Discrete Mathematics: Lecture 28

Homeomorphic, Kuratowski's Theorem, Graph Coloring, Tree

Xuming He
Associate Professor

School of Information Science and Technology
ShanghaiTech University

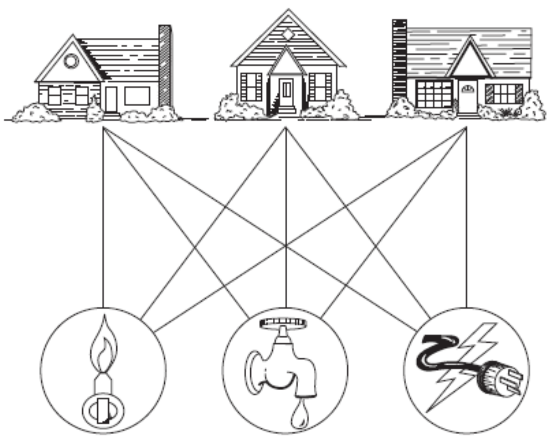
Spring Semester, 2022

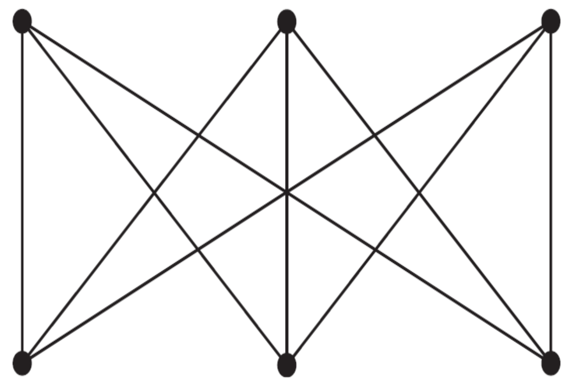
Notes by Prof. Liangfeng Zhang

Planar Graph

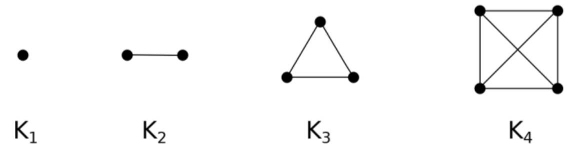
DEFINITION: Let $G = (V, E)$ be an undirected graph. G is called a **planar graph**平面图 if it can be drawn in the plane without any edges crossing.

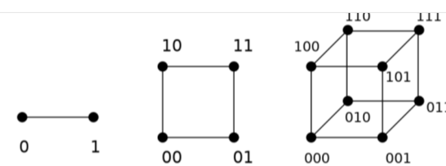
- Crossing of edges: an intersection other than endpoints (vertices)
- **planar representation**平面表示: a drawing w/o edge crossing; **nonplanar**非平面的





- K_1, K_2, K_3, K_4 are planar graphs
- $K_{1,n}, K_{2,n}$ are planar graphs
- $C_n\ (n \geq 3), W_n (n \geq 3)$ are planar graphs
- Q_1, Q_2, Q_3 are planar graphs

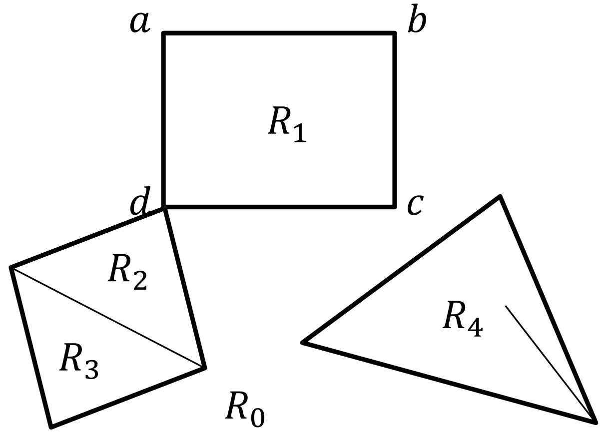




Regions

DEFINITION: Let $G = (V, E)$ be a planar graph. Then the plane is divided into several **regions**面 by the edges of G.

- The infinite region is **exterior region**外部面. The others are **interior regions**内部面.
- The **boundary**边界 of a region is a subset of E.
- The **degree**度数 of a region is the number of edges on its boundary.
  - If an edge is shared by R_i, R_j, then it contributes 1 to $\deg(R_i)$, $\deg(R_j)$
  - If an edge is on the boundary of a single region R_i, then it contributes 2 to $\deg(R_i)$



- The plane is divided into 5 regions R_0, R_1, R_2, R_3, R_4
  - R_0 is the exterior region
  - R_1, R_2, R_3, R_4 are interior regions
- The boundary of R_1; $\deg(R_1) = 4$
- There are 4 edges on the boundary of R_4
  - $\deg(R_4) = 1 + 1 + 1 + 2 = 5$ because one of the edges contribute 2 to $\deg(R_4)$
- $\deg(R_0) = 11, \deg(R_1) = 4, \deg(R_2) = 3, \deg(R_3) = 3, \deg(R_4) = 5$

Euler's Formula

THEOREM: Let $G = (V, E)$ be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

THEOREM: Let G be a planar simple graph with p connected components. Then $|V(G)| - |E(G)| + |R(G)| = p + 1$.

