Discrete Mathematics: Lecture 17

translation, precedence, truth table, tautology, contradiction, contingency, satisfiable, rule of substitution, logically equivalent, rule of replacement

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Review

Proposition: a declarative sentence that is either true or false.

- simple, compound, propositional constant/variable

Logical Connectives: $\neg$ (unary), $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ (binary)

propositional

- Truth table

Well-Formed Formulas (WFFs): formulas

- propositional constants (T, F) and propositional variables are WFFs
- $\neg A, (A \wedge B), (A \vee B), (A \leftarrow B), (A \leftrightarrow B)$
- Finite

Review: Proposition

Simple Proposition: cannot be broken into 2 or more propositions

- $\sqrt{2}$ is irrational.

Compound Proposition: not simple

- 2 is rational and $\sqrt{2}$ is irrational.

Propositional Constant: a concrete proposition (truth value fixed)

- Every even integer $n > 2$ is the sum of two primes.

Propositional variables: a variable that represents any proposition

- Lower-case letters denote proposition variables: $p, q, r, s, \ldots$
- Truth value is not determined until it is assigned a concrete proposition

Propositional Logic: the area of logic that deals with propositions

Review

Definition: Let p be any proposition.

- The **negation** of p is the statement "It is not the case that p"
- Notation: $\neg p$; read as "not p"
- **True table:**

p	$\neg p$
T	**F**
F	**T**

Definition: Let p, q be any propositions.

- The **conjunction** of p and q is the statement "p and q". Notation: $p \wedge q$
- The **disjunction** of p and q is the statement "p or q". Notation: $p \vee q$
- **True table:**

p	q	$p \wedge q$	$p \vee q$
T	**T**	**T**	**T**
T	**F**	**F**	**T**
F	**T**	**F**	**T**
F	**F**	**F**	**F**

Review

Definition: Let p, q be any propositions.

- The **conditional statement** $p \to q$ is the proposition "if p, then q."
  - p: hypothesis; q: conclusion; read as "p implies q", or "if p, then q"
- True table:

p	q	$p \to q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

- The **biconditional statement** $p \leftrightarrow q$ is the proposition "p if and only if q."
  - read as"p if and only if q"
- True table:

p	q	$p \leftrightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Review: WFFs & Precedence

DEFINITION: recursive definition of **well-formed formulas** (**WFFs**)

① propositional constants (**T, F**) and propositional variables are WFFs

② If A is a WFF, then $\neg A$ is a WFF

③ If A, B are WFFs, then $(A \land B), (A \lor B), (A \to B), (A \leftrightarrow B)$ are WFFs

④ WFFs are results of finitely many applications of ①, ② , and ③

Precedence(优先级)**:** $\neg$, $\land$, $\lor$, $\to$, $\leftrightarrow$

- formulas inside () are computed firstly
- different connectives: $\neg$, $\land$, $\lor$, $\to$, $\leftrightarrow$
- same connectives: from left to the right
    - $\underline{\underline{\underline{(p \to q)} \land \underline{(q \to r)}} \leftrightarrow \underline{(p \to r)}}$

From Natural Language to WFFs

The Method of Translation:

- Introduce symbols p, q, r … to represent simple propositions
- Connect the symbols with logical connectives to obtain WFFs

EXAMPLE:

- "It is not the case that snow is black."
  - p:"Snow is black" Translation: $\neg p$
- "π and e are both irrational."
  - p:"π is irrational."; q:"e is irrational." Translation: $p \wedge q$
- "If π is irrational, then 2π is irrational"
  - p:"π is irrational"; q:"2π is irrational" Translation: $p \rightarrow q$
- "$e^{\pi} > \pi^{e}$ if and only if $\pi > e \ln \pi$."
  - p:"$e^{\pi} > \pi^{e}$"; q:"$\pi > e \ln \pi$" Translation: $p \leftrightarrow q$

