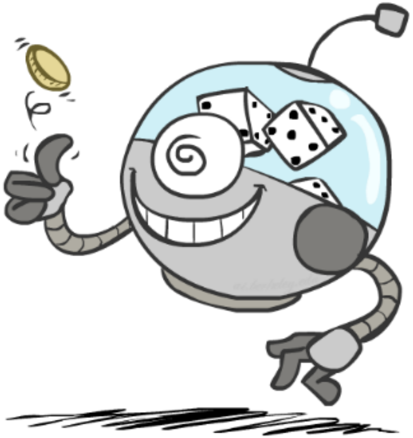
Probability

AIMA Chapter 13

[Adapted from slides by Dan Klein and Pieter Abbeel at UC Berkeley (ai.berkeley.edu).]

Outline

- Probability
  - Random Variables
  - Joint and Marginal Distributions
  - Conditional Distributions
  - Inference
  - Product Rule, Chain Rule, Bayes' Rule
  - Independence.



Uncertainty

- My flight to New York is scheduled to leave at 11:25
  - Let action A_t = leave home t minutes before flight and drive to the airport
  - Will A_t ensure that I catch the plane?
- Problems:
  - noisy sensors (radio traffic reports, Google maps)
  - uncertain action outcome (car breaking down, accident, etc.)
  - partial observability (other drivers' plans, etc.) (not known)
  - immense complexity of modelling and predicting traffic, security line, etc.

Responses to uncertainty

- Ignore it – map directly from percept stream (known) to actions
  - Hopeless!
- Some sort of softening of logical rules (容差 ***fudge factors***)
  - $A_{1440} \rightarrow_{0.9999}$ *CatchPlane*
  - *CatchPlane* $\rightarrow_{0.95}$ $\neg$ *MajorTrafficJam*
  - Hence, chaining these together, $A_{1440} \rightarrow_{0.949}$ $\neg$ *MajorTrafficJam*
  - Oops
- Probability (Mahaviracarya (9th C.), Cardamo (1565))
  - Given the available evidence and the choice A_{120}, I will catch the plane with probability 0.92

Probability

- Probability
  - Given the available evidence and the choice A_{120}, I will catch the plane with probability 0.92
- ***Subjective*** or ***Bayesian*** probability:
  - Probabilities relate propositions to one's own state of knowledge
    - ignorance: lack of relevant facts, initial conditions, etc.
    - laziness: failure to list all exceptions, compute detailed predictions, etc.
  - Not claiming a "probabilistic tendency" in the actual situation (traffic is not like quantum mechanics)

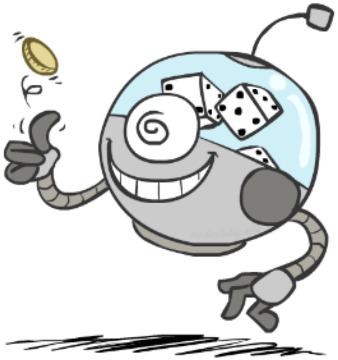
Decisions

- Suppose I believe
  - $P(CatchPlane \mid A_{60}, \text{all my evidence...}) = 0.51$
  - $P(CatchPlane \mid A_{120}, \text{all my evidence...}) = 0.97$
  - $P(CatchPlane \mid A_{1440}, \text{all my evidence...}) = 0.9999$
- Which action should I choose?
- Depends on my ***preferences*** for, e.g., missing flight, airport food, etc.
- ***Utility theory*** is used to represent and infer preferences
- ***Decision theory*** = utility theory + probability theory
- ***Maximize expected utility*** : $a^* = \mathrm{argmax}_a \sum_s P(s \mid a)\, U(s)$

Random Variables

utility

- A random variable is some aspect of the world (formally a ***deterministic function*** of ω) about which we (may) be uncertain
  - *Odd* = Is the dice roll an odd number?
  - *T* = Is it hot or cold?
  - *D* = How long will it take to get to the airport?
- Random variables have domains
  - *Odd* in {true, false} e.g. *Odd*(1)=true, *Odd*(6) = false
    - often write the *event* Odd=true as odd, Odd=false as $\neg$odd
  - *T* in {hot, cold}
  - *D* in $[0, \infty)$

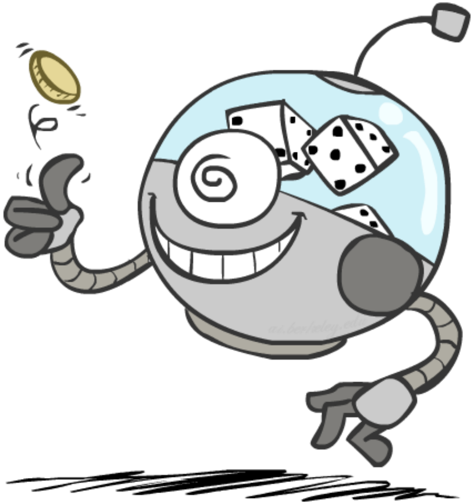


Random Variables

- A random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the pacman?

- We denote random variables with capital letters

- Like variables in a CSP, random variables have domains
  - R in {true, false}   (often write as {+r, -r})
  - T in {hot, cold}
  - D in $[0, \infty)$
  - L in possible locations, maybe {(0,0), (0,1), …}



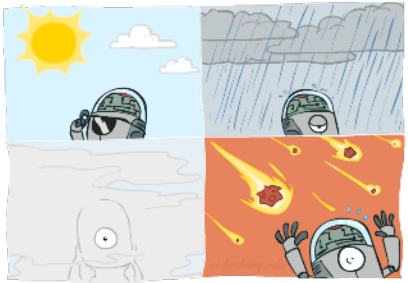
Probability Distributions

- Associate a probability with each value of a random variable
  - Temperature:

$$P(T)$$

T	P
hot	0.5
cold	0.5

  - Weather:



$$P(W)$$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

0.000...01

better

- A probability is a single number

$$P(W = rain) = 0.1 \qquad \text{Shorthand notation: } P(rain) = P(W = rain),$$

- Must have: $\forall x \;\; P(X = x) \geq 0$ and $\sum_x P(X = x) = 1$

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n)$$

  - Must obey: $P(x_1, x_2, \ldots x_n) \geq 0$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

T, w joined

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- Size of distribution for n variables with domain size d? d^n
  - For all but the smallest distributions, cannot write out by hand!

