Discrete Mathematics
Lecture 3

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Summary of Lecture 2

Sum of ideals: $a\mathbb{Z} + b\mathbb{Z} = \gcd(a, b)\mathbb{Z}$. $\{a, b\} \neq \{0\}$

Greatest common divisor: $\gcd(a, b) = as + bt$.

- $c|ab, \gcd(c, a) = 1 \Rightarrow c|b$ 互质
- p is a prime, $p|ab \Rightarrow p|a$ or $p|b$, induction
- Uniqueness proof for FTA
- Infinity of primes

Equivalence relation: a binary relation R on a set A

- R, aRa aRb bRa ab bc ac
- reflexive, symmetric, transitive
- equivalence class $[a]_R$

同余

Congruence: $R = \{(a, b) \in \mathbb{Z}^2 : n|(a - b)\}$

- $a \equiv b \pmod{n}$: $(a, b) \in R$
- $a = bq + r$: $a \bmod n = r$ 前提下
- $[a]_n$ equivalence class of a under mod n.

$= a + n\mathbb{Z} = \{a + nx : x \in \mathbb{Z}\}$

$[0]6 = \{0 \pm 6 \pm 12 \cdots\}$ $[1]6 = \{\cdots -11\ -5, 1, 7, \cdots\}$

Residue Class

DEFINITION: Let $\alpha \in \mathbb{R}$.

- $\lfloor \alpha \rfloor$: **floor** of α, the largest integer $\leq \alpha$
- $\lceil \alpha \rceil$: **ceiling** of α, the smallest integer $\geq \alpha$
  - If $a = bq + r$, then $q = \lfloor a/b \rfloor$ and $r = a - bq$

DEFINITION: Let $a \in \mathbb{Z}, n \in \mathbb{Z}^+$. We denote the equivalence class of a under the equivalence relation mod n with $[a]_n$ and call it the **residue class of** a mod n.

- $[a]_n = a + n\mathbb{Z} = \{a + nx : x \in \mathbb{Z}\}$
  - any element of $[a]_n$ is a **representative** of $[a]_n$

EXAMPLE: $[0]_6 = \{0, \pm 6, \pm 12, \dots\}$; $[1]_6 = \{\dots, -11, -5, 1, 7, 13, \dots\}$; ...

Residue Class

THEOREM: Let $n \in \mathbb{Z}^+, a, b \in \mathbb{Z}$. Then

$$[a]_n \cap [b]_n = \emptyset \text{ or } [a]_n = [b]_n.$$

- $[a]_n \cap [b]_n = \emptyset$: done
- $[a]_n \cap [b]_n \neq \emptyset$
  - $\exists c \in [a]_n \cap [b]_n$
  - $c \equiv a \pmod{n}, c \equiv b \pmod{n}$
  - $a \equiv b \pmod{n}$
  - $\exists t \in \mathbb{Z}$ such that $a = b + nt$
  - $[a]_n = \{a + nx : x \in \mathbb{Z}\} = \{b + nt + nx : x \in \mathbb{Z}\} = [b]_n$

COROLLARY: $[a]_n = [b]_n$ iff $a \equiv b \pmod{n}$.

COROLLARY: $\{[0]_n, [1]_n, \dots, [n-1]_n\}$ is a partition of $\mathbb{Z}$.

- $[a]_n \cap [b]_n = \emptyset$ for all $a, b \in \{0, 1, \dots, n-1\}$
- $\mathbb{Z} = [0]_n \cup [1]_n \cup \cdots \cup [n-1]_n$

$\mathbb{Z}_n$

DEFINITION: Let n be any positive integer. We define $\mathbb{Z}_n$ to be set of all residue classes modulo n.