- Let $G_1, G_2, \ldots, G_p$ be the connected components of G.
  - By Euler's formula, $|R(G_i)| = |E(G)_i| - |V(G_i)| + 2$ for all $i \in [p]$
- $|V(G)| = |V(G_1)| + |V(G_2)| + \cdots + |V(G_p)|$
- $|E(G)| = |E(G_1)| + |E(G_2)| + \cdots + |E(G_p)|$
- $|R(G)| = |R(G_1)| + |R(G_2)| + \cdots + \left|R\left(G_p\right)\right| - p + 1$
- $|V(G)| - |E(G)| + |R(G)| = \sum_{i=1}^{p}(|V(G_i)| - |E(G_i)| + |R(G_i)|) - p + 1$
  $= 2p - p + 1 = p + 1$

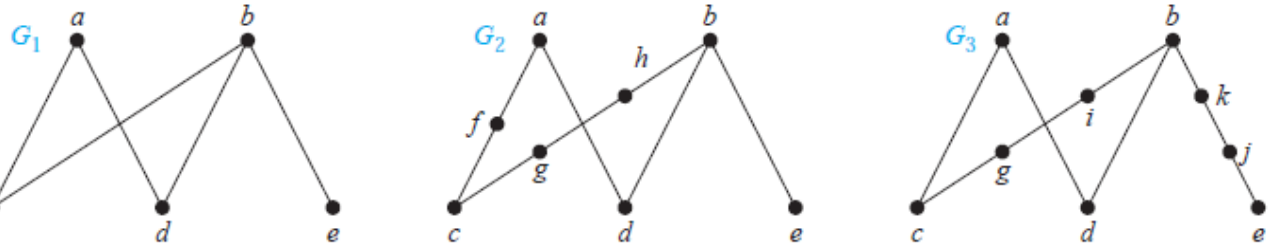
Homeomorphic

DEFINITION: Let $G = (V, E)$ be a graph and $\{u, v\} \in E$.

- **elementary subdivision**初等细分: $G' = (V \cup \{w\}, E - \{u, v\} + \{u, w\} + \{v, w\})$

  旧　新

- Two graphs are **homeomorphic**同胚的 if they can be obtained from the same graph via elementary subdivisions



G_2 and G_3 are homeomorphic

Kuratowski's Theorem

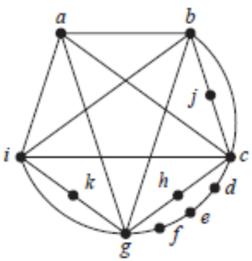
IFF

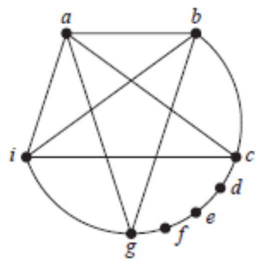
THEOREM: A graph G is nonplanar if and only if it has a subgraph homeomorphic to $K_{3,3}$ or K_5.

degree match!

EXAMPLE: The following graph is nonplanar.

K5





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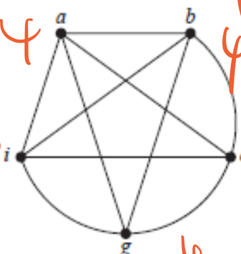
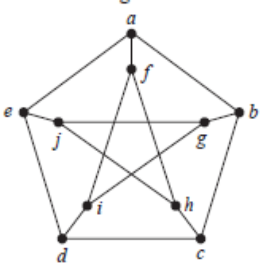
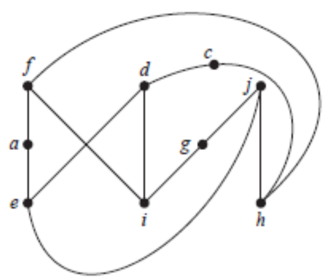


There is a subgraph homoemorphic to K_5





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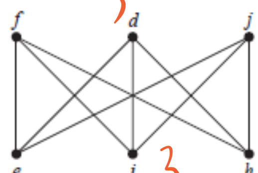
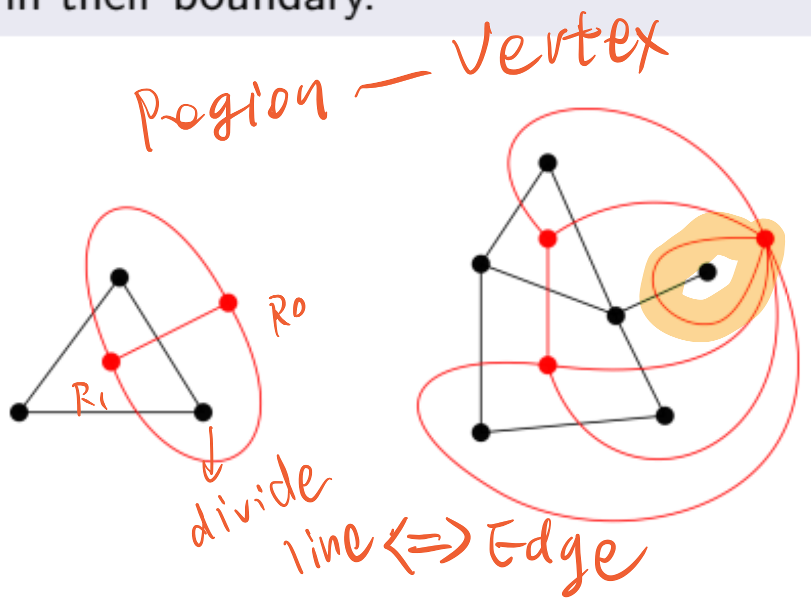


There is a subgraph homoemorphic to $K_{3,3}$

K3,3.

