Discrete Mathematics: Homework 7

(Deadline: April 15, 2022)

1. (20 points) Show that if $n > 6$, then $p_3(n) = p_3(n-6) + n - 3$.

2. (20 points) Suppose that $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where $k \geq 2$, $e_i \geq 1$ for all $i \in [k]$, and $p_1, p_2, \ldots, p_k$ are k distinct primes. Show that $\phi(n) = n(1 - 1/p_1) \cdots (1 - 1/p_k)$ by using the principle of inclusion-exclusion, where $\phi(n)$ is Euler's Phi function.

   (**Hint**: Calculate the number of integers in $[n] = \{1, 2, \ldots, n\}$ that can be divided by at least one of the primes. Define $A_i = \{x : x \in [n], p_i | x\}$ for all $i \in [k]$ and consider $\cup_{i=1}^{k} A_i$.)

3. (20 points) Let $a \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Show that there exist $p, q \in \mathbb{Z}$ such that $p \in [n]$ and

$$\left| a - \frac{q}{p} \right| < \frac{1}{n}.$$

   (**Hint**: Consider the set $A = \{k \cdot a - \lfloor k \cdot a \rfloor : k = 0, 1, \ldots, n\}$. In fact, by using the Pigeonhole principle, you may prove the stronger result that $|pa - q| < \frac{1}{n}$.)

4. (20 points) Solve $a_n = 8a_{n-2} - 16a_{n-4}$ with $a_0 = 3, a_1 = 6, a_2 = 44$, and $a_3 = 56$.

5. (20 points) Solve $a_n = 3a_{n-1} - 2a_{n-2} + n \cdot 2^n$ with $a_0 = 1$ and $a_1 = -1$.