Discrete Mathematics

number of T-Routes, parenthesization

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Summary of Lecture 10

infinite　　　　　　　　　　　　　　　　　　　　　　uncountable

Countable: A is **countable** if $|A| < \infty$ or $|A| = |\mathbb{Z}^+|$　　$|A|=|B|$

- A is countably infinite $\Leftrightarrow A = \{a_1, a_2, \dots\}$
- A is countably infinite $\Rightarrow$ so is any infinite subset of A　　$f: A \to B$
- A is uncountable $\Rightarrow$ so is any super set of A　　bijection
- A, B are countably infinite $\Rightarrow$ so are $A \cup B$ and $A \times B$

Schröder-Bernstein: $|A| \le |B|$ and $|B| \le |A| \Rightarrow |A| = |B|$

　　∃ injection　　⇒ injection　　⇒ bijection

- $|\mathcal{P}(\mathbb{Z}^+)| = |[0,1)|$　　$2^{\aleph_0} = c$
- Cantor　$\aleph_0$　$|\mathbb{Z}^+| < |\mathcal{P}(\mathbb{Z}^+)| = |[0,1)| = |(0,1)| = |\mathbb{R}|$　　-- 推广　$\aleph_0 < c \cancel{<} 2^{\aleph_0} = c$

Basic Rules of Counting: Sum, Product, Bijection

　　$A = A_1 + A_2 + A_3$　　$A_{all} = A_{s1} \cdot A_{s2} \cdot A_{s3}$　　finite

Permutation of Set: r-permutation (w/o repetition)

Permutation of Multiset: r-permutation　　$|A| = |A_1| \times |A_2| \cdots$

- $A = \{n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k\}$ has $\frac{(n_1+n_2+\cdots+n_k)!}{n_1! n_2! \cdots n_k!}$ permutations.　　(1: number

{ r permution
　r permutation with rep

$\{2a, 3b, 1c\}$

$r \in N^+$

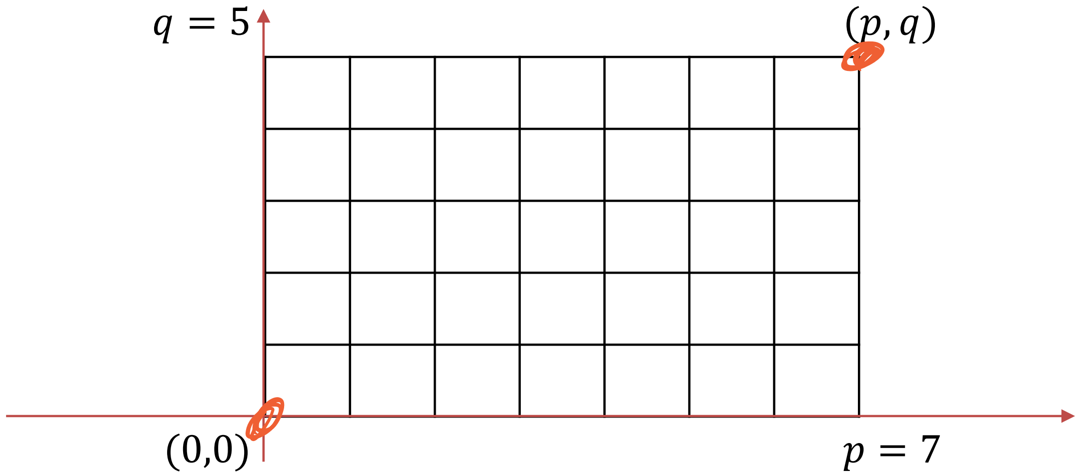
r子集：元素数为r

多重集r排列：

r子集的全排　　全排

Shortest Path

DEFINITION: A $\boldsymbol{p \times q}$**-grid** is a collection of pq squares of side length 1, organized as a rectangle of side length p and q.



全排列

THEOREM: # of shortest paths from (0,0) to (p, q) is $\dfrac{(p+q)!}{p!q!}$.

- Let $A = \{p \cdot \rightarrow, q \cdot \uparrow\}$ be a $(p+q)$-multiset.
- # of shortest paths=# of permutations of A.

