Discrete Mathematics: Lecture 26

Paths and Isomorphism, Counting Paths, Euler Paths and Circuits

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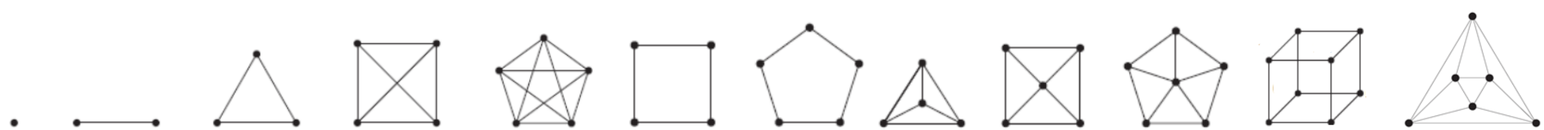
Vertex Connectivity

DEFINITION: A connected undirected graph $G = (V, E)$ is said to be **nonseparable**不可分的 if G has no cut vertex.

DEFINITION: Let $G = (V, E)$ be a connected simple graph.

- **vertex cut**点割集: A subset $V' \subseteq V$ such that $G - V'$ is disconnected
- **vertex connectivity**点连通度 $\kappa(G)$: the minimum number of vertices whose removal disconnect G or results in K_1; equivalently,
  - if G is disconnected, $\kappa(G) = 0$; //additional definition
  - if $G = K_n, \kappa(G) = n - 1$ //K_n has no vertex cut
  - else, $\kappa(G)$ is the minimum size of a vertex cut of G



0 1 2 3 4 2 2 3 3 3 3 4

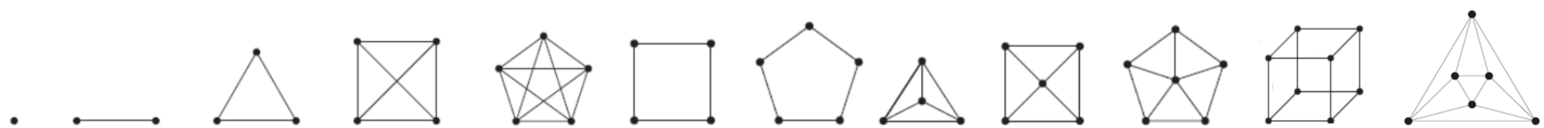
These graphs are all nonseparable

Edge Connectivity

DEFINITION: Let $G=(V,E)$ be a connected simple graph. $E' \subseteq E$ is an **edge cut**边割集 of G if $G-E'$ is disconnected.

DEFINITION: Let $G=(V,E)$ be a simple graph.
The **edge connectivity**边连通度 $(\lambda(G))$ of G is defined as below:

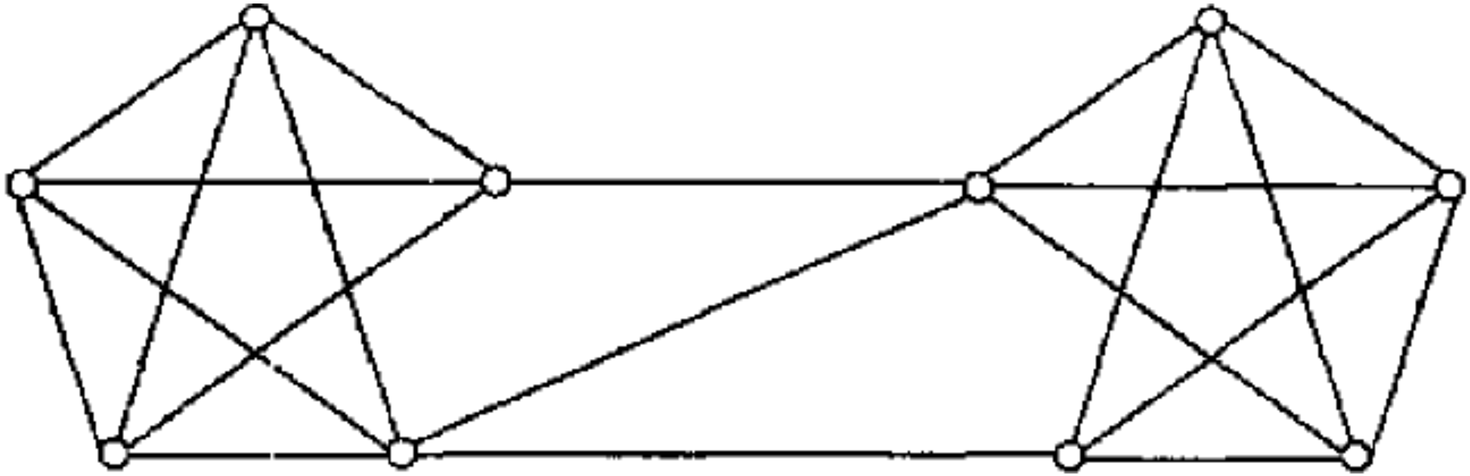
- G disconnected: $\lambda(G)=0$
- G connected:
  - $|V|=1$: $\lambda(G)=0$
  - $|V|>1$: $\lambda(G)$ is the minimum size of edge cuts of G.