- $\mathbb{Z}_n = \{[0]_n, [1]_n, \dots, [n-1]_n\}$
  - $\mathbb{Z}_n = \{0,1, \dots, n-1\}$;
- $\mathbb{Z}_n = \{[1]_n, [2]_n, \dots, [n]_n\}$
  - $\mathbb{Z}_n = \{1,2, \dots, n\}$

EXAMPLE: Two representations of the set $\mathbb{Z}_6$

- $\mathbb{Z}_6 = \{[0]_6, [1]_6, [2]_6, [3]_6, [4]_6, [5]_6\}$
  $= \{0,1,2,3,4,5\}$
- $\mathbb{Z}_6 = \{[-3]_6, [-2]_6, [-1]_6, [0]_6, [1]_6, [2]_6\}$
  $= \{-3,-2,-1,0,1,2\}$

$\mathbb{Z}_n$

DEFINITION: Let $n \in \mathbb{Z}^+$. For all $[a]_n, [b]_n \in \mathbb{Z}_n$, define

- **addition**: $[a]_n + [b]_n = [a+b]_n$
- **subtraction**: $[a]_n - [b]_n = [a-b]_n$
- **multiplication**: $[a]_n \cdot [b]_n = [a \cdot b]_n$

Well-defined? If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$, then

$a \pm b \equiv a' \pm b' \pmod{n}$ and $ab \equiv a'b' \pmod{n}$.

- Hence, $[a]_n \pm [b]_n = [a']_n \pm [b']_n$; $[a]_n \cdot [b]_n = [a']_n \cdot [b']_n$
  - $a \equiv a' \pmod{n} \Rightarrow n|(a-a') \Rightarrow \exists x$ such that $a - a' = nx$
  - $b \equiv b' \pmod{n} \Rightarrow n|(b-b') \Rightarrow \exists y$ such that $b - b' = ny$
    - $(a+b) - (a'+b') = nx + ny$
    - $(a-b) - (a'-b') = nx - ny$
    - $ab - a'b' = a(b-b') + b'(a-a') = any + b'nx$

$\mathbb{Z}_n^*$

DEFINITION: Let $n \in \mathbb{Z}^+$ and $[a]_n \in \mathbb{Z}_n$. $[s]_n \in \mathbb{Z}_n$ is called an **inverse** of $[a]_n$ if $[a]_n[s]_n = [1]_n$.

- **division**: If $[a]_n\ [s]_n = [1]_n$, define $\frac{[b]_n}{[a]_n} = [b]_n \cdot [s]_n$

THEOREM: Let $n \in \mathbb{Z}^+$. $[a]_n \in \mathbb{Z}_n$ has an inverse iff $\gcd(a, n) = 1$.

- Only if: $\exists\ s$ s.t. $[a]_n[s]_n \equiv [1]_n$; $\exists\ t, as - 1 = nt$; $\gcd(a, n) = 1$
- If: $\exists\ s, t$ s.t. $as + nt = 1$; $as \equiv 1 \pmod{n}$

DEFINITION: Let $n \in \mathbb{Z}^+$. Define $\mathbb{Z}_n^* = \{[a]_n \in \mathbb{Z}_n\colon \gcd(a, n) = 1\}$

- If n is prime, then $\mathbb{Z}_n^* = \{1,2, \ldots, n-1\}$
- If n is composite, then $\mathbb{Z}_n^* \subset \mathbb{Z}_n$

EXAMPLE: $\mathbb{Z}_5^* = \{1,2,3,4\}$; $\mathbb{Z}_6^* = \{1,5\}$; $\mathbb{Z}_8^* = \{1,3,5,7\}$

Euler's Phi Function

QUESTION: How many elements are there in $\mathbb{Z}_n^*$?

- $|\mathbb{Z}_n^*|$ is the number of integers $a \in [n]$ such that $\gcd(a, n) = 1$

DEFINITION: (Euler's Phi Function) $\phi(n) = |\mathbb{Z}_n^*|, \forall n \in \mathbb{Z}^+$.

- $\phi(n)$ is the number of integers $a \in [n]$ such that $\gcd(a, n) = 1$

THEOREM: Let p be a prime. Then $\forall e \in \mathbb{Z}^+, \phi(p^e) = p^{e-1}(p-1)$.

- Let $x \in [p^e]$.
- $\gcd(x, p^e) \neq 1$ iff $p|x$

  iff $x = p, 2p, \dots, p^{e-1} \cdot p$

  - $\phi(p^e) = p^e - p^{e-1} = p^{e-1}(p-1)$

EXAMPLE: $\phi(3^2) = 3(3-1) = 6$

- $\mathbb{Z}_9^* = \{1,2,\cancel{3},4,5,\cancel{6},7,8,\cancel{9}\}$

EXAMPLE: $\phi(p) = p - 1$

- $\mathbb{Z}_p^* = \{1,2, \dots, p-1\}$

Euler's Phi Function

QUESTION: Formula of $\phi(n)$ for general integer n?

