Discrete Mathematics

combinations, inverse binomial transform,
distributing objects into boxes

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Summary of Lecture 11

THEOREM: There is a T-route from $A = (a, \alpha)$ to $B = (b, \beta)$ iff (1) $b > a$; (2) $b - a \geq |\beta - \alpha|$; and (3) $2 \mid (b + \beta - a - \alpha)$.

THEOREM: If $A = (a, \alpha), B = (b, \beta)$ satisfy the T-condition.

- \# of T-routes from A to B is $\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta-\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta-\alpha}{2}\right)!}$

$\frac{(p+q)!}{p!\,q!}$

- $\alpha, \beta > 0$: # of T-Routes intersecting the x-axis is $\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta+\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta+\alpha}{2}\right)!}$

THEOREM: The number of solutions of the equation system

$$\begin{cases} x_1 + x_2 + \cdots + x_{2n+1} = n \\ x_1 + x_2 + \cdots + x_i < i/2, i = 1,2, \dots, 2n+1 \\ x_i \in \{0,1\}, i = 1,2, \dots, 2n+1 \end{cases}$$

is $C_n = \frac{(2n)!}{n!(n+1)!}$

Catalan Number: # of ways of parenthesizing $a_1 * a_2 * \cdots * a_n * a_{n+1}$

Combinations of Sets

$0 \le r \le n$.

DEFINITION: Let $A = \{a_1, \dots, a_n\}$ and let $r \in \{0,1,\dots,n\}$.

- ***r*-combination of *A***: an r-subset of A.
  - Notation: $\{a_{i_1}, \dots, a_{i_r}\}$ with $1 \le i_1 < \cdots < i_r \le n$
  - $\binom{n}{r}$: the number of r-combinations of an n-element set

n 选 r

THEOREM: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for all $n \in \mathbb{Z}^+$ and $r \in \{0,1,\dots,n\}$.

DEFINITION: Let $A = \{a_1, \dots, a_n\}$ and let $r \ge 0$.

无 upper limit

- ***r*-combination of *A* with repetition**: a multiset $\{x_1 \cdot a_1, \dots, x_n \cdot a_n\}$ of r elements, where $x_1, \dots, x_n \ge 0$ are integers and $x_1 + \cdots + x_n = r$.
  - Notation: $\{a_{i_1}, \dots, a_{i_r}\}$ with $1 \le i_1 \le i_2 \le \cdots \le i_r \le n$

THEOREM: The number of r-combinations of an n element set with repetition is $\binom{n+r-1}{r}$

Combinations of Sets

- $\mathcal{U}$: the set of all r-combinations of A with repetition
- $\mathcal{V}$: the set of all r-combinations of $[n+r-1]$ without repetition
  - Let $U = \{u_1, u_2, \ldots, u_r\} \in \mathcal{U}$ and $1 \le u_1 \le u_2 \le \cdots \le u_r \le n$.
    - $1 \le u_1 < u_2 + 1 < u_3 + 2 < \cdots < u_r + r - 1 \le n + r - 1$ (平移)
      - $\{u_1, u_2 + 1, \ldots, u_r + r - 1\} \in \mathcal{V}$
      - $f\colon \mathcal{U} \to \mathcal{V}\ \ \{u_1, u_2, \ldots, u_r\} \mapsto \{u_1, u_2 + 1, \ldots, u_r + r - 1\}$
    - f is bijective. Hence, $|\mathcal{U}| = |\mathcal{V}| = \binom{n+r-1}{r}$

THEOREM: The number of natural number solutions of the equation $x_1 + x_2 + \cdots + x_n = r$ is $\binom{n+r-1}{r}$.

解个数

- $\mathcal{X} = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{N} \text{ and } x_1 + \cdots + x_n = r\}$
- $\mathcal{Y}$: the set of all r-combinations of $[n]$ with repetition
- $f\colon \mathcal{X} \to \mathcal{Y}\quad \underbrace{(x_1, \ldots, x_n)}_{\text{解}} \mapsto \underbrace{\{x_1 \cdot 1, x_2 \cdot 2, \ldots, x_n \cdot n\}}_{r\text{ 多重集}}$
  - f is bijective. Hence, $|\mathcal{X}| = |\mathcal{Y}| = \binom{n+r-1}{r}$. => 多重集个数

Application

n is limiting # of layers. So go through all possible r-combination rep

EXAMPLE: What is the value of k after the program execution?