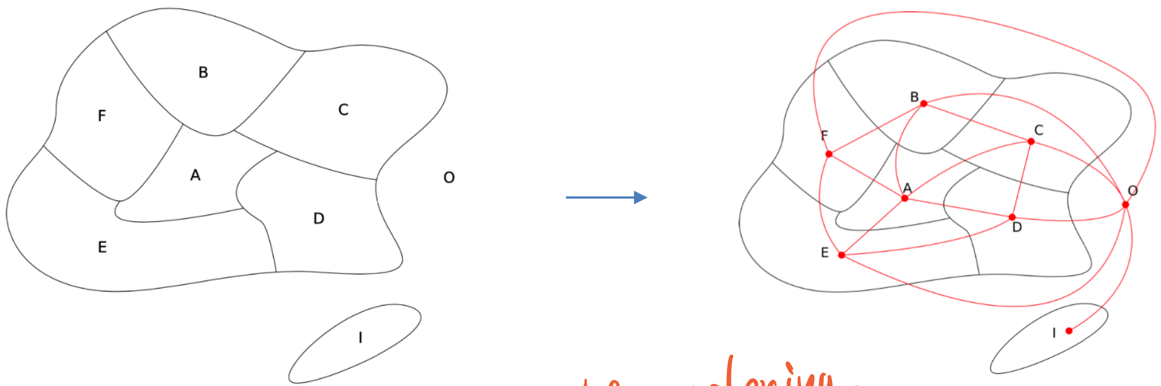
Dual Graph

Let G be a planar graph and assume we take a planar representation of G that we denote also G. The **dual of** G is the graph G^* that has a vertex for each face of G and an edge connecting two vertices if the corresponding faces in G have a common edge in their boundary.

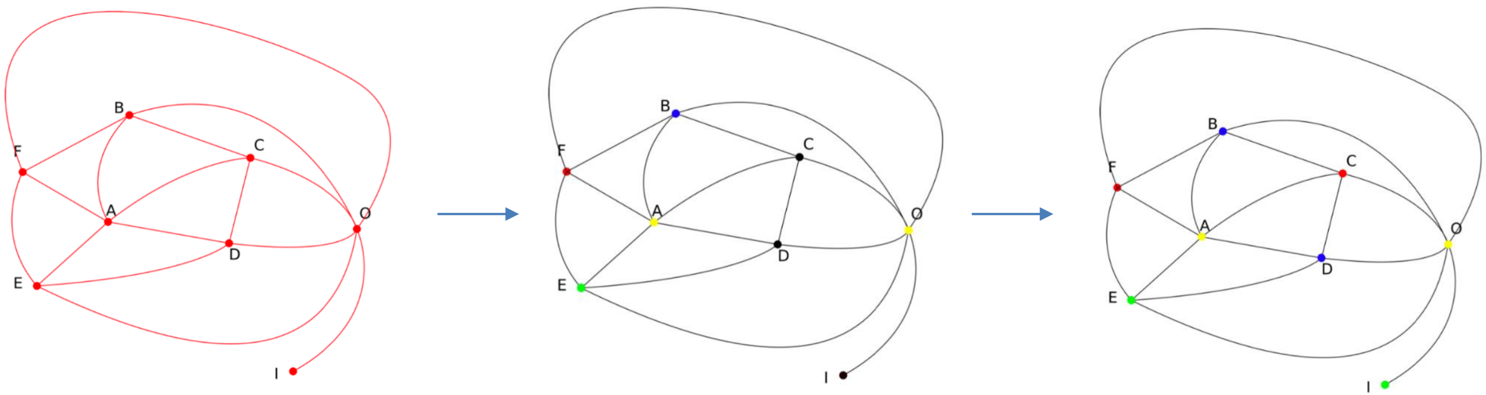


Remark: The dual of a planar simple graph is not necessarily simple.

Coloring a Map



vertex coloring.

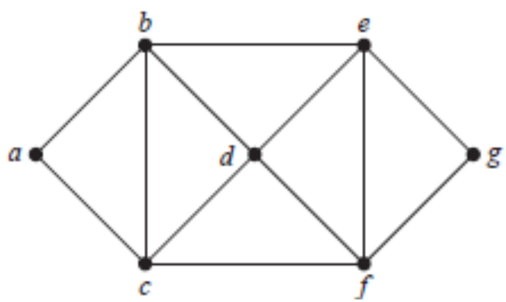


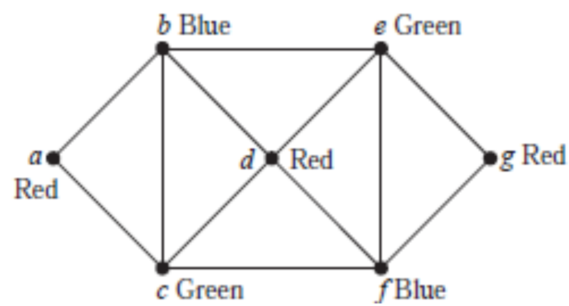
Coloring regions of the map ⇔ Coloring vertices of the dual graph

Graph Coloring

DEFINITION: Let $G = (V, E)$ be a simple graph. A ***k*-coloring** k-着色 of G is a map $f: V \to [k]$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E$.