THEOREM: If $n = p_1^{e_1} \cdots p_k^{e_k}$ for distinct primes $p_1, \dots, p_k$ and integers $e_1, \dots, e_k \geq 1$, then $\phi(n) = \phi\left(p_1^{e_1}\right) \cdots \phi\left(p_k^{e_k}\right)$.

Hence, $\phi(n) = n(1 - p_1^{-1}) \cdots \left(1 - p_k^{-1}\right)$.

- There are many proofs. We will see in the future.

COROLLARY: If $n = pq$ for two different primes p and q, then $\phi(n) = (p - 1)(q - 1)$.

EXAMPLE: $\phi(10) = (2 - 1)(5 - 1) = 4; n = 10; p = 2, q = 5$

- $\mathbb{Z}_{10}^* = \{1, \cancel{2}, 3, \cancel{4}, \cancel{5}, \cancel{6}, 7, \cancel{8}, 9, \cancel{10}\}$

Euler's Theorem

THEOREM (Euler) Let $n \geq 1$ and $\alpha \in \mathbb{Z}_n^*$. Then $\alpha^{\phi(n)} = 1$.

- $\alpha^{\phi(n)}, 1$ are both residue classes modulo n
- Suppose that $\alpha = [a]_n$ for $a \in \mathbb{Z}$. Then $\alpha^{\phi(n)} = 1$ is $([a]_n)^{\phi(n)} = [1]_n$
- How to prove?
  - Consider the map $f\colon \mathbb{Z}_n^* \to \mathbb{Z}_n^* \quad [x]_n \mapsto [a]_n \cdot [x]_n$
  - We show that f is injective
    - $f([x]_n) = f([y]_n)$
    - $[a]_n \cdot [x]_n = [a]_n \cdot [y]_n$
    - $[ax]_n = [ay]_n$
    - $n | a(x - y)$
    - $n | (x - y)$, this is because $\gcd(n, a) = 1$
      - $[x]_n = [y]_n$

Euler's Theorem

THEOREM (Euler) Let $n \geq 1$ and $\alpha \in \mathbb{Z}_n^*$. Then $\alpha^{\phi(n)} = 1$.

- $\alpha^{\phi(n)}, 1$ are both residue classes modulo n
- Suppose that $\alpha = [a]_n$ for $a \in \mathbb{Z}$. Then $\alpha^{\phi(n)} = 1$ is $([a]_n)^{\phi(n)} = [1]_n$
- How to prove?
  - Consider the map $f\colon \mathbb{Z}_n^* \to \mathbb{Z}_n^* \quad [x]_n \mapsto [a]_n \cdot [x]_n$
  - Suppose that $\mathbb{Z}_n^* = \left\{[x_1]_n, \ldots, \left[x_{\phi(n)}\right]_n\right\}$.
    - $f([x_1]_n) \cdots f\left(\left[x_{\phi(n)}\right]_n\right) = [x_1]_n \cdots \left[x_{\phi(n)}\right]_n$
    - $[ax_1]_n \cdots \left[ax_{\phi(n)}\right]_n = [x_1]_n \cdots \left[x_{\phi(n)}\right]_n$
    - $\left[a^{\phi(n)} x_1 \cdots x_{\phi(n)}\right]_n = \left[x_1 \cdots x_{\phi(n)}\right]_n$
      - $n \mid (a^{\phi(n)} - 1) x_1 \cdots x_{\phi(n)}$
        - $n \mid (a^{\phi(n)} - 1)$, this is because $\gcd\left(n, x_1 \cdots x_{\phi(n)}\right) = 1$
          - $\left[a^{\phi(n)}\right]_n = [1]_n$, i. e., $([a]_n)^{\phi(n)} = [1]_n$

Fermat's Little Theorem

EXAMPLE: Understand Euler's theorem with $\mathbb{Z}_{10}^* = \{1,3,7,9\}$.