Probabilistic Models

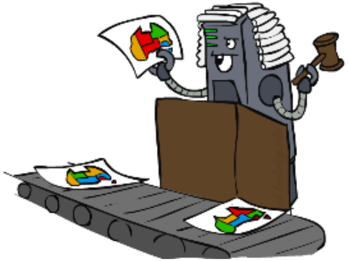
- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
  - (Random) variables with domains
  - Joint distributions: say whether assignments (outcomes) are likely
  - Ideally: only certain variables directly interact
- Constraint satisfaction problems:
  - Variables with domains
  - Constraints: state whether assignments are possible
  - Ideally: only certain variables directly interact

Distribution over T,W

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Constraint over T,W

T	W	P
hot	sun	T
hot	rain	F
cold	sun	F
cold	rain	T



Probabilities of events

- An *event* is a set E of outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- Given a joint distribution over all variables, we can compute any event probability!
  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR sunny?

$$P(T, W)$$

T	W	P
hot	sun	$\boxed{0.4}$
hot	rain	$\boxed{0.1}$
cold	sun	$\boxed{0.2}$
cold	rain	0.3

Quiz: Events

- P(+x, +y) ?

- P(+x) ?

- P(-y OR +x) ?

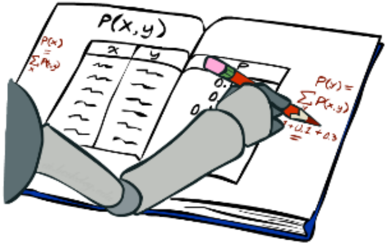
$$P(X, Y)$$

"consistent"
with +x
(match)

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables (unnecessary)
- Marginalization (summing out): Combine collapsed rows by adding



$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(t) = \sum_w P(t, w)$$

$P(T)$

T	P
hot	0.5
cold	0.5

$$P(w) = \sum_t P(t, w)$$

$P(W)$

W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

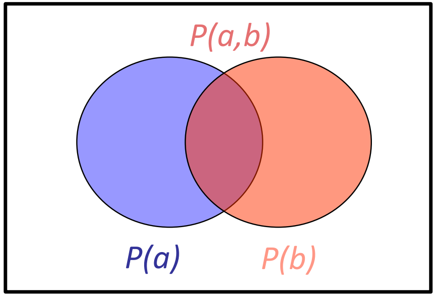
Conditional Probabilities

- The probability of an event given that another event has occurred

a and b.

a on b.

$$P(a|b) = \frac{P(a,b)}{P(b)}$$



$$P(T,W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = 0.4$$

$$\boxed{\begin{aligned} &= P(W=s,T=c) + P(W=r,T=c) \\ &= 0.2 + 0.3 \;\; = 0.5 \end{aligned}}$$

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Joint Distribution

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

size to 1

Conditional Distributions

$$P(W|T)$$

$$P(W|T = hot)$$

W	P
sun	0.8
rain	0.2

$$P(W|T = cold)$$

W	P
sun	0.4
rain	0.6

Normalization Trick

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$\begin{aligned} P(W = s|T = c) &= \frac{P(W = s, T = c)}{P(T = c)} \\ &= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\ &= \frac{0.2}{0.2 + 0.3} = 0.4 \end{aligned}$$

$$\begin{aligned} P(W = r|T = c) &= \frac{P(W = r, T = c)}{P(T = c)} \\ &= \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\ &= \frac{0.3}{0.2 + 0.3} = 0.6 \end{aligned}$$

$$P(W|T = c)$$

W	P
sun	0.4
rain	0.6

Normalization Trick

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)}$$
$$= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.2}{0.2+0.3} = 0.4$$

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence

$P(c,W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)

$P(W|T=c)$

W	P
sun	0.4
rain	0.6

$$P(W=r|T=c) = \frac{P(W=r,T=c)}{P(T=c)}$$
$$= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.3}{0.2+0.3} = 0.6$$

Normalization Trick

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence

$$P(c, W)$$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)

$$P(W|T = c)$$

W	P
sun	0.4
rain	0.6

- Why does this work? Sum of selection is P(evidence)! (P(T=c), here)

$$P(x_1|x_2) = \frac{P(x_1, x_2)}{P(x_2)} = \frac{P(x_1, x_2)}{\sum_{x_1} P(x_1, x_2)}$$

Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities (e.g. conditional from joint)

compute desired from known

- We generally compute conditional probabilities
  - P(on time | no reported accidents) = 0.90
  - These represent the agent's *beliefs* given the evidence

- Probabilities change with new evidence:
  - P(on time | no accidents, 5 a.m.) = 0.95
  - P(on time | no accidents, 5 a.m., raining) = 0.80
  - Observing new evidence causes *beliefs to be updated*