- **chromatic number** **($\chi(G)$)**色数: the least k s.t. G has a k-coloring.





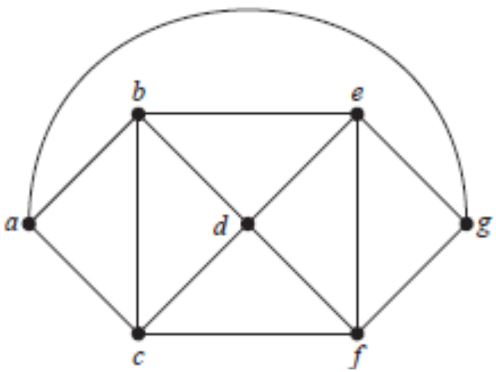
$\chi(G) = 3$

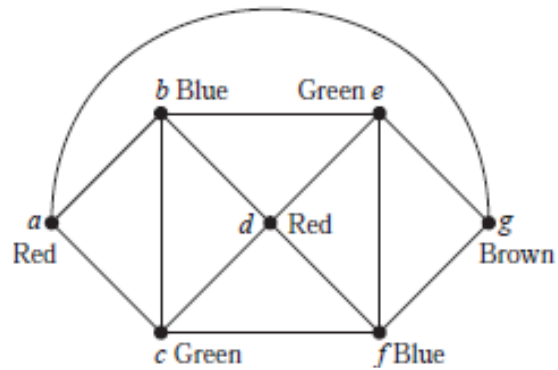
The chromatic number is at least 3 because *a; b; c* is a circuit of length 3

Graph Coloring

DEFINITION: Let $G = (V, E)$ be a simple graph. A ***k*-coloring**$_{k\text{-着色}}$ of G is a map $f: V \to [k]$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E$.

- **chromatic number ($\chi(G)$)**$_{\text{色数}}$**:** the least k s.t. G has a k-coloring.





$\chi(G) = 4$

constraint

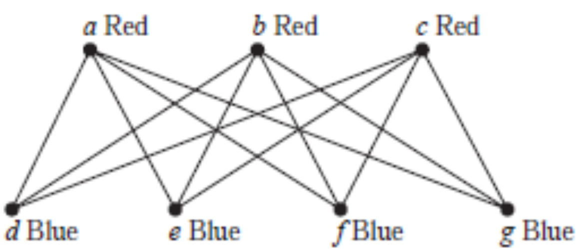
Graph Coloring

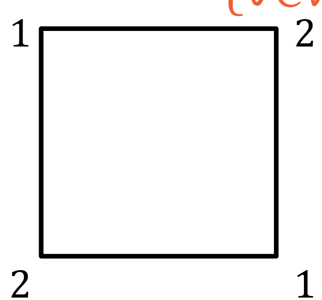
satisfy (CSP)

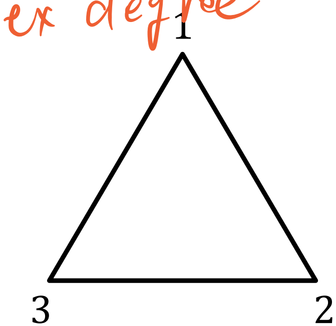
THEOREM: Let $G = (V, E)$ be a simple graph.

- $1 \leq \chi(G) \leq |V|$
- $\chi(G) = 1$ iff $E = \emptyset$
- $\chi(G) = 2$ iff G is bipartite and $|E| \geq 1$.
- $\chi(K_n) = n$ for every integer $n \geq 1$. full connected
  - $\chi(G) \geq n$ if G has a subgraph isomorphic to K_n
- $\chi(C_n) = 2$ if $2|n$; $\chi(C_n) = 3$ if $2|(n-1)$; $(n \geq 3)$ 环
- $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G) = \max\{\deg(v) : v \in V\}$.

(vertex degree







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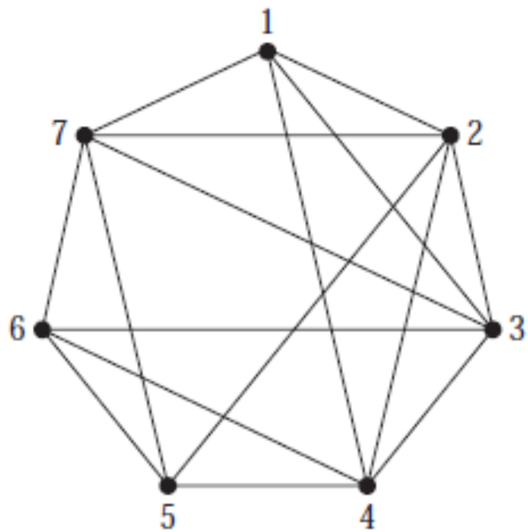
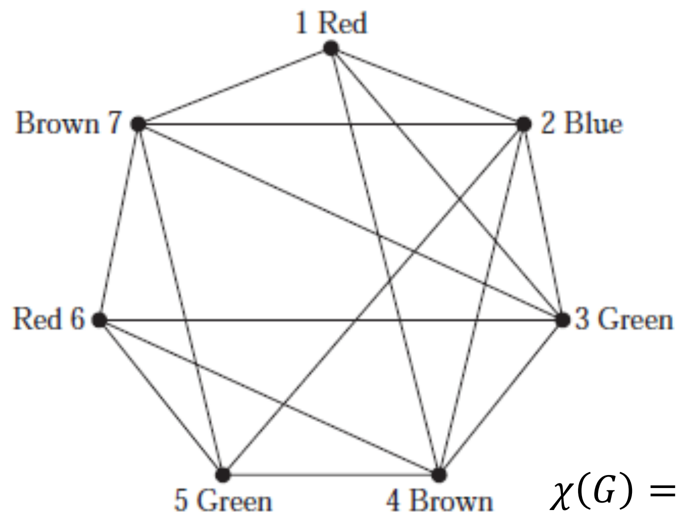
Application

PROBLEM: How can the final exams at a university be scheduled so that no student has two exams at the same time?