0 1 2 3 4 2 2 3 3 3 3 4

Connectivity

THEOREM: Let $G = (V, E)$ be a simple graph. Then $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G) = \min_{v \in V} \deg(v)$ is the least degree of $G's$ vertices.



- $\kappa(G) = 2$
- $\lambda(G) = 3$
- $\delta(G) = 4$

https://cp-algorithms.com/graph/edge_vertex_connectivity.html
http://www.math.caltech.edu/~2014-15/2term/ma006b/05%20connectivity%201.pdf

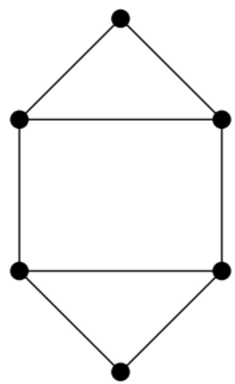
Paths and Isomorphism

(Simple graph)

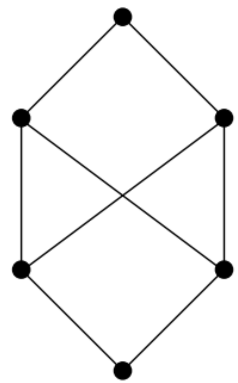
Theorem

The existence of a simple circuit of length k, $k \geq 3$ is an isomorphism invariant for simple graphs.

3 edge circuit



G$_1$



G$_2$

4 edge circuit

→ not iso-

Circuit — iso mo. Var

6 vertices, 8 edges

Degree sequence: 3, 3, 3, 3, 2, 2

Paths and Isomorphism*

Theorem

The existence of a simple circuit of length k, $k \geq 3$ is an isomorphism invariant for simple graphs.

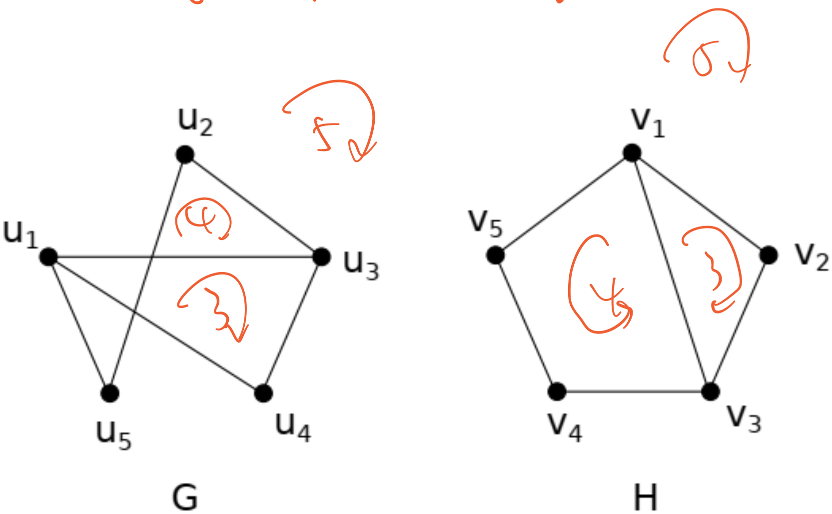
Proof: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isomorphic graphs: there is a bijective function $f : V_1 \to V_2$ respecting adjacency conditions.
Assume G_1 has a simple circuit of length k: $u_0, u_1, \ldots, u_k = u_0$, with $u_i \in V_1$ for $0 \leq i \leq k$. Let's denote $v_i = f(u_i)$, for $0 \leq i \leq k$.

$$(u_i, u_{i+1}) \in E_1 \ \Rightarrow\ (f(u_i), f(u_{i+1})) = (v_i, v_{i+1}) \in E_2, \text{ for } 0 \leq i \leq k-1.$$

So $v_0, \ldots, v_k$ is a path of length k in G_2.
It is a circuit because $v_k = f(u_k) = f(u_0) = v_0$.
It is simple: if not, at least one edge is traversed more than once, so it would mean that there exist $0 \leq i \neq j \leq k-1$ such that $(v_i, v_{i+1}) = (v_j, v_{j+1})$. But this implies $(u_i, u_{i+1}) = (u_j, u_{j+1})$ by bijectivity of f. This is impossible because $u_0, u_1, \ldots, u_k$ is simple.

A way of finding BIJ

BIJ NOT only 1



5 vertices, 6 edges

Degree sequence: 3, 3, 2, 2, 2

1 simple circuit of length 3,

1 simple circuit of length 4,

1 simple circuit of length 5.

Isomorphic graphs ?