Enumeration

Enumer

Inference by Enumeration

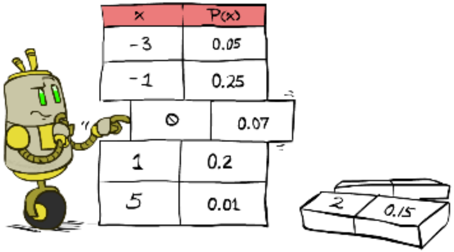
- General case:
  - Evidence variables: $E_1 \dots E_k = e_1 \dots e_k$
  - Query variable: Q
  - Hidden variables: $H_1 \dots H_r$

$$\left.\begin{array}{l} E_1 \dots E_k = e_1 \dots e_k \\ Q \\ H_1 \dots H_r \end{array}\right\} \begin{array}{c} X_1, X_2, \dots X_n \\ \textit{All variables} \end{array}$$

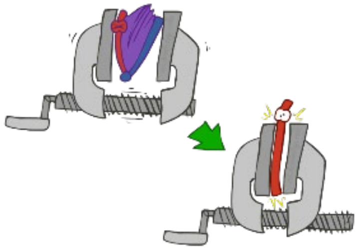
- We want:

$$P(Q|e_1 \dots e_k)$$

- Step 1: Select the entries consistent with the evidence



- Step 2: Sum out H to get joint of Query and evidence



$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots X_n}$$

- Step 3: Normalize

$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

Inference by Enumeration

1. Select the entries consistent with the evidence
2. Sum out H to get joint of Query and evidence
3. Normalize

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

.15+.10 .20+.05

- P(W | winter)? sun: 0.5, rain: 0.5
- P(W | winter, hot)? sun: 0.67, rain: 0.33

.10 .05

Inference by Enumeration

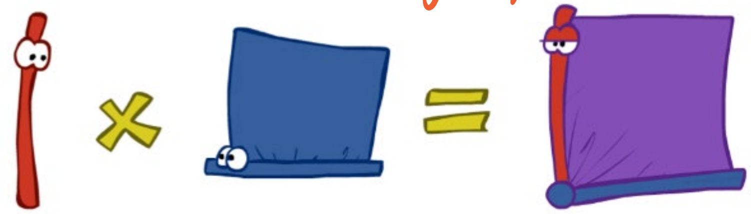
- Obvious problems:
  - Worst-case time complexity $O(d^n)$
  - Space complexity $O(d^n)$ to store the joint distribution

The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(y)P(x|y) = P(x, y) \quad \Longleftrightarrow \quad P(x|y) = \frac{P(x, y)}{P(y)}$$

$p(x,y) = p(x|y) \cdot p(y)$



The Product Rule

$$P(y)P(x|y) = P(x, y)$$

- Example:

$P(W)$

W	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

×0.8

×0.2

$P(D, W)$

D	W	P
wet	sun	0.08
dry	sun	0.72
wet	rain	0.14
dry	rain	0.06

The Chain Rule

- More generally, can always write any joint distribution as an incremental product of conditional distributions

$P(x_1, x_2)$

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

Bayes' Rule

- Two ways to factor a joint distribution over two variables:

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems we'll see later

- In the running for most important AI equation!

Quiz: Bayes' Rule

- Given:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

dry: 0.6 wet: 0.4

- What is P(W | dry) ? $= \dfrac{P(dry|W)}{P(dry)} \cdot P(W)$

when do normalize
no need to comput

P(sun|dry) ~ P(dry|sun)P(sun) = .9*.8 = .72
P(rain|dry) ~ P(dry|rain)P(rain) = .3*.2 = .06
P(sun|dry)=12/13
P(rain|dry)=1/13

Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- Example:
  - M: meningitis, S: stiff neck

$$\left.\begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned}\right\} \text{Example givens}$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m) + P(+s|-m)P(-m)} = \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.9999}$$

stiffneck 是病

$$P(s) = P(s,-m) + P(s,+m)$$

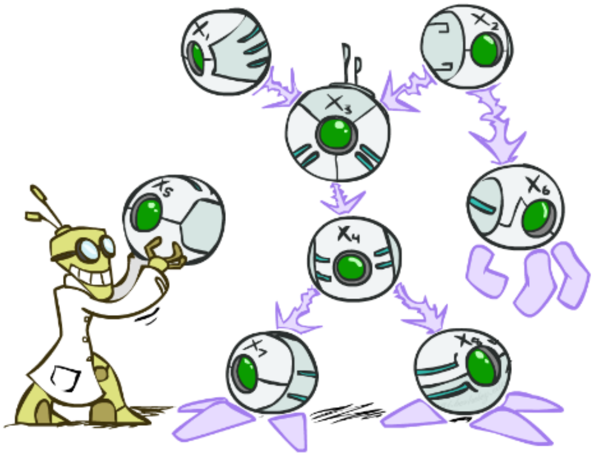
$$P(s,-m) = P(s|-m)P(-m);$$

$P(s,-m) = P(s|-m)P(-m)$

$P(a) = P(a,b_1) + P(a,b_2)$

$P(a,b) = P(a|b)\,P(b)$

Bayesian Networks



AIMA Chapter 14.1, 14.2

[Adapted from slides by Dan Klein and Pieter Abbeel at UC Berkeley (ai.berkeley.edu) and from slides by Markus Svensén & Christopher M Bishop]

Additional Reference

- [PRML] Pattern Recognition and Machine Learning, Christopher Bishop, Springer 2006.
  - Chapter 8.1 - 8.3

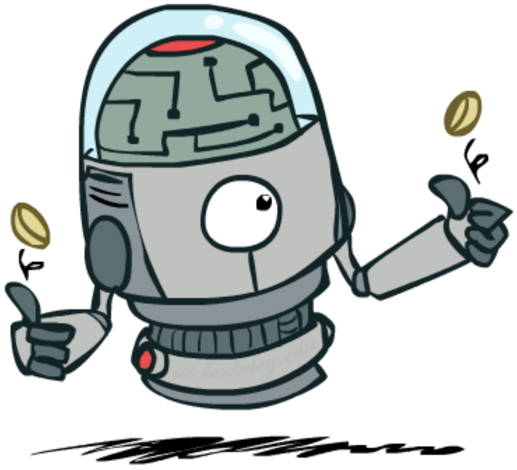
Probabilistic Models

- Models describe how (a portion of) the world works
  - Models are always simplifications
  - May omit some variables and interactions
  - "All models are wrong; but some are useful."
    – George E. P. Box

- What do we do with probabilistic models?
  - We (or our agents) need to reason about unknown variables, given evidence
  - Example: explanation (diagnostic reasoning)
  - Example: prediction (causal reasoning)
  - Example: making decisions based on expected utility

- How do we build models, avoiding the d^n blowup?