$\chi(G) = 4$

- There are 7 different courses, they are vertices of a graph.
- Two courses are adjacent if there is a student registered both courses.
- Choose time slots for the courses such that no two adjacent courses take place at the same time.
  - $\chi(G)$ time slots is needed.

$1 \le \chi(G) \le 7$

$1 \le \chi(G) \le \Delta(G) + 1 = 6$

$\chi(G) \ge 4$: G has a subgraph isomorphic to K_4

$\to$ find χ

4-coloring Theorem

Theorem (Four coloring Theorem)

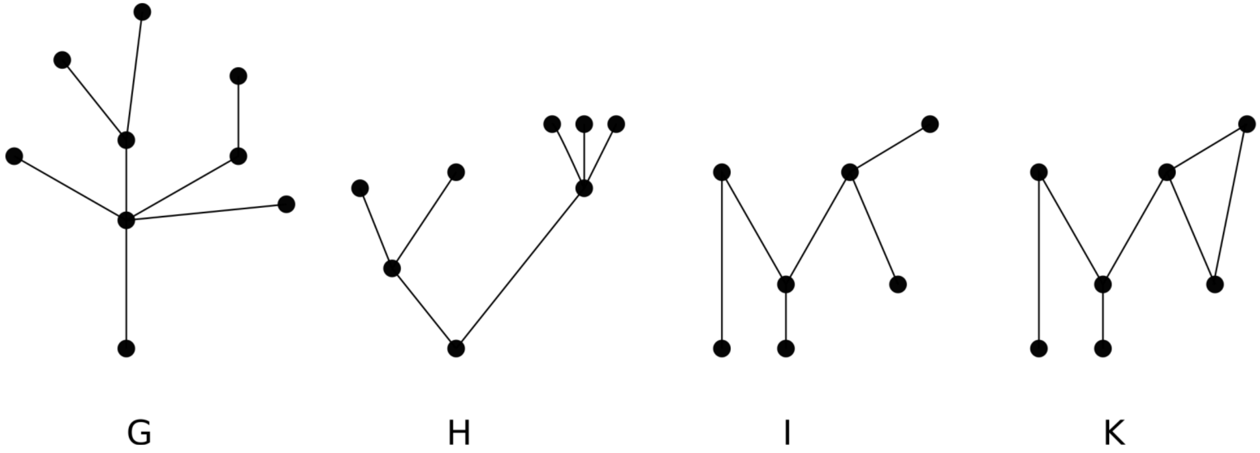
The chromatic number of a simple planar graph is no greater than 4.

Remarks: The proof of the 4-coloring Theorem depends on a computer. The two previous theorems are true for planar graphs only. A non planar graph can have an arbitrarily large chromatic number.

Tree

Definition

- A **tree** is a connected undirected graph with no simple circuits.
- A **forest** is an graph such that each of its connected components is a tree.



G, H, I are trees, but K is not a tree.

Characterization of Tree

Theorem

An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

iff

Proof: ($\Rightarrow$) Assume T is a tree and let u and v be two vertices. T is connected so there is a *simple path* P_1 from u to v. Assume there is a second simple path P_2 from u to v.

Claim: There is a simple circuit in T.
Let $u = x_0, x_1, \ldots, x_n = v$ denote the vertices of P_1 and $u = y_0, y_1, \ldots,$ $y_m = v$ the vertices of P_2.
P_1 and P_2 start at u but are not equal so must diverge at some point.

- If they diverge after one of them has ended, then the remaining part of the other path is a circuit from v to v.

Characterization of Tree

P_1 $\quad x_0, x_1 \dots x_i$ $\quad x_{i+1}$

P_2 $\quad y_{i+1}$

- Otherwise, we can assume

$$x_0 = y_0, x_1 = y_1, \dots, x_i = y_i$$

and $x_{i+1} \neq y_{i+1}$.

We follow then $y_{i+1}, y_{i+2}, \dots$ until we reach a vertex of P_1.
Then go back to x_i following P_1 forwards or backwards.

This gives a circuit which is simple because P_1 and P_2 are, and we stop using edges of P_2 as soon as we hit P_1.

