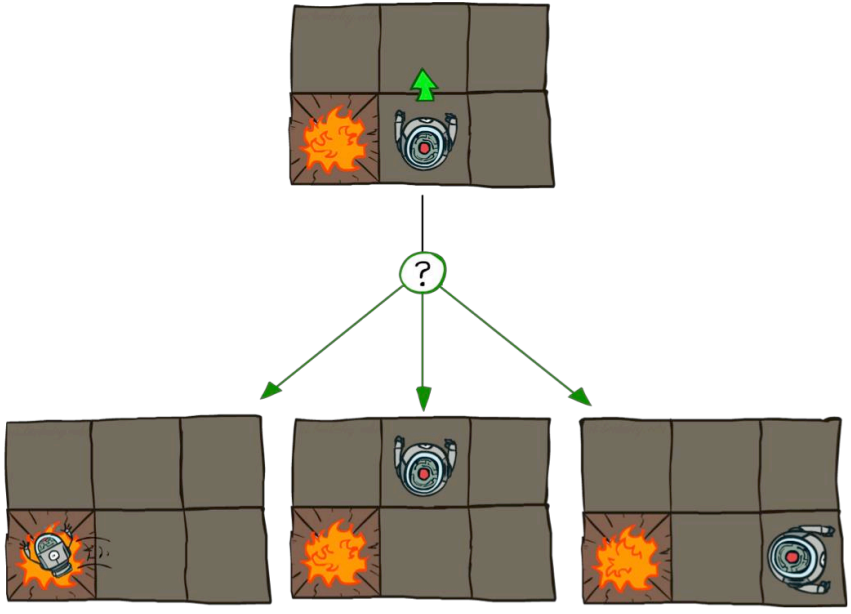


# Grid World Actions

Deterministic Grid World

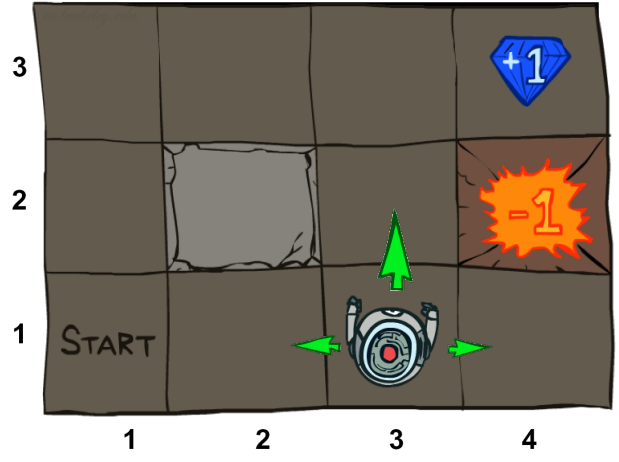


Stochastic Grid World



# Markov Decision Processes

- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition function $T(s, a, s') = P(s' | s, a)$
    - Probability that a from s leads to s', i.e., P(s' | s, a)
    - Also called the model or the dynamics
  - A reward function R(s, a, s')
    - Sometimes just R(s) or R(s') same time inde from actions.
  - A start state
  - Maybe a terminal state




- MDPs are non-deterministic search problems
  - One way to solve them is with expectimax search
  - We'll have a new tool soon

# What is Markov about MDPs?

- "Markov" generally means that given the present state, the future and the past are independent

- For Markov decision processes, "Markov" means action outcomes depend only on the current state

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \ldots S_0 = s_0)$$

$$= P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

- This is just like search, where the successor function could only depend on the current state (not the history)

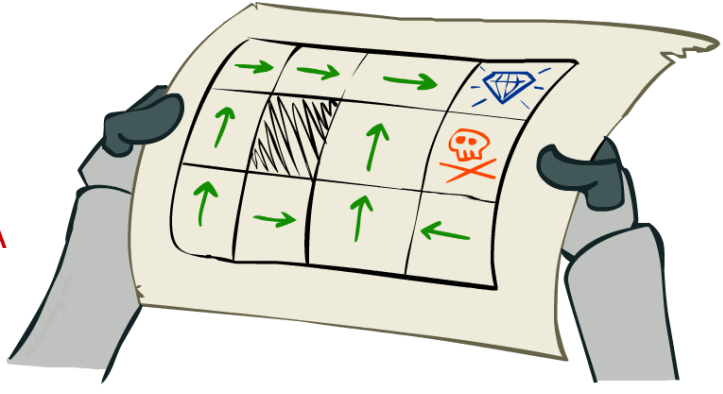
Andrey Markov
(1856-1922)

# Policies

- In deterministic single-agent search problems, we wanted an optimal plan, or sequence of actions, from start to a goal

"Rule"

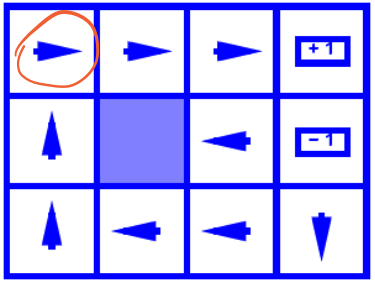
- For MDPs, we want an optimal policy $\pi^*: S \rightarrow A$
  - A policy $\pi$ gives an action for each state
  - An optimal policy is one that maximizes expected utility if followed
  - An explicit policy defines a reflex agent



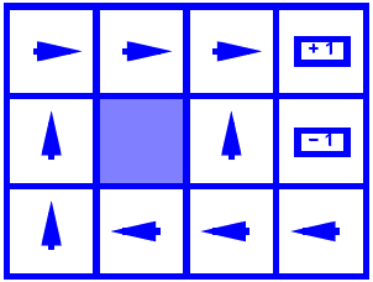
how to find best action

# Optimal Policies

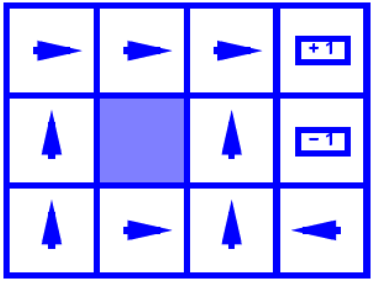
R(s) = “living reward”



