1. $P_3(n) = P_1(n-3) + P_2(n-3) + P_3(n-3)$

$= 1 + P_2(n-3) + P_1(n-6) + P_2(n-6) + P_3(n-6)$

$= 2 + P_2(n-3) + P_2(n-6) + P_3(n-6)$

① if n is odd: $(n-3)$ even, $(n-6)$ odd.

$P_2(n-3) = \frac{n-3}{2}$ $\quad P_2(n-6) = \frac{n-7}{2}$

$P_3(n) = \frac{n-3}{2} + \frac{n-7}{2} + 2 + P_3(n-6) = P_3(n-6) + n-3$

② if n is even: $(n-3)$ odd, $(n-6)$ even.

$P_2(n-3) = \frac{n-4}{2}$ $\quad P_2(n-6) = \frac{n-6}{2}$

$P_3(n) = \frac{n-4}{2} + \frac{n-6}{2} + 2 + P_3(n-6) = P_3(n-6) + n-3$

In sum, $P_3(n-6) + n - 3 = P_3(n)$

2. Define $A_i = \{x : x \in [n], p_i | x\}$. $i \in [k]$ $\quad A_i \subseteq [n]$.

Since $n = p_1^{e_1} \cdots p_k^{e_k}$, $k \geq 2$, $e_i \geq 1$, $i \in [k]$.

~~$\phi(n) = \phi(p_1^{e_1}) \cdots \phi(p_k^{e_k})$.~~

$\phi(n)$ is number of all $m \in [n]$ that $\gcd(m,n) = 1$; if $d \in A_i$, $\gcd(d,n) \neq 1$ since $p_i | d$, $p_i | n$

$\phi(n) = n - \left|\bigcup_{i=1}^{k} A_i\right| = n - \sum_{t=1}^{k} (-1)^{t-1} \sum_{1 \leq i_1 < \cdots < i_t \leq k} |A_{i_1} \cap \cdots \cap A_{i_t}|$

Include-exclude

$= n - |C_1 - C_2 + \cdots + (-1)^{k-1} C_k|$

$C_1 = n\left(\frac{1}{p_1} + \cdots + \frac{1}{p_k}\right)$

$C_2 = n\left(\frac{1}{p_1p_2} + \frac{1}{p_2p_3} + \cdots \frac{1}{p_{k-1}p_k}\right) \cdots$

$\Rightarrow \phi(n) = n - n\left[\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots \frac{1}{p_k}\right) - \left(\frac{1}{p_1p_2} + \frac{1}{p_2p_3} + \cdots \frac{1}{p_{k-1}p_k}\right) + (-1)^{k} \frac{1}{p_1 \cdots p_k}\right)$

$= n\left(1 - \left(\frac{1}{p_1} + \cdots \frac{1}{p_k}\right) + \left(\frac{1}{p_1p_2} + \cdots \frac{1}{p_{k-1}p_k}\right) + (-1)^{k} \frac{1}{p_1 \cdots p_k}\right)$

$= n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right)$.

3. Let $A = \{k \cdot a - \lfloor k \cdot a \rfloor : k = 0, 1, \ldots n\}$.

let $A_1 = \{a - \lfloor a \rfloor\}$, $k=1$

$A_2 = \{2a - \lfloor 2a \rfloor\}$, $k=2$

$\ldots$

$A_n = \{n \cdot a - \lfloor n a \rfloor\}$. $k=n$.

$A_{n+1} = \{(n+1)a - \lfloor (n+1) \cdot a \rfloor\}$, $k = n+1$

number: $(n+1)$

let's put $A_1 \cdots A_{n+1}$ into $[0, \frac{1}{n})$ $[\frac{1}{n}, \frac{2}{n})$ $\cdots$ $[\frac{(n-1)}{n}, 1)$ (n buckets)

from pigeon-hole.

There always exists. a interval that got A_i, A_j $\in [\frac{k}{n}, \frac{k+1}{n})$ $[\frac{k}{n}, \frac{k+1}{n})$

$A_i = (ia - \lfloor ia \rfloor)$
$A_j = (ja - \lfloor ja \rfloor)$ $\in [\frac{k}{n}, \frac{k+1}{n})$.