If there is an iso $f : V_G \to V_H$, the simple circuit of length 5 u_1, u_4, u_3, u_2, u_5 must be sent to the simple circuit of length 5 in H, respecting the degrees of vertices.

Check that $f(u_1) = v_1, f(u_4) = v_2, f(u_3) = v_3, f(u_2) = v_4, f(u_5) = v_5$ is an isomorphism by writing adjacency matrices.

Counting Paths Between Vertices

Theorem

Let G be a graph with adjacency matrix A with respect to the ordering of vertices $v_1, \ldots, v_n$. The number of different paths of length $r \geq 1$ from v_i to v_j equals the (i, j) entry of the matrix A^r.

A^r

Proof: By induction

- $r = 1$: the number of paths of length 1 from v_i to v_j is equal to the (i, j) entry of A by definition of A, as it corresponds to the number of edges from v_i to v_j.

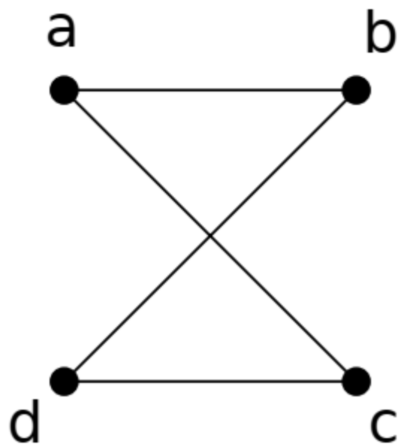
• Assume the (i, j) entry of the matrix A^r is the number of different paths of length r from v_i to v_j.
We can write $A^{r+1} = A^r A$
Let's denote $A^r = (b_{ij})_{1 \leq i,j \leq n}$, and $A = (a_{ij})_{1 \leq i,j \leq n}$. The (i, j) entry of A^{r+1} is given by:

$$\sum_{k=1}^{n} b_{ik} a_{kj} = b_{i1} a_{1j} + b_{i2} a_{2j} + \cdots + b_{in} a_{nj} \tag{1}$$

By hypothesis: b_{ik} equals the number of paths of length r from v_i to v_k.
"Path of length $r+1$ from v_i to v_j = path of length r from v_i to any vertex v_k + an edge from v_k to v_j."
This is equal to the sum (1).

Example

How many paths of length four are there from a to d in the simple graph G



G

with ordering of vertices $(a, b, c, d, \cancel{e})$:

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

r=1 path

r=2 path.

a—b—a

a—c—a.

$$A_G^2 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \quad A_G^3 = \begin{pmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{pmatrix} \quad A_G^4 = \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix}$$

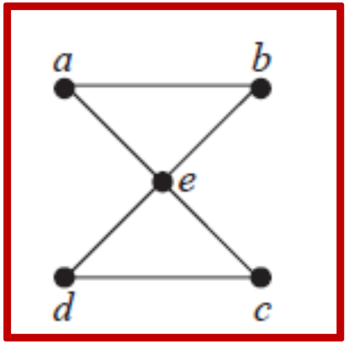
Euler Paths and Circuits

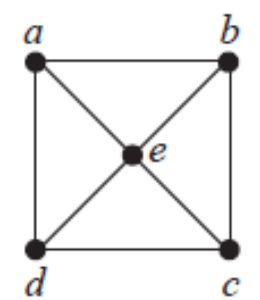
一E一次

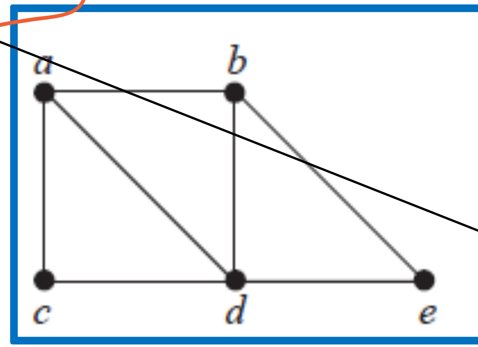
回路

DEFINITION: Let $G = (V, E)$ be a graph.

- **Euler Path**欧拉路径: a simple path that traverses every edge of G.
- **Euler Circuit**欧拉回路: a simple circuit that traverses every edge of G.