- $k := 0;$
- for $i_1 := 1$ to n do
  - for $i_2 := 1$ to i_1 do
    - $\vdots$
      - for $i_r := 1$ to i_{r-1} do
        - $k := k + 1;$

r次循环

可重复! $i_r \le i_{r-1}$

Analysis:

- Loop variables: $1 \boxed{\le i_r \le i_{r-1} \le \cdots \le i_1 \le} n$
- The number of iterations is equal to the number of r-combinations of the set $[n]$ with repetition
- In every iteration, k increases by 1.
  - After the program execution, $k = \binom{n+r-1}{r}$

# of 多重集 [n].

Combinations of Multiset

$n_1 + n_2 + \cdots n_k = n$

DEFINITION: Let $A = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$ be an n-multiset. Let $r \in \{0,1,\ldots,n\}$.

- ***r*-combination of *A***: an r-subset (multiset) of A
  - Notation: $\{x_1 \cdot a_1, x_2 \cdot a_2, \ldots, x_k \cdot a_k\}$, where $0 \le x_i \le n_i$ for every $i \in [k]$ and $x_1 + x_2 + \cdots + x_k = r$.

EXAMPLE: $A = \{1 \cdot \mathrm{a}, 2 \cdot \mathrm{b}, 3 \cdot \mathrm{c}\}$

- $\{1 \cdot b, 2 \cdot c\}$ is a 3-combination of A; a 3-subset of A

REMARK:

- For every $r \in \{0,1,\ldots,n\}$, an r-combination of $A = \{a_1, a_2, \ldots, a_n\}$ without repetition is an r-combination of $\{1 \cdot a_1, 1 \cdot a_2, \ldots, 1 \cdot a_n\}$.
- For every $r \ge 0$, an r–combination of $A = \{a_1, a_2, \ldots, a_n\}$ with repetition is an r–combination of $\{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$.

Trans

$$\begin{cases} x_1 a_1, x_2 a_2 \cdots x_n a_n \\ 0 \le x_i \le 1 \\ x_1 + x_2 + \cdots x_n = r \end{cases}$$

$$\begin{cases} x_1 a_1, x_2 a_2 \cdots x_n a_n . \\ x_1 + x_2 + \cdots + x_n = r \\ x_i \geq 0 . \end{cases}$$

二项式反演 $\begin{cases} \text{Binomial trans} \\ \text{Inverse Bin trans} \end{cases}$

Inverse Binomial Transform

DEFINITION: The **binomial transform** of $\{a_n\}_{n\geq s}$ is a sequence $\{b_n\}_{n\geq s}$ such that

$a_s, a_{s+1}, \ldots$

$$b_n = \sum_{k=s}^{n} \binom{n}{k} a_k \qquad (1)$$

DEFINITION: The **inverse binomial transform** of $\{a_n\}_{n\geq s}$ is a sequence $\{b_n\}_{n\geq s}$ such that

$$b_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k \qquad (2)$$

QUESTION: Given (1), how to find the sequence $\{a_n\}$?

- Answer: $\{a_n\}$ is the inverse binomial transform of $\{b_n\}$
- Application: determine $\{a_n\}$ via $\{b_n\}$
- Proof?

t8

$$b_{s+1} = \sum_{k=s}^{s+1} \binom{s+1}{k} \cdot a_k$$

$$= \binom{s+1}{s} \cdot a_s + \binom{s+1}{s+1} \cdot a_{s+1}$$

Method:

某set元素个数

Method:

Combinatorial Proofs

组合数相关恒等式

DEFINITION: A **combinatorial proof** of an identity $L = R$ is

- **a double counting proof,** which shows that L, R count the same set of objects but in different ways:
  - $L = |X| = R$ and L, R count $|X|$ in different ways.

L, R: 用两种不同的方法计算 |X| 中的元素个数

- **a bijective proof,** which shows a bijection between the sets of objects counted by L and R:
  - $L = |X|, R = |Y|$ and there is a bijection $f: X \to Y$.

EXAMPLE: $\binom{n}{r} = \binom{n}{n-r}$

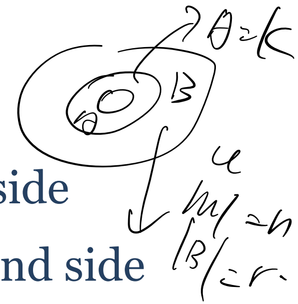
构造 $X = \{s \in \{0,1\}^n : s \text{ contains } r \text{ 0s}\} = \{s \in \{0,1\}^n : s \text{ contains } n - r \text{ 1s}\}$

长度为n的二元字符串中恰好有r个0的串之集合.

- $\binom{n}{r} = |X|$
- $\binom{n}{n-r} = |X|$

Inverse Binomial Transform

LEMMA: $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ for any $n, k, r \in \mathbb{N}$ such that $n \geq k \geq r$.