Independence



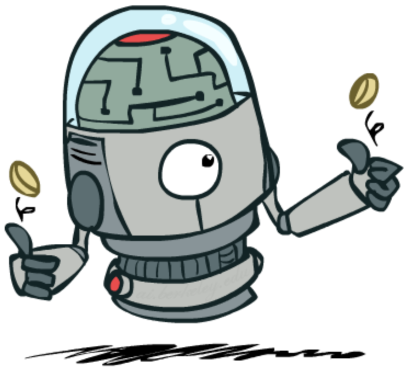
Independence

- Two variables X and Y are (absolutely) ***independent*** if

$$\forall x,y \qquad P(x,y) = P(x)P(y) \qquad \qquad X \perp\!\!\!\perp Y$$

  - This says that their joint distribution ***factors*** into a product of two simpler distributions
  - Combine with product rule $P(x,y) = P(x|y)P(y)$ we obtain another form:

$$\forall x,y\ P(x|y) = P(x) \quad \text{or} \quad \boxed{\forall x,y\ P(y|x) = P(y)}$$



- Example: two dice rolls $Roll_1$ and $Roll_2$
  - $P(Roll_1=5, Roll_2=5) = P(Roll_1=5)P(Roll_2=5) = 1/6 \times 1/6 = 1/36$
  - $P(Roll_2=5 \mid Roll_1=5) = P(Roll_2=5)$

Independence in the real world

- Independence is a simplifying *modeling assumption*
  - Sometimes it's reasonable for real-world variables
  - What could we assume for {Weather, Temperature, Cavity, Toothache}?
    - Cavity and Toothache are ***not*** independent of each other
      - Ditto for hundreds of dentistry variables
    - Weather and Temperature are ***not*** independent of each other
      - Ditto for hundreds of meteorological variables
    - Cavity and Toothache are ***roughly*** independent of Weather and Temperature

Conditional Independence

- Unconditional (absolute) independence is rare
- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.
- X is conditionally independent of Y given Z $X \perp\!\!\!\perp Y | Z$

  if and only if:

  $\forall x,y,z \quad P(x \mid y,z) = P(x|z)$

  or, equivalently, if and only if

  $\forall x,y,z \quad P(x,y \mid z) = P(x|z)P(y|z)$

"z前提下，x，y独立"

Conditional Independence

- P(Toothache, Cavity, Catch)

Independent = Σ 1 depedent

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  - P(+catch | +toothache, +cavity) = P(+catch | +cavity)
- The same independence holds if I don't have a cavity:
  - P(+catch | +toothache, -cavity) = P(+catch| -cavity)
- Catch is *conditionally independent* of Toothache given Cavity:
  - P(Catch | Toothache, Cavity) = P(Catch | Cavity)

(A|C)

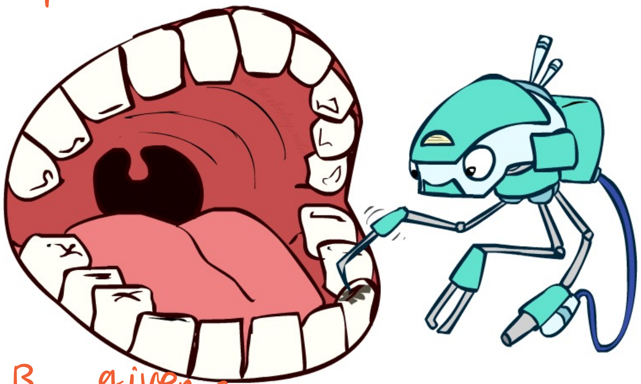
(A|B,C) A conditional indepedent on B given C

B c inde on A given C.

- Equivalent statements:
  - P(Toothache | Catch , Cavity) = P(Toothache | Cavity)
  - P(Toothache, Catch | Cavity) = P(Toothache | Cavity) P(Catch | Cavity)
  - One can be derived from the other easily