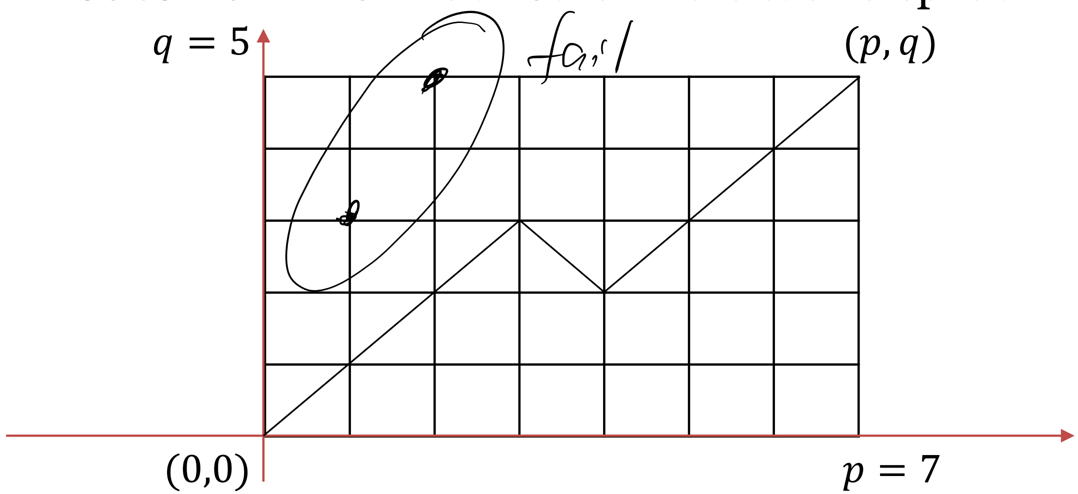
T-Route

type 2

type 1

DEFINITION: Let $A = (x, y), B \in \mathbb{Z}^2$. //**integral points**整点

- A **T-Step** at A is a segment from A to $(x+1, y+1)$ or $(x+1, y-1)$.
- A **T-Route** from A to B is a route where each step is a T-step.



T-Route

THEOREM: There is a T-route from $A = (a, \alpha)$ to $B = (b, \beta)$ only if (1) $b > a$; (2) $b - a \geq |\beta - \alpha|$; and (3) $2 | (b + \beta - a - \alpha)$.

- Let $A = P_0, P_1, \ldots, P_k = B$ be a T-route from A to B, where $P_i = (x_i, y_i)$.
  - $x_0 = a, y_0 = \alpha; x_k = b, y_k = \beta$;
  - $x_i - x_{i-1} = 1; y_i - y_{i-1} \in \{\pm 1\}$ for every $i = 1,2, \ldots, k$
- $b - a = x_k - x_0 = (x_k - x_{k-1}) + (x_{k-1} - x_{k-2}) + \cdots + (x_1 - x_0) = k > 0$
- $\beta - \alpha = y_k - y_0 = (y_k - y_{k-1}) + (y_{k-1} - y_{k-2}) + \cdots + (y_1 - y_0)$
  - $|\beta - \alpha| \leq |y_k - y_{k-1}| + |y_{k-1} - y_{k-2}| + \cdots + |y_1 - y_0| = k = b - a$
- $b + \beta - a - \alpha = \sum_{i=1}^{k} (y_i - y_{i-1} + x_i - x_{i-1})$
  - $y_i - y_{i-1} + x_i - x_{i-1} \in \{0,2\}$
  - $2 | (b + \beta - a - \alpha)$

