Discrete Mathematics: Lecture 24

Degree, Handshaking Theorem, Graph Transform, Graph Isomorphism, Bipartite Graph, Matching

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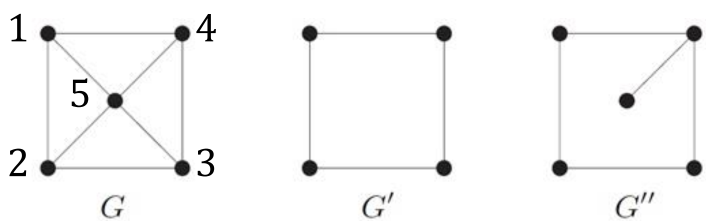
Subgraph

DEFINITION: Let $G = (V, E)$ be a simple graph. $H = (W, F)$ is a **subgraph**子图 of G if $W \subseteq V$ and $F \subseteq E$.

- **proper subgraph**真子图: H is a subgraph of G and $H \neq G$. all
- The **subgraph induced**导出子图 by $W \subseteq V$ is (W, F), where $F = \{e: e \in E, e \subseteq W\}$.
  //Notation: $G[W]$
  顶点 induce
- The **subgraph induced**导出子图 by $F \subseteq E$ is (W, F), where $W = \{v: v \in V, v \in e \text{ for some } e \in F\}$. //Notation: $G[F]$
  边 induce

EXAMPLE: Let G, G', G'' be three graphs as below.

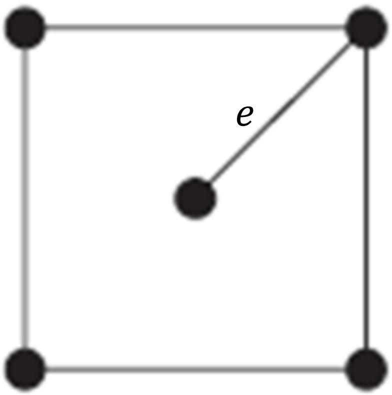


- G', G'' are subgraphs of G; G', G'' are proper subgraphs of G
- G' is a subgraph induced by $W = \{1,2,3,4\}$, i.e., $G' = G[W]$
- G'' is a subgraph induced by $F = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\},\{4,5\}\}$, i.e., $G'' = G[F]$

