$P(A)=P(B)$

1. for $\forall x \in A$ $\{x\} \subseteq A \to \{x\} \in P(A) \xrightarrow{\Downarrow} \{x\} \in P(B) \to \{x\} \subseteq B$

$\Rightarrow A \subseteq B$.

for $\forall x \in B$ $\{x\} \subseteq B \to \{x\} \in P(B) \xrightarrow{P(A)=P(B)} \{x\} \in P(A) \to \{x\} \subseteq A$

$\Rightarrow B \subseteq A$

$\Rightarrow A = B$

Discrete Mathematics: Homework 5

(Deadline: March 25, 2022)

1. (20 points) Let A and B be any sets. Show that if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.
   (**Remark:** $\mathcal{P}(A)$ is the power set of A, i.e., the set of all subsets of A)
2. (20 points) Construct a bijection from $A = (0, 1) \cup [2, 3) \cup (4, 5]$ to $B = (6, 7) \cup [8, +\infty)$.
3. (20 points) Prove or disprove $|\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}| = |\mathbb{R}|$.
4. (20 points) Prove or disprove $|\{(a_1, a_2, a_3, \ldots) : a_i \in \{1, 2, 3\} \text{ for all } i = 1, 2, 3, \ldots\}| = |\mathbb{Z}^+|$.
5. (20 points) Find a countably infinite number of subsets of $\mathbb{Z}^+$, say $A_1, A_2, \ldots \subseteq \mathbb{Z}^+$ such that the following requirements are simultaneously satisfied:
   - $|A_i| = |\mathbb{Z}^+|$ for all $i = 1, 2, \ldots$;
   - $A_i \cap A_j = \emptyset$ for all $i \neq j$;
   - $\cup_{i=1}^{\infty} A_i = \mathbb{Z}^+$.

2. $f: A \to B$:

$$\begin{cases} x \in (0,1) \to f(x) = x+5 \in (6,7). \\ x \in [2,3) \to f(x) = x+6 \in [8,9). \\ x \in (4,5] \to f(x) = \frac{9}{x-4} \in [9,+\infty). \end{cases}$$

f is a bijection.

3. let $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ $B = [0,1)$.

1° def $f = (\cos(2n\pi), \sin(2n\pi))$. $f: B \to A$.

for f, it is one-to-one and onto.

f is bijection

$\Rightarrow |A| = |B|$.

2° proof: $|B| = |[0,1)| = C$ $\left(C \overset{def:}{=} |(0,1)|\right)$

$f_1: B \to C$

$f_1(0) = 2^{-1}$ $f_1(2^{-n}) = 2^{-n-1}$, $n = 1, 2, 3, \ldots$

$f_1(x) = x$ for other x.

f_1 is bijection $\Rightarrow |[0,1)| = |(0,1)|$

3° $|(0,1)| = |\mathbb{R}|$

$f_2: C \to \mathbb{R}$

$f_2(x) = \tan(\pi(x - \frac{1}{2}))$. f_2 is bijection

$\Rightarrow |(0,1)| = |\mathbb{R}|$

$\Rightarrow \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\} = |[0,1)| = |(0,1)| = |\mathbb{R}|$

4. let

$A=\{(a_1,a_2,a_3\ldots): a_1,a_2,a_3\ldots\in Z^+\}$

assume: $|A|=|Z^+|$, then A is countable:

Its element can be arrange as: $A_1, A_2, A_3\ldots$ $(A_i(a_{i1},a_{i2}\ldots))$

but we can create sequence A_0:

$$A_{0i}=\begin{cases}4 & A_{ii}\neq 4\\ 5 & A_{ii}=4\end{cases}$$

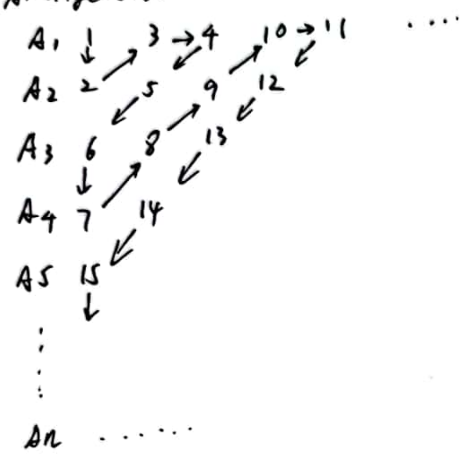
A_0 is differnt from $A_1, A_2, A_3\ldots$

$A_0 \in Z$. $A \notin |A|$

=> contradict. $|\{(a_1, a_2, a_3\ldots): a_1,a_2\ldots\in Z^+\}|\neq|Z^+|$

Arrangement:

5.



Satisfies:

A_i is countably infinite.

$A_i \cap A_j=\phi$ for all $i\neq j$

$\bigcup_{i=1}^{\infty} A_i = Z^+$