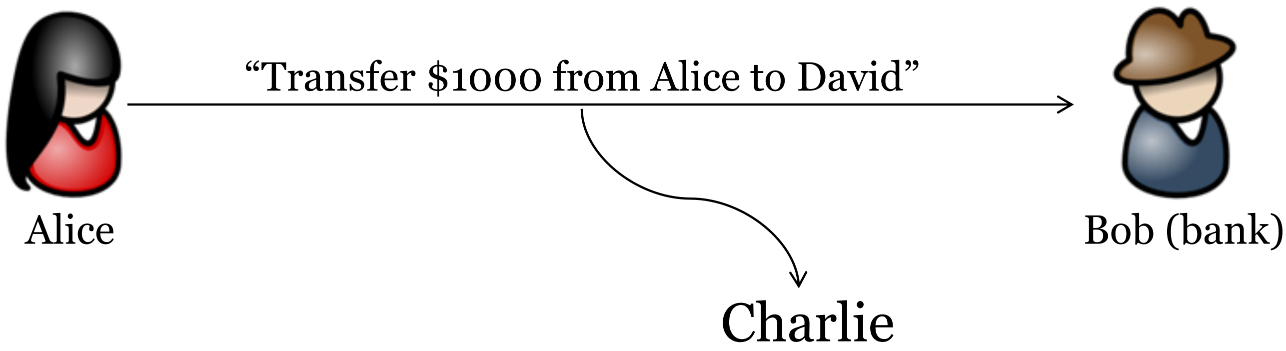
- $n = 10, \phi(n) = 4,$
- $1^4 \equiv 1 \pmod{10} \Rightarrow ([1]_{10})^4 = [1]_{10}$
- $3^4 = 81 \equiv 1 \pmod{10} \Rightarrow ([3]_{10})^4 = [1]_{10}$
- $7^4 = 2401 \equiv 1 \pmod{10} \Rightarrow ([7]_{10})^4 = [1]_{10}$
- $9^4 = 6561 \equiv 1 \pmod{10} \Rightarrow ([9]_{10})^4 = [1]_{10}$
  - Consider the map $f\colon \mathbb{Z}_{10}^* \to \mathbb{Z}_{10}^* \quad [x]_n \mapsto [9]_n \cdot [x]_n$
  - $f([1]_{10}) = [9]_{10} \cdot [1]_{10} = [9]_{10}; f([3]_{10}) = [7]_{10}; f([7]_{10}) = [3]_{10}, f([9]_{10}) = [1]_{10}$
  - f is injective
  - $f([1]_{10})f([3]_{10})f([7]_{10})f([9]_{10}) = [9]_{10}[7]_{10}[3]_{10}[1]_{10}$

Fermat's Little Theorem: If p is a prime and $\alpha \in \mathbb{Z}_p$.

Then $\alpha^p = \alpha$.

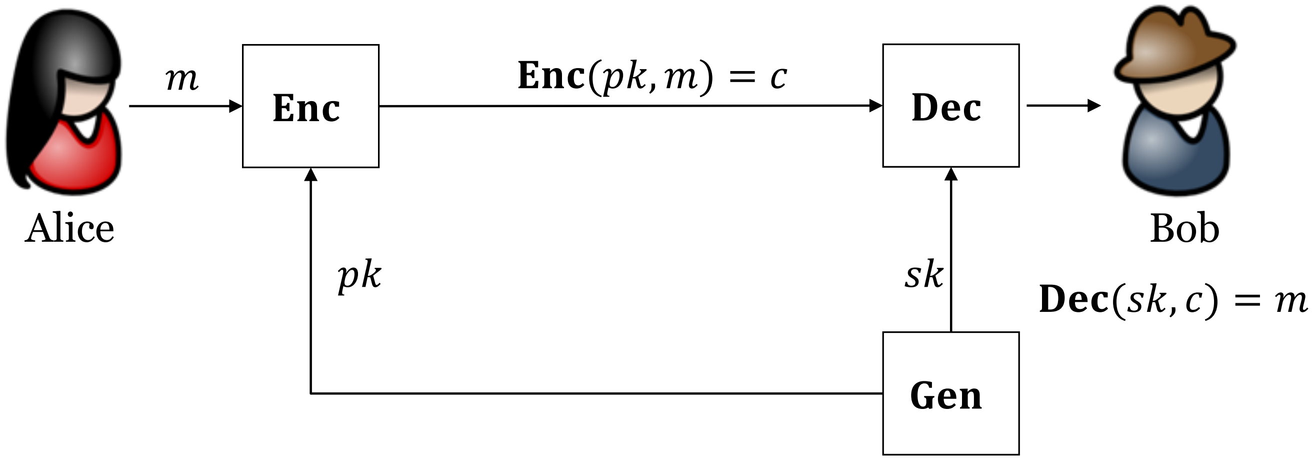
- This is a corollary of Euler's theorem for $n = p$
- By Euler's theorem, $\alpha^{p-1} = 1$
  - $\alpha^p = \alpha$