- Let $U = \{u_1, u_2, \dots, u_n\}$ be a finite set of n elements
- $S = \{(A, B): A \subseteq U, |A| = k, B \subseteq A, |B| = r\}$
  - choose A then choose B: $|S| = \binom{n}{k}\binom{k}{r}$, the left-hand side
  - choose B then choose A: $|S| = \binom{n}{r}\binom{n-r}{k-r}$, the right-hand side

B A

LEMMA: $\sum_{k=r}^{n}(-1)^{n-k}\binom{n}{k}\binom{k}{r} = \begin{cases} 1 & n = r \\ 0 & n > r \end{cases}$ when $n \geq r$.

$i: k: r \to n$
构造 $k-r: 0 \to n-r$

- $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ as $n \geq k \geq r \geq 0$
- **left** $= \sum_{k=r}^{n}(-1)^{n-k}\binom{n}{r}\binom{n-r}{k-r} = \binom{n}{r}\sum_{k=r}^{n}(-1)^{(n-r)-(k-r)}\binom{n-r}{k-r}$

  对k

  $= \binom{n}{r}\sum_{i=0}^{n-r}(-1)^{(n-r)-i}\binom{n-r}{i}$ 视为整体.

  $=$ **right** $\binom{n}{r}$ $(1-1)^{n-r}$

  $(a+b)^n = \sum_0^n C_n^r a^r b^{n-r}$

  二项式展开

Inverse Binomial Transform

交换求和顺序

LEMMA: Let $n, s \in \mathbb{N}, s \le n$. Then $\underbrace{\sum_{k=s}^{n} \sum_{i=s}^{k}}_{\cancel{\alpha_k}\ \alpha_k} a_{k,i} = \sum_{i=s}^{n} \underbrace{\sum_{k=i}^{n}}_{\beta_i} a_{k,i}$ β_i

$k \backslash i$	s	$s+1$	$s+2$	$\cdots$	n	row sum
s	$a_{s,s}$			$\cdots$		α_s
$s+1$	$a_{s+1,s}$	$a_{s+1,s+1}$		$\cdots$		α_{s+1}
$s+2$	$a_{s+2,s}$	$a_{s+2,s+1}$	$a_{s+2,s+2}$	$\cdots$		α_{s+2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n	$a_{n,s}$	$a_{n,s+1}$	$a_{n,s+2}$	$\cdots$	$a_{n,n}$	α_n
col sum	β_s	β_{s+1}	β_{s+2}	$\cdots$	β_n	$\Sigma\Sigma$

sum

THEOREM: Let $\{a_n\}, \{b_n\}$ be two sequences s.t. for all $n \ge s$,

$a_n = \sum_{k=s}^{n} \binom{n}{k} b_k$. Then $b_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k$ $(n \ge s)$.

k换成i

- $\sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} \sum_{i=s}^{k} \binom{k}{i} b_i$ $= \sum_{k=s}^{n} \sum_{i=s}^{k} (-1)^{n-k} \binom{n}{k} \binom{k}{i} b_i$

n换成k

$= \sum_{i=s}^{n} \sum_{k=i}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{i} b_i = b_n$

$$= \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

$$b_n \cdot 1 + b_{n+1} \cdot 0 + \cdots b_s \cdot 0$$

$$= b_n .$$

Distributing Objects into Boxes

The Problem Statement: distributing n objects into k boxes

- Objects may be distinguishable (**labeled** with numbers $1,2,\ldots,n$) or indistinguishable (**unlabeled**)
- Boxes may be distinguishable (**labeled** with numbers $1,2,\ldots,k$) or indistinguishable (**unlabeled**)
- ? What is the # of distributing n objects into k?

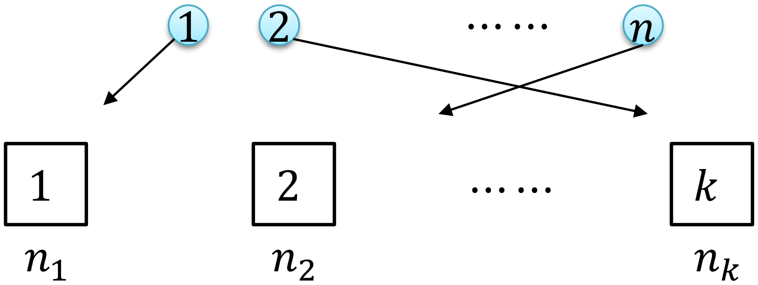
Problem Type	Objects	Boxes
1	labeled	labeled
2	unlabeled	labeled
3	labeled	unlabeled
4	unlabeled	unlabeled

Problem Classification

Type 1

Problem: distributing n labeled objects into k labled boxes

total # of objects



Classifications

$n_1 + n_2 + \cdots + n_k = n$

$n_1, n_2, \ldots, n_k \in \mathbb{N}$

# of item in box 1

$(n_1, n_2 \cdots n_k)$: method

THEOREM: The number of ways of distributing n labeled objects into k labeled boxes such that n_i objects are placed into box i for every $i \in [k]$ is $N_1 = n!/(n_1!\, n_2! \cdots n_k!)$.