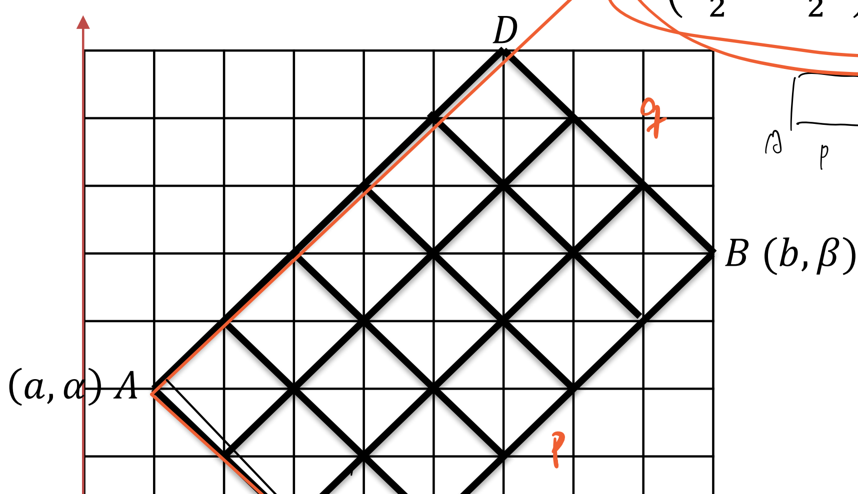
充要-

REMARK: The T-condition (1)+(2)+(3) is also sufficient for the existence of a T-route.

Number of T-Routes

THEOREM: If $A = (a, \alpha), B = (b, \beta)$ satisfy the T-condition. Then the number of T-routes from A to B is $\dfrac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta-\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta-\alpha}{2}\right)!}$.

$\frac{(p+q)!}{p!\,q!}$

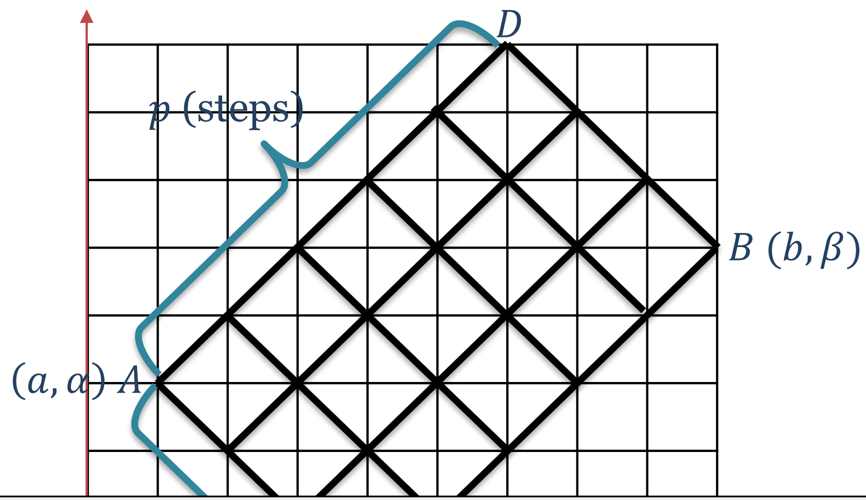


The number of T routes from A to B = the number of shortest paths from A to B on the $p \times q$-grid.

解析直线.

- $AC: y-\alpha=-(x-a)$; $AD: y-\alpha=x-a$;
- $BC: y-\beta=x-b$; $BD: y-\beta=-(x-b)$.
- $p=\frac{1}{2}\cdot(a+b-\alpha+\beta)-a=\frac{1}{2}\cdot(b-a)+\frac{1}{2}\cdot(\beta-\alpha)$
- $q=\frac{1}{2}\cdot(\alpha-\beta+a+b)-a=\frac{1}{2}\cdot(b-a)-\frac{1}{2}\cdot(\beta-\alpha)$