Cryptography



- **Confidentiality**: The property that sensitive information is not disclosed to unauthorized individuals, entities, or processes. --FIPS 140-2

Public-Key Encryption



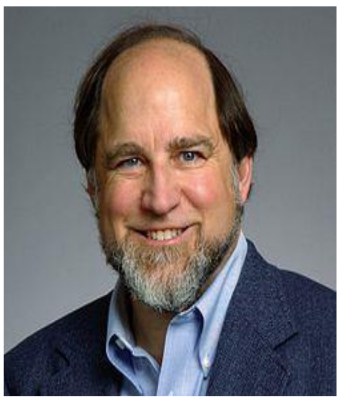
- **Gen**, **Enc**, **Dec**: key generation, encryption, decryption

  明文 密文 sk,
- m, c, pk, sk: plaintext (message), ciphertext, public key, private key
- $\mathcal{M}, \mathcal{C}$: plaintext space, ciphertext space
  - $\Pi = (\mathbf{Gen}, \mathbf{Enc}, \mathbf{Dec}) + \mathcal{M}, |\mathcal{M}| > 1$
    - **Correctness**: $\mathbf{Dec}\big(sk, \mathbf{Enc}(pk, m)\big) = m$ for any pk, sk, m
    - **Security**: if sk is not known, it's difficult to learn m from pk, c

RSA

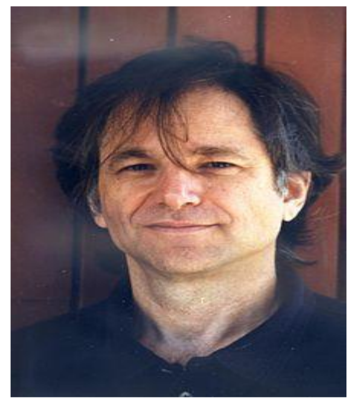
A method for obtaining digital signatures and public-key cryptosystem

- Ronald Rivest, Adi Shamir and Leonard Adleman (1977)-MIT
- Scientific Contributions: Turing Award (2002)
    - Public-Key Encryption: the first construction
    - Digital Signature: the first construction



Rivest

Shamir



Adleman

Plain RSA

Q

CONSTRUCTION: $\Pi = (\mathbf{Gen}, \mathbf{Enc}, \mathbf{Dec}) + \mathcal{M}$, the message space is $\mathcal{M} = \{m: m \in [N], \gcd(m, N) = 1\}$

- $(pk, sk) \leftarrow \mathbf{Gen}(1^n)$
  - choose two n-bit primes $p \neq q$;
  - $N = pq$; $\phi(N) = (p-1)(q-1)$
  - $[e]_{\phi(N)} \leftarrow \mathbb{Z}^*_{\phi(N)}$ [illegible]随机[illegible]
  - $[d]_{\phi(N)} = \left([e]_{\phi(N)}\right)^{-1}$
    - $0 \leq e, d < \phi(N)$
  - output $pk = (N, e)$ and $sk = (N, d)$
- $c \leftarrow \mathbf{Enc}(pk, m)$:
  - output $c = m^e \bmod N$
    - $0 \leq c < N$
- $m \leftarrow \mathbf{Dec}(sk, c)$:
  - output $m = c^d \bmod N$
    - $0 \leq m < N$

?Dec$(sk, \mathbf{Enc}(pk, m)) = m$

- $[d]_{\phi(N)} = \left([e]_{\phi(N)}\right)^{-1}$
- $\exists t \in \mathbb{Z}$ s.t. $ed = 1 + t \cdot \phi(N)$
- $$\begin{aligned} [c^d]_N &= ([c]_N)^d \\ &= ([m^e]_N)^d \\ &= \left(([m]_N)^e\right)^d \\ &= ([m]_N)^{ed} \\ &= ([m]_N)^{1+t\phi(N)} \\ &= [m]_N \cdot ([m]_N)^{\phi(N)t} \\ &= [m]_N \cdot [1]_N \\ &= [m]_N \end{aligned}$$
- $m = c^d \bmod N$

RSA is correct!