- S: the set of the expected distributing schemes
- $|S| = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$

REMARK: N_1 = # of permutations of $\{n_1 \cdot 1, \ldots, n_k \cdot k\}$.

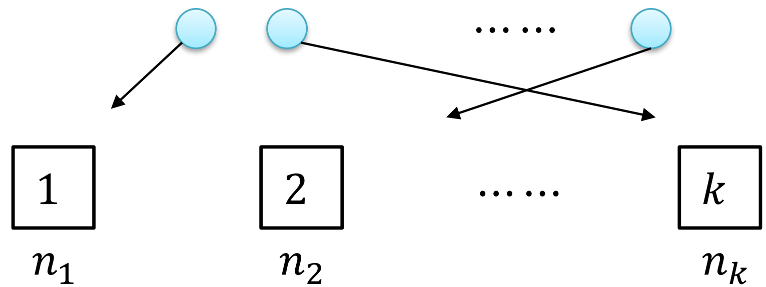
全排列个数

k steps.

Type 2

完全相同

Problem: distributing n unlabeled objects into k labled boxes



\# of soln

Classifications

$$n_1 + n_2 + \cdots + n_k = n$$
$$n_1, n_2, \dots, n_k \in \mathbb{N}$$

THEOREM: The number of ways of distributing n unlabeled objects into k labeled boxes is $N_2 = \binom{n+k-1}{n}$.

可重复

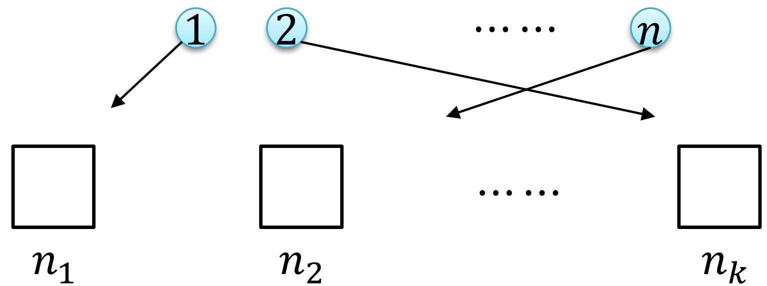
- S: the set of the expected distributing schemes
- $T = \{(n_1, n_2, \dots, n_k): n_1 + n_2 + \cdots + n_k = n; n_1, n_2, \dots, n_k \in \mathbb{N}\}$
- $f: T \to S \quad (n_1, n_2, \dots, n_k) \mapsto$ a scheme where n_i objects are put into box i
  - f is a bijection. Hence, $|S| = |T| = \binom{n+k-1}{n}$

REMARK: $N_2 = \#$ of n-combinations of $\{\infty \cdot 1, \dots, \infty \cdot k\}$

或

Type 3

Problem: distributing n labeled objects into k unlabled boxes



Classifications

$$n_1 + n_2 + \cdots + n_k = n$$
$$n_1, n_2, \ldots, n_k \in \mathbb{N}$$
$$n_1 \geq n_2 \geq \cdots \geq n_k$$

EXAMPLE: Assigning 4 employees {a, b, c, d} into 3 unlabeled offices. Each office can contain any number of employees.

- 4 0 0: [abcd – –]
- 3 1 0: [abc d –] [abd c –] [acd b –] [bcd a –]
- 2 2 0: [ab cd –] [ac bd –] [ad bc –]
- 2 1 1: [ab c d][ac b d] [ad b c] [bc a d] [bd a c] [cd a b]

方案分类.

classifying method

REMARK: The schemes can be classified with $\{n_1, \ldots, n_k\}$

Stirling numbers

Stirling number

$$S_2(n,j)$$

n 标物 in j 未标盒.

DEFINITION: $S_2(n,j)$, the **Stirling number of the second kind**, is defined as the number of different ways of distributing n labeled objects into j unlabeled boxes so that no box is empty.

THEOREM: $S_2(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n$ when $n \geq j \geq 1$.

THEOREM: The number of schemes of distributing n labeled objects into k unlabeled boxes is

占用至多k个盒

$$\sum_{j=1}^{k} S_2(n,j) = \sum_{j=1}^{k} \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n$$

- $S_2(n,j)$: the number of schemes that use exactly j boxes, $j = 1,2,\ldots,k$