Discrete Mathematics: Lecture 19

tautological implications, argument

Xuming He
Associate Professor

School of Information Science and Technology
ShanghaiTech University

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Notes by Prof. Liangfeng Zhang

Review: Types of WFFs

Tautology(重言式)**:** a WFF whose truth value is **T** for all truth assignment

- $p \vee \neg p$ is a tautology

Contradiction(矛盾式)**:** a WFF whose truth value is **F** for all truth assignment

- $p \wedge \neg p$ is a contradiction

Contingency(可能式)**:** neither tautology nor contradiction

- $p \rightarrow \neg p$ is a contingency

Satisfiable(可满足的):a WFF is satisfiable if it is true for at least one truth assignment

Rule of Substitution: (代入规则) Let B be a formula obtained from a tautology A by substituting a propositional variable in A with an arbitrary formula. Then B must be a tautology.

- $p \vee \neg p$ is a tautology: $(q \wedge r) \vee \neg(q \wedge r)$ is a tautology as well.

Review: Proving $A \equiv B$

Rule of Replacement: (替换规则) Replacing a sub-formula in a formula F with a logically equivalent sub-formula gives a formula logically equivalent to the formula F.

EXAMPLE: $P \to Q \equiv \neg Q \to \neg P$

$$P \to Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg(\neg Q) \vee \neg P \equiv \neg Q \to \neg P$$

EXAMPLE: $P \leftrightarrow Q \equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$

$$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P) \equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$$
$$\equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$$

EXAMPLE: $P \to (Q \to R) \equiv (P \wedge Q) \to R$

$$P \to (Q \to R) \equiv \neg P \vee (\neg Q \vee R) \equiv (\neg P \vee \neg Q) \vee R \equiv \neg(P \wedge Q) \vee R$$
$$\equiv (P \wedge Q) \to R$$

Tautological Implications

DEFINITION: Let A and B be WFFs in propositional variables $p_1, \ldots, p_n$.

- A **tautologically implies** (重言蕴涵) B if every truth assignment that causes A to be true causes B to be true.
  - Notation: $A \Rightarrow B$, called a **tautological implication**
  - $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$; $B^{-1}(\mathbf{F}) \subseteq A^{-1}(\mathbf{F})$

THEOREM: $A \Rightarrow B$ iff $A \to B$ is a tautology.

- $A \Rightarrow B$ iff $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$ iff $A \to B$ is a tautology

THEOREM: $A \Rightarrow B$ iff $A \wedge \neg B$ is a contradiction.

- $A \to B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$

Proving $\boldsymbol{A \Rightarrow B}$: (1) $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$; (2) $B^{-1}(\mathbf{F}) \subseteq A^{-1}(\mathbf{F})$;

(3) $A \to B$ is a tautology; (4) $A \wedge \neg B$ is a contradiction

Proving $A \Rightarrow B$

EXAMPLE: Show the tautological implication "$p \wedge (p \to q) \Rightarrow q$".

- Let $A = p \wedge (p \to q)$; $B = q$. Need to show that "$A \Rightarrow B$"
- $A^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T})\}$; $B^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T}), (\mathbf{F}, \mathbf{T})\}$: $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$.

p	q	$p \to q$	A	B
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**F**	**F**

- $$\begin{aligned} A \to B &\equiv \neg\big(p \wedge (p \to q)\big) \vee q \\ &\equiv \big(\neg p \vee \neg(p \to q)\big) \vee q \\ &\equiv (\neg p \vee q) \vee \neg(p \to q) \\ &\equiv (p \to q) \vee \neg(p \to q) \\ &\equiv \mathbf{T} \end{aligned}$$
- $$\begin{aligned} A \wedge \neg B &\equiv \big(p \wedge (p \to q)\big) \wedge \neg q \\ &\equiv (\neg q \wedge p) \wedge (p \to q) \\ &\equiv \neg(p \to q) \wedge (p \to q) \\ &\equiv \mathbf{F} \end{aligned}$$