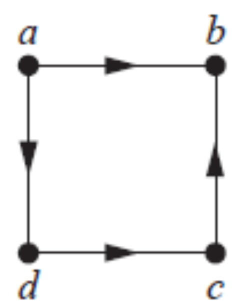


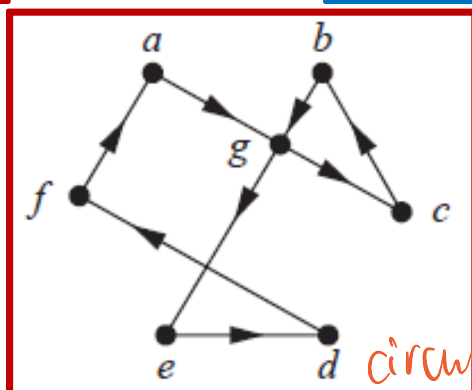


path

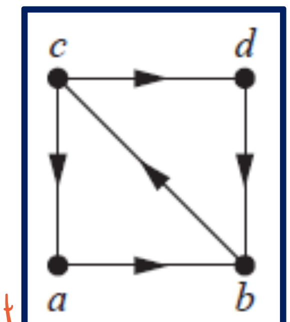
not simple

Remark: When G has multiple edges, these edges will be given different names and considered as different. This is implicit in the textbook.





circuit



Euler Circuits

Circuit

所有结点

THEOREM: Let $G = (V, E)$ be a connected multigraph of order ≥ 2.

Then G has an Euler circuit iff $2 | \deg(x)$ for every $x \in V$.

- $\Rightarrow$: Let P: $\{x_0, x_1\}, \dots, \{x_{i-1}, x_i\}, \dots, \{x_{n-1}, x_n\}$ be an Euler circuit, $x_0 = x_n$
  - Every occurrence of x_i in P contributes 2 to $\deg(x_i)$
    - Every vertex x_i has an even degree
- $\Leftarrow$: Let P: $\{x_0, x_1\}, \dots, \{x_{n-1}, x_n\}$ be a longest simple path in G.
  - Let $H = G[P]$, the subgraph of G induced by all edges in P
    - If $x_n \neq x_0$, then $\deg_H(x_n)$ is odd and so P cannot be longest.
      - $x_n = x_0$, P is a simple circuit, and $2 | \deg_H(x_i)$ for all i.
    - If $\exists i \in \{0,1, \dots, n-1\}$ such that $\deg_H(x_i) < \deg_G(x_i)$,
      - then $\exists y \in V$ such that $\{x_i, y\} \notin P$
        - $y, x_i, x_{i+1}, \dots, x_n, x_1, \dots, x_{i-1}, x_i$ is longer than P
    - Hence, $\deg_H(x_i) = \deg_G(x_i)$ for all $i \in \{0,1, \dots, n-1\}$.
      - $V = \{x_0, x_1, \dots, x_{n-1}\}$ and $H = G$.
        - P is an Euler circuit.

maximum path

$\Downarrow$

circuit

Remark: H contains all vertices of G. Otherwise, P can be extended.

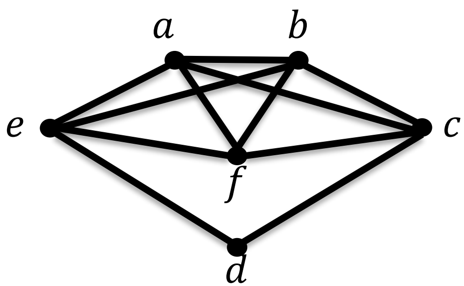
Construction

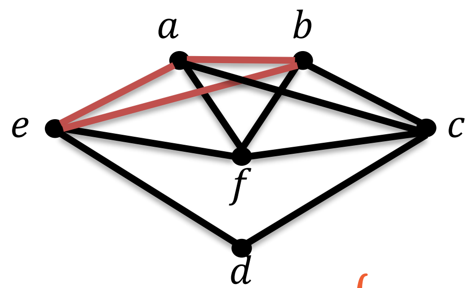
How to find?

ALGORITHM (Hierholzer):

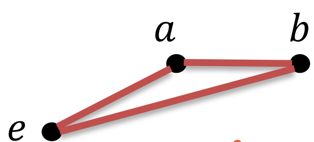
- **Input**: $G = (V, E)$, a connected multigraph, $2 | \deg(x), \forall x \in V$
- **Output**: an Euler circuit
  - **circuit**$:=$ a circuit in G
  - $H := G -$ **circuit** $-$ isolated vertices
  - while H has edges do
    - **subcircuit**$:=$ a circuit in H that intersects **circuit**
    - $H := H -$ **subcircuit** $-$ isolated vertices
    - **circuit**$:=$ **circuit** $\cup$ **subcircuit**
  - return **circuit**