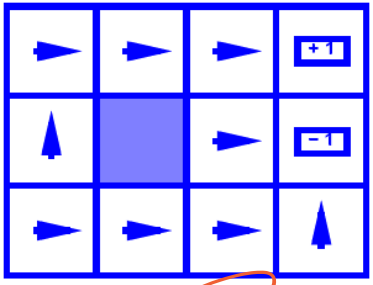

R(s) = -0.01




R(s) = -0.03

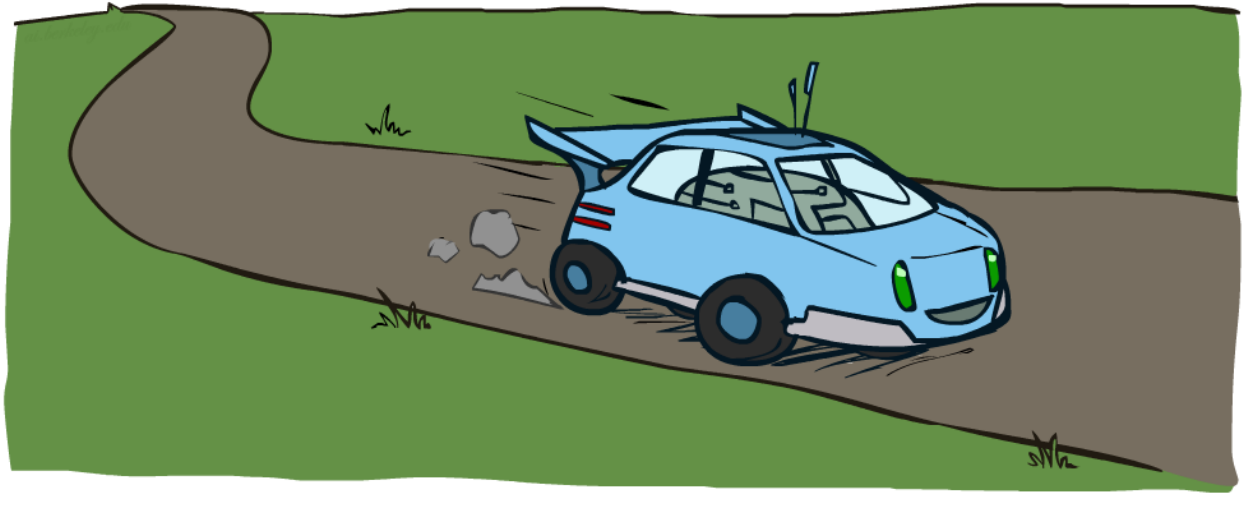



R(s) = -0.4




R(s) = -2.0

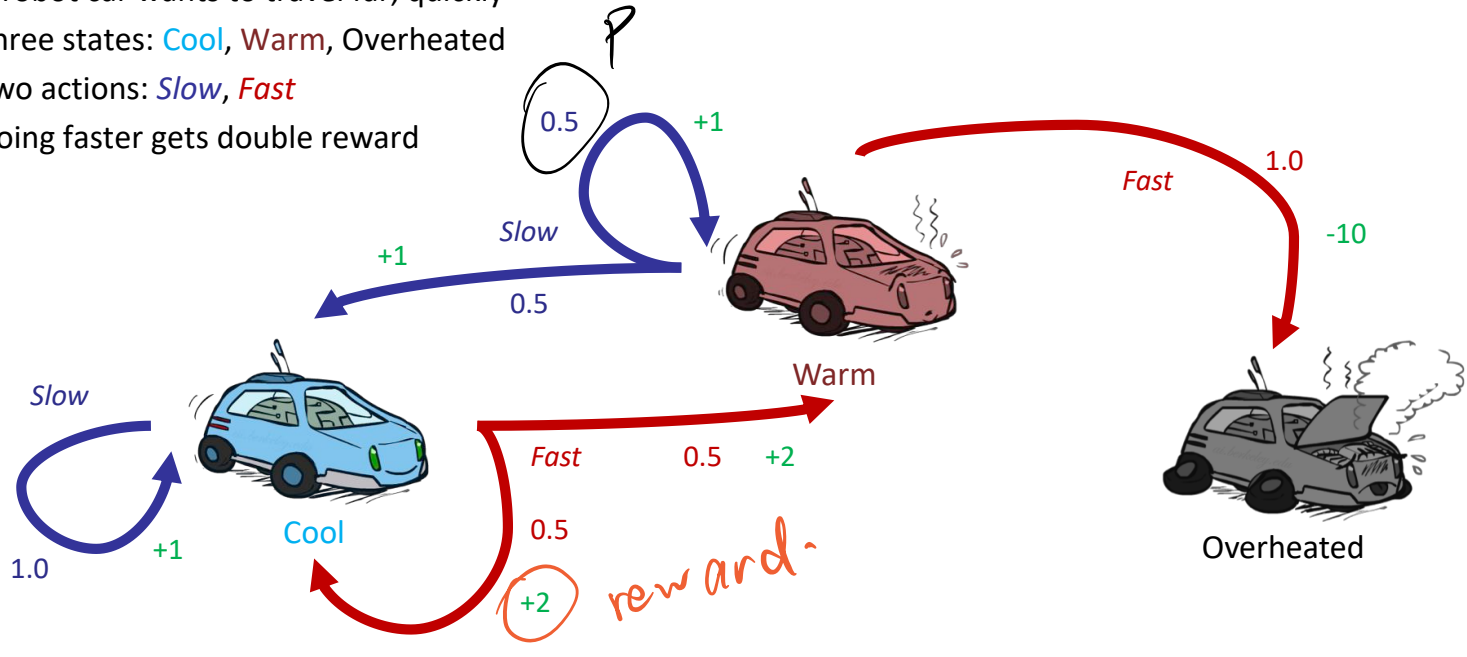
# Example: Racing



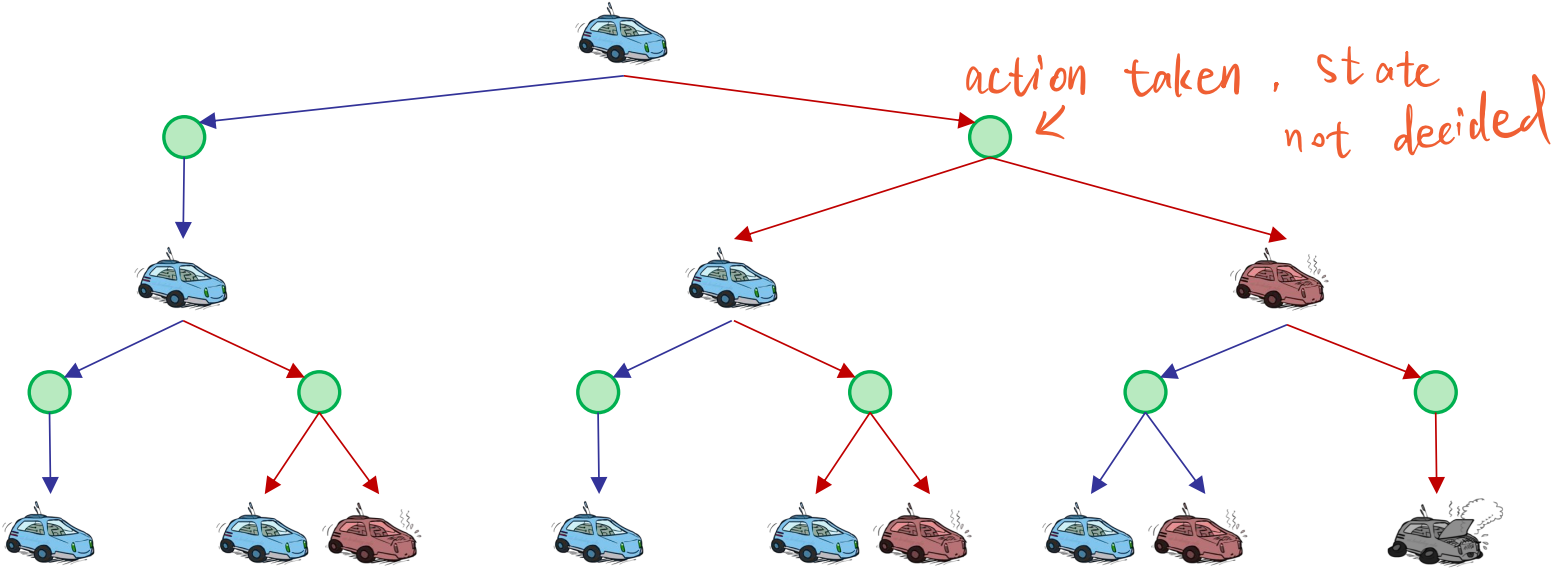
FSM

# Example: Racing

- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward

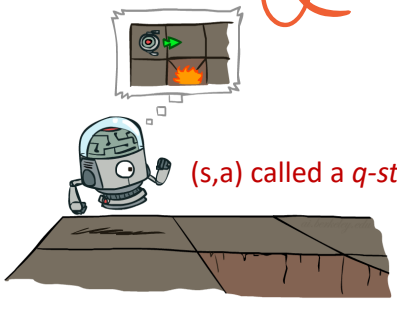


# Racing Search Tree

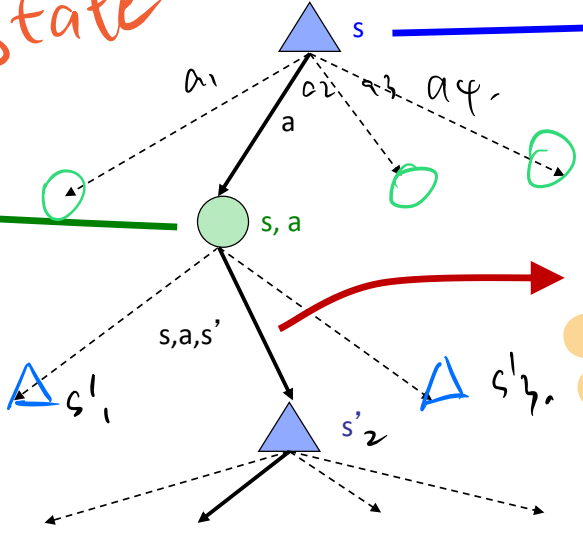


# MDP Search Trees

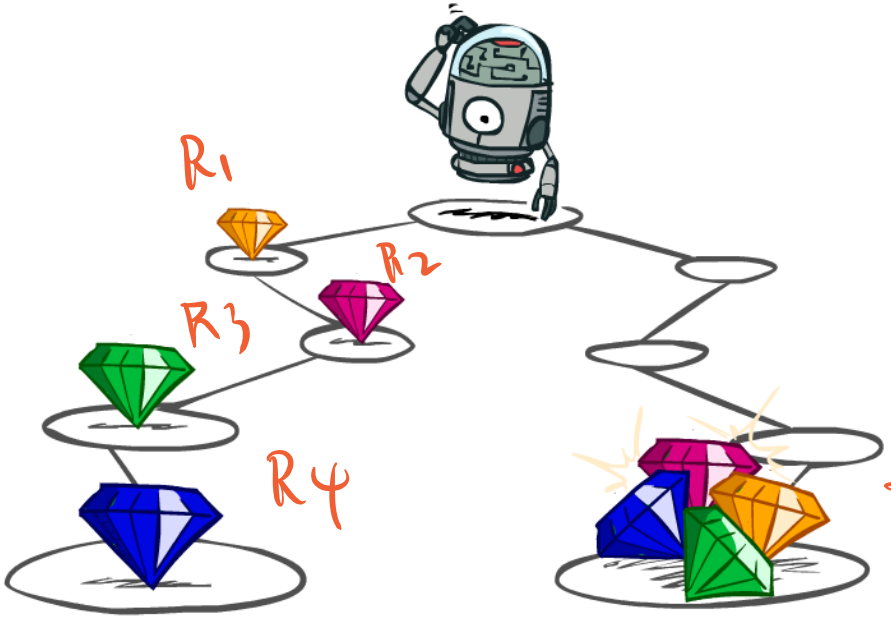
- Each MDP state projects an expectimax-like search tree



(s,a) called a *q-state*




# Utilities of Sequences




RSum=
$R_1+R_2+R_3$
$+R_4$

Which better?

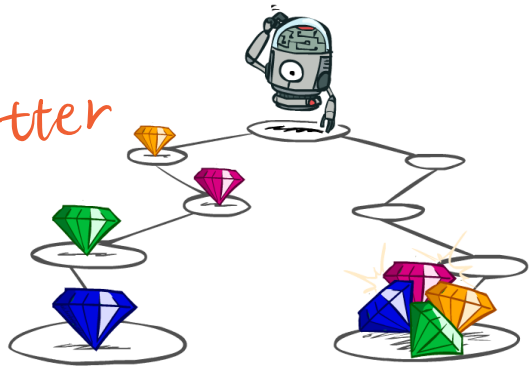
# Utilities of Sequences

- What preferences should an agent have over reward sequences?
- More or less? [1, 2, 2] or [2, 3, 4]
- Now or later? [0, 0, 1] or [1, 0, 0]

maybe ... better

speed!

faster



# Discounting

- It's reasonable to maximize the sum of rewards
- It's also reasonable to prefer rewards now to rewards later
- One solution: values of rewards decay exponentially

1

Worth Now

$\gamma$

Worth Next Step

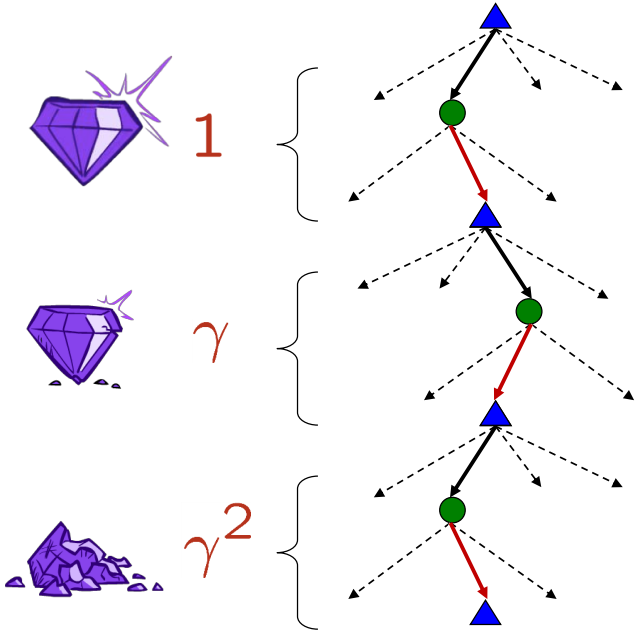
$\gamma^2$

Worth In Two Steps

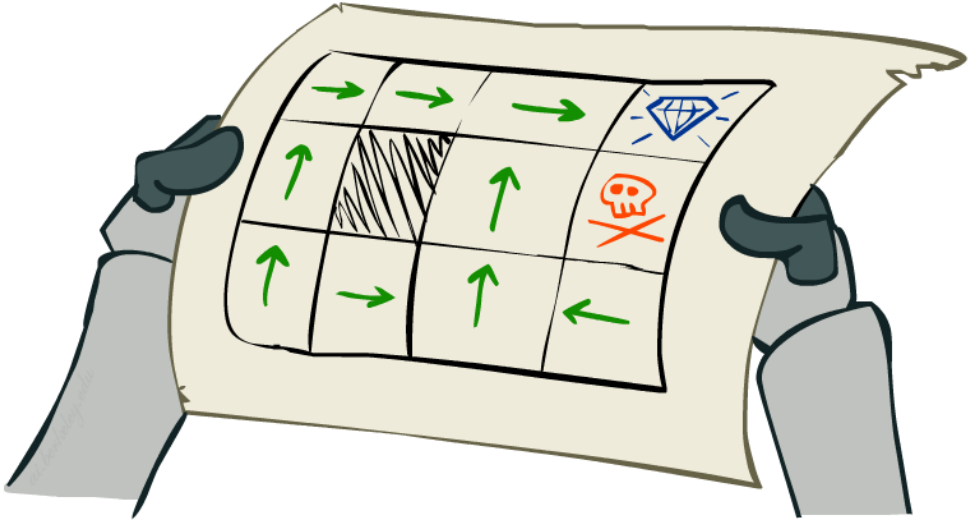
time ↑ weight ↓

# Discounting

- How to discount?
  - Each time we descend a level, we multiply in the discount once
- Why discount?
  - Sooner rewards probably do have higher utility than later rewards
  - Also helps our algorithms converge
- Example: discount of 0.5
  - U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3
  - U([1,2,3]) < U([3,2,1])

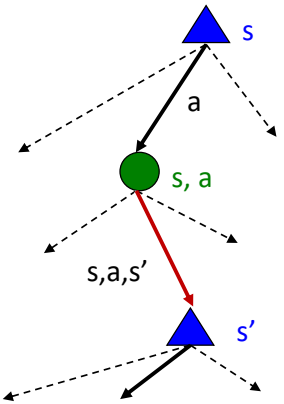


# Solving MDPs



# Optimal Quantities

- The value (utility) of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
  $\pi^*(s)$ = optimal action from state s