Tautological Implications

Name	Tautological Implication	NO.
Conjunction(合取)	$(P) \wedge (Q) \Rightarrow P \wedge Q$	1
Simplification(化简)	$P \wedge Q \Rightarrow P$	2
Addition(附加)	$P \Rightarrow P \vee Q$	3
Modus ponens(假言推理)	$P \wedge (P \rightarrow Q) \Rightarrow Q$	4
Modus tollens(拒取)	$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$	5
Disjunctive syllogism(析取三段论)	$\neg P \wedge (P \vee Q) \Rightarrow Q$	6
Hypothetical syllogism(假言三段论)	$(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R)$	7
Resolution (归结)	$(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$	8

Proofs for 5 and 6

EXAMPLE: $\neg Q \wedge (P \to Q) \Rightarrow \neg P$

- $A = \neg Q \wedge (P \to Q), B = \neg P.$
- $A \to B \equiv \neg\big(\neg Q \wedge (P \to Q)\big) \vee \neg P$

$$\equiv \big(Q \vee \neg(P \to Q)\big) \vee \neg P$$

$$\equiv (\neg P \vee Q) \vee \neg(P \to Q)$$

$$\equiv \mathbf{T}$$

EXAMPLE: $\neg P \wedge (P \vee Q) \Rightarrow Q$

- $A = \neg P \wedge (P \vee Q), B = Q.$
- $A \to B \equiv \neg\big(\neg P \wedge (P \vee Q)\big) \vee Q$

$$\equiv \big(P \vee \neg(P \vee Q)\big) \vee Q$$

$$\equiv (\neg(P \vee Q) \vee P) \vee Q$$

$$\equiv \neg(P \vee Q) \vee (P \vee Q)$$

$$\equiv \mathbf{T}$$

Proofs for 7 and 8

EXAMPLE: $(P \to Q) \land (Q \to R) \Rightarrow P \to R$

- $A = (P \to Q) \land (Q \to R);\ B = (P \to R).$
- $A \land \neg B \equiv (\neg P \lor Q) \land (\neg Q \lor R) \land (P \land \neg R)$

$$\equiv \big((\neg P \lor Q) \land P\big) \land \big((\neg Q \lor R) \land \neg R\big)$$

$$\equiv \big((\neg P \land P) \lor (Q \land P)\big) \land \big((\neg Q \land \neg R) \lor (R \land \neg R)\big)$$

$$\equiv (Q \land P) \land (\neg Q \land \neg R)$$

$$\equiv \mathbf{F}$$

EXAMPLE: $(P \lor Q) \land (\neg P \lor R) \Rightarrow Q \lor R$

- $A = (P \lor Q) \land (\neg P \lor R); B = (Q \lor R).$
- $A \land \neg B \equiv (P \lor Q) \land (\neg P \lor R) \land (\neg Q \land \neg R)$

$$\equiv \big((P \lor Q) \land \neg Q\big) \land ((\neg P \lor R) \land \neg R)$$

$$\equiv (P \land \neg Q) \land (\neg P \land \neg R)$$

$$\equiv \mathbf{F}$$

More Examples

EXAMPLE: $(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \Rightarrow (P \leftrightarrow R)$

- $A = (P \leftrightarrow Q) \wedge (Q \leftrightarrow R); B = (P \leftrightarrow R).$
- $A = \mathbf{T}$ iff $(P \leftrightarrow Q) = \mathbf{T}$ and $(Q \leftrightarrow R) = \mathbf{T}$ iff $P = Q$ and $Q = R$
  - $A^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T}, \mathbf{T}), (\mathbf{F}, \mathbf{F}, \mathbf{F})\}$
- $B = \mathbf{T}$ iff $P = R$
  - $B^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T}, \mathbf{T}), (\mathbf{T}, \mathbf{F}, \mathbf{T}), (\mathbf{F}, \mathbf{T}, \mathbf{F}), (\mathbf{F}, \mathbf{F}, \mathbf{F})\}$
- $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$