Just let $A_i < A_j$

let $p = j - i > 0$. $p \in Z^+$, $p \in [n]$

$A_j - A_i$:

$\Rightarrow 0 < (j-i)a - \lfloor (j-i)a \rfloor < \frac{1}{n}$

$\left|a - \frac{\lfloor (j-i)a \rfloor}{p}\right| < \frac{1}{pn}$ $\quad (q = \lfloor (j-i)a \rfloor \in \mathbb{Z})$

4. $a_n = 8a_{n-2} - 16a_{n-4}$.

characteristic equation:

$r^4 - 8r^2 + 16 = 0 \quad (r+2)^2(r-2)^2 = 0$

$r_1 = -2,\ m_1 = 2,\ r_2 = 2,\ m_2 = 2$

$a_n = (\alpha_{1,0} + \alpha_{1,1} n)\cdot(-2)^n + (\alpha_{2,0} + \alpha_{2,1} n)\cdot 2^n$

$a_0 = 3.\ a_1 = 6,\ a_2 = 44,\ a_3 = 56.$

$$\begin{cases} a_0 = 3 = \alpha_{1,0} + \alpha_{2,0} \\ a_1 = 6 = -2\alpha_{1,0} - 2\alpha_{1,1} + 2\alpha_{2,0} + 2\alpha_{2,1} \\ a_2 = 44 = 4\alpha_{1,0} + 8\alpha_{1,1} + 4\alpha_{2,0} + 8\alpha_{2,1} \\ a_3 = 56 = -8\alpha_{1,0} - 24\alpha_{1,1} + 8\alpha_{2,0} + 24\alpha_{2,1} \end{cases}$$

$\Rightarrow \alpha_{1,0} = 1 \quad \alpha_{1,1} = 1 \quad \alpha_{2,0} = 2 \quad \alpha_{2,1} = 3$

$\Rightarrow a_n = (n+1)\cdot(-2)^n + (3n+2)\cdot 2^n$

5. $a_n = 3a_{n-1} - 2a_{n-2} + n\cdot 2^n \quad a_0 = 1,\ a_1 = 1$

Associated LHRR: $a_n' = 3a_{n-1} - 2a_{n-2}$

$r^2 - 3r + 2 = 0 \quad r_1 = 1 \quad r_2 = 2 \quad m_1 = 1,\ m_2 = 1.$

Let $F(n) = n\cdot 2^n \quad f(n) = n.\ s = 2 = r_2,\ m_2 = 1.$

$\Rightarrow$ particular solution: $x_n = (p_1 n + p_0)\cdot 2^n \cdot n^1$

$= (p_1 n + p_0) n \cdot 2^n.$

for LHRR: $y_n = \alpha_{1,0}\cdot 1^n + \alpha_{2,0}\cdot 2^n$

$= \alpha_{1,0} + \alpha_{2,0}\cdot 2^n$

Total solution:

$\Rightarrow z_n = x_n + y_n = (p_1 n + p_0)\cdot n\cdot 2^n + \alpha_{1,0} + \alpha_{2,0}\cdot 2^n.$

$a_0 = 1 \quad a_1 = -1,\ a_2 = 3a_1 - 2a_0 + 2\cdot 2^2 = 3.$

$a_3 = 3a_2 - 2a_1 + 3\cdot 2^3 = 35.$

$$\Rightarrow \begin{cases} \alpha_{1,0} + \alpha_{2,0} = a_0 = 1 \\ \alpha_{1,0} + 2\alpha_{2,0} + 2p_1 + 2p_0 = a_1 = -1 \\ \alpha_{1,0} + 4\alpha_{2,0} + 16p_1 + 8p_0 = 3. \\ \alpha_{1,0} + 8\alpha_{2,0} + 72p_1 + 24p_0 = 35. \end{cases}$$

$\alpha_{1,0} = 3,\ \alpha_{2,0} = -2,\ p_1 = 1,\ p_0 = -1$

$\Rightarrow z_n = (n-1)n\cdot 2^n + 3 - 2^{n+1}$