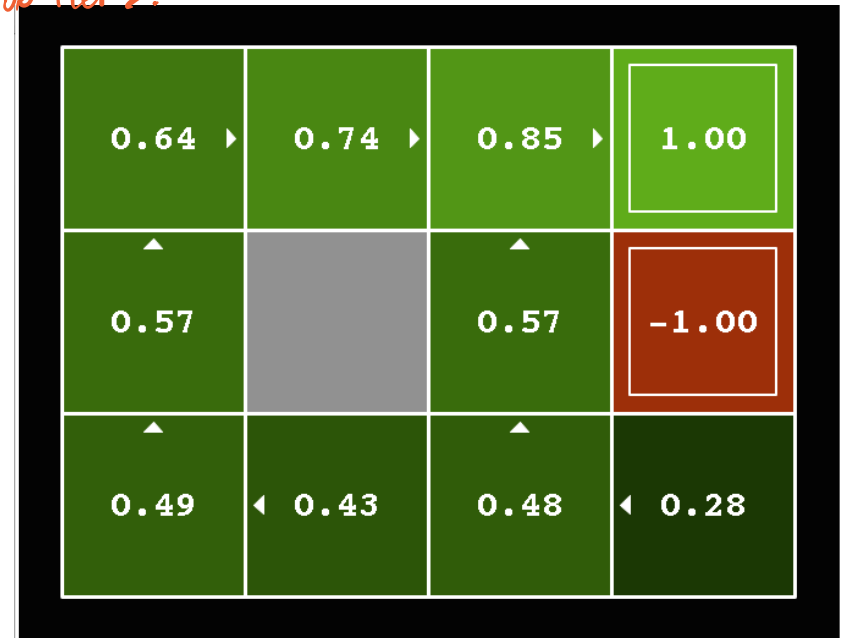



s is a *state*

(s, a) is a *q-state*

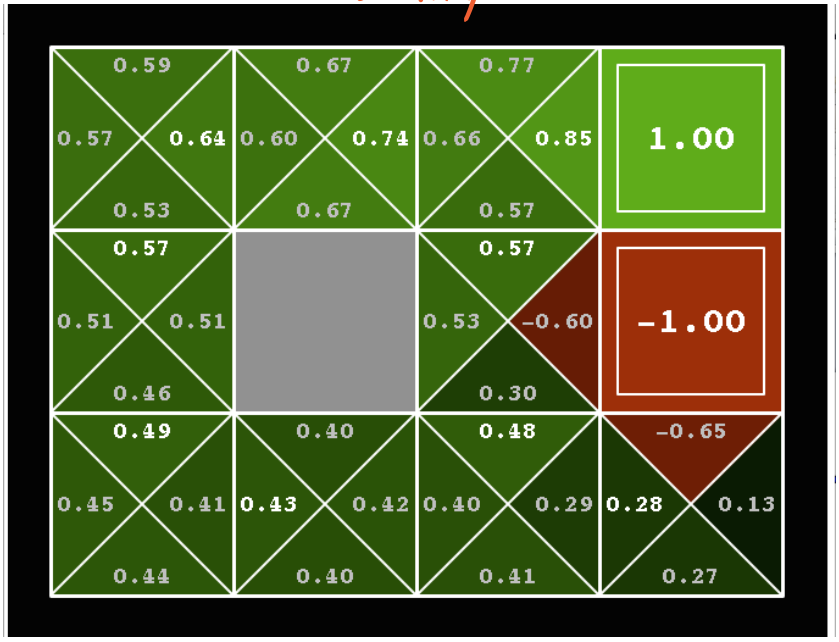
(s,a,s') is a *transition*

# Gridworld V Values




Noise = 0.2
Discount = 0.9
Living reward = 0

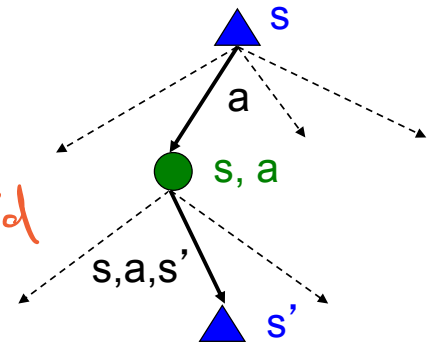
# Gridworld Q Values




Noise = 0.2
Discount = 0.9
Living reward = 0

# Values of States

- How to compute the value of a state
  - Expected utility under optimal action
  - This is just what expectimax computed!




- Recursive definition of value:

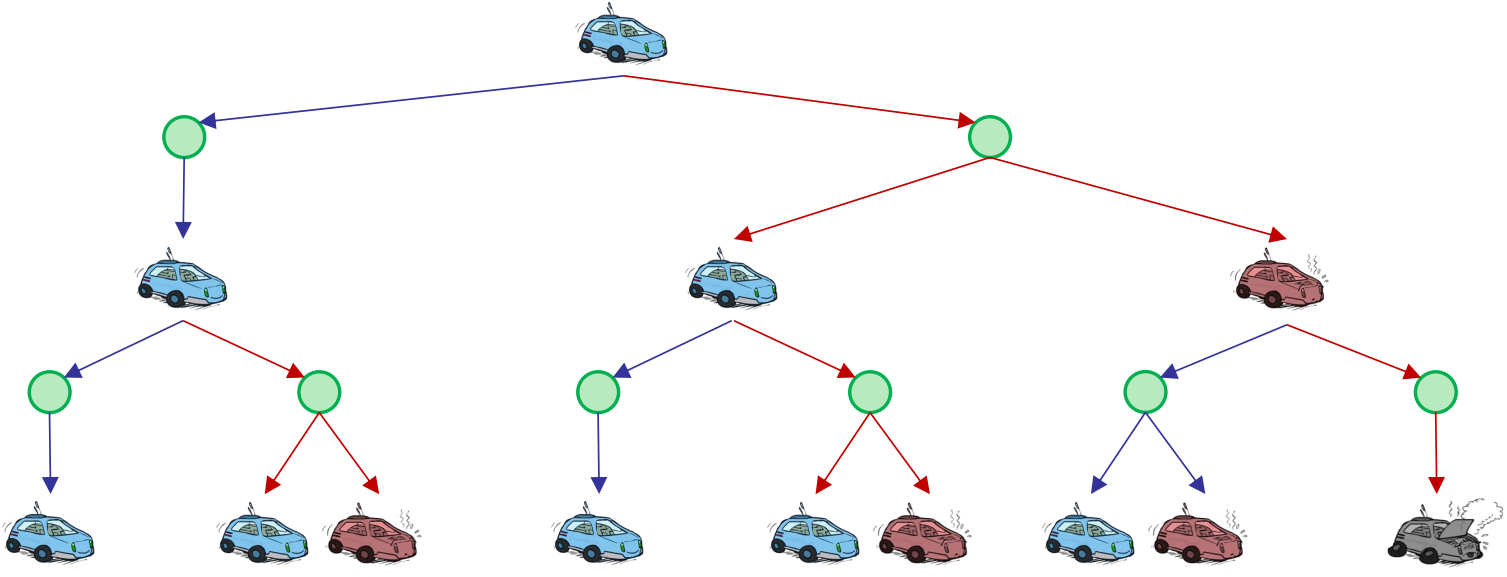
$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V^*(s') \right]$$

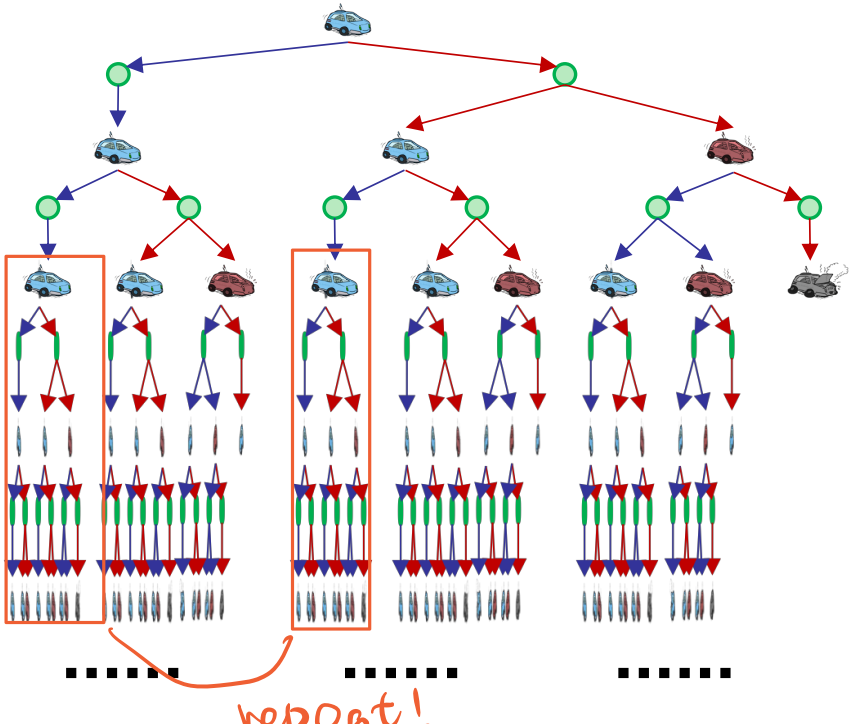
$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V^*(s') \right]$$

The Bellman Equation

# Racing Search Tree

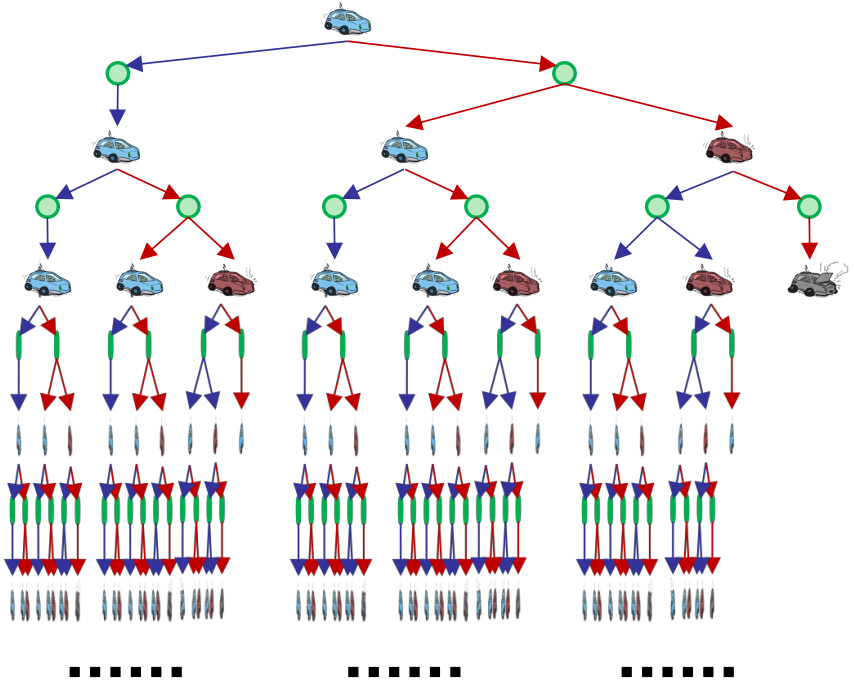


# Racing Search Tree

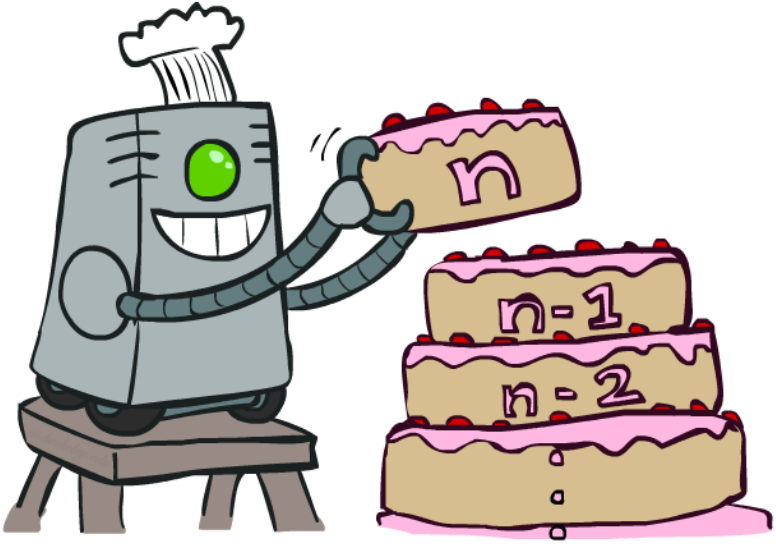


# Problems with Expectimax

- Problem 1: States are repeated
  - Idea: Only compute needed quantities once
- Problem 2: Tree goes on forever
  - Idea: Do a depth-limited computation, but with increasing depths until change is small
  - Note: deep parts of the tree eventually don't matter if $\gamma < 1$