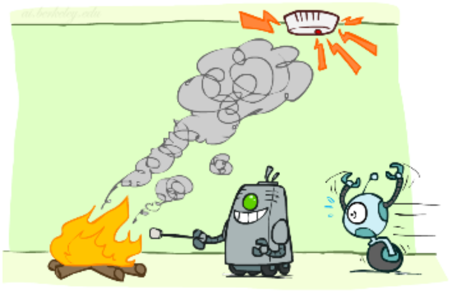
(A, B) happen given C

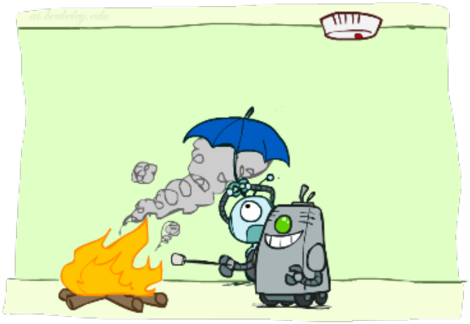
= A given C * B given C



Conditional Independence

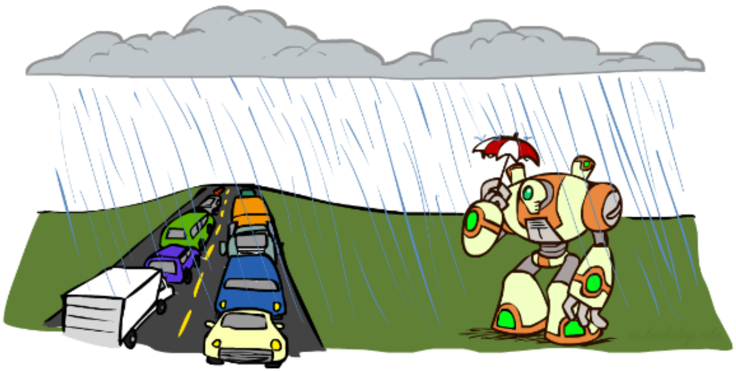
- What about this domain:
  - Fire
  - Smoke
  - Alarm (smoke detector)





Conditional Independence

- What about this domain:
  - Traffic
  - Umbrella
  - Raining

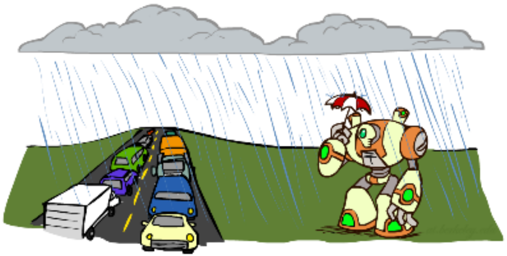


Conditional Independence and the Chain Rule

- Chain rule: $P(X_1, X_2, \ldots X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots$

- Trivial decomposition:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) = P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain}, \text{Traffic})$$



- With assumption of conditional independence:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) = P(\text{Rain})P(\text{Traffic}|\text{Rain})\boxed{P(\text{Umbrella}|\text{Rain})}$$

Requires less space to encode!

- BayesNets / graphical models help us express conditional independence assumptions

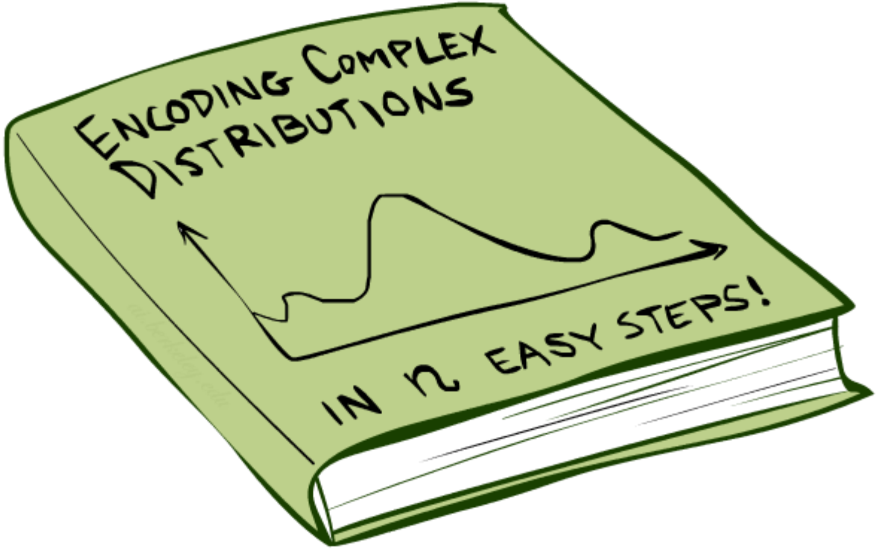
Ghostbusters Chain Rule

- Each sensor depends only on where the ghost is
- That means, the two sensors are conditionally independent, given the ghost position
- T: Top square is red
  B: Bottom square is red
  G: Ghost is in the top
- Givens:
  P(+g) = 0.5
  P(-g) = 0.5
  P(+t | +g) = 0.8
  P(+t | -g) = 0.4
  P(+b | +g) = 0.4
  P(+b | -g) = 0.8

P(T,B,G) = P(G) P(T|G) P(B|G)

T	B	G	P(T,B,G)
+t	+b	+g	0.16
+t	+b	-g	0.16
+t	-b	+g	0.24
+t	-b	-g	0.04
-t	+b	+g	0.04
-t	+b	-g	0.24
-t	-b	+g	0.06
-t	-b	-g	0.06