EXAMPLE: $(Q \rightarrow R) \Rightarrow ((P \vee Q) \rightarrow (P \vee R))$

- $A = Q \rightarrow R; B = ((P \vee Q) \rightarrow (P \vee R)).$
- $A = \mathbf{F}$ iff $(Q, R) = (\mathbf{T}, \mathbf{F})$
  - $A^{-1}(\mathbf{F}) = \{(\mathbf{T}, \mathbf{T}, \mathbf{F}), (\mathbf{F}, \mathbf{T}, \mathbf{F})\}$
- $B = \mathbf{F}$ iff $(P \vee Q, P \vee R) = (\mathbf{T}, \mathbf{F})$ iff $(P, Q) \neq (\mathbf{F}, \mathbf{F})$ and $(P, R) = (\mathbf{F}, \mathbf{F})$
  - $B^{-1}(\mathbf{F}) = \{(\mathbf{F}, \mathbf{T}, \mathbf{F})\}$
- $A^{-1}(\mathbf{F}) \supseteq B^{-1}(\mathbf{F})$

More Examples

EXAMPLE: $(P \rightarrow R) \wedge (Q \rightarrow S) \wedge (P \vee Q) \Rightarrow R \vee S$

- $A = (P \rightarrow R) \wedge (Q \rightarrow S) \wedge (P \vee Q); B = R \vee S$
- $A \wedge \neg B \equiv (P \rightarrow R) \wedge (Q \rightarrow S) \wedge (P \vee Q) \wedge \neg(R \vee S)$

$$\equiv (\neg P \vee R) \wedge (\neg Q \vee S) \wedge (P \vee Q) \wedge (\neg R \wedge \neg S)$$
$$\equiv ((\neg P \vee R) \wedge \neg R)) \wedge ((\neg Q \vee S) \wedge \neg S) \wedge (P \vee Q)$$
$$\equiv ((\neg P \wedge \neg R) \vee (R \wedge \neg R)) \wedge ((\neg Q \wedge \neg S) \vee (S \wedge \neg S)) \wedge (P \vee Q)$$
$$\equiv ((\neg P \wedge \neg R) \vee \mathbf{F}) \wedge ((\neg Q \wedge \neg S) \vee \mathbf{F}) \wedge (P \vee Q)$$
$$\equiv (\neg P \wedge \neg R) \wedge (\neg Q \wedge \neg S) \wedge (P \vee Q)$$
$$\equiv \neg R \wedge (\neg Q \wedge \neg S) \wedge (\neg P \wedge (P \vee Q))$$
$$\equiv \neg R \wedge (\neg Q \wedge \neg S) \wedge ((\neg P \wedge P) \vee (\neg P \wedge Q))$$
$$\equiv \neg R \wedge (\neg Q \wedge \neg S) \wedge (\mathbf{F} \vee (\neg P \wedge Q))$$
$$\equiv \neg R \wedge (\neg Q \wedge \neg S) \wedge (\neg P \wedge Q)$$
$$\equiv \neg R \wedge \neg S \wedge \neg P \wedge (\neg Q \wedge Q)$$
$$\equiv \neg R \wedge \neg S \wedge \neg P \wedge \mathbf{F}$$
$$\equiv \mathbf{F}$$

New part

Argument

proof = valid argument

DEFINITION: An **argument** (论证) is a sequence of propositions

- **Conclusion**(结论): the final proposition
- **Premises**(假设): all the other propositions
- **Valid**(有效): the truth of premises implies that of the conclusion
- **Proof** (证明): a valid argument that establishes the truth of a conclusion

EXAMPLE: a valid argument, a proof

- If $\{2^{-n}\}$ is convergent, then $\{2^{-n}\}$ has a convergent subsequence.
- $\{2^{-n}\}$ is convergent.
- $\{2^{-n}\}$ has a convergent subsequence.

Premises T

conclusion T

Argument Form

proposition : specific

variables !