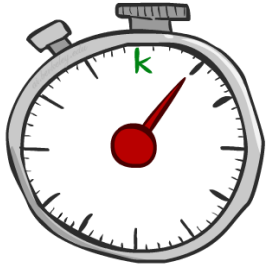
# Value Iteration

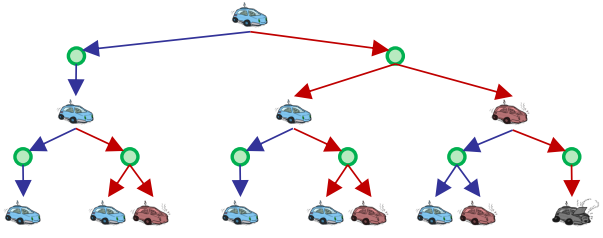


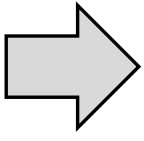
# Time-Limited Values

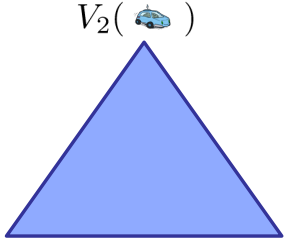
k step limit

- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
  - Equivalently, it's what a depth-k expectimax would give from s

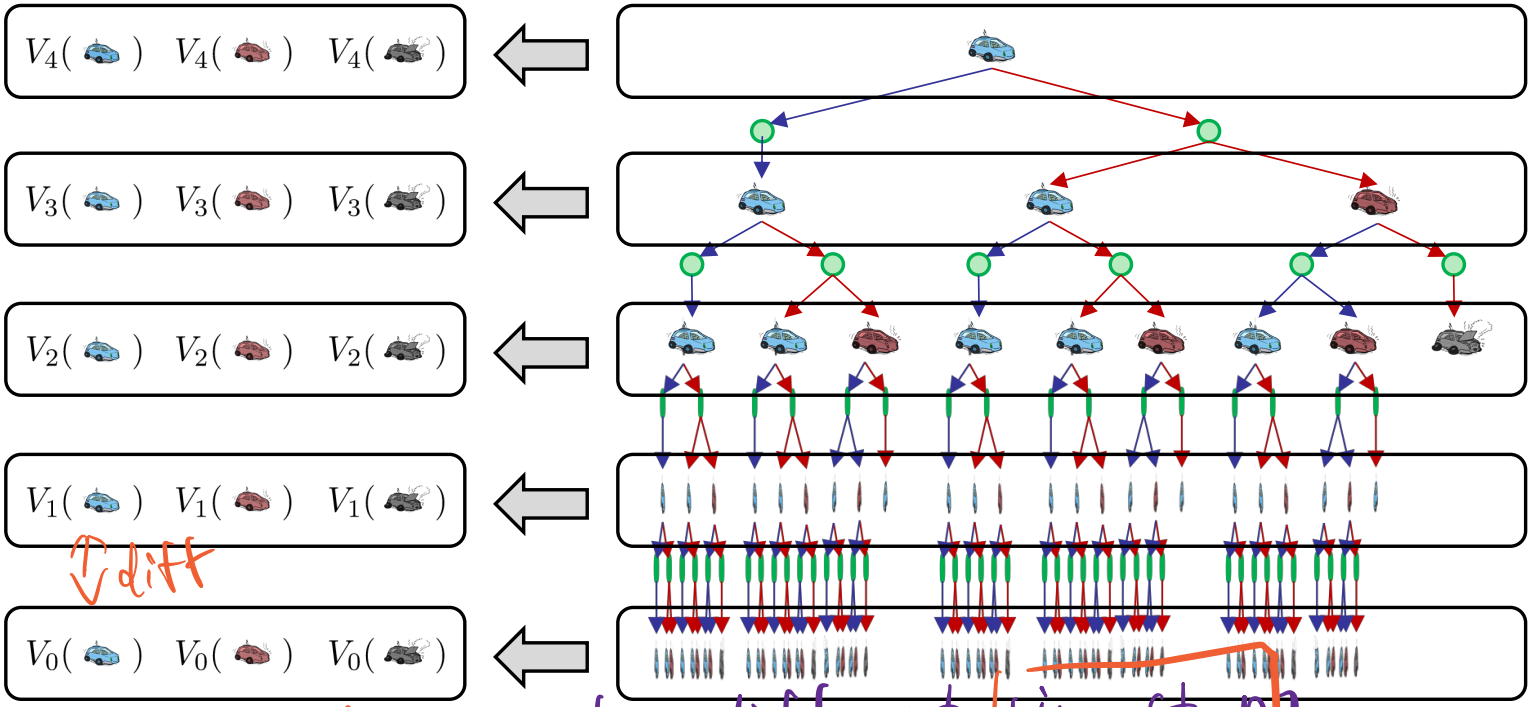






# Computing Time-Limited Values



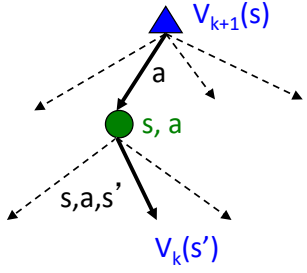
# Value Iteration

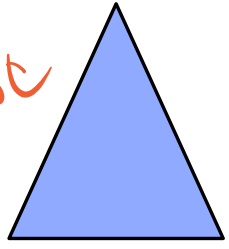
- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

time

- Repeat until convergence
- Complexity of each iteration: $O(S^2A)$
- Theorem: will converge to unique optimal values



# Example: Value Iteration

$V_2$

$V_1$

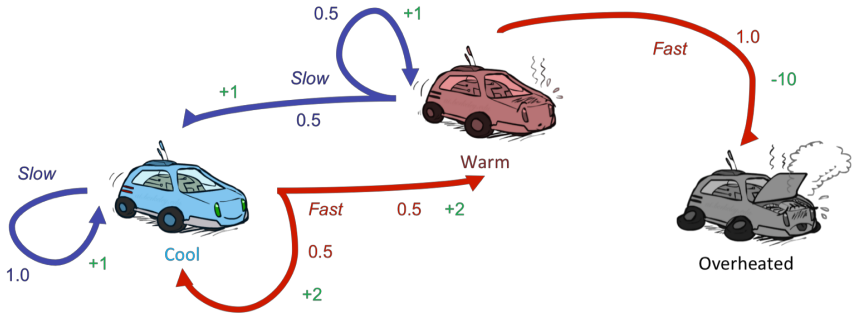
S: 1 + γ·0

F: .5*2+.5*2=2

.5×2+.5×2+ γ·0

$V_0$

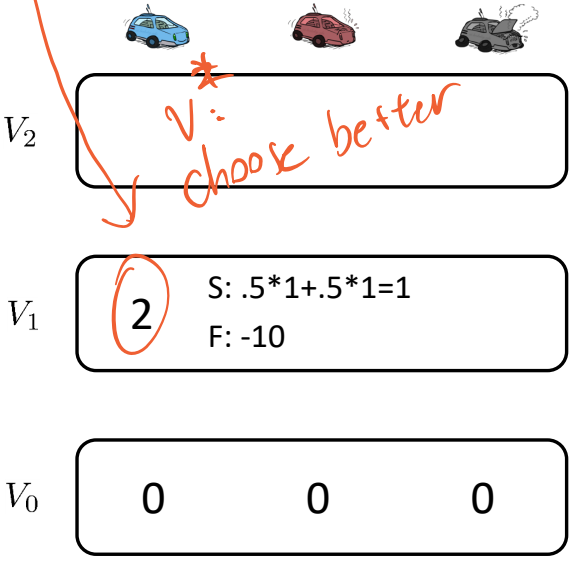
0 0 0

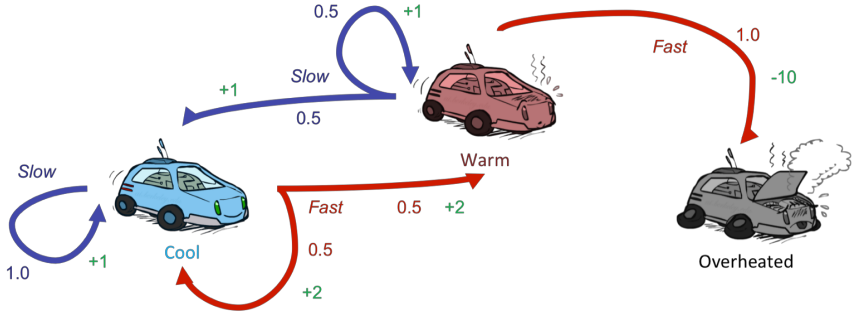



*Assume no discount!*

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

# Example: Value Iteration

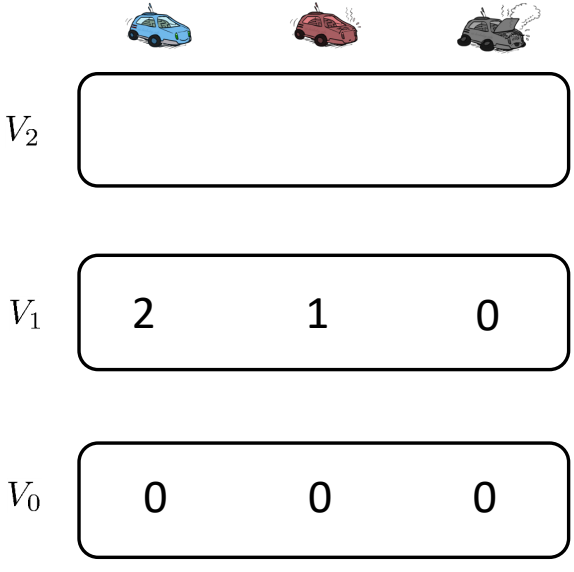


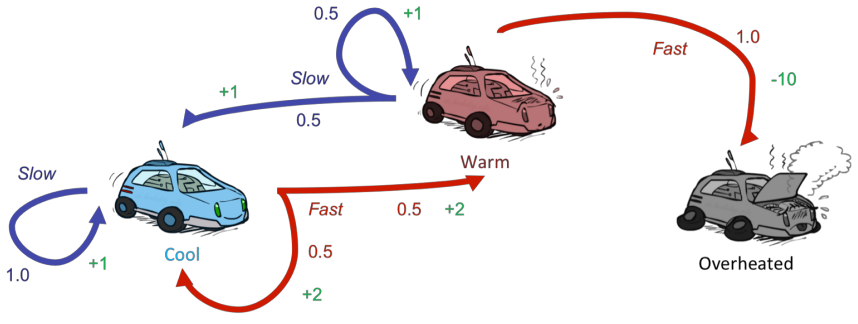




*Assume no discount!*

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

# Example: Value Iteration

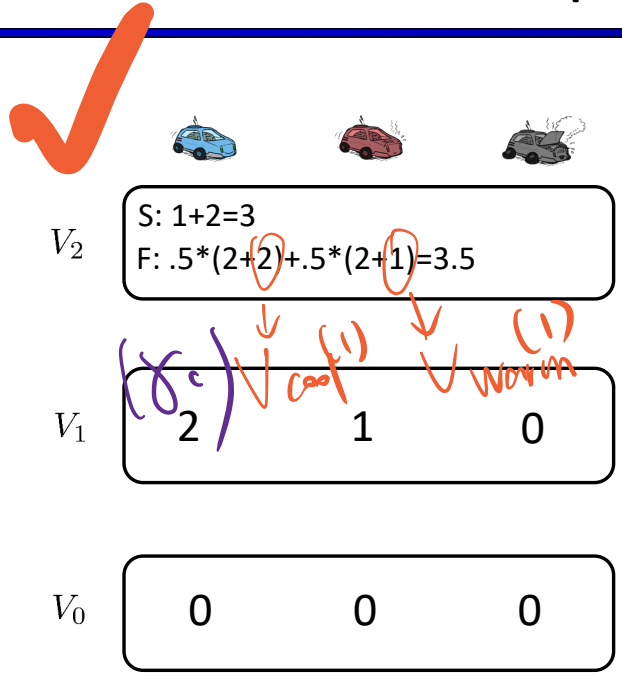


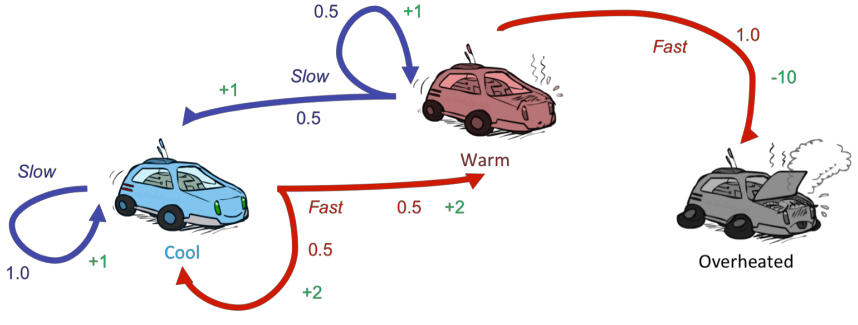




*Assume no discount!*

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

# Example: Value Iteration




*Assume no discount!*

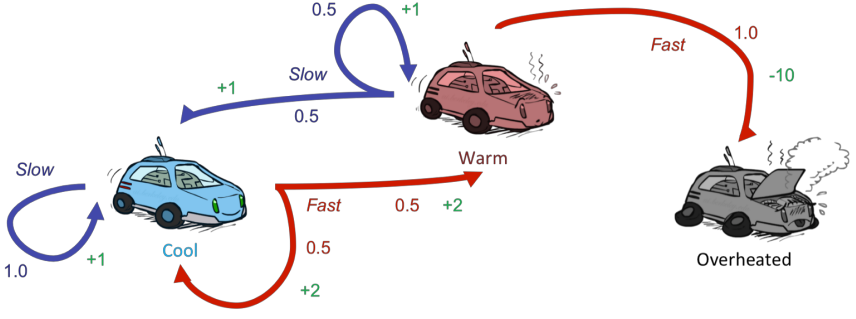
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

# Example: Value Iteration

$V_2$ | 3.5 | S: .5*(2+1)+.5*(1+1)=2.5<br>F: -10

$V_1$ | 2 | 1 | 0

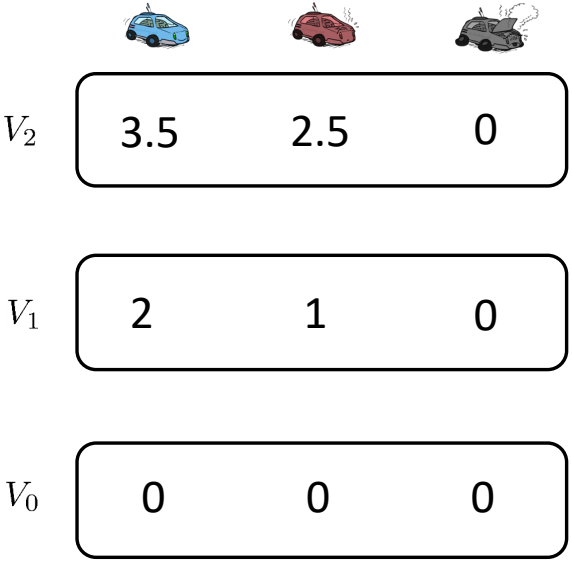
$V_0$ | 0 | 0 | 0

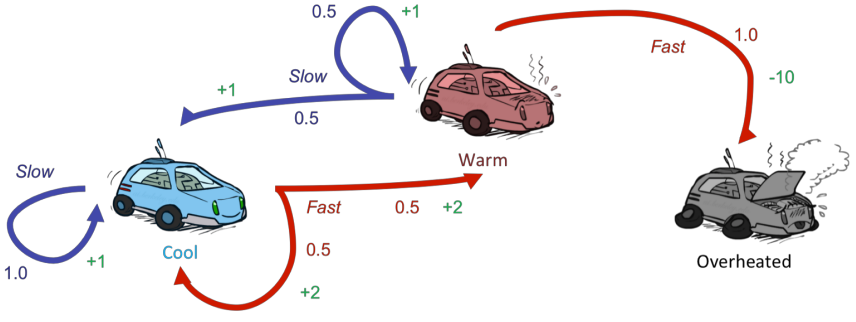



*Assume no discount!*

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

# Example: Value Iteration



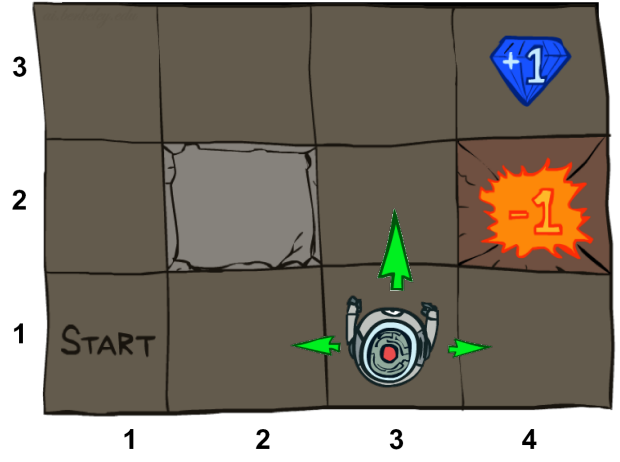




*Assume no discount!*

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

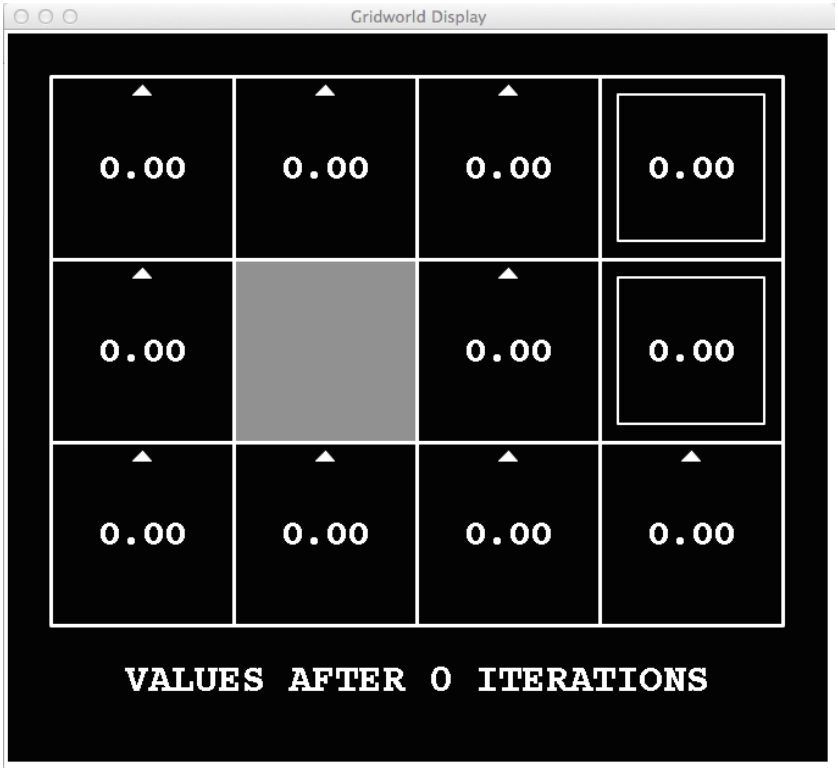
# Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
  - Small "living" reward each step (can be negative)
  - Big rewards come at the end (good or bad)



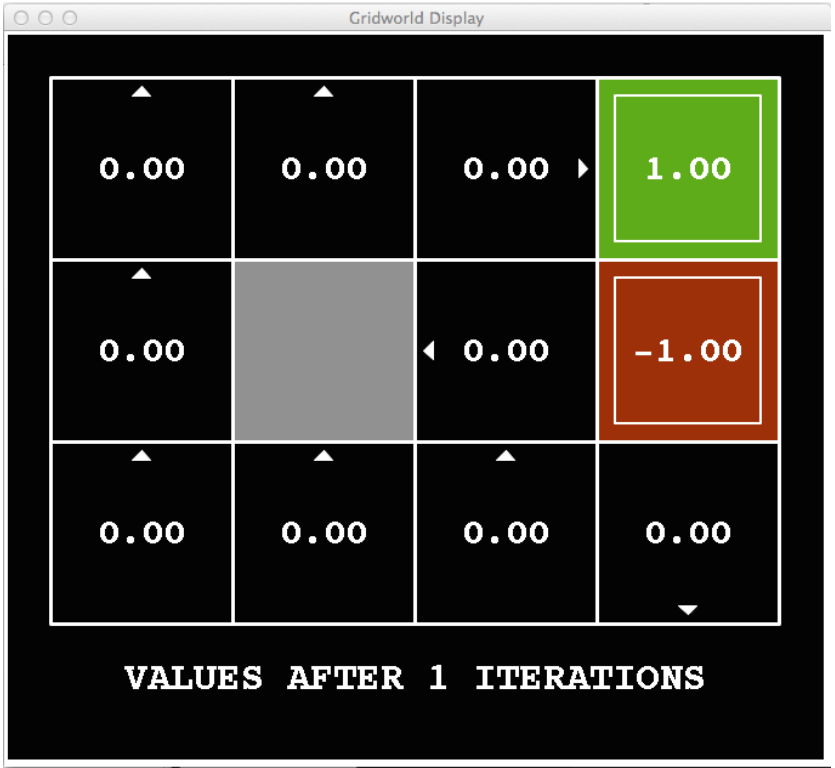

Suppose we get this reward by taking an "exit" action at a goal state

# k=0




Noise = 0.2
Discount = 0.9
Living reward = 0

# k=1




Noise = 0.2
Discount = 0.9
Living reward = 0

# k=2


Noise = 0.2
Discount = 0.9
Living reward = 0

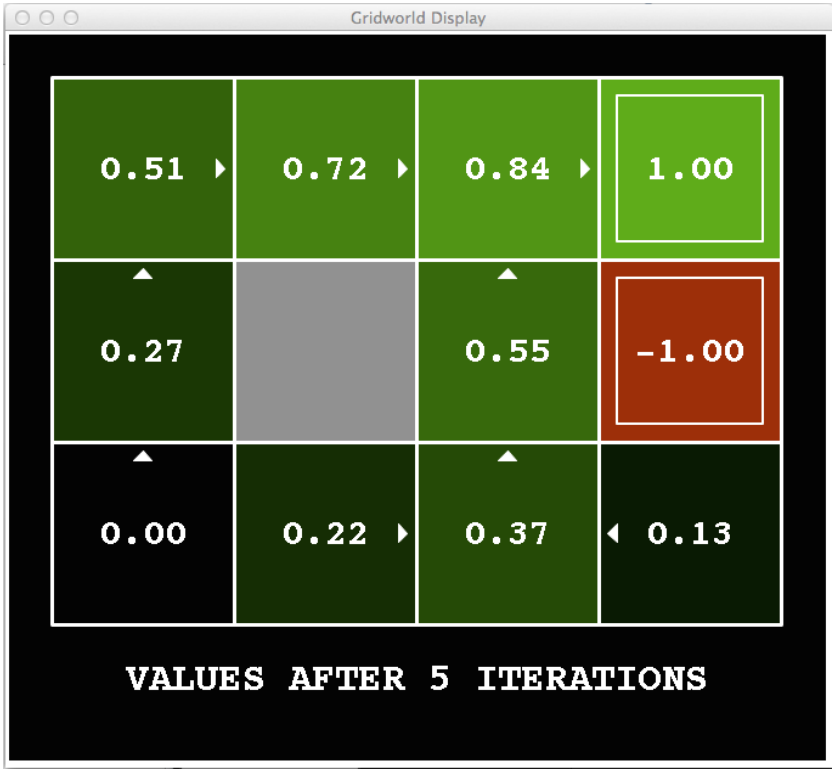
# k=3


Noise = 0.2
Discount = 0.9
Living reward = 0

# k=4


Noise = 0.2
Discount = 0.9
Living reward = 0

# k=5

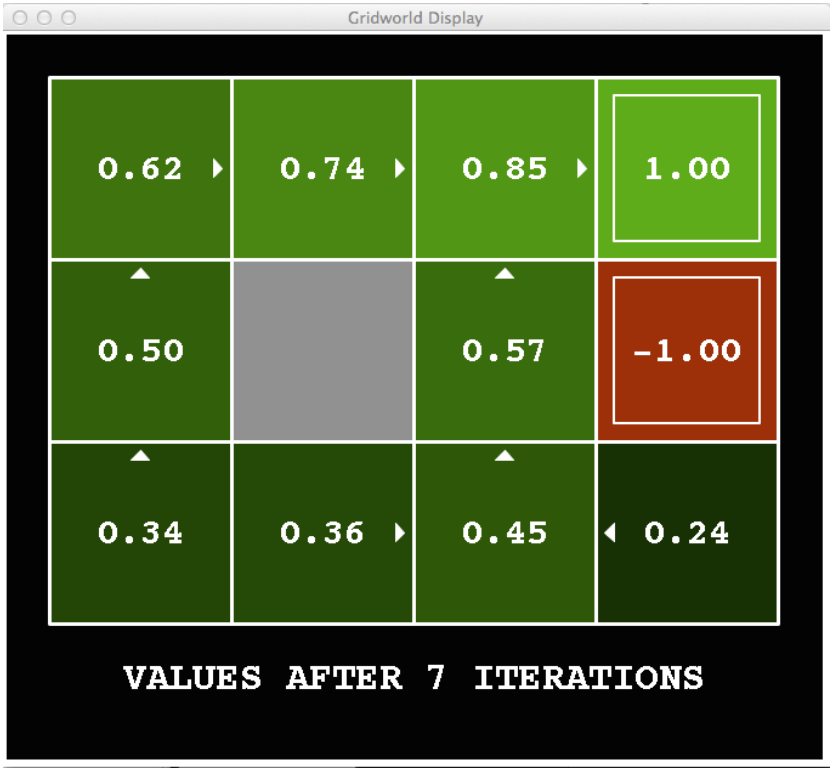



Noise = 0.2
Discount = 0.9
Living reward = 0

# k=6

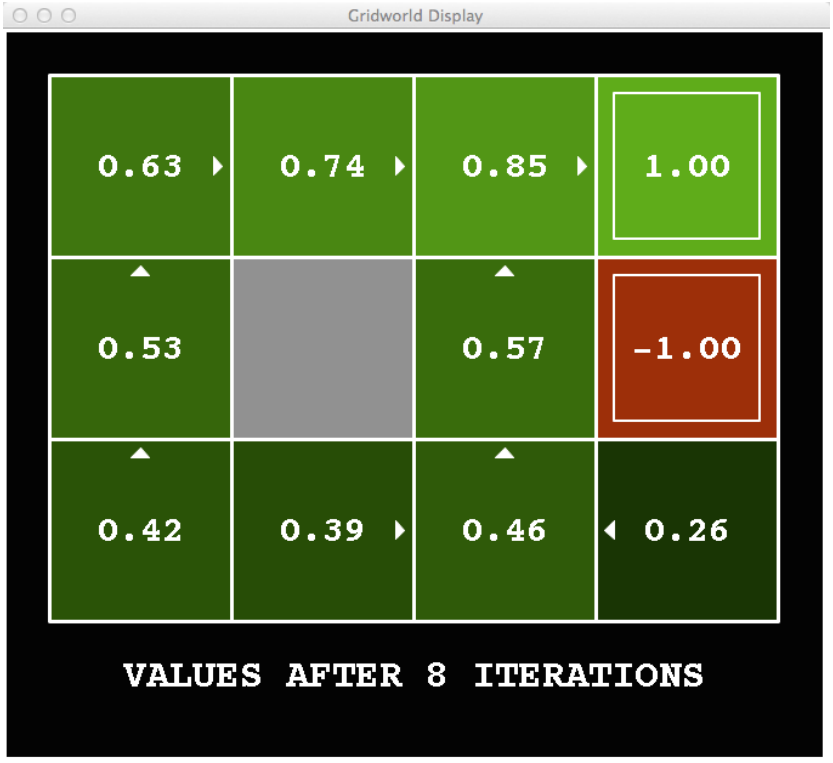

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=7



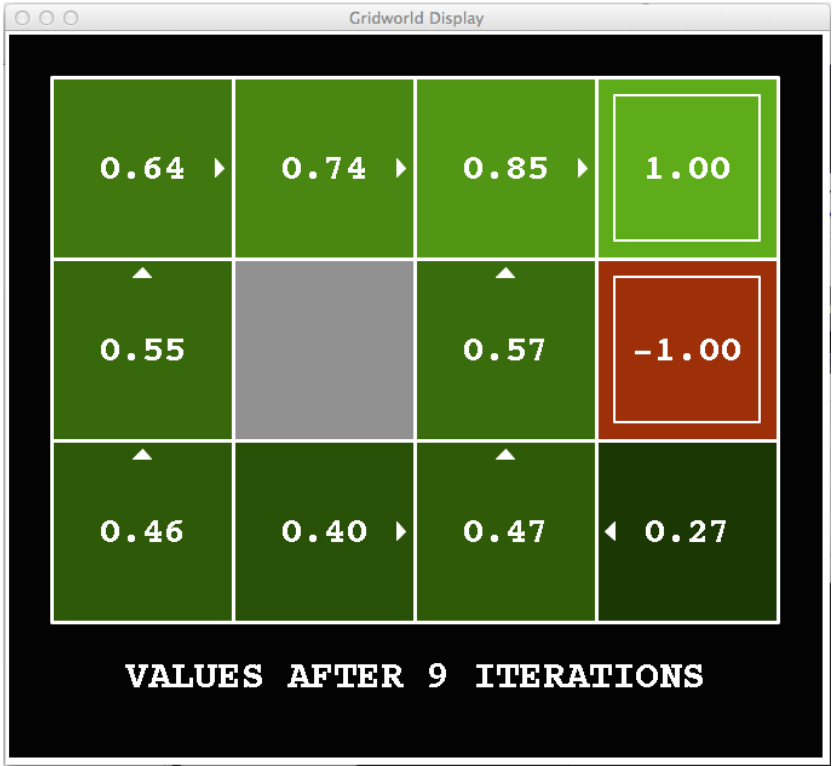

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=8




Noise = 0.2
Discount = 0.9
Living reward = 0

# k=9




Noise = 0.2
Discount = 0.9
Living reward = 0

# k=10


Noise = 0.2
Discount = 0.9
Living reward = 0

# k=11

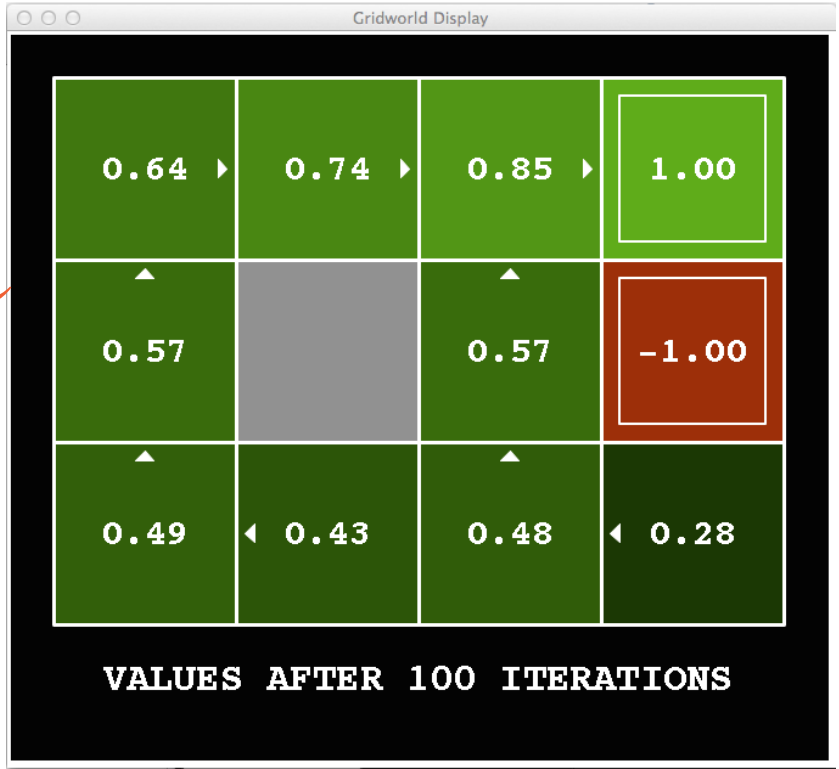

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=12


action convergence

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=100



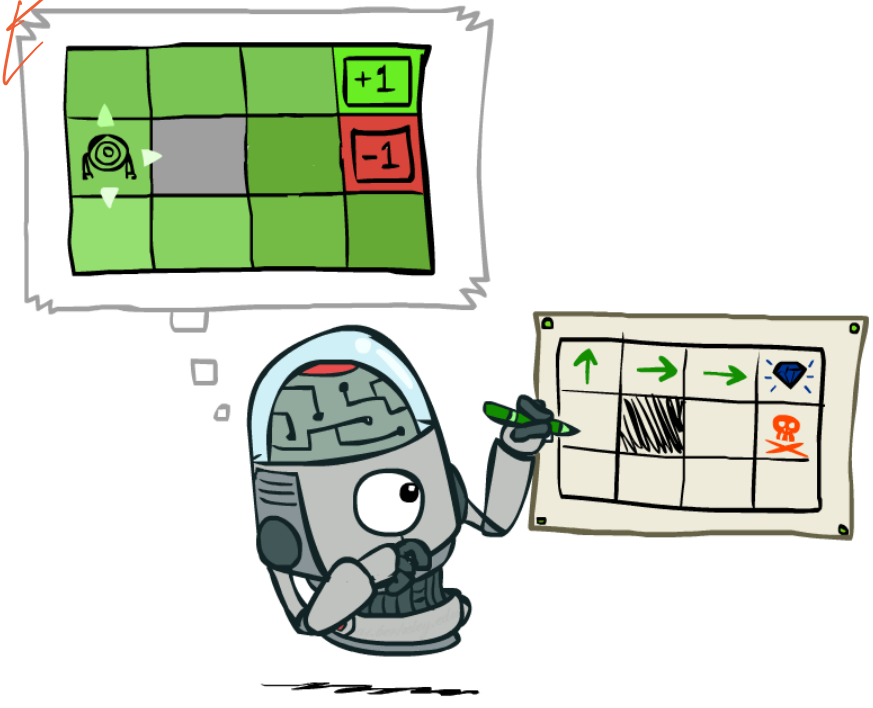

Value converge

Noise = 0.2
Discount = 0.9
Living reward = 0

from value

# Policy Extraction

find action (policy)

