1. Type 1 T-step: $P_{(i+1)} = (x_i+1, y_i+1)$ $P_i = (x_i, y_i)$

Type 2 T-step: $P_{(i+1)} = (x_i+1, y_i-1)$ $P_i = (x_i, y_i)$

Since $b > a$, $b, a \in \mathbb{Z}$, $(b-a) \in \mathbb{Z}^+$; $\beta, \alpha \in \mathbb{Z}$, $|\beta-\alpha| \in \mathbb{Z}^+$

Since $(b-a) \geq |\beta - \alpha|$ $\quad P_i(x_i, y_i)$

Let us choose. $(b-a)$ Tsteps ✓, Start at $A(\alpha, \beta)$. and ends at $P_s(x_s, y_s)$. $s = b-a$.

$\Rightarrow x_s = a + (b-a) = b$.

Lemma: $a+b$ is even. then $a-b$ is even. $(a, b \in \mathbb{Z})$.

proof: if $a-b$ odd, $(a+b)-(a-b) = 2a$ is odd then (even - odd = odd).

$a \in \mathbb{Z}$, $2a$ is even. contradicts

$\Rightarrow (a-b)$ is even.

① $\beta \geq \alpha$ $\quad |\beta-\alpha| = \beta-\alpha$.

from Lemma: $(b-a) + \beta - \alpha$ even

$\Rightarrow (b-a) - (\beta-\alpha)$ is even.

since $b-a \geq |\beta-\alpha|$

$(b-a) - |\beta-\alpha| \geq 0$ and is even.

So we can always. take $(\beta-\alpha)$ Type1 Tstep first.

And $\frac{(b-a)-(\beta-\alpha)}{2}$ Type1 Tsteps and $\frac{(b-a)-(\beta-\alpha)}{2}$ type2 T steps.

To ensure: $y_s = \alpha + (\beta-\alpha) + \frac{(b-a)-(\beta-\alpha)}{2}[1+(-1)]$

$= \beta \Rightarrow P_s = B$. Exist T-route from A to B with $\frac{(b-a)+(\beta-\alpha)}{2}$ type1 steps and $\frac{(b-a)-(\beta-\alpha)}{2}$ type 2 steps.

② $\alpha > \beta$. $|\beta-\alpha| = \alpha-\beta$.

$(b-a)+(\beta-\alpha)$ is even

$\Rightarrow (b-a) - |\beta-\alpha|$ is even and ≥ 0.

then Same with ① (with α, β switched)

Exist T-route from A to B

with $\frac{(b-a)+(\alpha-\beta)}{2}$ type 2 Tsteps.

And $\frac{(b-a)-(\alpha-\beta)}{2}$ type1 Tsteps.

from ①②.

We can always find a T-route under T-condition

1. (20 points) Let $A = (a, \alpha), B = (b, \beta) \in \mathbb{Z}^2$. Show that if A, B satisfy the T-condition, then there is a T-route from A to B. (**T-condition**: (1) $b > a$; (2) $b - a \geq |\beta - \alpha|$; (3) $b - a + \beta - \alpha$ is even.)

2. (20 points) At the end of a basketball match (*for simplicity, suppose that every successful shot gives a team 1 point*) between team A and team B, the result is 80:81. What is the number of possibilities that A's score is always less than B's score during the entire match? A possibility can be described with the sequence of intermediate results during the entire match. For example, $0:1, 0:2, \ldots, 0:81, 1:81, 2:81, \ldots, 80:81$ describes one of the possibilities that A's score is always less than B's score during the entire match. (**Hint:** Use the idea of counting T-routes.)

3. (20 points) Let n, r be positive integers such that $r \geq n$. Determine the number of vectors $(x_1, x_2, \ldots, x_n)$ such that $x_1 + x_2 + \cdots + x_n = r$ and $x_1, x_2, \ldots, x_n \in \mathbb{Z}^+$.

4. (20 points) Let $\{a_n\}_{n \geq s}, \{b_n\}_{n \geq s}$ be two sequences such that $a_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} b_k$ for all $n \geq s$. Show that $b_n = \sum_{k=s}^{n} \binom{n}{k} a_k$ for all $n \geq s$.