Example

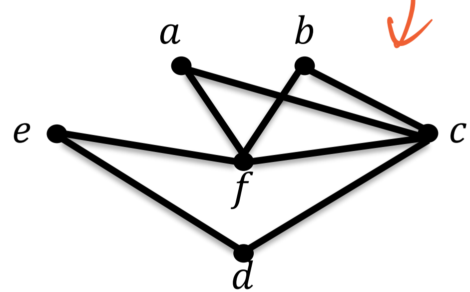




H



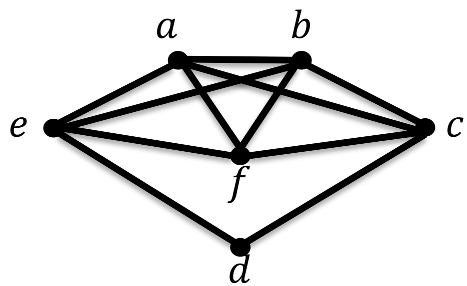
small circuit

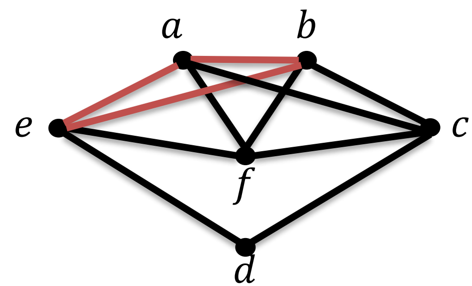


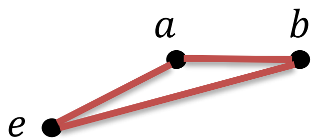
circuit $= a, b, e, a$

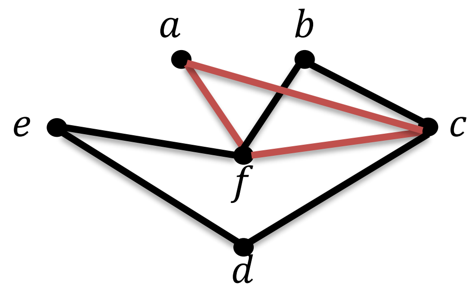
H

Example







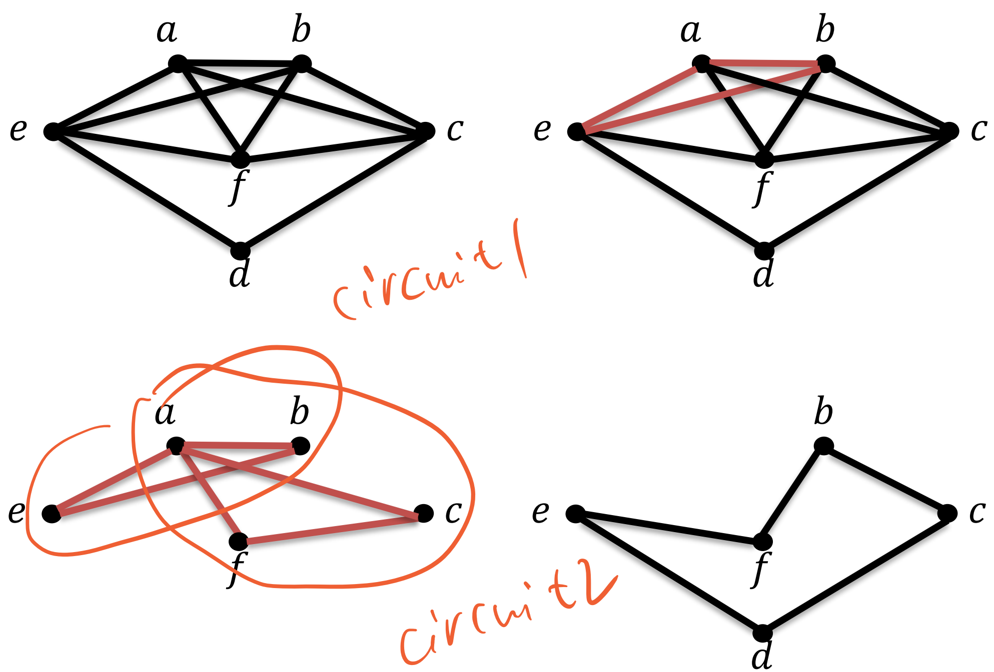


circuit $= a, b, e, a$

H

subcircuit $= a, c, f, a$

Example



circuit 1

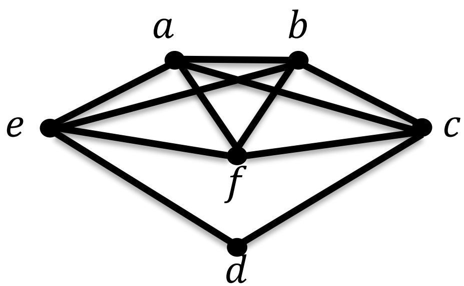
circuit 2

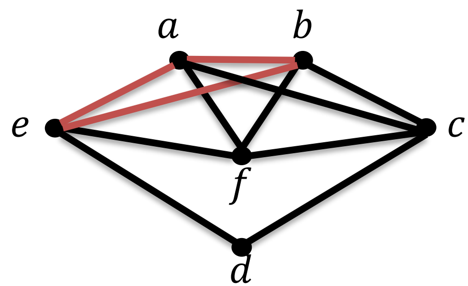
circuit $= a, b, e, a$

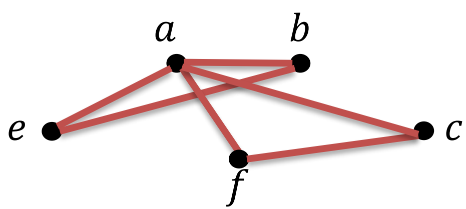