Example

- $\left(\sqrt{2}\right)^{\sqrt{2}}$ is rational or irrational. (ambiguity in natural language)
  - p: "$\left(\sqrt{2}\right)^{\sqrt{2}}$ is rational"; q: "$\left(\sqrt{2}\right)^{\sqrt{2}}$ is irrational"
  - **Explanation 1**: $\left(\sqrt{2}\right)^{\sqrt{2}}$ cannot be neither rational nor irrational.
    - Translation 1: $p \vee q$
      - We agree that $p \vee q$ is the correct translation of "$\left(\sqrt{2}\right)^{\sqrt{2}}$ is rational or irrational ."
  - **Explanation 2**: $\left(\sqrt{2}\right)^{\sqrt{2}}$ cannot be both rational and irrational.
    - Translation 2: $(p \wedge \neg q) \vee (q \wedge \neg p)$
      - We agree that $(p \wedge \neg q) \vee (q \wedge \neg p)$ is the translation of "$\left(\sqrt{2}\right)^{\sqrt{2}}$ is rational or irrational, but not both."

Example

- You are eligible to be President of the U.S.A, only if you are at least 35 years old, were born in the U.S.A, or at the time of your birth both of your parents were citizens, and you have lived at least 14 years in the country.
  - e: "You are eligible to be President of the U.S.A.,"
  - a: "You are at least 35 years old,"
  - b: "You were born in the U.S.A,"
  - p: "At the time of your birth, both of your parents were citizens,"
  - r: "You have lived at least 14 years in the U.S.A."
    - Translation: $e \rightarrow (a \wedge (b \vee p) \wedge r)$

A if B : $B \Rightarrow A$

A only if B $A \Rightarrow B$

Example

- A comes to the party if and only if B doesn't come, but, if B comes, then C doesn't come and D comes.
- A sufficient condition for A coming to the party is that, if B does not come, then at least one of C and D must come.
    - a: "A comes to the party."
    - b: "B comes to the party."
    - c: "C comes to the party."
    - d: "D comes to the party."

narrating part

        - Translation 1: $(a \leftrightarrow \neg b) \land \boxed{(b \rightarrow (\neg c \land d))}$
        - Translation 2: $(\neg b \rightarrow (c \lor d)) \rightarrow a$

Example

- **System Specifications**: Determine if there is a system that satisfy all of the following requirements.
  1. The diagnostic message is stored in the buffer or it is retransmitted.
  2. The diagnostic message is not stored in the buffer.
  3. If the diagnostic message is stored in the buffer, then it's retransmitted.
     - s: "The diagnostic message is stored in the buffer"
     - r: "The diagnostic message is retransmitted"
       - $s \vee r; \neg s; s \rightarrow r$

         $s \; \neg r$ T

         F T
         - There is a system that satisfies 1, 2 and 3. ($s = \mathbf{F}, r = \mathbf{T}$)
  4. Add one more requirement "The diagnostic message is not retransmitted"
     - $s \vee r; \neg s; s \rightarrow r; \neg r$
       - There is no system that satisfies 1, 2, 3 and 4.

Truth Table

DEFINITION: Let F be a WFF of $p_1, \ldots, p_n$, n propositional variables

- A **truth assignment** (真值指派) for F is a map $\alpha: \{p_1, \ldots, p_n\} \to \{\mathbf{T}, \mathbf{F}\}$.
  - There are 2^n different truth assignments.

p_1	p_2	$\cdots$	p_n	F
T	**T**	$\cdots$	**T**	$\cdot$
T	**T**	$\cdots$	**F**	$\cdot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F	**F**	$\cdots$	**F**	$\cdot$

EXAMPLE: Truth tables of $A = p \vee \neg p, B = p \wedge \neg p, C = p \to \neg p$

p	$\neg p$	A
T	**F**	T
F	**T**	T

p	$\neg p$	B
T	**F**	F
F	**T**	F

p	$\neg p$	C
T	**F**	F
F	**T**	T

Truth Table

DEFINITION: Let F be a WFF of $p_1, \ldots, p_n$, n propositional variables

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  - There are 2^n different truth assignments.

