Discrete Mathematics

generating functions

Liangfeng Zhang
School of Information Science and Technology
ShanghaiTech University

Summary of Lecture 14

LHRR of degree k with constant coefficients:

$$a_n = \sum_{i=1}^{k} c_i a_{n-i}$$

- Existence and uniqueness: given k initial terms
- characteristic equation: $r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$
  - 根全互异 k distinct roots $r_1, r_2, \dots, r_k$: $x_n = \sum_{j=1}^{k} \alpha_j r_j^n$
  - 根不全互异 Roots $\{m_1 \cdot r_1, \dots, m_t \cdot r_t\}$: $x_n = \sum_{j=1}^{t} \left(\sum_{\ell=0}^{m_j-1} \alpha_{j,\ell}\, n^{\ell} \right) r_j^n$

重数

$m r_j$

$a_{j,0} + a_{j,1} \cdot n + \cdots$

(需要给定 k 个初始值)

LNRR of degree k with constant coefficients:

$$a_n = \sum_{i=1}^{k} c_i a_{n-i} + \underline{F(n)}$$

- Existence and uniqueness: given k initial terms
- General solutions: $z_n = x_n + y_n$ LHRR
- Particular solutions: if $F(n) = \underline{(f_l n^l + \cdots + f_1 n + f_0)} \underline{s^n}$ and s is a ~~root~~ of multiplicity m, $x_n = (p_l n^l + \cdots + p_1 n + p_0) s^n n^m$

of

对应 LHRR

生成函数

Generating Functions

DEFINITION: The **generating function** of a sequence $\{a_r\}_{r=0}^{\infty}$ is defined as $G(x) = \sum_{r=0}^{\infty} \boxed{a_r} x^r$.

0开始

- Generating functions are **formal power series**.

形式级

- We do not discuss their convergence.

EXAMPLE: generating functions of sequences

- $a_r = 3, G(x) = 3(1 + x + \cdots + x^r + \cdots)$
- $a_r = 2^r, G(x) = 1 + 2x + \cdots + (2x)^r + \cdots$ $2^r \cdot x^r$
- $a_r = \binom{n}{r}, G(x) = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n$ $r>n$ $\binom{n}{r}=0$.

n选r

DEFINITION: Let $A(x) = \sum_{r=0}^{\infty} a_r x^r$, $B(x) = \sum_{r=0}^{\infty} b_r x^r$

- $A(x) = B(x)$ if $a_r = b_r$ for all $r = 0,1,2, \ldots$

Equal

Operations

DEFINITION: Let $A(x) = \sum_{r=0}^{\infty} a_r x^r$, $B(x) = \sum_{r=0}^{\infty} b_r x^r$

- $A(x) + B(x) = \sum_{r=0}^{\infty}(a_r + b_r)\, x^r$
- $A(x) - B(x) = \sum_{r=0}^{\infty}(a_r - b_r)\, x^r$
- $A(x) \cdot B(x) = \sum_{r=0}^{\infty}(\sum_{j=0}^{r} a_j b_{r-j})\, x^r$
- $\lambda \cdot A(x) = \sum_{r=0}^{\infty} \lambda a_r x^r$ for any constant $\lambda \in \mathbb{R}$
- We say that $B(x)$ is an **inverse** of $A(x)$ if $A(x)B(x) = 1$.
  - The inverse of $A(x)$: $A^{-1}(x)$
  - When $A(x)$ has an inverse, define $\frac{C(x)}{A(x)} = A^{-1}(x) \cdot C(x)$

$A(x) = a_0 + a_1 x + a_2 x^2 \ldots$

$B(x) = b_0 + b_1 x + b_2 x \ldots$

$C_r x^r$

$a_0 \cdot b_r x^r + a_1 b_{r-1} x^r \ldots$

$a_r \cdot b_0 x^r$

$a_0 b_r$ $\leftrightarrow$ $a_r b_0$.

逆

divide $\rightleftarrows$ multi

Operations

THEOREM: $A(x) = \sum_{r=0}^{\infty} a_r x^r$ has an inverse iff $a_0 \neq 0$.

EXAMPLE: Let $A(x) = \sum_{r=0}^{\infty} x^r$. Find $A^{-1}(x)$.