$1/2\cdot(a+b-\alpha+\beta,\alpha+\beta-a+b)$



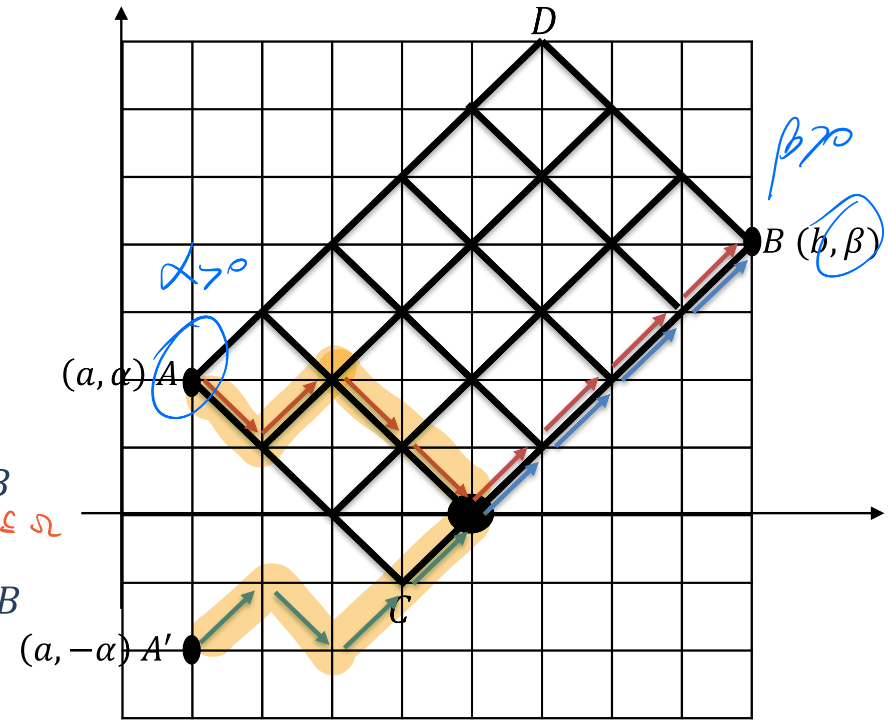
The number of T routes from A to B= the number of shortest paths from A to B on the $p\times q$-grid. This number is $\frac{(p+q)!}{p!q!}=\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta-\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta-\alpha}{2}\right)!}$

Number of T-Routes

THEOREM: Let $A=(a,\alpha), B=(b,\beta)$ satisfy the T-condition, where $\alpha,\beta>0$. Then # of T-routes from A to B that intersect the x-axis=# of T routes from $A'(a,-\alpha)$ to B. And this number is $\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta+\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta+\alpha}{2}\right)!}$.

- Ω: the set of T-routes from A to B
- 相交的 $U=\{\omega\in\Omega\colon \omega$ intersects y=0$\}$ $U\subseteq\Omega$
- V: the set of T-routes from A' to B
- $f\colon U\to V\quad u\mapsto f(u)$
  - u: the brown T route
  - $f(u)$: the blue T route
  - f is a bijection

bijection



$$|U|=|V|=\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta+\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta+\alpha}{2}\right)!}$$

Number of T Routes

THEOREM: Let $A = (a, \alpha), B = (b, \beta) \in \mathbb{Z}^2$ satisfy the T-condition, where $\alpha, \beta > 0$. Then # of T routes from A to B that do not intersect the x-axis is

$A'(a, -\alpha)$

$$\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta-\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta-\alpha}{2}\right)!} - \frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta+\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta+\alpha}{2}\right)!}$$

T-route usage

Parenthesization : ( )

PROBLEM: Let $a_1, a_2, \ldots, a_n, a_{n+1}$ be $n + 1$ numbers. Let $*$ be any binary operator. Let C_n be the number of different ways of parenthesizing $a_1 * a_2 * \cdots * a_n * a_{n+1}$ such that the calculation is not ambiguous. What is C_n?

bijection

C_3.

- $n = 3$: there are 5 different ways of parenthesizing the expression

A3

中缀	后缀		
$((a_1 * a_2) * a_3) * a_4$	a_1 ↲ a_2 ↲$*$ a_3 ↲$*$ a_4 ↲$*$	↲↲$*$↲$*$↲$*$	0010101
$(a_1 * a_2) * (a_3 * a_4)$	a_1 ↲ a_2 ↲$*$ a_3 ↲ a_4 ↲$**$	↲↲$*$↲↲$**$	0010011
$(a_1 * (a_2 * a_3)) * a_4$	a_1 ↲ a_2 ↲ a_3 ↲$**$ a_4 ↲$*$	↲↲↲$**$↲$*$	0001101
$a_1 * ((a_2 * a_3) * a_4)$	a_1 ↲ a_2 ↲ a_3 ↲$*$ a_4 ↲$**$	↲↲↲$*$↲$**$	0001011
$a_1 * (a_2 * (a_3 * a_4))$	a_1 ↲ a_2 ↲ a_3 ↲ a_4 ↲$***$	↲↲↲↲$***$	0000111