p_1	p_2	$\cdots$	p_n	F
T	**T**	$\cdots$	**T**	$\cdot$
T	**T**	$\cdots$	**F**	$\cdot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F	**F**	$\cdots$	**F**	$\cdot$

EXAMPLE: Truth tables of $A = p \vee \neg p, B = p \wedge \neg p, C = p \to \neg p$

p	$\neg p$	A
T	**F**	**T**
F	**T**	**T**

p	$\neg p$	B
T	**F**	**F**
F	**T**	**F**

p	$\neg p$	C
T	**F**	**F**
F	**T**	**T**

Truth Table

EXAMPLE: Truth table of $F = (p \to q) \land (q \to r) \leftrightarrow (p \to r)$

- $A = p \to q$; $B = q \to r$; $C = p \to r$
- $F = A \land B \leftrightarrow C$

p	q	r	A	B	C	$A \land B$	F
T	**T**	**T**					
T	**T**	**F**					
T	**F**	**T**					
T	**F**	**F**					
F	**T**	**T**					
F	**T**	**F**					
F	**F**	**T**					
F	**F**	**F**					

Truth Table

EXAMPLE: Truth table of $F = (p \to q) \wedge (q \to r) \leftrightarrow (p \to r)$

- $A = p \to q$; $B = q \to r$; $C = p \to r$
- $F = A \wedge B \leftrightarrow C$

p	q	r	A	B	C	$A \wedge B$	F
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**F**	**F**	**F**	**T**
T	**F**	**T**	**F**	**T**	**T**	**F**	**F**
T	**F**	**F**	**F**	**T**	**F**	**F**	**T**
F	**T**	**T**	**T**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**F**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**	**T**	**T**	**T**
F	**F**	**F**	**T**	**T**	**T**	**T**	**T**

Types of WFFs

all assignment

Tautology(重言式): a WFF whose truth value is **T** for ~~all truth assignment~~

- $p \vee \neg p$ is a tautology

Contradiction(矛盾式): a WFF whose truth value is **F** for all truth assignment

- $p \wedge \neg p$ is a contradiction

Contingency(可能式): neither tautology nor contradiction

- $p \rightarrow \neg p$ is a contingency

Satisfiable(可满足的):a WFF is satisfiable if it is true for at least one truth assignment

Rule of Substitution: (代入规则) Let B be a formula obtained from a tautology A by substituting a propositional variable in A with an arbitrary formula. Then B must be a tautology.

- $p \vee \neg p$ is a tautology: $(q \wedge r) \vee \neg(q \wedge r)$ is a tautology as well.

A B

Logically Equivalent

DEFINITION: Let A and B be WFFs in propositional variables $p_1, \ldots, p_n$.

- A and B are **logically equivalent** (等值) if they always have the same truth value for every truth assignment (of $p_1, \ldots, p_n$)
  - Notation: $A \equiv B$

THEOREM: $A \equiv B$ if and only if $A \leftrightarrow B$ is a tautology.

- $A \equiv B$
- iff for any truth assignment, A, B take the same truth values
- iff for any truth assignment, $A \leftrightarrow B$ is true
- iff $A \leftrightarrow B$ is a tautology

THEOREM: $A \equiv A$; If $A \equiv B$, then $B \equiv A$; If $A \equiv B, B \equiv C$, then $A \equiv C$

QUESTION: How to prove $A \equiv B$?

Proving $A \equiv B$

EXAMPLE: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ **//distributive law**

- <u>Idea: Show that A, B have the same truth table.</u>

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	**T**	**T**					
T	**T**	**F**					
T	**F**	**T**					
T	**F**	**F**					
F	**T**	**T**					
F	**T**	**F**					
F	**F**	**T**					
F	**F**	**F**					

Proving $A \equiv B$

EXAMPLE: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ **//distributive law**

- Idea: Show that A, B have the same truth table.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**T**	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**F**	**F**	**F**	**F**	**F**

REMARK: $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ can be shown similarly.

