Discrete Mathematics: Homework 6

(Deadline: April 1, 2022)

1. (20 points) Let $A = (a, \alpha), B = (b, \beta) \in \mathbb{Z}^2$. Show that if A, B satisfy the T-condition, then there is a T-route from A to B. (**T-condition**: (1) $b > a$; (2) $b - a \geq |\beta - \alpha|$; (3) $b - a + \beta - \alpha$ is even.)

2. (20 points) At the end of a basketball match (*for simplicity, suppose that every successful shot gives a team 1 point*) between team A and team B, the result is 80:81. What is the number of possibilities that A's score is always less than B's score during the entire match? A possibility can be described with the sequence of intermediate results during the entire match. For example, $0:1, 0:2, \ldots, 0:81, 1:81, 2:81, \ldots, 80:81$ describes one of the possibilities that A's score is always less than B's score during the entire match. (**Hint:** Use the idea of counting T-routes.)

3. (20 points) Let n, r be positive integers such that $r \geq n$. Determine the number of vectors $(x_1, x_2, \ldots, x_n)$ such that $x_1 + x_2 + \cdots + x_n = r$ and $x_1, x_2, \ldots, x_n \in \mathbb{Z}^+$.

4. (20 points) Let $\{a_n\}_{n \geq s}, \{b_n\}_{n \geq s}$ be two sequences such that $a_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} b_k$ for all $n \geq s$. Show that $b_n = \sum_{k=s}^{n} \binom{n}{k} a_k$ for all $n \geq s$.

5. (20 points) Suppose that $n + 1 \geq k \geq 2$. Provide a combinatorial proof of $S_2(n+1, k) = S_2(n, k-1) + k \cdot S_2(n, k)$. (**Hint**: Interpret both sides of the equation as the number of elements in a set X)