($\Leftarrow$) Assume there is a unique simple path between any two vertices of the graph T. Then:

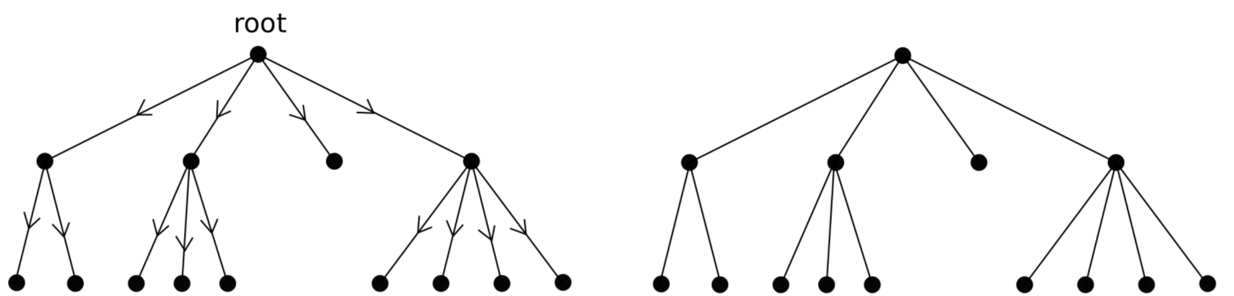
- T is connected (by definition)
- if T has a simple circuit containing the vertices x and y $\rightsquigarrow$ two simple paths between x and y. $\square$

Rooted Tree

Definition

A **rooted tree** is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

Remarks: • A rooted tree is a directed graph.
• We usually draw a rooted tree with its root at the top of the graph.
• We usually omit the arrows on the edges to indicate the direction because it is uniquely determined by the choice of the root.
• Any non rooted tree can be changed to a rooted tree by choosing a vertex for the root.

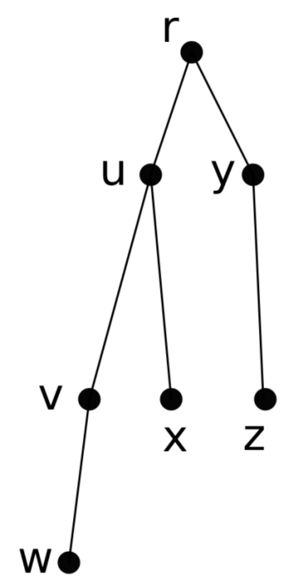


Rooted Tree

Definition

Let T be a rooted tree and v a vertex which is not the root. We call

- **parent** of v the *unique* vertex u such that there is an edge from u to v,
- **child** of v a vertex w such that there is an edge from v to w,
- **siblings** vertices with the same parent,
- **ancestors** of v all vertices in the path from the root to v,
- **descendants** of v all vertices that have v as an ancestor,
- **leaf** a vertex which has no children,
- **internal vertex** a vertex that has children,
- **subtree with** v **at its root** the subgraph of T consisting of v and its descendants and the edges incident to them.



- r is the root
- v is child of u and parent of w
- v and x are siblings

Rooted Tree

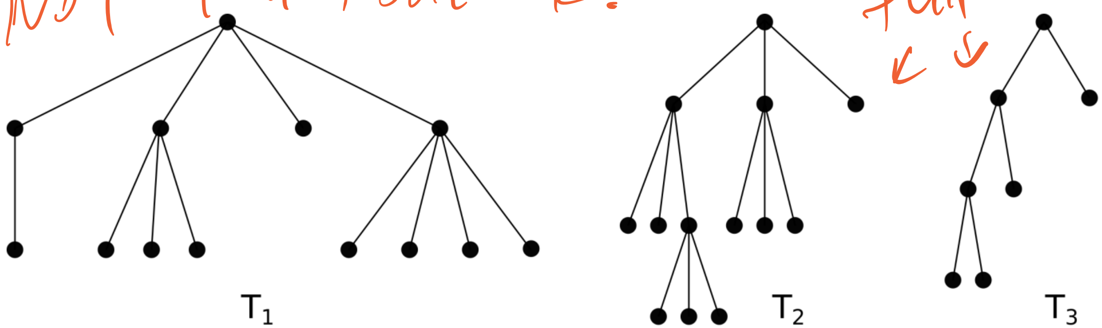
$\leq$!

Definition

- A rooted tree is called an *m*-**ary tree** if every internal vertex has no more than *m* children.
- A rooted tree is called a **full m-ary tree** if every internal vertex has exactly *m* children.
- An *m*-ary tree with $m = 2$ is called a **binary tree**. In this case if an internal vertex has two children, they are called **left child** and **right child**. The subtree rooted at the left (resp. right) child of a vertex is called the **left (resp. right) subtree** of this vertex.

Not full, but ok!

full



T_1 is a 4-ary tree, T_2 a full 3-ary tree, T_3 a full binary tree.

Properties of Tree

Theorem

A tree with n vertices has $n-1$ edges.

Properties of Tree

Theorem

A tree with n vertices has $n-1$ edges.

Proof: By induction on the number of vertices. 数归

- $n = 1$: A tree with one vertex has no edge.

- $k \rightsquigarrow k+1$: Assume every tree with k vertices has $k-1$ edges.

Let T be a tree with $k+1$ vertices, and v a leaf (which exists because the tree has a finite number of vertices).

Let T' be the tree obtained from T by removing v (and the edge incident to it). T' is a connected tree with k vertices $\Rightarrow$ it has $k-1$ edges by induction hypothesis.

$\Rightarrow$ T has $k+1$ vertices and k edges.

Properties of Tree

Tree = connected with no simple circuit (definition)

(1) connected
(2) no simple circuit #
(3) $(n-1)$ edges (n=nb of vertices)

3 pick 2

Previous theorem: (1) + (2) $\Rightarrow$ (3)

We also have: (1) + (3) $\Rightarrow$ (2)
(2) + (3) $\Rightarrow$ (1)

Example: For what value of m, n the complete bipartite graph $K_{m,n}$ is a tree?