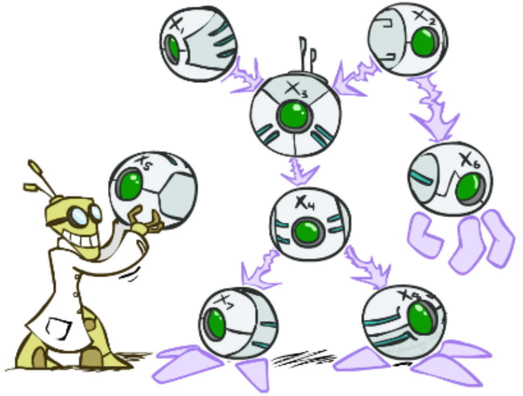
Bayesian Networks: Big Picture



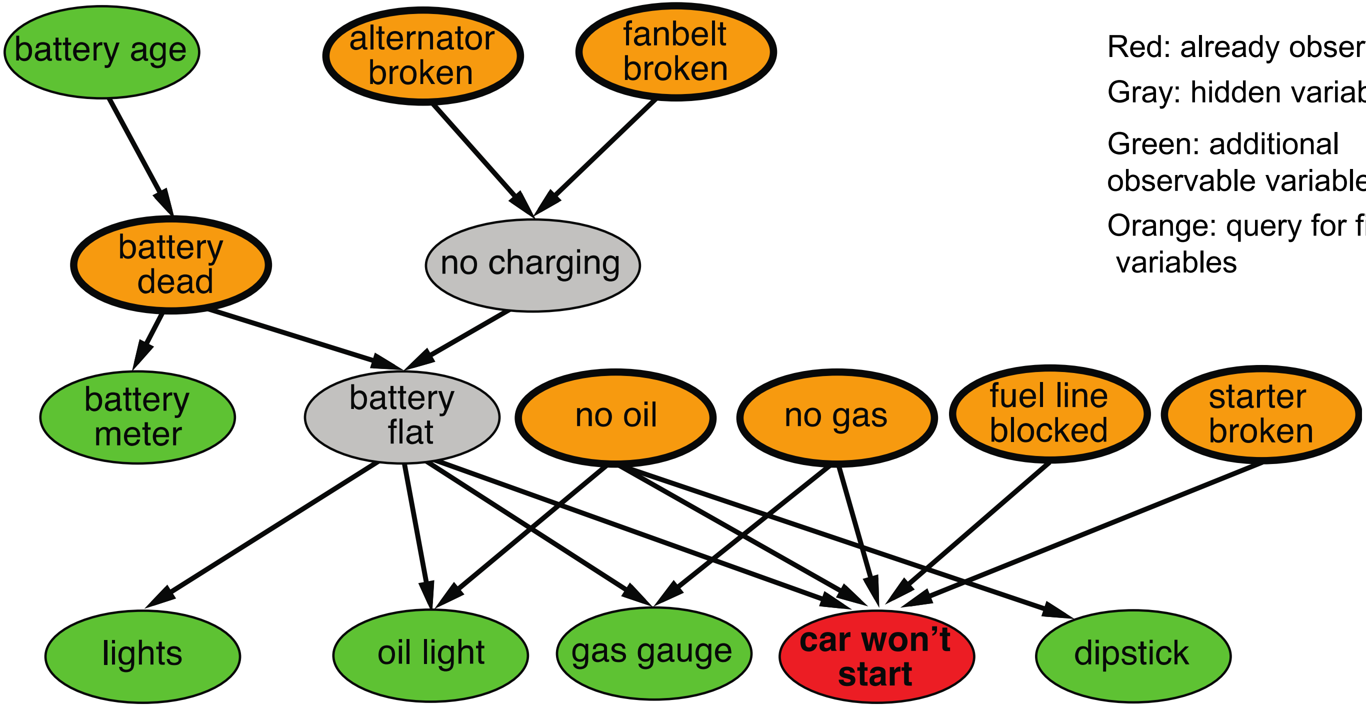
Bayesian Networks: Big Picture

- Full joint distribution tables answer every question, but:
  - Size is exponential in the number of variables
  - Need gazillions of examples to learn the probabilities
  - Inference by enumeration (summing out hiddens) is too slow

- Bayesian networks:
  - Express all the conditional independence relationships in a domain
  - Factor the joint distribution into a product of small (inde) conditionals
  - Often reduce size from exponential to linear
  - Faster learning from fewer examples
  - Faster inference (linear time in some important cases)



Example Bayes Net: Amateur Car Mechanic



Red: already observed

Gray: hidden variables

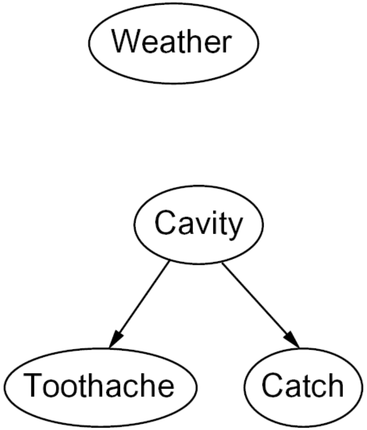
Green: additional observable variables

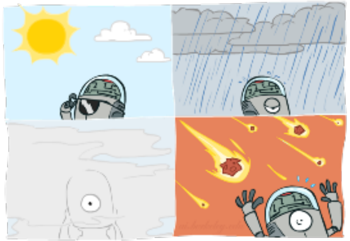
Orange: query for fixable variables

Bayesian Networks Syntax

Bayesian Networks Syntax

- Nodes: variables (with domains)
- Arcs: interactions
  - Indicate "direct influence" between variables
  - For now: imagine that arrows mean direct causation (in general, they may not!)
  - Formally: encode conditional independence (more later)
- No cycle is allowed!