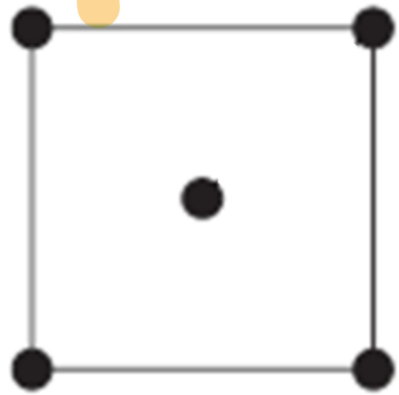
Removing An Edge

DEFINITION: Let $G = (V, E)$ be a simple graph and $e \in E$. Define

$$G - e = (V, E - \{e\})$$



$G = (V, E)$

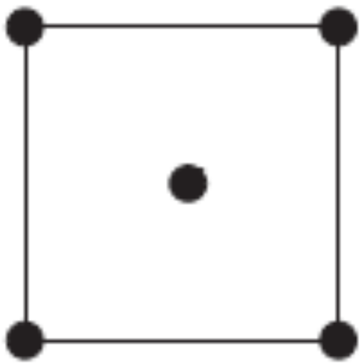


$G - e = (V, E - \{e\})$

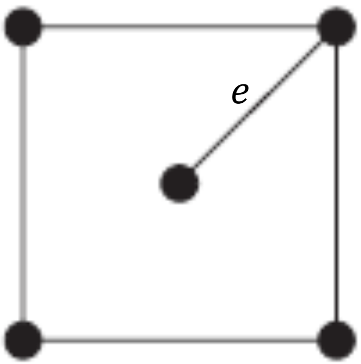
Adding An Edge

DEFINITION: Let $G = (V, E)$ be a simple graph and $e \notin E$. Define

$$G + e = (V, E \cup \{e\})$$



$G = (V, E)$



$G + e = (V, E \cup \{e\})$

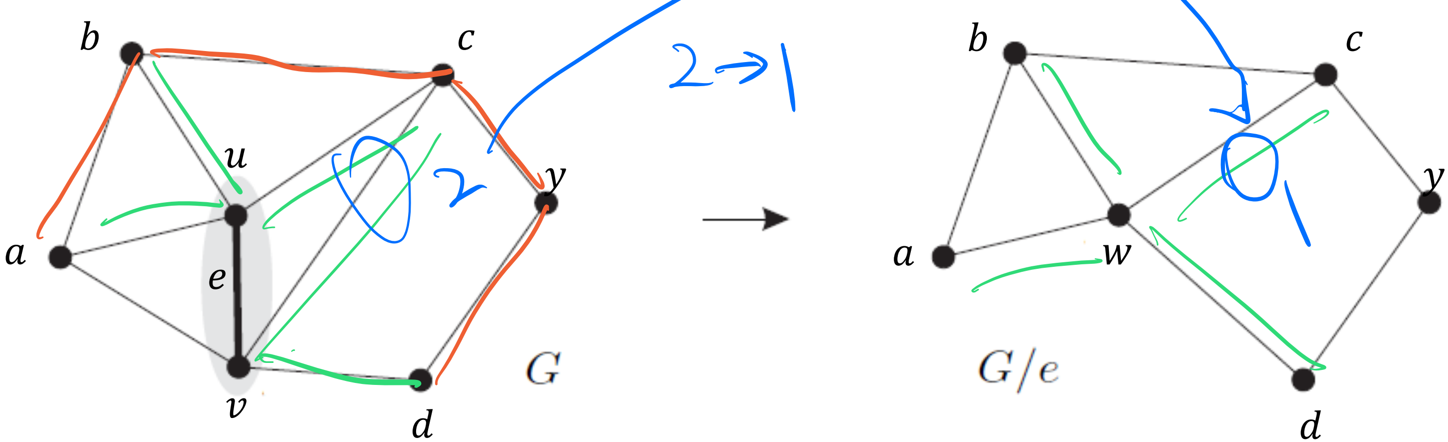
Edge Contraction

压缩

DEFINITION: Let $G = (V, E)$ be a simple graph and $e = \{u, v\} \in E$.

Define $G/e = (V', E')$, where $V' = (V - \{u, v\}) \cup \{w\}$ and

$$E' = \{e' \in E : e' \cap e = \emptyset\} \cup \{\{w, x\} : \{u, x\} \in E \text{ or } \{v, x\} \in E\}$$