Logical Equivalences

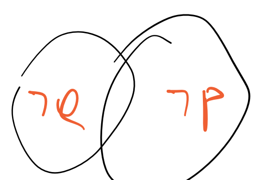
Name	**Logical Equivalences**	**NO.**
Double Negation Law 双重否定律	$\neg(\neg P) \equiv P$	1
Identity Laws 同一律	$P \wedge \mathbf{T} \equiv P$	2
	$P \vee \mathbf{F} \equiv P$	3
Idempotent Laws 等幂律	$P \vee P \equiv P$	4
	$P \wedge P \equiv P$	5
Domination Laws 零律	$P \vee \mathbf{T} \equiv \mathbf{T}$	6
	$P \wedge \mathbf{F} \equiv \mathbf{F}$	7
Negation Laws 补余律	$P \vee \neg P \equiv \mathbf{T}$	8
	$P \wedge \neg P \equiv \mathbf{F}$	9

Logical Equivalences

Name	Logical Equivalences	NO.
Commutative Laws 交换律	$P \vee Q \equiv Q \vee P$	10
	$P \wedge Q \equiv Q \wedge P$	11
Associative Laws 结合律	$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$	12
	$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$	13
Distributive Laws 分配律	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	14
	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	15
De Morgan's Laws 摩根律	$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$	16
	$\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$	17
Absorption Laws 吸收律	$P \vee (P \wedge Q) \equiv P$	18
	$P \wedge (P \vee Q) \equiv P$	19

Logical Equivalences

Name	Logical Equivalences	NO.
Laws Involving Implication $\rightarrow$	$P \rightarrow Q \equiv \neg P \vee Q$	20
	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$	21
	$(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \vee Q) \rightarrow R$	22
	$P \rightarrow (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$	23
	$P \rightarrow (Q \rightarrow R) \equiv Q \rightarrow (P \rightarrow R)$	24
Laws Involving Bi-Implication $\leftrightarrow$	$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$ by def	25
	$P \leftrightarrow Q \equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$	26
	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	27
	$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$	28



Proving $A \equiv B$

P Q

<u>Rule of Replacement:</u> (替换规则) Replacing a sub-formula in a formula F with a logically equivalent sub-formula gives a formula logically equivalent to the formula F.

EXAMPLE: $P \to Q \equiv \neg Q \to \neg P$

(交换

$$P \to Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg(\neg Q) \vee \neg P \equiv \neg Q \to \neg P$$

EXAMPLE: $P \leftrightarrow Q \equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$

$$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P) \equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$$
$$\equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$$

EXAMPLE: $P \to (Q \to R) \equiv (P \wedge Q) \to R$

$$P \to (Q \to R) \equiv \neg P \vee (\neg Q \vee R) \equiv (\neg P \vee \neg Q) \vee R \equiv \neg(P \wedge Q) \vee R$$
$$\equiv (P \wedge Q)^{1} \to R^{2}$$

$\vee\ \vee$ 要[illegible]!

A if B $\quad B \to A$.

A only if B, $A \to B$,

A iff B $\quad A \leftrightarrow B$,

Variations of "only"

Now that you can see how "only if" can be understood, let's cover a few variations of "only" that you're likely to encounter on the LSAT:

- I only wear a hat if it's sunny.
- I wear a hat only when it's sunny.
- The only time I wear a hat is if it's sunny.
- Only sunny days will get me to wear a hat.

Notice that the placement of "only" in relation to "sunny" is quite different in each statement, and the order of the elements "hat" and "sunny" are different as well. However, ***logically***, all four of these statements mean the same thing!

- if **I wear a hat** $\longrightarrow$ **sunny**

Top Tip: Therefore, it can be very helpful to **rephrase an "only" statement** as either "X only if Y" or "If X, then Y", so that you don't confuse the elements involved. Each of the four statements above can be rephrased as: "I wear a hat only if it's sunny" or "If I'm wearing a hat, then it's sunny".