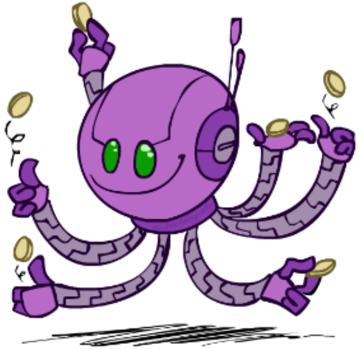




Example: Coin Flips

- N independent coin flips

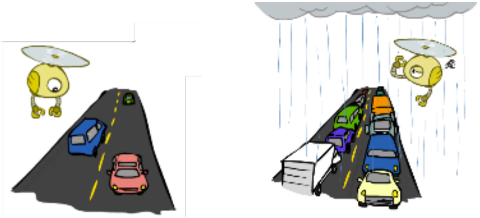
X_1 X_2 $\cdots$ X_n



- No interactions between variables: absolute independence

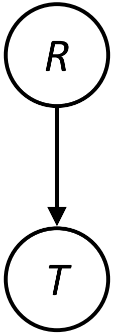
Example: Traffic

- Variables:
  - R: It rains
  - T: There is traffic



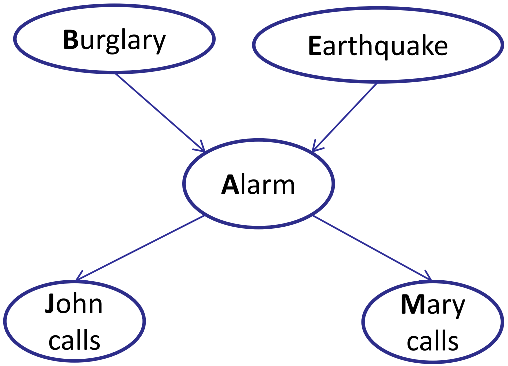
- Model 1: independence

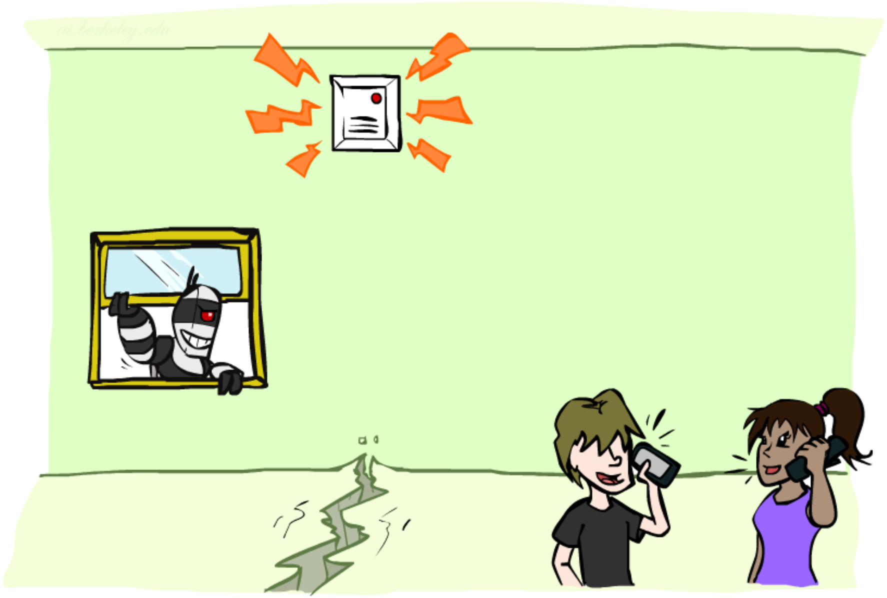
- Model 2: rain causes traffic



Example: Alarm Network

- Variables
  - B: Burglary
  - A: Alarm goes off
  - M: Mary calls
  - J: John calls
  - E: Earthquake!



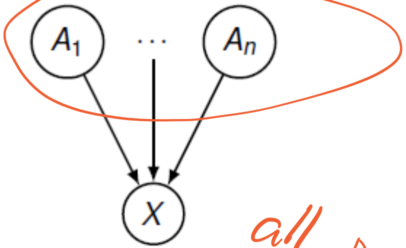


Bayesian Networks Syntax

- A directed, acyclic graph

无环

- Conditional distributions for each node given its ***parent variables*** in the graph
  - ***CPT***: conditional probability table: each row is a distribution for child given a configuration of its parents
  - Description of a noisy "causal" process



all parents

$$P(X|A_1, \cdots, A_n)$$

A Bayes net = Topology (graph) + Local Conditional Probabilities

Probabilities in BNs

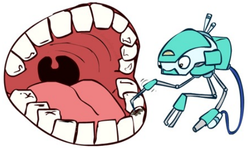
- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

all x_i's parent.

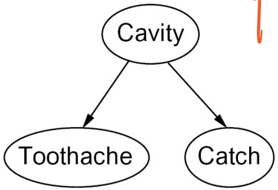
$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \textit{parents}(X_i))$$

  - Example:

eg



P(+cavity)



P(+catch|+cavity)

$P(+cavity,\ +catch,\ -toothache)$ P(+tooth|cavity)

=P(-toothache|+cavity)P(+catch|+cavity)P(+cavity)

(BN)

Probabilities in BNs

- Why are we guaranteed that setting

  $$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \mathit{parents}(X_i))$$

  results in a proper joint distribution?

- Chain rule (valid for all distributions): $$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | x_1 \ldots x_{i-1})$$

- Assume conditional independences: $$P(x_i | x_1, \ldots x_{i-1}) = P(x_i | \mathit{parents}(X_i))$$

  → Consequence: $$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \mathit{parents}(X_i))$$

- Not every BN can represent every joint distribution
  - The topology enforces certain conditional independencies

Example: Coin Flips

X_1 X_2 $\cdots$ X_n

$P(X_1)$

h	0.5
t	0.5

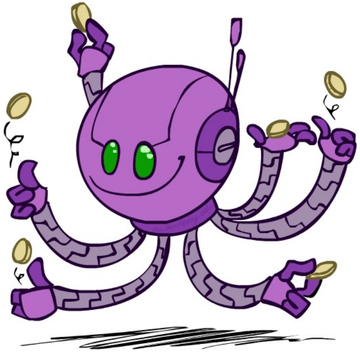
$P(X_2)$

h	0.5
t	0.5

$\cdots$

$P(X_n)$

h	0.5
t	0.5

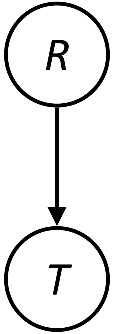


$$P(h, h, t, h) = \text{P(h)P(h)P(t)P(h)}$$

arc: dependence

Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.