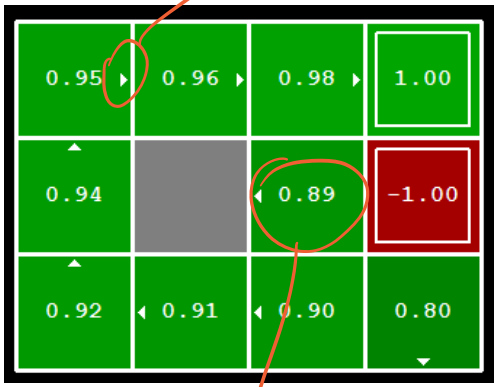


# Computing Actions from Values

- Let's imagine we have the optimal values V*(s)
- How should we act?
  - It's not obvious!
- We need to do a mini-expectimax (one step)

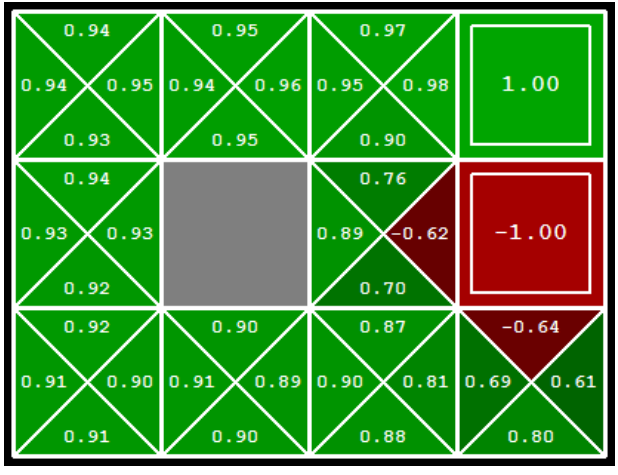
$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

- This is called policy extraction, since it gets the policy implied by the values



# Computing Actions from Q-Values

- Let's imagine we have the optimal q-values:




- How should we act?
  - Completely trivial to decide!

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

Q values : Easie

- Important: actions are easier to select from q-values than values!
- Q-values can also be computed in value iteration

# Q-Value Iteration

- Value iteration: find successive (depth-limited) values
  - Start with $V_0(s) = 0$
  - Given $V_k$, calculate the depth k+1 values for all states:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

t+1 t

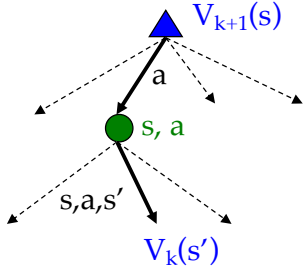
- But Q-values are more useful, so compute them instead
  - Start with $Q_0(s,a) = 0$
  - Given $Q_k$, calculate the depth k+1 q-values for all q-states:

$$Q_{k+1}(s,a) \leftarrow \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma \max_{a'} Q_k(s',a') \right]$$

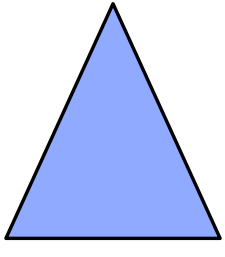
# Value Iteration

- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$



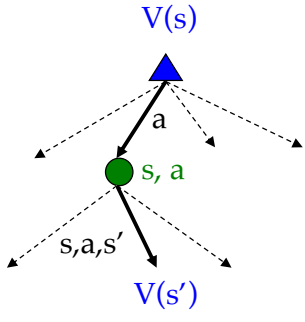

- Repeat until convergence, which yields V*
- Complexity of each iteration: $O(S^2A)$
- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do



# Value Iteration

- Bellman equations characterize the optimal values:

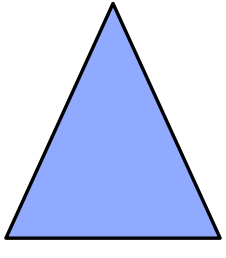
$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V^*(s') \right]$$



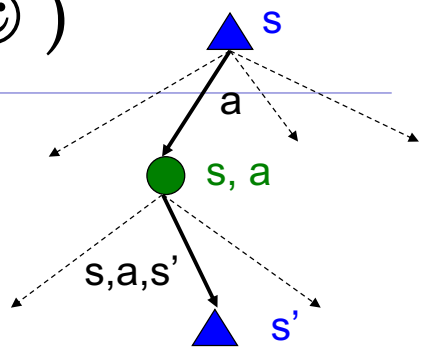

- Value iteration computes them:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

- Value iteration is just a fixed point solution method
  - … though the $V_k$ vectors are also interpretable as time-limited values



# Value Iteration (again ☺ )




- Init:

$$\forall s: \quad V(s) = 0$$

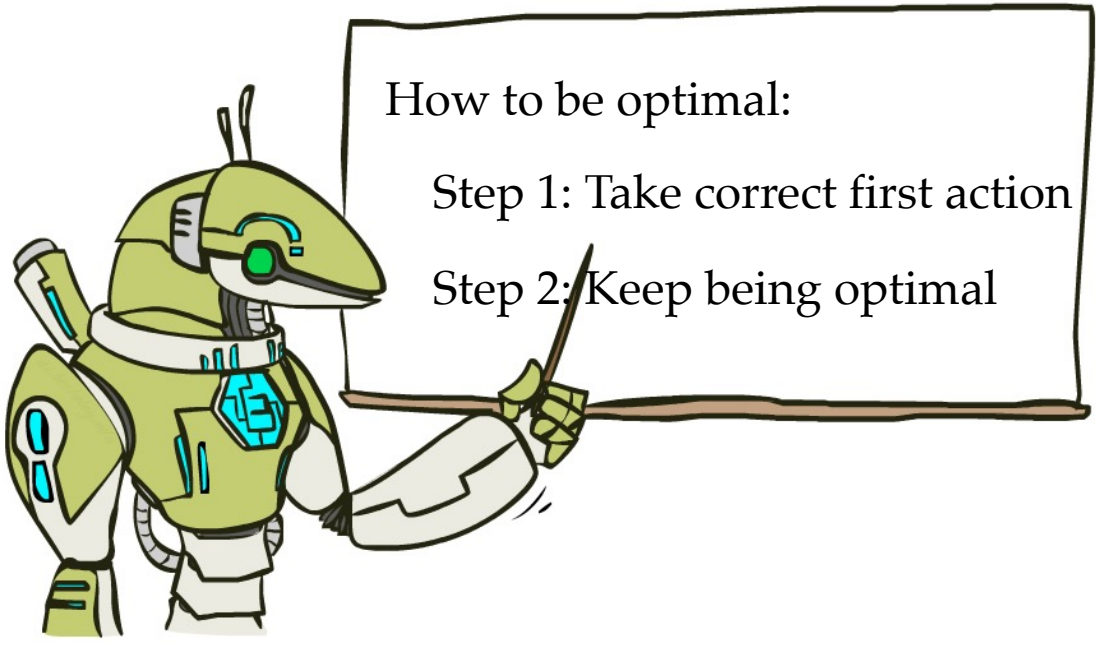
- Iterate:

$$\forall s: \quad V_{new}(s) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V(s')]$$

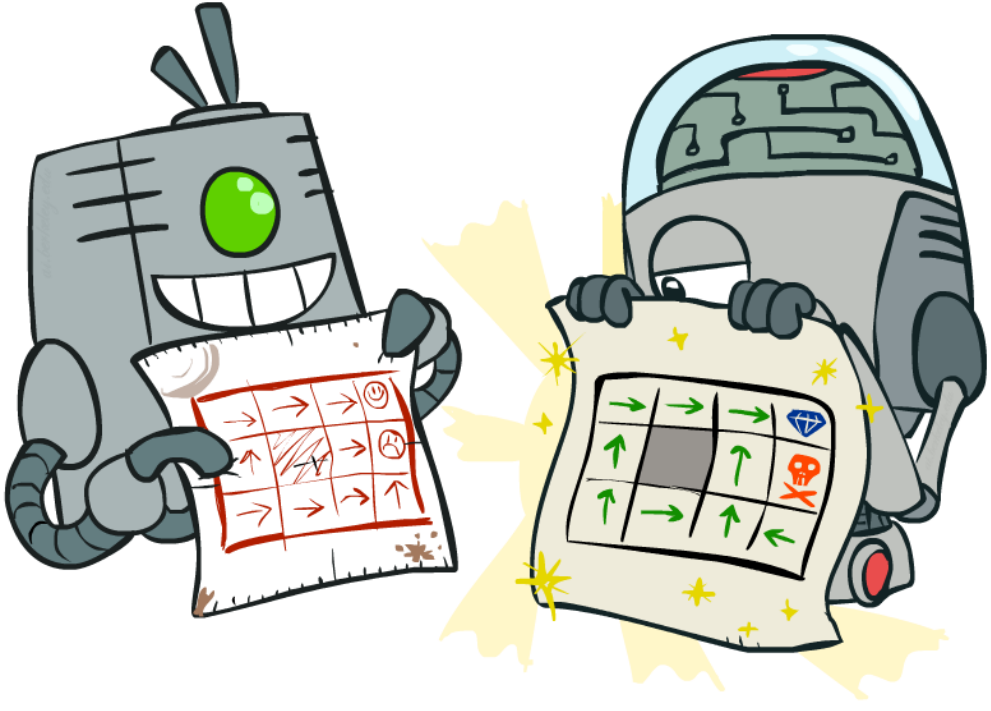
$$V = V_{new}$$

Note: can even directly assign to *V(s)*, which will not compute the sequence of $V_k$ but will still converge to $V^*$