$a_r = 1$

- $a_0 = 1 \neq 0$: $A^{-1}(x)$ exists
- Denote $A^{-1}(x) = \sum_{r=0}^{\infty} b_r x^r$; $b_0, b_1, \ldots$ are undetermined coefficients
- $A(x)A^{-1}(x) = 1$:
  - $(1 + x + x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) = 1 + 0 \cdot x + 0 \cdot x^2 + \cdots$

    带x全0!

    Compare Explicit x^r

    - Coefficient of x^0: $b_0 = 1$
    - Coefficient of x^1: $b_1 + b_0 = 0$
    - Coefficient of x^2: $b_2 + b_1 + b_0 = 0$
    - Coefficient of x^r: $b_r + b_{r-1} + \cdots + b_0 = 0$
      - $b_1 = -1, b_2 = 0, \ldots, b_r = 0$
      - $A^{-1}(x) = 1 - x$

改写成 sequence 形式

Operations

DEFINITION: $A(x) = \sum_{r=0}^{\infty} a_r x^r$

- $A'(x) = \sum_{r=1}^{\infty} r a_r x^{r-1}$
  - $A^{(0)}(x) = A(x)$
  - k阶 $A^{(k)}(x) = (A^{(k-1)}(x))'$ for all integers $k \geq 1$
- 积 $\int A(x)\mathrm{d}x = \sum_{r=0}^{\infty} \frac{1}{r+1} a_r x^{r+1} + C$, where C is a constant

THEOREM: Let $A(x) = \sum_{r=0}^{\infty} a_r x^r$ and $B(x) = \sum_{r=0}^{\infty} b_r x^r$.

- $\big(\alpha A(x) + \beta B(x)\big)' = \alpha A'(x) + \beta B'(x)$
- $\big(A(x)B(x)\big)' = A'(x)B(x) + A(x)B'(x)$
- $\big(A^k(x)\big)' = kA^{k-1}(x)\, A'(x)$ 链

$(1 + \alpha x)^u$ Common

扩展二项式

DEFINITION: Let $u \in \mathbb{R}$ and $r \in \mathbb{N}$. The **extended binomial coefficient** $\binom{u}{r} = \begin{cases} u(u-1)\cdots(u-r+1)/r! & r > 0 \\ 1 & r = 0 \end{cases}$

$u \in R$ 有理、无理

THEOREM: Let x be a real number with $|x| < 1$ and let u be a real number. Then $(1+x)^u = \sum_{r=0}^{\infty} \binom{u}{r} x^r$.

$$\frac{u!}{r!\,(u-r)!} = \frac{u \cdots (u-r+1)}{r!}$$

EXAMPLE: $u = -1$

- $(1 - \alpha x)^{-1} = \sum_{r=0}^{\infty} \alpha^r x^r$

  $x : (-\alpha x)$

- $(1 - \alpha x)^{-n} = \sum_{r=0}^{\infty} \binom{r+n-1}{r} \alpha^r x^r$

$r \uparrow$ $-1 \times r$

$$\sum_{0}^{\infty} \binom{-1}{r} (-\alpha x)^r$$

$$\frac{(-1)\times(-2)\cdots\times(-1-r+1)}{r!} \cdot (-\alpha x)^r$$

$(-\alpha)^r x^r$

$$= \frac{r!}{r!} (- - \alpha x)^r$$

$$\sum_{0}^{\infty} \binom{-n}{r} (-\alpha x)^r$$

$$\frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} (+\alpha x)^r$$

$n \quad n+1 \quad n+r-1$

$n = n+r-1 \left(- r+1\right)$

CF 收

Counting Combinations with GFs

$\{\infty\cdot 1, \infty\cdot 2, \ldots \infty\cdot n\}$ 的 r 组合 $\Longleftrightarrow$ [n] 的 r 可重组合

QUESTION: Let $n > 0, R_1, \ldots, R_n \subseteq \mathbb{N}$. For every $r \geq 0$, let a_r be the number of r-combinations of $[n]$ with repetition where every $i \in [n]$ appears R_i times.