Plain RSA

5是随便选了一个

EXAMPLE: this is a toy example; all numbers are very small

- $(pk, sk) \leftarrow \mathbf{Gen}(1^n)$
  - $p = 7, q = 13,$
  - $N = 91, \phi(N) = 72$
  - $[e]_{72} = [5]_{72}$
  - $[d]_{72} = [29]_{72}$ 怎么算出来
  - $pk = (91, 5);\ sk = (91, 29)$
- $c \leftarrow \mathbf{Enc}(pk, m)$: $m = 2$ 信息
  - 加密 $c = (2^5 \bmod 91) = 32$ 密文
- $m \leftarrow \mathbf{Dec}(sk, c)$: $c = 10$
  - 解密 $m = (32^{29} \bmod 91) = 2$ 明文

m

- $32^{29} = (2^5)^{29} = 2^{145}$ 解法
- $2^{145} \equiv ?\ (\text{mod } 91)$
- $[2^{145}]_{91} = [?\,]_{91}$
- 换顺 $([2]_{91})^{145} = [?\,]_{91}$
- $[2]_{91} \in \mathbb{Z}_{91}^*$
- $([2]_{91})^{\phi(91)} = [1]_{91}$
- $([2]_{91})^{145} = ([2]_{91})^{72}([2]_{91})^{72}[2]_{91}$

$$= [1]_{91}[1]_{91}[2]_{91}$$
$$= [2]_{91}$$

Security

Security: If sk is not known, it's difficult to learn m from pk, c

- At least, it should be difficult to learn d from pk

Plain RSA and Integer Factoring (given N, find p, q**):**

- "Factoring is easy" ⇒ "Plain RSA is not secure"
  - $N \to (p, q) \to \phi(N) \to d$: computable with EEA
- "Plain RSA is secure" ⇒ "Factoring is hard"
- "Factoring is hard"⇏ "Plain RSA is secure"
- It is likely that "Factoring is hard"⇒ "Plain RSA is secure"
  - The best known method of computing d is via factoring N

How Large is the N in practice?

- $|N| = 2048$ is recommended from present to 2030
- $|N| = 3072$ is recommended after 2030

RSA

EXAMPLE: A sample execution of the RSA public-key encryption.

- p =179769313486231590772930519078902473361797697894230657273430081157732675805500963132708477322407536021120113879871393357658789768814416622492847430639474124377767893424865485276302219601246094119453082952085005768838150682342462881473913110540827237163350510684586298239947245938479716304835356329624225795083
- q =179769313486231590772930519078902473361797697894230657273430081157732675805500963132708477322407536021120113879871393357658789768814416622492847430639474124377767893424865485276302219601246094119453082952085005768838150682342462881473913110540827237163350510684586298239947245938479716304835356329624227077847
- N =32317006071311007300714876688669951960444102669715484032130345427524655138867890893197201411522913463688717960921898019494119559150490921095088152386448283120630877367300996091750197750389652106796057638384067568276792218642619756161838094338476170470581645852036305042887575891541065808607552399123931212190742861198666048560131098081430518774846347259215332611759149330725252437276424147817808729273755165527379964561074264587032664709511346018327798373715290148129504141795132314929388992688247440232727539575514688633282447719228530664706520939357878528540284184156513405575872085703420500969966917951381310826301
- $\phi(N)$ =32317006071311007300714876688669951960444102669715484032130345427524655138867890893197201411522913463688717960921898019494119559150490921095088152386448283120630877367300996091750197750389652106796057638384067568276792218642619756161838094338476170470581645852036305042887575891541065808607552399123931212190383322571693585378585237043272713828122751863426871297212289168409787085665422221552391774628940093485139736801331477871715085171882512773342103512436341899373968354945401344376784553485755251993821373671344677095606146354543604901758694718276224054213583162787340809095977593826461068360296205292132857953372

RSA

EXAMPLE: A sample execution of the RSA public-key encryption.