Example: Traffic



$$P(R)$$

+r	1/4
-r	3/4

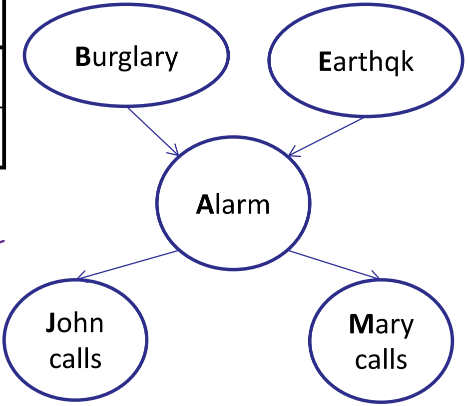
$$P(T|R)$$

+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

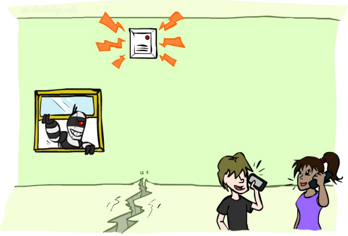
$P(+r, -t) =$ P(+r)P(-t|+r) = ¼*1/4

Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



E	P(E)
+e	0.002
-e	0.998



A	J	P(J|A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

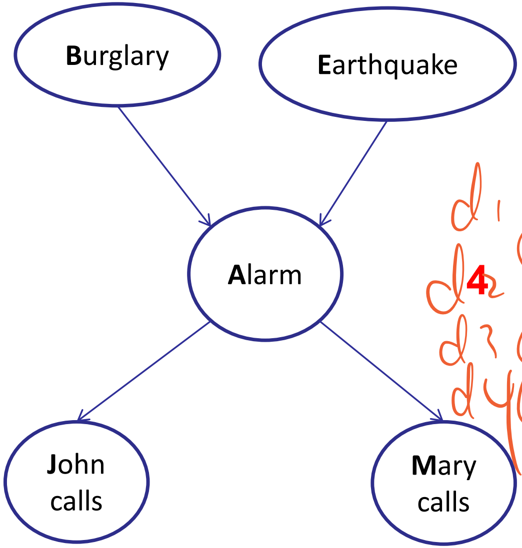
B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

$P(M|A)P(J|A)$
$P(A|B,E)P(E)$
$P(B)$

Example: Alarm Network

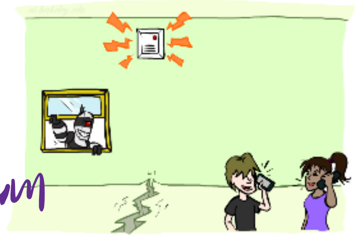
P(B)	
true	false
0.001	0.999

1



1

P(E)	
true	false
0.002	0.998



Sum

B	E	P(A|B,E)	
		true	false
true	true	0.95	0.05
true	false	0.94	0.06
false	true	0.29	0.71
false	false	0.001	0.999

d_1 d_2 d_3 d_4

4

= 1
= 1
= 1
= 1

table

Number of free parameters in each CPT:

- Parent domain sizes $d_1, \ldots, d_k$
- Child domain size d
- Each table row must sum to 1

$(d-1)\ \Pi_i\ d_i$

A	P(J|A)	
	true	false
true	0.9	0.1
false	0.05	0.95

2

A	P(M|A)	
	true	false
true	0.7	0.3
false	0.01	0.99

2

General formula for sparse BNs

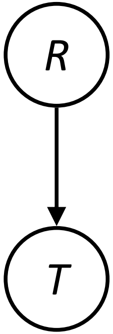
- Suppose
  - n variables
  - Maximum domain size is d
  - Maximum number of parents is k
- Full joint distribution has size $O(d^n)$
- Bayes net has size $O(n \cdot d^{k+1})$
  - Linear scaling with n as long as causal structure is local

$n \cdot d^{k}$ parent

variabe domain

Example: Traffic

- Causal direction



$P(R)$

+r	1/4
-r	3/4

$P(T|R)$

+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

$P(T,R)$

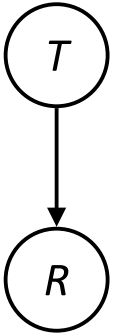
+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

Example: Reverse Traffic

- Reverse causality?

$P(T)\,P(R|T)$

$= P(R)\,P(T|R)$



$P(T)$

+t	9/16
-t	7/16

$P(R|T)$

+t	+r	1/3
	-r	2/3
-t	+r	1/7
	-r	6/7

$P(T, R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

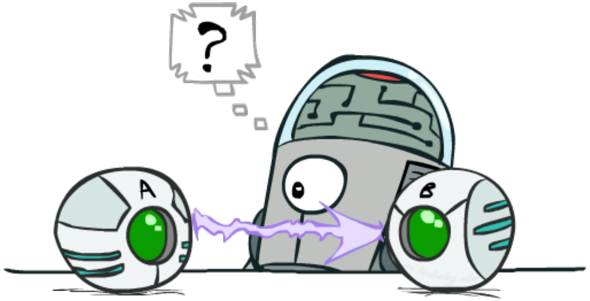
Causality? 无关因果

- When Bayes' nets reflect the true causal patterns:
  - Often simpler (nodes have fewer parents)
  - Often easier to think about
  - Often easier to elicit from experts

- BNs need not actually be causal
  - Sometimes no causal net exists over the domain (especially if variables are missing)
  - E.g. consider the variables *Traffic* and *Drips*
  - End up with arrows that reflect correlation, not causation

- What do the arrows really mean?
  - Topology may happen to encode causal structure
  - Topology really encodes conditional independence

$$P(x_i|x_1, \ldots x_{i-1}) = P(x_i|parents(X_i))$$



arc (arrow)

parents → children.

chain law.

$A \to B$.

$P(B|A)$.