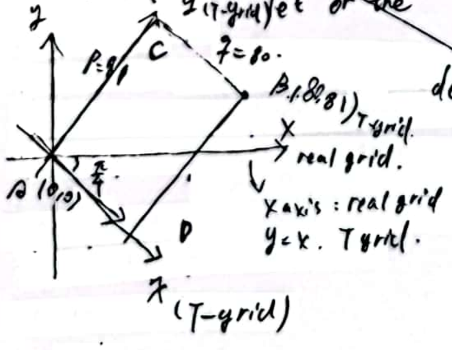
5. (20 points) Suppose that $n + 1 \geq k \geq 2$. Provide a combinatorial proof of $S_2(n+1, k) = S_2(n, k-1) + k \cdot S_2(n, k)$. (**Hint**: Interpret both sides of the equation as the number of elements in a set X)

2. the game start from (0:0) and ends at (80:81) all possible score change way (step): $\begin{cases} x+1 \ y+0 \text{ type1} \\ x+0 \ y+1 \text{ type2} \end{cases}$

Take A (0,0) B (80,81) Take this grid T-grid

The route on it from A to B → possible score way., Put the T-grid in a normal grid. all types change are T-steps.

Shortest y (T-grid) yet or the



the routes go through y=x in T grid ⟺. the routes in sets x-axis in real grid.

define C, D. as the edge point of (0,81), (80,81) in T grid

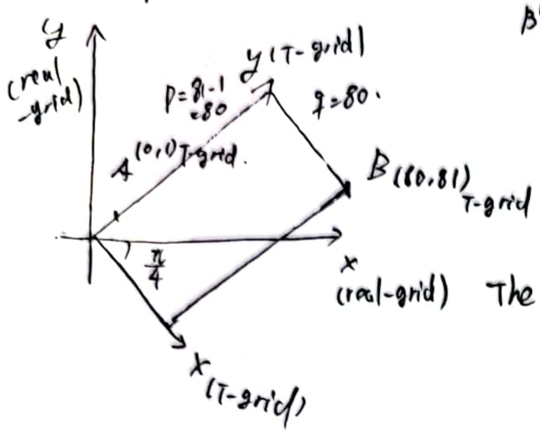
p, q as numbers of type1. type2 T-steps.

The number of all routes is: $\frac{(80+81)!}{80!\,81!} = \frac{161!}{80!\,81!}$

for B. it is $(80+81, 81 \times 1 + 80 \times (-1)) = (161, 1)$ in real grid.

So the number of T routes that path through x-axis (means they go through y=x in T-grid) is: $\frac{(80-0)!}{\left(\frac{80-0}{2} + \frac{81+0}{2}\right)!\,(}$

2. Since we want A's score always less than B. we can take the game start at $(0:1)$ and ends at $(80:81)$.
define $A(0,1)$ $B(80,81)$ on the x, y T-grid. The transistor of game has two types: $\begin{cases} A \text{ score} +1 \\ B \text{ score} +0 \end{cases}$ $\begin{cases} B \text{ score} +1 \\ A \text{ score} +0 \end{cases}$.
So we match them with 2 types. The number of possible score change to a T-route problem:



A's normal grid place: $(1,1)$
B's normal grid place: $(80+81,\ 81-80) = (161,1)$.

In T-grid. we find all the Troute number $= \dfrac{(80+80)!}{80!\,80!} = \dfrac{160!}{80!\,80!}$

To make score A always > B. means $y > x$ in T-grid.
mean the T-route from A to B should not pass through X-axis in real grid.

The number of T-route pass through ~~real~~ X-axis from A to B:

$$\text{number} = \frac{(161-1)!}{\left(\frac{161-1}{2}+\frac{1+1}{2}\right)!\left(\frac{161-1}{2}-\frac{1+1}{2}\right)!} = \frac{160!}{81!\,79!}$$

$$\Rightarrow \text{possible Outcome} = \frac{160!}{80!\,80!} - \frac{160!}{81!\,79!}$$

(number of T-routes not pass X-axis $\Leftrightarrow$ number of possible score outcomes).