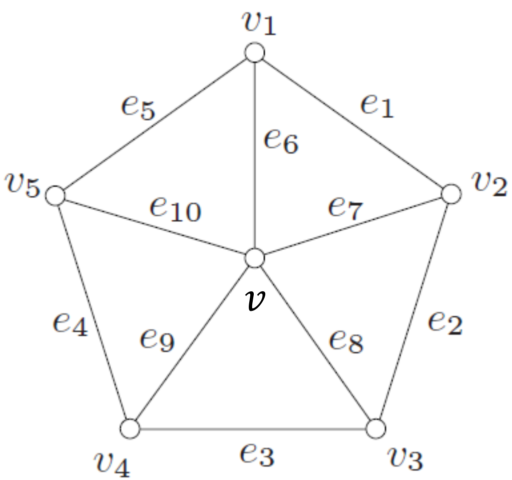


Graph trans: define V′, E′

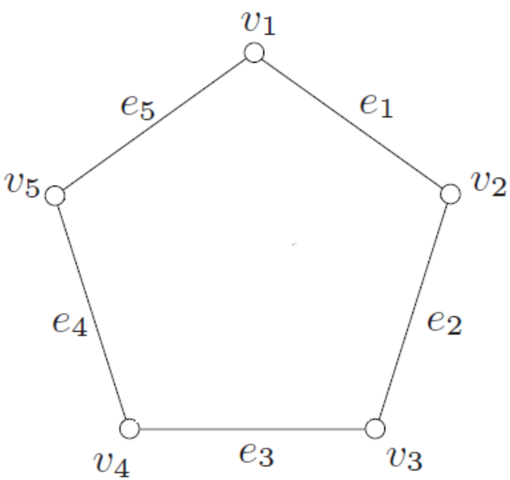
Removing A Vertex

DEFINITION: Let $G = (V, E)$ be a simple graph and let $v \in V$. Define

$$G - v = (V - \{v\}, E'), \text{ where } E' = \{e \in E : v \notin e\}$$



$G = (V, E)$



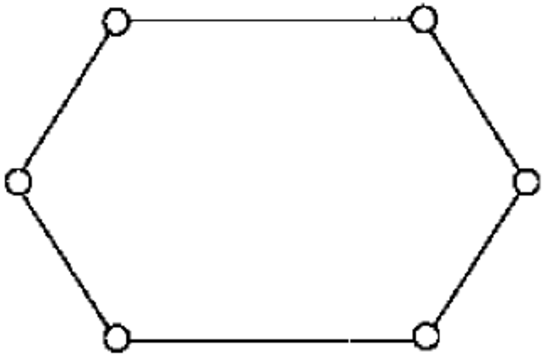
$G - v$

Complement

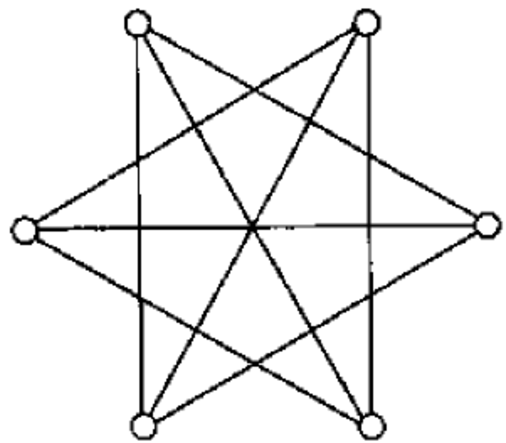
$V' = V$

DEFINITION: Let $G = (V, E)$ be a simple graph of order n. Define the **complement graph** 补图 of G as $\bar{G} = (V, E')$, where

$$E' = \{\{u, v\}: u, v \in V,\ u \neq v, \{u, v\} \notin E\}$$



G



$\bar{G}$

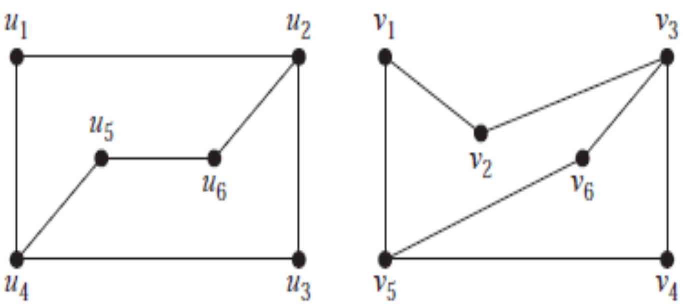
complete − origin
of origin

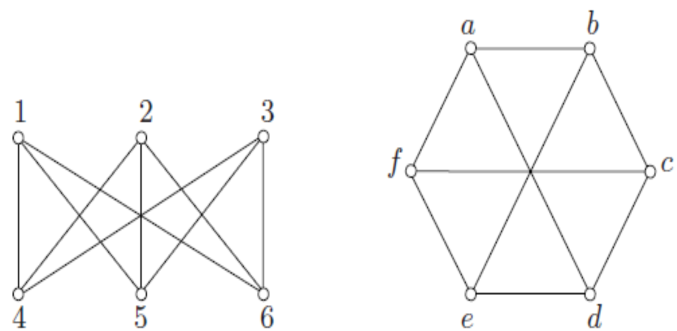
Graph Isomorphism

同构

DEFINITION: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic**同构 if there is a bijection $\sigma: V_1 \to V_2$ such that $\{u, v\} \in E_1 \Leftrightarrow \{\sigma(u), \sigma(v)\} \in E_2$.