subcircuit $= a, c, f, a$

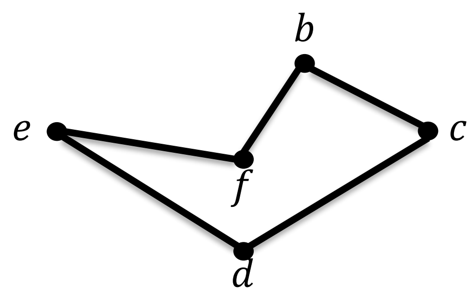
H

Example







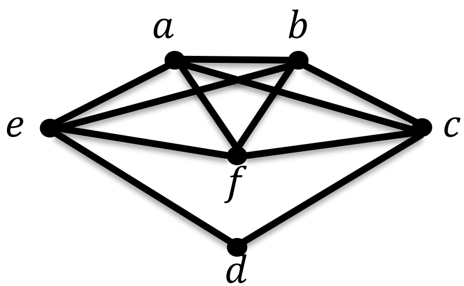


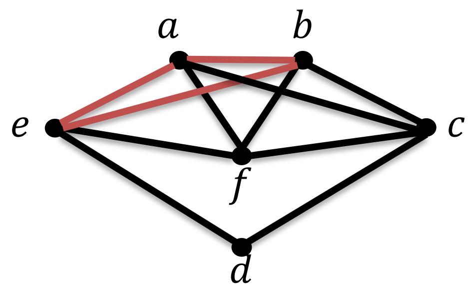
circuit $= a, b, e, a, c, f, a$

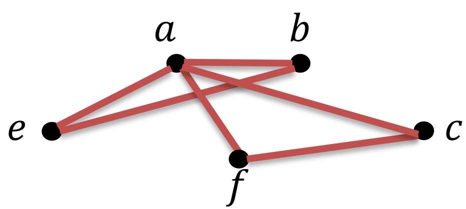
1 2

H

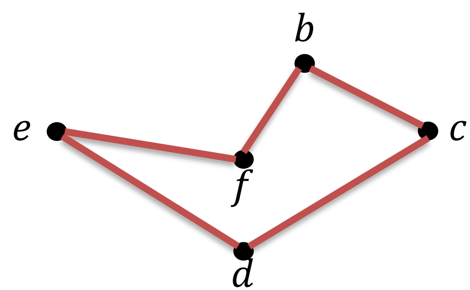
Example







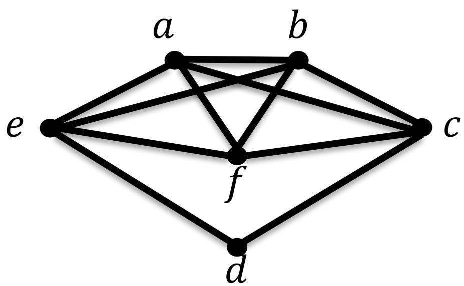
circuit $= a, b, e, a, c, f, a$

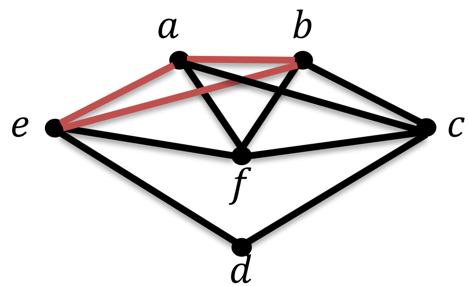


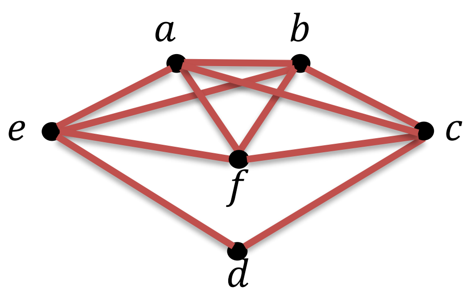
H

subcircuit $= c, d, e, f, b, c$

Example







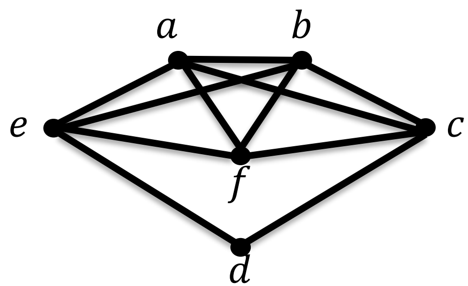
"Insert"

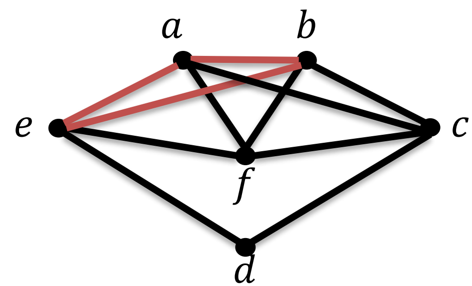
circuit $= a, b, e, a, c, f, a$

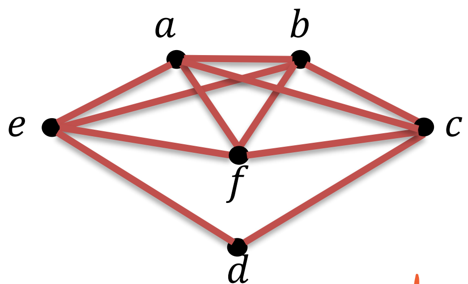
H

subcircuit $= c, d, e, f, b, c$

Example







circuit $= a, b, e, a, c, d, e, f, b, c, f, a$

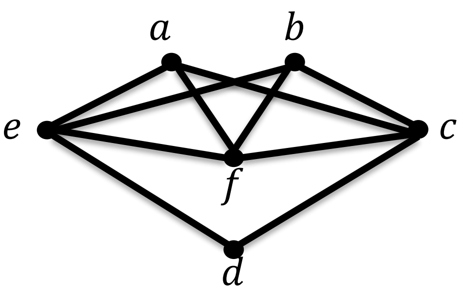
Euler Paths

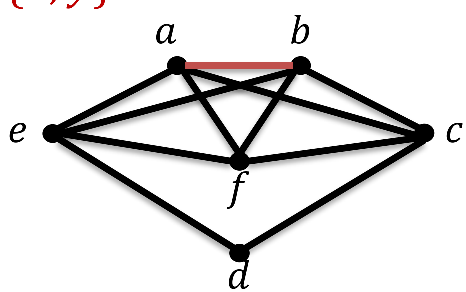
THEOREM: Let $G = (V, E)$ be a connected multigraph of order ≥ 2.
Then G has an Euler path (not Euler circuit) iff G has exactly 2 vertices of odd degree.

ALGORITHM:

- **Input**: $G = (V, E)$, a connected multigraph, $x, y \in V$ have odd degrees
- **Output**: an Euler path
  - $H := G + \{x, y\}$
  - find an Euler circuit using Hierholzer's algorithm
  - remove the edge $\{x, y\}$ from the circuit

circuit - additional edge of 2 odd = path





$a, c, d, e, f, b, a, e, b, c, f, a$

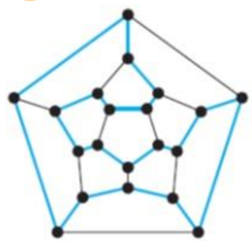
$a, c, d, e, f, b, a, e, b, c, f, a$

$a, e, b, c, f, a, c, d, e, f, b$

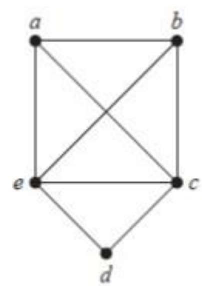
Hamilton Paths and Circuits

DEFINITION: Let $G = (V, E)$ be a graph.

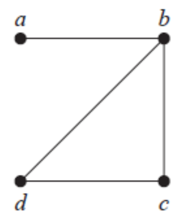
- **Hamilton Path**: A simple path that passes through every vertex exactly once.
- **Hamilton Circuit:** A simple circuit that passes through every vertex exactly once.



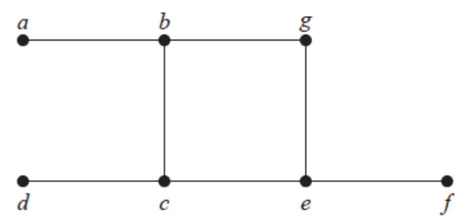
√ Hamilton path
√ Hamilton circuit



√ Hamilton path
√ Hamilton circuit



√ Hamilton path
×Hamilton circuit



× Hamilton path
× Hamilton circuit

Hamilton Circuits

NP-C

Determine if there is a Hamilton circuit in a given graph G?

- This problem is NP-Complete. //that means very difficult

Necessary conditions on Hamilton circuit.

- If G has a vertex of degree 1, then G cannot have a Hamilton circuit.
- If G has a vertex of degree 2, then a Hamilton circuit of G traverses both edges.