$K_{m,n}$ is connected, has $m+n$ vertices and $m \times n$ edges.
It is a tree if:

$$m \times n = m + n - 1 \iff (n-1)m = n - 1$$

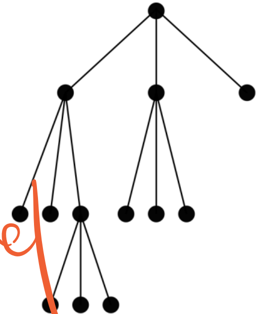
If $n \neq 1$: $m = 1$
If $n = 1$: $m \in \mathbb{N}^*$

Properties of Tree

Theorem

A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.

Proof: Each vertex (except the root) is the child of an internal vertex.
There are i internal vertices, each with m children
$\Rightarrow mi$ vertices + root $= mi + 1$ vertices



Compute internal

A full m-ary tree with

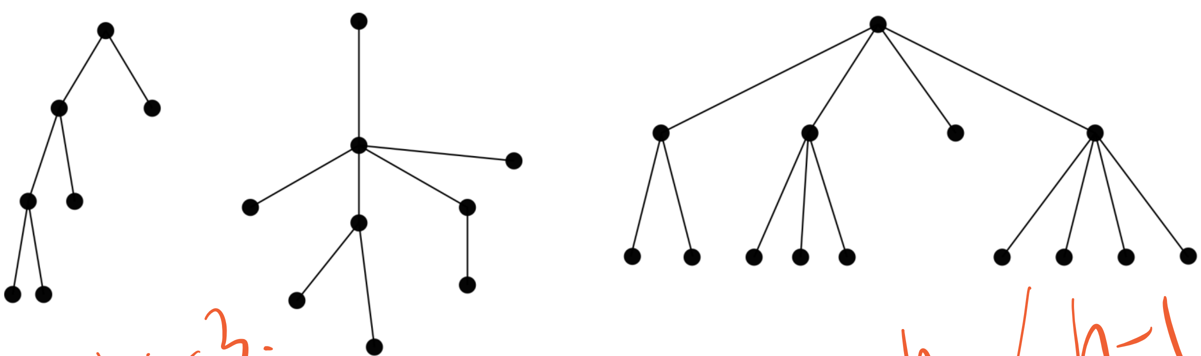
1. *n vertices has $i = (n-1)/m$ internal vertices and $\ell = \big((m-1)n + 1\big)/m$ leaves,*
2. *i internal vertices has $n = mi + 1$ vertices and $\ell = (m-1)i + 1$ leaves,*
3. *ℓ leaves has $n = (m\ell - 1)/(m-1)$ vertices and $i = (\ell - 1)/(m-1)$ internal vertices.*

full binary: n vertex. $\frac{\quad}{2}$ internal $\frac{n+1}{2}$ leaves.

Balanced m-ary Tree

Definition

- The **level** of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.
- The **height** of a rooted tree is the maximum of the levels of its vertices.
- A rooted m-ary tree of height h is **balanced** if all leaves are at levels h or $h-1$.



height = 3.

leave h / h-1

Balanced m-ary Tree

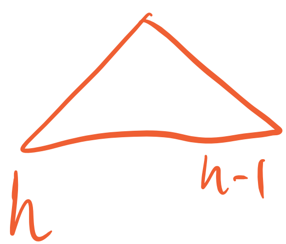
Theorem

There are at most m^h leaves in an m-ary tree of height h.

Proof: Induction again!

Corollary

If an m-ary tree of height h has l leaves, then $h \geq \lceil \log_m l \rceil$. If moreover the m-ary tree is full and balanced, then $h = \lceil \log_m l \rceil$.



also full and balance.

Balanced m-ary Tree*

Theorem

There are at most m^h leaves in an m-ary tree of height h.

Proof: Induction again!

- An m-ary tree of height 1 consists of a root and its children (at most m) that are leaves. So the tree has at most $m^1 = m$ leaves.
- Assume all m-ary tree of height less or equal to h have at most m^h leaves.

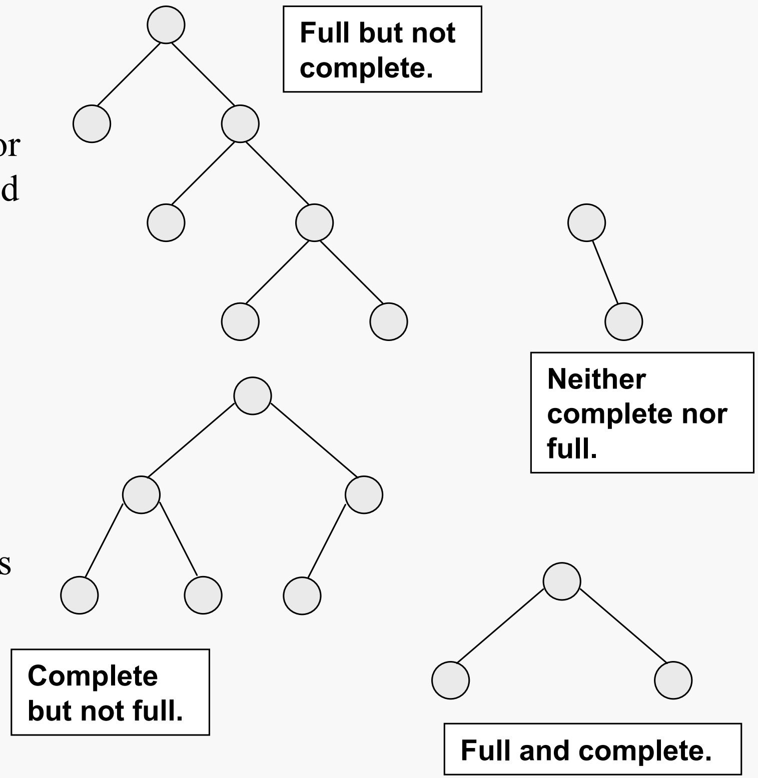
Let T be an m-ary tree of height $h + 1$ and denote r its root.
Consider the subtrees rooted at the children of r. Each of them is an m-ary tree of height less or equal to h, so by inductive hypothesis they have at most m^h leaves.
There are at most m of such trees because r has at most m children.
So in total T has at most $m \times m^h$ leaves.