# The Bellman Equations = Dp.



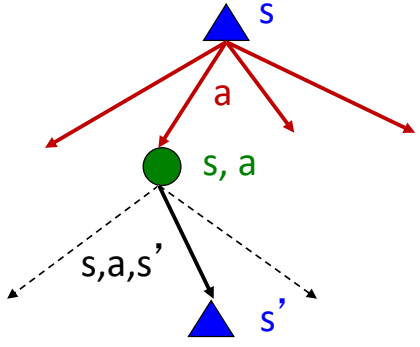
# Policy Methods



# Problems with Value Iteration

- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$




- Problem 1: It's slow – O(S²A) per iteration

- Problem 2: The "max" at each state rarely changes
  - The policy often converges long before the values

policy faster than value

# k=12


Noise = 0.2
Discount = 0.9
Living reward = 0

# k=100

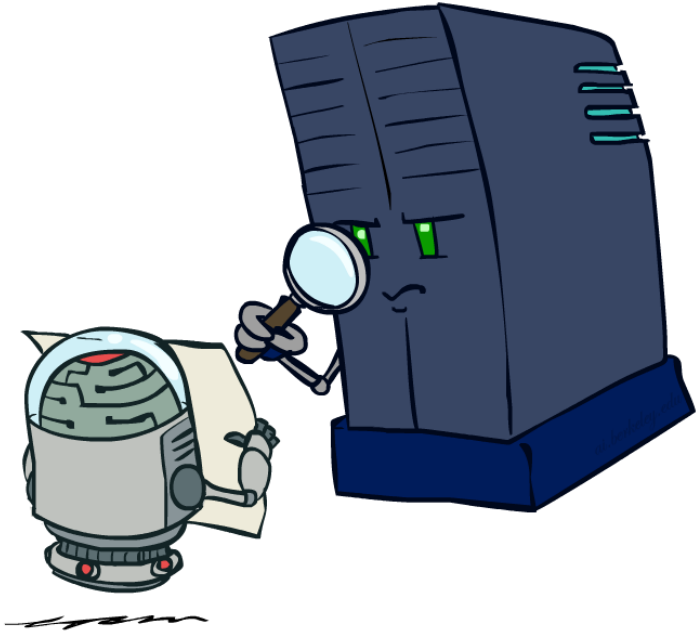

Noise = 0.2
Discount = 0.9
Living reward = 0

# Policy Iteration

- Policy iteration: an alternative approach for value iteration
  - Step 1: Policy evaluation: calculate utilities for some fixed (not optimal) policy
  - Step 2: Policy improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges

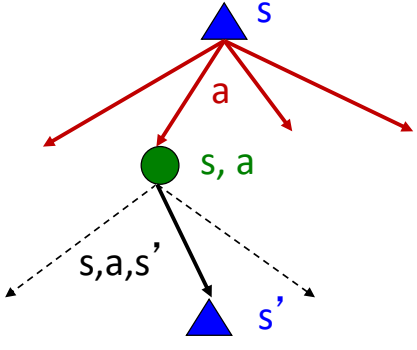
- It's still optimal!
- Can converge (much) faster under some conditions

# Step 1: Policy Evaluation

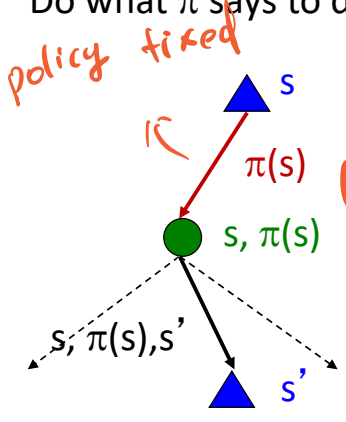


# Fixed Policies

Do the optimal action




Do what π says to do




( action fixed)

- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy π(s), then the tree would be simpler – only one action per state

# Utilities for a Fixed Policy

- The utility of a state s, under a fixed policy π:

  $V^{\pi}(s)$ = expected utility starting in s and following π

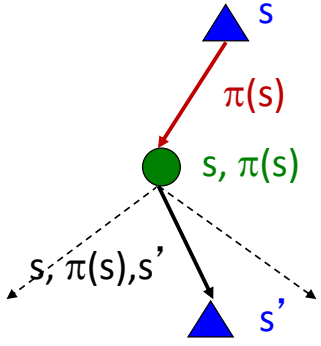
  given policy π

- Recursive relation (one-step look-ahead):

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

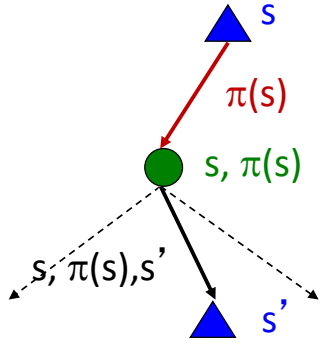
(transition func)

Bellman func



# Policy Evaluation

- How do we calculate the values under a fixed policy π?




- Idea 1: Iterative updates (like value iteration)
  - Start with $V_0^\pi(s) = 0$
  - Given $V_k^\pi$, calculate the depth k+1 values for all states:

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$

  - Repeat until convergence
  - Efficiency: $O(S^2)$ per iteration

$O(S^2 A)$

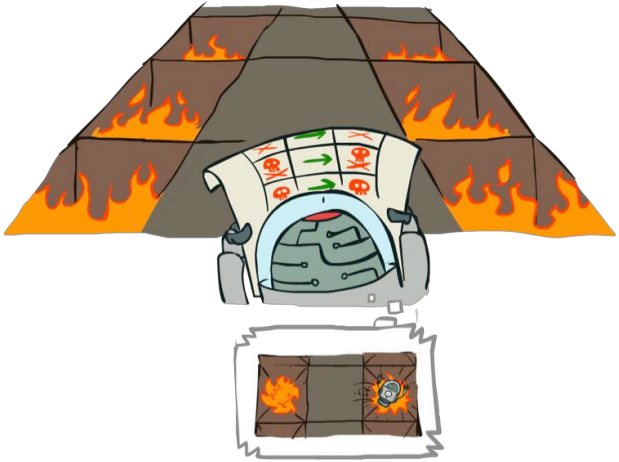
- Idea 2: Without the maxes, the Bellman equations are just a linear system

$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^\pi(s')]$$

  - Solvable with a linear system solver

# Example: Policy Evaluation

Policy: Always Go Right



| | | |
|---|---|---|
| -10.00 | 100.00 | -10.00 |
| -10.00 | 1.09 $V_1$ | -10.00 |
| -10.00 | -7.88 $V_2$ | -10.00 |
| -10.00 | -8.69 $V_3$ | -10.00 |

$V_1 = f(V_2)$

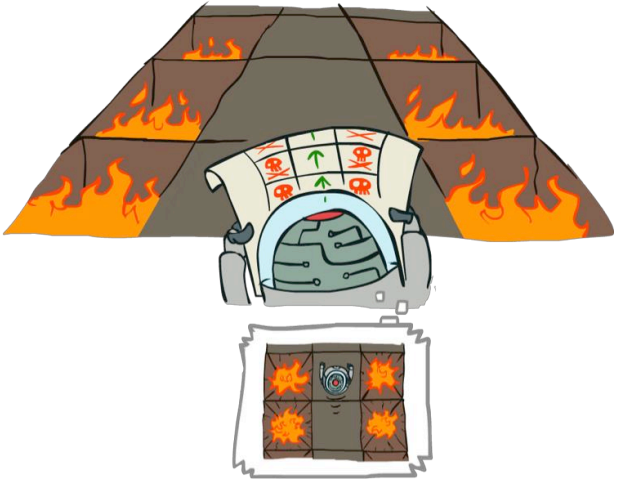
$V_2 = f(V_1, V_3)$

$V_3 = f(V_2)$

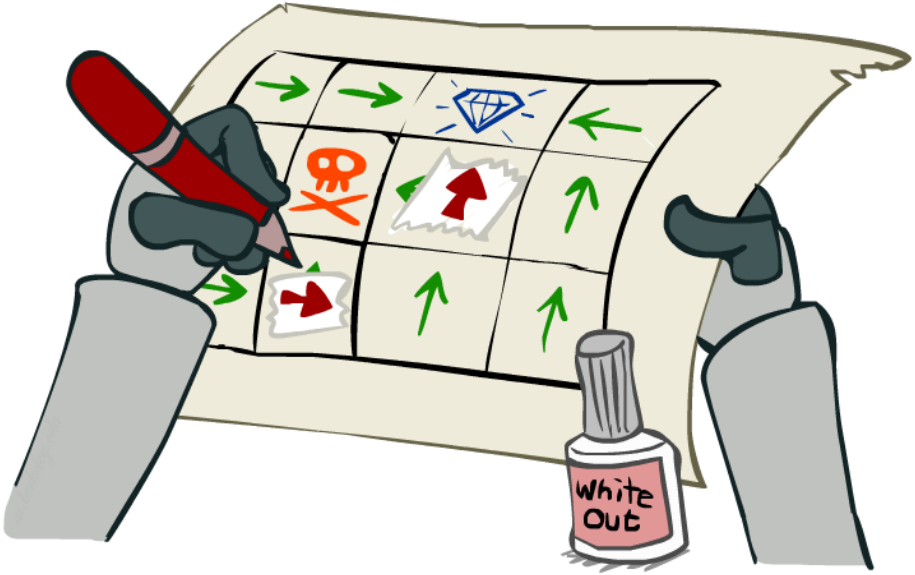
$V_1 = 0.1 \times 100 \times \gamma + 0.1 \times V_2 \times \gamma$

# Example: Policy Evaluation

Always Go Forward



# Step 2: Policy Improvement




Actions.

Value ⇒ Policy

# Policy Improvement

- Step 2: Improvement: For fixed values, get a better policy using policy extraction
  - One-step look-ahead:

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V^{\pi_i}(s') \right]$$

value of previous round

Just like value iter / expect max.

- Policy Iteration: repeat the two steps until policy converges

from Policy

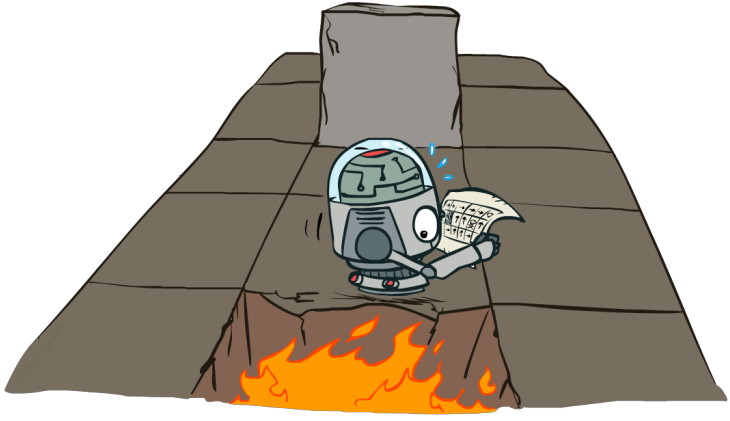
V (max)

# Value Iteration vs. Policy Iteration

- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
  - Every iteration updates both the values and (implicitly) the policy
  - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
  - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
  - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
  - May converge faster
- Both are dynamic programs for solving MDPs

# Summary

- Markov Decision Process
  - States S, Actions A, Transitions P(s'|s,a), Rewards R(s,a,s')
- Quantities:
  - Policy, Utility, Values, Q-Values
- Solve MDP
  - Value iteration
  - Policy iteration



# Summary: MDP Algorithms

- So you want to….
  - Compute optimal values: use value iteration or policy iteration
  - Compute values for a particular policy: use policy evaluation
  - Turn your values into a policy: use policy extraction (one-step lookahead)

just compute values.

- These all look the same!
  - They basically are – they are all variations of Bellman updates
  - They all use one-step lookahead expectimax fragments
  - They differ only in whether we plug in a fixed policy or max over actions