- σ is called an **isomorphism**同构映射
- **nonisomorphic:** not isomorphic





u_1	u_2	u_3	u_4	u_5	u_6
v_6	v_3	v_4	v_5	v_1	v_2

Isomorphism σ

1	2	3	4	5	6
a	c	e	b	d	f

Isomorphism σ

Graph Invariants

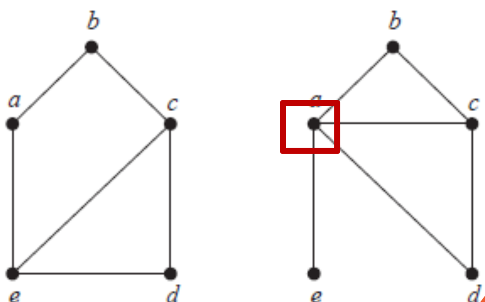
不变量

DEFINITION: **Graph invariants** are properties preserved by graph isomorphism. For example,

同构图中不变量

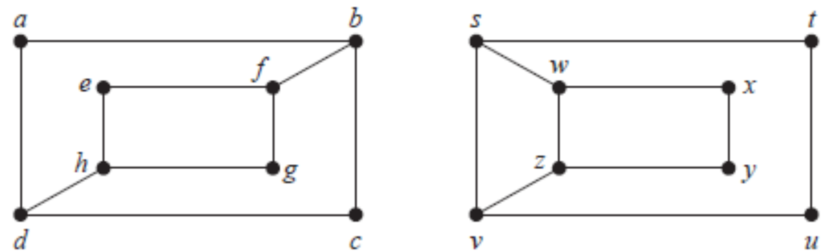
① • The number of vertices
② • The number of edges
③ • The number of vertices of each degree

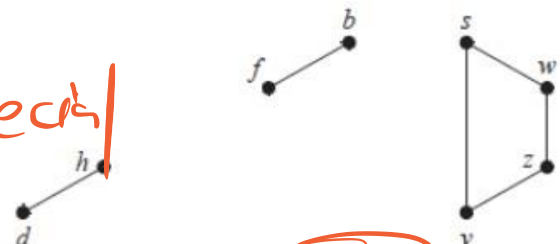
REAMRKS: The graph invariants can be used to determine if two graphs are isomorphic or not.



There is no vertex of degree 4 in the 1st graph

④ Special





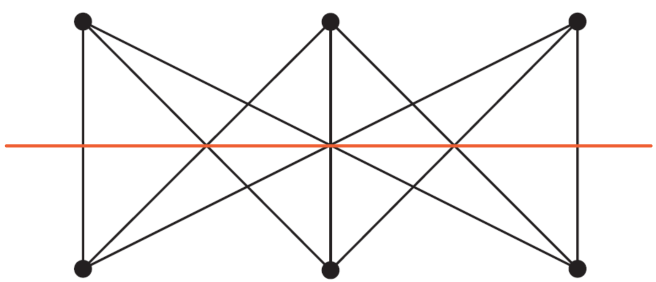
Induced subgraphs are not isomorphic. Original graphs are not isomorphic.

The subgraphs induced by the vertices of degree 3 must be isomorphic to each other.

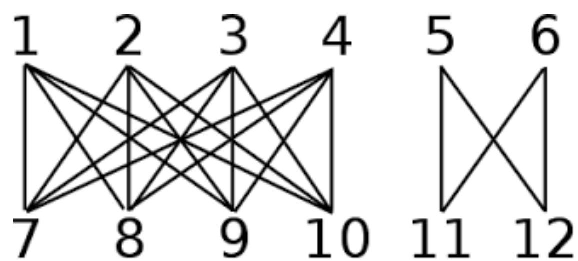
Bipartite Graph

DEFINITION: $G = (V, E)$ is a **bipartitie graph**二分图、二部图 if V has a partition $\{V_1, V_2\}$ such that $E \subseteq \{\{u_1, u_2\}: u_1 \in V_1, u_2 \in V_2\}$.