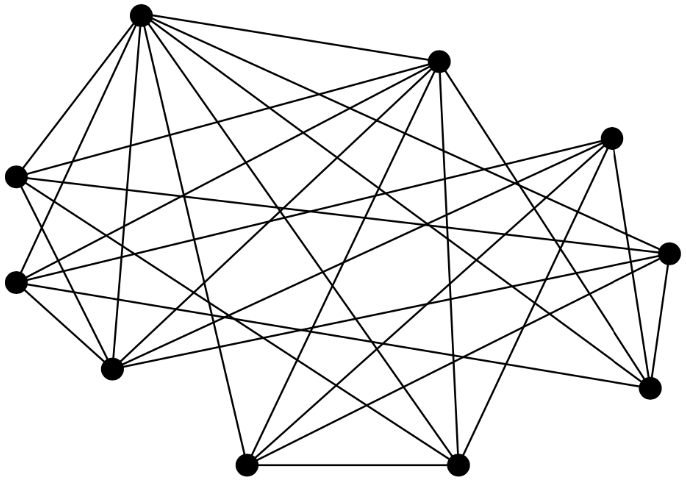
Sufficient conditions on Hamilton circuit. 不直接连的u,v

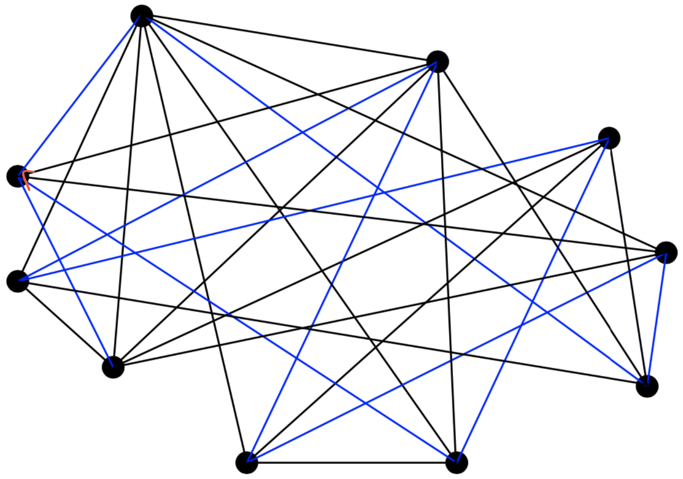
- **Ore's Theorem:** Let $G = (V, E)$ be a simple graph of order $n \geq 3$. If $\deg(u) + \deg(v) \geq n$ for all $\{u, v\} \notin E$, then G has a Hamilton circuit.
- **Dirac's Theorem:** Let $G = (V, E)$ be a simple graph of order $n \geq 3$. If $\deg(u) \geq n/2$ for every $u \in V$, then G has a Hamilton circuit.
  - This is a corollary of Ore's Theorem
    - $\forall u \in V,\ \deg(u) \geq n/2 \Rightarrow \forall u, v \in V, \deg(u) + \deg(v) \geq n$

判定

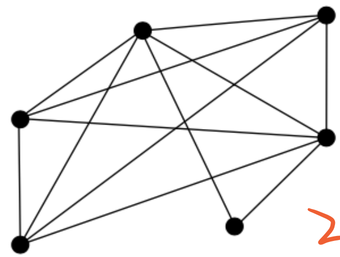
Hamilton Circuits

- Examples (sufficient condition)





Hamilton Circuits



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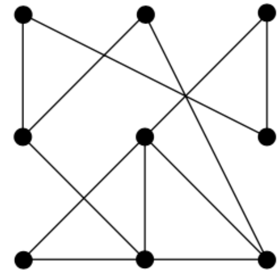
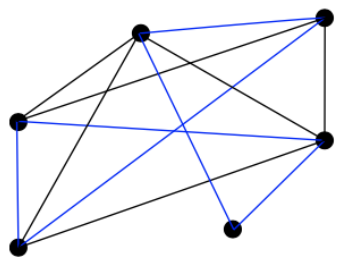
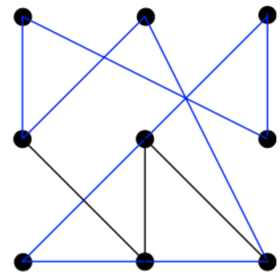






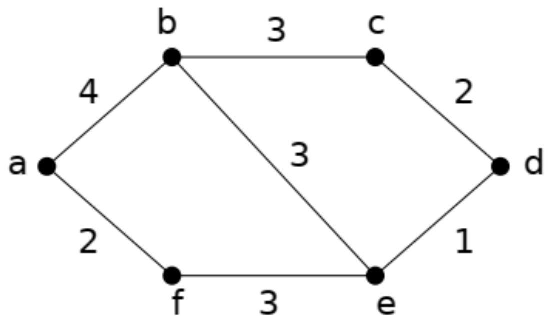
Remark: Dirac's and Ore's Theorems do not give a necessary condition for the existence of a Hamilton circuit!

Dirac, Ore just 充分

Shortest Path Problem

Definition

A **weighted graph** is a graph $G = (V, E)$ such that each edge is assigned with a strictly positive number.
The **length** of a path in weighted graph is the sum of the weights of the edges of this path.



a, b, c is a path of length 7 and b, e, d, c is a path of length 6

Remark: Observe that in a non-weighted graph the length of a path is the number of edges in the path!