bij ⇒ bij ⇒

- $\mathcal{A}_3$: the set of all different parenthesizations of $a_1 * a_2 * a_3 * a_4$ bij
- $\mathcal{C}_3$: the set of all $x = x_1x_2x_3x_4x_5x_6x_7 \in \{0,1\}^7$ such that 4 0 3 1
  - There are exactly three 1's in x
  - In any prefix of x, the number of 1's < the number of 0's

0101 前缀：(不是) $\left\{ \begin{array}{l} 0 \\ 01 \\ 001 \\ 0010 \end{array} \right.$

Parenthesization

THEOREM: C_n is the number of solutions of the equation system

$$\begin{cases} x_1 + x_2 + \cdots + x_{2n+1} = n \\ x_1 + x_2 + \cdots + x_i < i/2, i = 1,2,\ldots,2n+1 \\ x_i \in \{0,1\}, i = 1,2,\ldots,2n+1 \end{cases}$$

1的个数少于一半

方程的解

In particular, $C_n = \frac{(2n)!}{n!(n+1)!}$

- $\mathcal{A}_n$: the set of all different parenthesizations of $a_1 * a_2 * \cdots * a_n * a_{n+1}$
- $\mathcal{C}_n$: the set of all $x = x_1x_2\cdots x_{2n+1} \in \{0,1\}^{2n+1}$ such that
    - The number of 1's in x is exactly equal to n
    - In any prefix of x, the number of 1's < the number of 0's
- There is a bijection f: $\mathcal{A}_n \to \mathcal{C}_n$
- $C_n = |\mathcal{A}_n| = |\mathcal{C}_n|$
- $\mathcal{C}_n$ is the set of all solutions of the equation system
- $\mathcal{T}_n$: the set of all T-routes from (1,1) to $(2n+1,1)$ above the x-axis

equation set

$P_2 = (2, 1-2x_1+1-2x_2)$

Parenthesization

- From $\mathcal{C}_n$ to $\mathcal{T}_n$: Given a solution $(x_1, x_2, \ldots, x_{2n+1})$ of the equation system

bijection

proce

  - Let $P_i = (i, 1 - 2x_1 + \cdots + 1 - 2x_i)$ for all $i = 1,2, \ldots, 2n+1$
  - $1 - 2x_1 + \cdots + 1 - 2x_i > 0$ for $i = 1,2, \ldots 2n+1$
  - $P_1 = (1, 1 - 2x_1) = (1,1)$; $P_{2n+1} = (2n+1, 1)$ 相邻2点1步
  - $P_1, P_2, \ldots, P_{2n+1}$ is a T-route above the x-axis

$P_{i-1} = (i-1, 1-2x_1 + \cdots + 1-2x_{i-1})$

$P_i = (i, \ldots + 1-2x_i)$

- From $\mathcal{T}_n$ to $\mathcal{C}_n$: Let $\{P_i = (u_i, v_i): 1 \le i \le 2n+1\}$ be the points on a T-Route from $P_1 = (1,1)$ to $P_{2n+1} = (2n+1, 1)$, where the T-Route is above the x-axis
  - $x_1 = (1 - v_1)/2 = 0$
  - $x_i = (1 - (v_i - v_{i-1}))/2 \in \{0,1\}, i = 2, \ldots, 2n+1$
  - $x_1 + x_2 + \cdots + x_{2n+1} = (2n + 1 - v_{2n+1})/2 = n$
  - $x_1 + x_2 + \cdots + x_i = (i - v_i)/2 < i/2, i = 1,2, \ldots, 2n+1$

$=0 \to 1$

$=1 \to -1$

- $A = P_1 = (1,1)$: $a = 1, \alpha = 1$; $B = P_{2n+1} = (2n+1, 1)$: $b = 2n+1, \beta = 1$
  - $|\mathcal{C}_n| = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} = \frac{(2n)!}{n!(n+1)!}$

T route number

Catalan number