- (V_1, V_2) is a **bipartition**二划分 of the vertex set V.



A bipartite graph of order 6



A bipartite graph of order 12

- $V_1 = \{1,2,3,4,5,6\}$
- $V_2 = \{7,8,9,10,11,12\}$

# of V

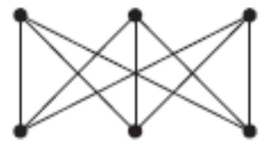
Size: # of E

Complete Bipartite Graph

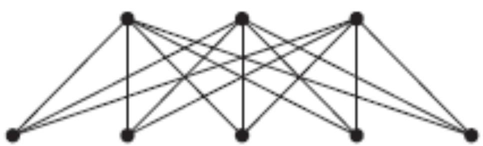
DEFINITION: A **complete bipartite graph**完全二部图 is a graph $K_{m,n} = (V, E)$ with $V = \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$ and $E = \left\{\{x_i, y_j\}: i \in [m], j \in [n]\right\}$

- Every vertex in V_1 is adjacent to every vertex in V_2

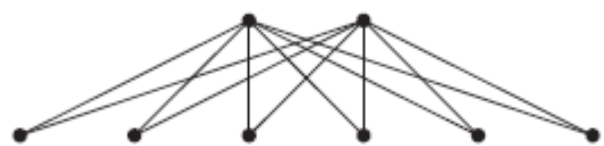
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



$K_{2,6}$

Bipartite Graph

Theorem

A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex such that no two adjacent vertices have the same color.

Proof:

- If $G = (V, E)$ is bipartite, $V = V_1 \cup V_2$. Assign color c_1 to vertices of V_1 and color c_2 to vertices of V_2.
- Reversely, suppose we can assign colors c_1 and c_2 to the vertices such that no two adjacent have the same. Let V_i be the set of vertices of color c_i, for $i = 1, 2$. Then $V = V_1 \cup V_2$. By assumption there are no edges connecting two vertices of V_1 or two vertices of V_2, so each edge connects one vertex of V_1 with one vertex of V_2. $\square$

Bipartite Graph*

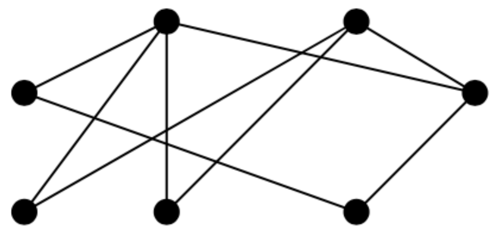
(formal proof)

THEOREM: A simple graph $G = (V, E)$ is a bipartite graph iff there is a map $f: V \to \{1,2\}$ such that "$\{x, y\} \in E \Rightarrow f(x) \neq f(y)$"

- Only if: $G = (V_1 \cup V_2,\ E)$, where $V_1 \cap V_2 = \emptyset$.
  - Define $f: V \to \{1,2\}$ such that $f(x) = \begin{cases} 1 & \text{if } x \in V_1 \\ 2 & \text{if } x \in V_2 \end{cases}$
  - $\{x, y\} \in E \Rightarrow x \in V_1, y \in V_2$ or $x \in V_2, y \in V_1$
    - $f(x) \neq f(y)$
- If: $f: V \to \{1,2\}$ is a map such that "$\{x, y\} \in E \Rightarrow f(x) \neq f(y)$"
  - Let $V_1 = f^{-1}(1), V_2 = f^{-1}(2)$
    - $V = V_1 \cup V_{2,} V_1 \cap V_2 = \emptyset$
      - $\{V_1, V_2\}$ is a bipartition of V
  - $\{x, y\} \in E \Rightarrow f(x) \neq f(y) \Rightarrow x \in V_1, y \in V_2$ or $x \in V_2, y \in V_1$
    - G is a bipartite graph.