Full and Complete Binary Trees

Here are two important types of binary trees. Note that the definitions, while similar, are logically independent.

<u>Definition</u>: a binary tree T is *full* if each node is either a leaf or possesses exactly two child nodes.

<u>Definition</u>: a binary tree T with n levels is *complete* if all levels except possibly the last are completely full, and the last level has all its nodes to the left side.



Full Binary Tree Theorem

<u>Theorem</u>: Let T be a nonempty, full binary tree Then:

(a) If T has I internal nodes, the number of leaves is $L = I + 1$.

(b) If T has I internal nodes, the total number of nodes is $N = 2I + 1$.

(c) If T has a total of N nodes, the number of internal nodes is $I = (N - 1)/2$.

(d) If T has a total of N nodes, the number of leaves is $L = (N + 1)/2$.

(e) If T has L leaves, the total number of nodes is $N = 2L - 1$.

(f) If T has L leaves, the number of internal nodes is $I = L - 1$.

Basically, this theorem says that the number of nodes N, the number of leaves L, and the number of internal nodes I are related in such a way that if you know any one of them, you can determine the other two.

Proof of Full Binary Tree Theorem

proof of (a): We will use induction on the number of internal nodes, I. Let S be the set of all integers $I \geq 0$ such that if T is a full binary tree with I internal nodes then T has $I + 1$ leaf nodes.

For the base case, if $I = 0$ then the tree must consist only of a root node, having no children because the tree is full. Hence there is 1 leaf node, and so $0 \in S$.

Now suppose that for some integer $K \geq 0$, every I from 0 through K is in S. That is, if T is a nonempty binary tree with I internal nodes, where $0 \leq I \leq K$, then T has $I + 1$ leaf nodes.

Let T be a full binary tree with $K + 1$ internal nodes. Then the root of T has two subtrees L and R; suppose L and R have I_L and I_R internal nodes, respectively. Note that neither L nor R can be empty, and that every internal node in L and R must have been an internal node in T, and T had one additional internal node (the root), and so $K + 1 = I_L + I_R + 1$.

Now, by the induction hypothesis, L must have I_L+1 leaves and R must have I_R+1 leaves. Since every leaf in T must also be a leaf in either L or R, T must have $I_L + I_R + 2$ leaves.

Therefore, doing a tiny amount of algebra, T must have $K + 2$ leaf nodes and so $K + 1 \in S$. Hence by Mathematical Induction, $S = [0, \infty)$.

QED

Limit on the Number of Leaves

Theorem: Let T be a binary tree with λ levels. Then the number of leaves is at most $2^{\lambda-1}$.

proof: We will use strong induction on the number of levels, λ. Let S be the set of all integers $\lambda \geq 1$ such that if T is a binary tree with λ levels then T has at most $2^{\lambda-1}$ leaf nodes.

For the base case, if $\lambda = 1$ then the tree must have one node (the root) and it must have no child nodes. Hence there is 1 leaf node (which is $2^{\lambda-1}$ if $\lambda = 1$), and so $1 \in S$.

Now suppose that for some integer $K \geq 1$, all the integers 1 through K are in S. That is, whenever a binary tree has M levels with $M \leq K$, it has at most 2^{M-1} leaf nodes.

Let T be a binary tree with K + 1 levels. If T has the maximum number of leaves, T consists of a root node and two nonempty subtrees, say S_1 and S_2. Let S_1 and S_2 have M_1 and M_2 levels, respectively. Since M_1 and M_2 are between 1 and K, each is in S by the inductive assumption. Hence, the number of leaf nodes in S_1 and S_2 are at most 2^{K-1} and 2^{K-1}, respectively. Since all the leaves of T must be leaves of S_1 or of S_2, the number of leaves in T is at most $2^{K-1} + 2^{K-1}$ which is 2^K. Therefore, K + 1 is in S.

Hence by Mathematical Induction, $S = [1, \infty)$.

QED

More Useful Facts

Theorem: Let T be a binary tree. For every $k \geq 0$, there are no more than 2^k nodes in level k.

Theorem: Let T be a binary tree with λ levels. Then T has no more than $2^\lambda - 1$ nodes.

Theorem: Let T be a binary tree with N nodes. Then the number of levels is at least $\lceil \log (N + 1) \rceil$.

Theorem: Let T be a binary tree with L leaves. Then the number of levels is at least $\lceil \log L \rceil + 1$.