可以是≥，≤等条件

是个集合，一些数

r 无标物，n个有标盒，i盒收 R_i 个物体，

- $a_r = |\{(r_1, \ldots, r_n): r_1 \in R_1, \ldots, r_n \in R_n, r_1 + \cdots + r_n = r\}|$
  - This is also the number of ways of distributing r unlabeled objects into n labeled boxes such that R_i objects are sent to box i

THEOREM: $\sum_{r=0}^{\infty} a_r x^r = \prod_{i=1}^{n} \sum_{r_i \in R_i} x^{r_i}$.

- $\prod_{i=1}^{n} \sum_{r_i \in R_i} x^{r_i} = \underbrace{\sum_{r_1 \in R_1} x^{r_1}}_{1} \cdot \underbrace{\sum_{r_2 \in R_2} x^{r_2}}_{2} \cdots \underbrace{\sum_{r_n \in R_n} x^{r_n}}_{n}$

  $= \sum_{r=0}^{\infty} \left(\sum_{r_1 \in R_1, \ldots, r_n \in R_n, r_1 + \cdots + r_n = r} 1\right) x^r$

  $= \sum_{r=0}^{\infty} a_r x^r$

分类：标准：r 的次数

Counting Combinations with GFs

EXAMPLE: Let a_r be the number of ways of distributing r identical books to 5 persons such that person 1, 2, 3, and 4 receive $\geq 3, \geq 2, \geq 4, \geq 6$ books, respectively. Calculate a_{20}.

- $a_r = |\{(r_1, \dots, r_5): r_1 \geq 3, r_2 \geq 2, r_3 \geq 4, r_4 \geq 6, r_5 \geq 0, r_1 + \dots + r_5 = r\}|$ 分配 implicit
  - $R_1 = \{3,4,\dots\}; R_2 = \{2,3,\dots\}; R_3 = \{4,5,\dots\};$
    $R_4 = \{6,7,\dots\}; R_5 = \{0,1,2\dots\}$
  - $\sum_{r=0}^{\infty} \boxed{a_r x^r} = \prod_{i=1}^{5} \sum_{r_i \in R_i} x^{r_i}$

$\in$!

$$= (x^3 + \cdots)(x^2 + \cdots)(x^4 + \cdots)(x^6 + \cdots)(1 + x + \cdots)$$

$$= \frac{x^3}{1-x}\frac{x^2}{1-x}\frac{x^4}{1-x}\frac{x^6}{1-x}\frac{1}{1-x} = \frac{x^{15}}{(1-x)^5}$$

$$= x^{15}\sum_{r=0}^{\infty}\binom{-5}{r}(-1)^r x^r$$

- $a_{20} = \binom{-5}{5}(-1)^5 = 126$

$(1+x)^u = \sum_{r=0}^{\infty} \binom{u}{r} x^r$

since

$(1-x)^{-5}$

x^0

提出

$= 1-x$

除: $1-x$

取系数

a_{20}: x^{20}

$r=5$

$= x^3(1+x+\cdots)$

$x^2(1+x+\cdots)$

$x^4((1+x+\cdots$

Counting Permutations with GFs

QUESTION: Let $n > 0, R_1, \ldots, R_n \subseteq \mathbb{N}$. For every $r \geq 0$, let a_r be the number of r-permutations of $[n]$ with repetition where every $i \in [n]$ appears R_i times.

- $a_r = \sum_{r_1 \in R_1, r_2 \in R_2, \ldots, r_n \in R_n,\, r_1 + r_2 + \cdots + r_n = r} \frac{r!}{r_1! r_2! \cdots r_n!}$
  - This is the number of ways of distributing r labeled objects into n labeled boxes such that R_i objects are sent to box i for all $i \in [n]$

type2

THEOREM: $\boxed{\sum_{r=0}^{\infty} \frac{a_r}{r!} x^r} = \prod_{i=1}^{n} \sum_{r_i \in R_i} \frac{x^{r_i}}{r_i!}$.