DEFINITION: An **argument form**(论证形式) is a sequence of formulas.

- **Valid**(有效): no matter which propositions are substituted for the propositional variables, the truth of conclusion follows from the truth of premises
  - Fixed **Rules of inference**(推理规则): valid argument forms (relatively simple)

EXAMPLE: a valid argument form and an invalid argument form

formulas

$p \to q$ premises

p

q → conclusion

valid

p: $\{(-1)^n\}$ is convergent. (F)

q: $\{(-1)^n\}$ has a convergent subsequence. (T)

$p \to q$: If $\{(-1)^n\}$ is convergent, then $\{(-1)^n\}$ has a convergent subsequence. (T)

$p \to q$

$\neg p$

$\neg q$

invalid

$\neg p$: $\{(-1)^n\}$ is not convergent. (T)

$\neg q$: $\{(-1)^n\}$ does not have a convergent subsequence. (F)

The truth of $\neg p$ and $p \to q$ does not imply that of $\neg q$

Building Arguments

QUESTION: Given the premises $P_1, \dots, P_n$, show a conclusion Q, that is, show that $P_1 \land \cdots \land P_n \Rightarrow Q$.

假设前提

助记 Name	Operations
Premise	Introduce the given formulas $P_1, \dots, P_n$ in the process of constructing proofs.
Conclusion	Quote the intermediate formula that have been deducted. 中间结论
Rule of replacement	Replace a formula with a logically equivalent formula.
Rules of Inference	Deduct a new formula with a tautological implication.
Rule of substitution	Deduct a formula from a tautology.

Just for Remembering

Rules

8 rules

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Building Arguments

EXAMPLE: Show that the premises 1, 2, 3, and 4 lead to conclusion 5.

1. "It is not sunny this afternoon and it is colder than yesterday,"
2. "We will go swimming only if it is sunny,"
3. "If we do not go swimming, then we will take a canoe trip,"
4. "If we take a canoe trip, then we will be home by sunset"
5. "We will be home by sunset."

- **Translating the premises and the conclusion into formulas. Let**
  - p: "It is sunny this afternoon"
  - q: "It is colder than yesterday"
  - r: "We will go swimming"
  - s: "We will take a canoe trip"
  - t: "We will be home by sunset"
    - The premises are $\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s$, and $s \rightarrow t$.
    - The conclusion is t.
- **Question: ?**$(\neg p \wedge q) \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t) \Rightarrow t$
  - Can be proven with truth table. 32 rows!

Building Arguments

EXAMPLE: Show that the premises 1, 2, 3, and 4 lead to conclusion 5.

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5. "We will be home by sunset."

- **Show that** $(\neg p \wedge q) \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t) \Rightarrow t$

(1)	$\neg p \wedge q$	Premise
(2)	$\neg p$	Simplification using (1)
(3)	$r \rightarrow p$	Premise
(4)	$\neg r$	Modus tollens using (2) and (3)
(5)	$\neg r \rightarrow s$	Premise
(6)	s	Modus ponens using (4) and (5)
(7)	$s \rightarrow t$	Premise
(8)	t	Modus ponens using (6) and (7)

choose p for p behind

$A \cap B \Rightarrow A$

$A \cap (A \rightarrow B) \Rightarrow B$

Building Arguments

EXAMPLE: Show that $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S) \Rightarrow S \vee R$

(1)	$P \vee Q$	Premise
(2)	$\neg P \rightarrow Q$	Rule of replacement applied to (1)
(3)	$Q \rightarrow S$	Premise
(4)	$\neg P \rightarrow S$	Hypothetical syllogism applied to (2) and (3)
(5)	$\neg S \rightarrow P$	Rule of replacement applied to (4)
(6)	$P \rightarrow R$	Premise
(7)	$\neg S \rightarrow R$	Hypothetical syllogism applied to (5) and (6)
(8)	$S \vee R$	Rule of replacement applied to (7)

$A \rightarrow B$

$B \rightarrow C$