Bipartite Graph

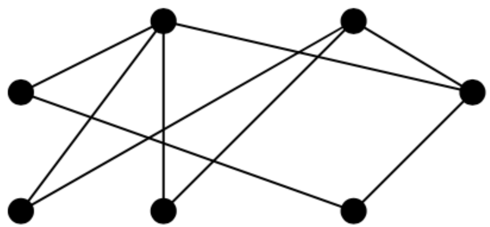
Example: Is the graph G bipartite?



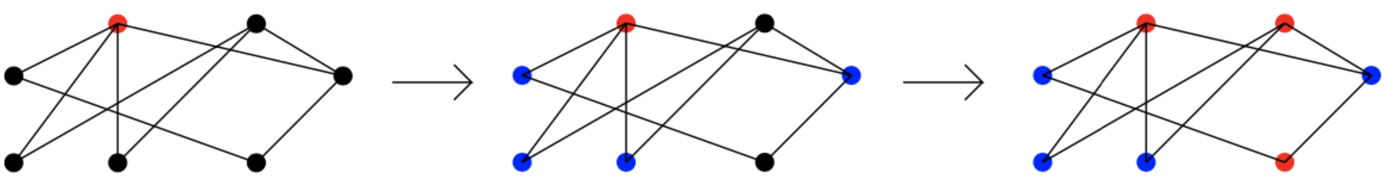
G

Bipartite Graph

Example: Is the graph G bipartite?



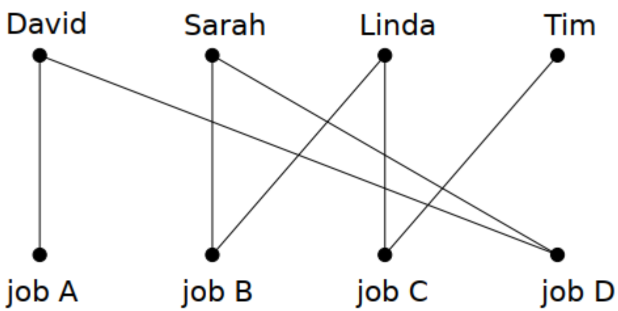
G



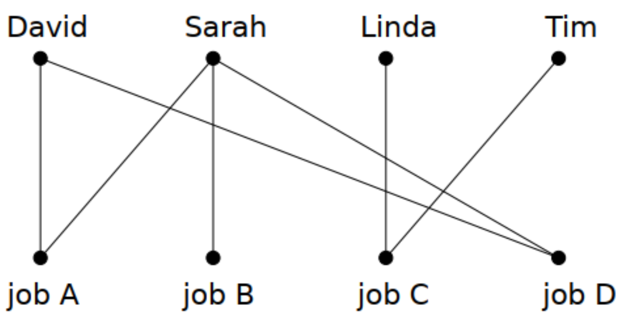
Motivation: Job Assignment

Suppose there are m employees and n different jobs to be done, with $m \geq n$.

1



2

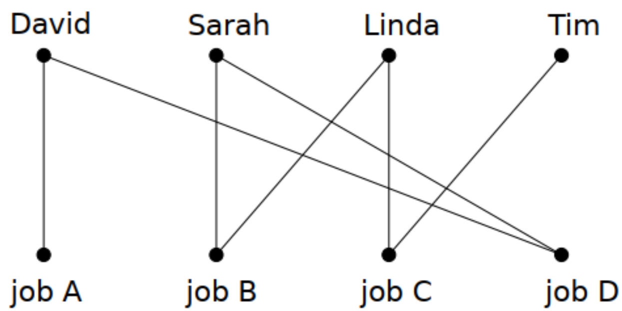


Common model

Motivation: Job Assignment

Suppose there are m employees and n different jobs to be done, with $m \geq n$.

1



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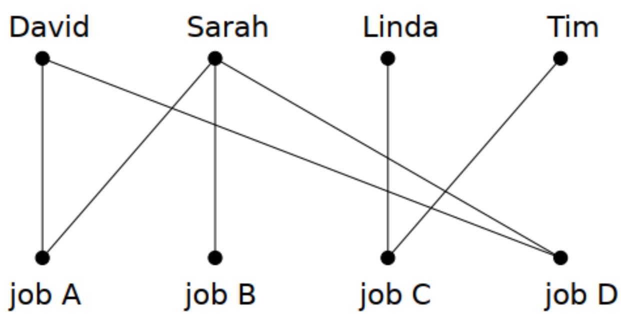
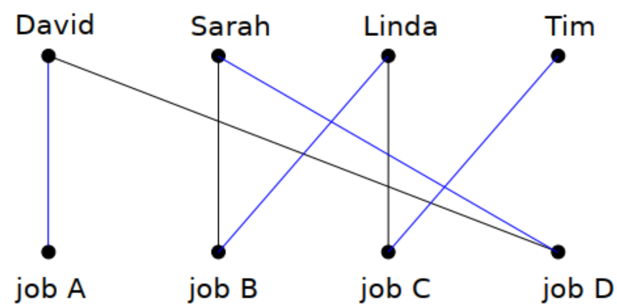




Possible solution for situation 1

一一对应

Matching

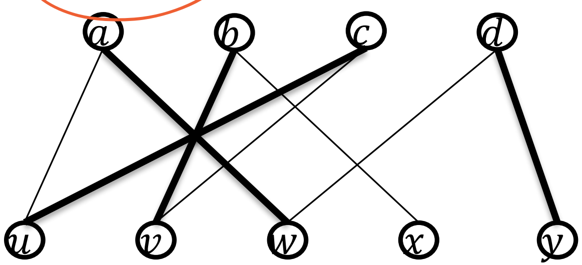
DEFINITION: Let $G = (V, E)$ be a simple graph. $M \subseteq E$ is a **matching**匹配 if $e \cap e' = \emptyset$ for every $e, e' \in M$. A vertex $v \in V$ is **matched** in M if $\exists e \in M$ such that $v \in e$, otherwise, v is **not matched.**

- **maximum matching**最大匹配: a matching with largest number of edges.
- In a bipartite graph $G = (A \cup B, E)$, $M \subseteq E$ is a **complete matching**完全匹配 from A to B if every $u \in A$ is matched.

A

B

ok. com



- $M = \{au, bv\}$ is a matching
  - a, b, u, v are matched in M
  - c, d, x, y are not matched in M
  - M is not a maximum matching
- $M' = \{aw, bv, cu, dy\}$ is a maximum matching
- M' is a complete matching from V_1 to V_2

- $V = \{a, b, c, d, u, v, w, x, y\}$
- $V_1 = \{a, b, c, d\}$;
- $V_2 = \{u, v, w, x, y\}$
- $E = \{au, aw, bv, bx, cu, cv, dw, dy\}$

Matching

DEFINITION: Let $G = (V, E)$ be a simple graph. $M \subseteq E$ is a **matching**匹配 if $e \cap e' = \emptyset$ for every $e, e' \in M$. A vertex $v \in V$ is **matched** in M if $\exists e \in M$ such that $v \in e$, otherwise, v is **not matched.**

- **maximum matching**最大匹配**:** a matching with largest number of edges.
- In a bipartite graph $G = (A \cup B, E)$, $M \subseteq E$ is a **complete matching**完全匹配 from A to B if every $u \in A$ is matched.

Example: Marriages. Suppose there are *m* men and *n* women on an island. Each person has a list of people of the opposite gender acceptable as a spouse $\Rightarrow$ bipartite graph.

- matching $\Leftrightarrow$ marriages
- maximum matching $\Leftrightarrow$ largest possible number of marriages
- complete matching from women to men $\Leftrightarrow$ marriages such that every women is married but possibly not all men.

N: neighbour

Hall's Theorem

EXAMPLE: Marriage on an Island

- There are m boys $X = \{x_1, \dots, x_m\}$ and n girls $Y = \{y_1, \dots, y_n\}$
- $G = \left(X \cup Y, E = \left\{\{x_i, y_j\}: x_i \text{ and } y_j \text{ are willing to get married}\right\}\right)$
- What is the largest number of couples that can be formed?

N(A) (in Y)

THEOREM (Hall 1935): A bipartitie graph $G = (X \cup Y, E)$ has a complete matching from X to Y iff $|N(A)| \geq |A|$ for any $A \subseteq X$.

- $\Rightarrow$: Let $\{\{x_1, y_1\}, \dots, \{x_m, y_m\}\}$ be a complete matching from X to Y
  - For any $A = \{x_{i_1}, \dots, x_{i_s}\} \subseteq X$, $N(A) \supseteq \{y_{i_1}, \dots, y_{i_s}\}$
    - $|N(A)| \geq s = |A|$
- $\Leftarrow$: suppose that $|N(A)| \geq |A|$ for any $A \subseteq X$. Find a complete matching M.
  - By induction on $|X|$
  - $|X| = 1$: Let $X = \{x\}$.
    - $|N(X)| \geq 1$
      - $\exists y \in Y$ such that $e = \{x, y\} \in E$.
        - $M = \{e\}$ is a complete matching from X to Y

Hall's Theorem

数归

- **Induction hypothesis**: "$\forall A \subseteq X, |N(A)| \geq |A| \Rightarrow \exists$ complete matching" is true when $|X| \leq k$
- Prove that "$\forall A \subseteq X, |N(A)| \geq |A| \Rightarrow \exists$ complete matching" when $|X| = k + 1$
  - Let $X = \{x_1, \dots, x_k, x_{k+1}\}$.
  - **Case 1:** $\forall\ A \subseteq X$ with $1 \leq |A| \leq k$, $|N_G(A)| \geq |A| + 1$
    - $N_G(A)$: A's neighborhood in G
    - Say $y_{k+1} \in N_G(\{x_{k+1}\})$.

subgraph

    - Let $V' = (X \setminus \{x_{k+1}\}) \cup (Y \setminus \{y_{k+1}\})$; $E' = \{e \in E: e \subseteq V' \times V'\}$
    - Let $G' = (V', E') = G - \{x_{k+1}\} - \{y_{k+1}\}$.
      - $\forall\ A \subseteq \{x_1, \dots, x_k\}, |N_{G'}(A)| \geq |N_G(A)| - |\{y_{k+1}\}| \geq |A| + 1 - 1 = |A|$
        - $\exists$ a complete matching M' from $X - \{x_{k+1}\}$ to $Y - \{y_{k+1}\}$ in G' (IH)
    - $M = M' \cup \{\{x_{k+1}, y_{k+1}\}\}$ is a complete matching from X to Y in G

Hall's Theorem

separate into 2

exactly

- **Case 2:** $\exists A \subseteq X, 1 \leq |A| \leq k$ such that $|N_G(A)| = |A|$
  - Say $A = \{x_1, \dots, x_j\}$ and $N_G(A) = \{y_1, \dots, y_j\}$, where $1 \leq j \leq k$
  - Let $V' = A \cup N_G(A), E' = \{e \in E: e \subseteq V' \times V'\}$ and $G' = (V', E')$
    - $\forall A' \subseteq A,\ |N_{G'}(A')| = |N_G(A')| \geq |A'|$
    - There is a complete matching M' from A to $N_G(A)$ in G' (IH)
  - Let $V'' = (X \setminus A) \cup \left(Y \setminus N_G(A)\right), E'' = \{e \in E: e \subseteq V'' \times V''\}$,
  - Let $G'' = (V'', E'') = G - A - N_G(A)$
    - Then $\forall\, A'' \subseteq X \setminus A, |N_{G''}(A'')| \geq |A''|$.
      - Otherwise, $|N_G(A'' \cup A)| = |N_{G''}(A'')| + |N_G(A)| < |A''| + |A|$
        - $\exists$ a complete matching M'' from $X \setminus A$ to $Y \setminus N_G(A)$ (IH)
  - $M = M' \cup M''$ is a complete matching from X to Y

complete + complete

A A