GF

- $\prod_{i=1}^{n} \sum_{r_i \in R_i} \frac{x^{r_i}}{r_i!} = \sum_{r_1 \in R_1} \frac{x^{r_1}}{r_1!} \cdot \sum_{r_2 \in R_2} \frac{x^{r_2}}{r_2!} \cdots \sum_{r_n \in R_n} \frac{x^{r_n}}{r_n!}$

$$= \sum_{r=0}^{\infty} \left(\sum_{r_1 \in R_1, r_2 \in R_2, \ldots, r_n \in R_n,\, \underline{r_1 + r_2 + \cdots + r_n = r}} \frac{r!}{r_1! r_2! \cdots r_n!} \right) \frac{x^r}{r!}$$

a_r

$$= \sum_{r=0}^{\infty} \frac{a_r}{r!} x^r$$

Counting Permutations with GFs

$s_1, s_2 \ldots s_r \quad s_i \in \{1,2,3,4\}$

序列中偶个1

EXAMPLE: Find $a_r = \{s \in \{1,2,3,4\}^r : s \text{ has an even number of } 1s\}$

- $a_r =$ the number of r-permutations of $\{1,2,3,4\}$ with repetition where 1 appears an even number of times N^+
- $R_1 = \{0,2,4,\ldots\}, R_2 = R_3 = R_4 = \{0,1,2,\ldots\}$

1出现的次数

R_2, R_3, R_4

- $\sum_{r=0}^{\infty} \frac{a_r}{r!} x^r = \underbrace{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)}_{R_1}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)^3$

$\prod_{i=1}^{n} \sum_{r_i \in R_i} \frac{x^{r_i}}{r_i!}$

$$= \frac{e^x + e^{-x}}{2} \cdot e^{3x}$$

去奇

$$= \frac{e^{4x} + e^{2x}}{2}$$

$$= \frac{1}{2} \sum_{r=0}^{\infty} \left(\frac{(4x)^r}{r!} + \frac{(2x)^r}{r!}\right)$$

$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

- $\frac{a_r}{r!} = \frac{1}{2} \cdot \left(\frac{4^r}{r!} + \frac{2^r}{r!}\right)$
  - $a_r = \frac{4^r + 2^r}{2}$

对比 x^r 系数

LNRR

Partial Fraction Decomposition

$n \geq 1$

EXAMPLE: Solve the LNRR $a_n = 8a_{n-1} + 10^{n-1}$ with the initial condition $a_0 = 1$ using generating function.

P(x)

- $A(x) = \sum_{n=0}^{\infty} a_n x^n$ use Type 1 $\qquad A(x) = \dfrac{1-9x}{(1-8x)(1-10x)}$

Q(x)

$= 1 + \sum_{n=1}^{\infty}(8a_{n-1} + 10^{n-1})x^n$

$n < k$ part

$a_0 x^0$

$8a_{n-1}x^{n-1}$ $x^n \to x \cdot x^{n-1} \to n-1 \geq 0 : A(x)$ $\sum_0^{\infty} 10x$

$= 1 + 8xA(x) + \dfrac{x}{1-10x}$ $\qquad 10^{n-1} \cdot x^{n-1} \cdot x \cdot = (10x)^{n-1} \sum_1^{\infty}$ 化简

LEMMA: Let $Q(x), P(x)$ be two polynomials s.t. $\deg(Q) > \deg(P)$. If $Q(x) = (1 - r_1 x)^{m_1} \cdots (1 - r_t x)^{m_t}$ for distinct non-zero numbers $r_1, \ldots, r_t$ and integers $m_1, \ldots, m_t \geq 1$, then there exist unique coefficients $\{\alpha_{j,u} : j \in [t], u \in [m_j]\}$ such that

$$\frac{P(x)}{Q(x)} = \sum_{j=1}^{t} \sum_{u=1}^{m_j} \frac{\alpha_{j,u}}{(1 - r_j x)^u} .$$

here:

$t=2$ $\quad r_1 = 8, r_2 = 10.$ $\quad m_1 = 1$, $m_2 = 1$

Solving LNRR with GFs

EXAMPLE: Solve the LNRR $a_n = 8a_{n-1} + 10^{n-1}$ with the initial condition $a_0 = 1$ using generating function.

- $A(x) = \frac{1-9x}{(1-8x)(1-10x)}$
- $A(x) = \frac{\alpha_{1,1}}{1-8x} + \frac{\alpha_{2,1}}{1-10x}$
    - $\alpha_{1,1} = \alpha_{2,1} = \frac{1}{2}$
- $A(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$ 目标：原式 $\frac{P(x)}{Q(x)}$

  $= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$
- $a_n = \frac{1}{2}(8^n + 10^n) \quad (n \geq 0)$

比较系数