- e =15
- d =4308934142841467640095316891822660261392547022628731204284046057003287351849052119092960
1882030551284918290614562530692658826078867321228126784203181931044160841169823067994789000
2636671862028090614101845120900910357229581901596748824507924513015606274421944693817400571
8343452205475441147673653216524161625384443009559144717144698272436361843749700248456916172
9616385557879716114220562962069855699505253457980186315735108637162286780229176683697789471
3499151225324986244732605351258357127379810070026584284982284595694608081951393914732023449
2629103496540561811088371645441212797012510194809114706160705617714393783
- m =1060492175475872144576165469414485300895277760828043761504547236562152874067991556927005
1503191522500036448557172487959011926112038398359402756573149541644330968641767630622070720
6300611302597838253559482233713309491580368127421870570456049345468117909489758782001441890
4834424987320032029927723446568903940998962231923268398424184371118321200199145779352875281
2978134072787404790207031482099444968252108690296363773578594703102617386738297675080295774
0914472401975212215460354590300865381144285160786447331806555401091337782416072602736553356
61777894173665137928787960365220712025120785257907244561721692764755210375
- c =1052638995813896291959559409341115889309974350846590234712847813990877461431177809735479
5345791726768384252751637693995592403757856185437083738829836072472243389583367910268799453
3780394197213455665495167301873084368644600883966117266700507232420801391760803347202941953
0404891500380565634181654830724988604902791048824931866006271433570305757657601698851348414
8308512574950252535463185824865665499749033598201370342142901944632549253564037639312442875
0397358269093293568406659937836951014476104859227269159699679685846612404304259821941895044
00469889762574275824269475495394920107921066723277769226199475558068627049

Questions from RSA

CONSTRUCTION: $\Pi = (\mathbf{Gen}, \mathbf{Enc}, \mathbf{Dec}) + \mathcal{M}$, the message space is $\mathcal{M} = \{m: m \in [N], \gcd(m, N) = 1\}$

- $(pk, sk) \leftarrow \mathbf{Gen}(1^n)$
  - choose two n-bit primes $p \neq q$
  - $N = pq$; $\phi(N) = (p-1)(q-1)$
  - $[e]_{\phi(N)} \leftarrow \mathbb{Z}^*_{\phi(N)}$
  - $[d]_{\phi(N)} = \left([e]_{\phi(N)}\right)^{-1}$
    - $0 \leq e, d < \phi(N)$
  - output $pk = (N, e)$ and $sk = (N, d)$
- $c \leftarrow \mathbf{Enc}(pk, m)$:
  - output $c = m^e \bmod N$
    - $0 \leq c < N$
- $m \leftarrow \mathbf{Dec}(sk, c)$:
  - output $m = c^d \bmod N$
    - $0 \leq m < N$

How to choose p, q EFFICIENTLY?

How to compute d EFFICIENTLY?

How to compute c EFFICIENTLY?

How to compute m EFFICIENTLY?

Implementation Issues

CONSTRUCTION: $\Pi = (\mathbf{Gen}, \mathbf{Enc}, \mathbf{Dec}) + \mathcal{M}$, the message space is $\mathcal{M} = \{m: m \in [N], \gcd(m, N) = 1\}$

- $(pk, sk) \leftarrow \mathbf{Gen}(1^n)$
  - choose two n-bit primes $p \neq q$
  - $N = pq$; $\phi(N) = (p-1)(q-1)$
  - $[e]_{\phi(N)} \leftarrow \mathbb{Z}^*_{\phi(N)}$
  - $[d]_{\phi(N)} = \left([e]_{\phi(N)}\right)^{-1}$
    - $0 \leq e, d < \phi(N)$
  - output $pk = (N, e)$ and $sk = (N, d)$
- $c \leftarrow \mathbf{Enc}(pk, m)$:
  - output $c = m^e \bmod N$
    - $0 \leq c < N$
- $m \leftarrow \mathbf{Dec}(sk, c)$:
  - output $m = c^d \bmod N$
    - $0 \leq m < N$

Questions

- Choose p, q efficiently?
  - Prime number generation
- Compute d efficiently?
  - Square-and-multiply
- Compute c/m efficiently?
  - Square-and-multiply