3.
$r \geq n$
$x_1, x_2 \cdots x_n \in \mathbb{Z}^+$
let's define: $\mathcal{X} = \{(x_1, \ldots x_n) : x_1, \cdots x_n \in \mathbb{Z}^+,\ x_1 + x_2 \cdots + x_n = r\}$ (all vectors satisfy the target).
$\mathcal{Y}$: the set of r-combination of $[n]$ with repetition.

$f: \mathcal{X} \to \mathcal{Y}$ $(x_1, \cdots x_n) \mapsto \{x_1 \cdot 1, x_2 \cdot 2 \cdots x_n \cdot n\}$. is bijection.

$|\mathcal{X}| = |\mathcal{Y}|$

define $\mathcal{V}$: the set of all combination of $[n+r-1]$ without repetition.

let $U = \{y_1, y_2 \ldots y_r\} \in \mathcal{Y}$ and $1 \leq y_1 \leq y_2 \leq \cdots y_r \leq n$.

$1 \leq y_1 < y_2 + 1 < y_3 + 2 \cdots < y_r + r - 1 \leq n + r - 1$

$\Rightarrow \{y_1, y_2 + 1 \cdots y_r + r - 1\} \in \mathcal{V}$

$f_1: \mathcal{Y} \to \mathcal{V}: \{y_1, y_2 \cdots y_r\} \mapsto \{y_1, y_2 + 1 \cdots y_r + r - 1\}$

f_1 is bijection

$|\mathcal{Y}| = |\mathcal{V}|$

$\Rightarrow |\mathcal{X}| = |\mathcal{Y}| = |\mathcal{V}| = \binom{n+r-1}{r} = \dfrac{(n+r-1)!}{(n-1)!\,r!}$

4.

$a_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} b_k$

$$\sum_{k=s}^{n} \binom{n}{k} a_k = \sum_{k=s}^{n} \binom{n}{k} \sum_{i=s}^{k} (-1)^{k-i} \binom{k}{i} b_i$$

$$= \sum_{k=s}^{n} \sum_{i=s}^{k} (-1)^{k-i} \binom{n}{k}\binom{k}{i} b_i$$

Lemma: $\sum_{k=s}^{n}\sum_{i=s}^{k} a_{ki} = \sum_{i=s}^{n}\sum_{k=i}^{n} a_{ki}$

$$\overset{\text{lemma}}{=\!=} \sum_{i=s}^{n} \sum_{k=i}^{n} (-1)^{k-i} \binom{n}{k}\binom{k}{i} b_i$$

lemma2: $\sum_{k=r}^{n} (-1)^{n-k} \binom{n}{k}\binom{k}{r} = \begin{cases} 1 & n=r \\ 0 & n>r \end{cases}$

$$= b_n \cdot 1 + b_{n-1} \cdot 0 + \dots b_s \cdot 0$$

$$= b_n$$

$\Rightarrow b_n = \sum_{k=s}^{n} \binom{n}{k} a_k$ for $(n \geq s)$. (lemma, lemma 2 proved in the lesson).

~~5. when we have put n labeled objects in k box and to put the n+1 th object.~~

~~The is a partition:~~

~~① have covered all k boxes,~~

~~②~~

5. let A to be the set of methods of partition $[n+1]$ into k subsets.

① if we have partitioned $[n]$ into k sets. then. the $(n+1)$ can be in any of those sets. there is $k \cdot S_2(n,k)$ method

② if we have partitioned $[n]$ into $k-1$ sets, then the $(n+1)$ can only be to form the k th set, there is $S_2(n, k-1)$

③ partitioned $[n]$ into less than $k-1$ sets before is not possible since we only have $(n+1)$ and can only form 1 new set

from ①, ②, ③.

$$S_2(n+1, k) = |A| = S_2(n, k-1) + k\, S_2(n,k)$$

5: