

Discrete Mathematics: Lecture 16

proposition, truth value, propositional constant/variable, negation, truth table, conjunction, disjunction, implication, bi-implication, formula

Xuming He^{*}
Associate Professor

^{*}School of Information Science and Technology, ShanghaiTech University

Spring Semester, 2022

- *Combinatorics: complexity analysis, etc*
- *Number theory: cryptography*
- Logic: software engineering, artificial intelligence, database theory, programming language, etc
- Graph theory: software engineering, theoretical computer science
- ...

Textbook: Discrete Mathematics and Its Applications (7th edition)
Kenneth H. Rosen, William C Brown Pub, 2011.

Mathematical Logic

Logic: the study of reasoning, the basis of all mathematical reasoning.

Mathematical logic: the mathematical study of reasoning and the study of mathematical reasoning // foundation of mathematics

- Leibniz: introduced the idea of mathematical logic in “Dissertation on the Art of Combinations” in 1666
- Universal system of reasoning: reasoning based on symbols+calculations
- **Contributors:** Boole, De Morgan, Frege, Peano, Russell, Hilbert, Gödel,...
- **Areas:** (1) set theory, (2) proof theory, (3) recursion theory, (4) model theory, and their foundation (5) propositional logic and predicate logic

Our focus: propositional logic and predicate logic, (naive) set theory

Proposition



Definition: A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false.

- Lower-case letters represent propositions: p, q, r, \dots
- **Truth value:** The truth value of p is true (**T**) if p is a true proposition. The truth value of p is false (**F**) if p is a false proposition.

Example:

- Washington, D.C, is the capital of the United States of America.
(**T**)
- $1 + 1 = 3$
(**F**)
- $(x^2)' = 2x$
(**T**)

Proposition

Example:

- Every even integer $n > 2$ is the sum of two primes.
 - Proposition?:
Yes!
 - Goldbach's conjecture
 - A proposition whose truth value is not known now
- What time is it?
 - Proposition?:
No!. It's not declarative.
- Do not smoke!
 - Proposition?:
No!. It's not declarative.
- $x + 1 = 2$.
 - Proposition?:
No!. It's neither true nor false.

Proposition

Simple Proposition: cannot be broken into 2 or more propositions

- $\sqrt{2}$ is irrational.

Compound Proposition: not simple

- 2 is rational and $\sqrt{2}$ is irrational.

Propositional Constant: a concrete proposition (truth value **fixed**)

- Every even integer $n > 2$ is the sum of two primes.

Propositional variables: a variable that represents any proposition

- Lower-case letters denote proposition variables: p, q, r, s, \dots
- Truth value is not determined until it is assigned a concrete proposition

Propositional Logic: the area of logic that deals with propositions

Negation: \neg

Definition: Let p be any proposition.

- The **negation** of p is the statement “It is not the case that p ”
- Notation: $\neg p$; read as “not p ”
- **True table:**

p	$\neg p$
T	F
F	T

Negation: \neg

Example:

- $p = \text{"Snow is black"}$
 - $\neg p = \text{"It is not the case that snow is black."}$
 - $\neg p = \text{"Snow is not black."}$
 - $\neg p \neq \text{"Snow is white."}$
- $p = \text{"Amy's smartphone has at least 32 GB of memory."}$
 - $\neg p = \text{"It is not the case that Amy's smartphone has at least 32 GB of memory."}$
 - $\neg p = \text{"Amy's smartphone does not have at least 32 GB."}$
 - $\neg p = \text{"Amy's smartphone has less than 32 GB."}$

Conjunction: \wedge

Definition: Let p, q be any propositions.

- The **conjunction** of p and q is the statement “ p and q ”
- Notation: $p \wedge q$; read as “ p and q ”
- True table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example:

- $p = “2 < 3”$; $q = “2^2 < 3^3”$
 - $p \wedge q = “2 < 3 \text{ and } 2^2 < 3^3.”$
(T)
- $p = “\text{Dog can fly}”$; $q = “\text{Eagle can fly}”$
 - $p \wedge q = “\text{Dog can fly and Eagle can fly.}”$
(F)

Disjunction: \vee

Definition: Let p, q be any propositions.

- The **disjunction** of p and q is the statement “ p or q ”
- Notation: $p \vee q$; read as “ p or q ”
- True table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example:

- $p = “2 > 3”$; $q = “2^2 > 3^3”$
 - $p \vee q = “2 > 3 \text{ or } 2^2 > 3^3.”$
(F)
- $p = “\text{Dog can fly}”$; $q = “\text{Eagle can fly}”$
 - $p \vee q = “\text{Dog can fly or Eagle can fly.}”$
(T)

Implication: \rightarrow

Definition: Let p, q be any propositions.

- The **conditional statement** $p \rightarrow q$ is the proposition “if p , then q .”
 - p : hypothesis; q : conclusion; read as “ p implies q ”, or “if p , then q ”
- True table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example:

- p = “you get 100 on the final”; q = “you will receive A+”
 - $p \rightarrow q$ = “If you get 100 on the final, you will receive A+.” (T)
 - It is false when you get 100 on the final but don’t receive A+, which is “when p is true but q is false.” (F)

Bi-Implication: \leftrightarrow

Definition: Let p, q be any propositions.

- The **biconditional statement** $p \leftrightarrow q$ is the proposition “ p if and only if q .”
 - read as “ p if and only if q ”
- True table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example:

- p = “you can take the flight”; q = “you buy a ticket”
 - $p \leftrightarrow q$ = “You can take the flight if and only if you buy a ticket.”
 - False when $(p, q) = (T, F)$ or (F, T)

Well-Formed Formulas (propositional)

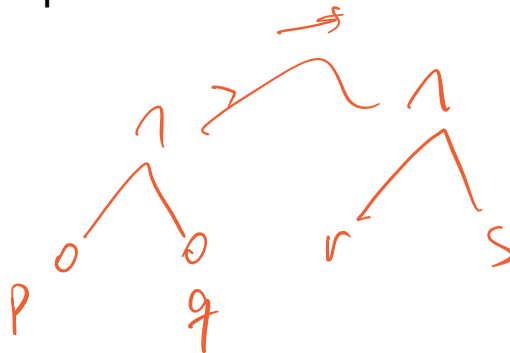
Definition: recursive definition of **well-formed formulas (WFFs)**

- ① propositional constants (**T**, **F**) and propositional variables are WFFs
- ② If A is a WFF, then $\neg A$ is a WFF.
- ③ If A, B are WFFs, then $(A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B)$ are WFFs
- ④ WFFs are results of finitely many applications of 1, 2, 3.

Remark: well-formed formulas = propositional formulas = formulas

Use tree structure to check

- $\neg(p \wedge q) \rightarrow (r \wedge s)$
(T)
- $(p \wedge q) \neg r$
(F)
- $m \leftrightarrow ((p \wedge q) \rightarrow (\neg r \wedge s))$
(T)



Summary

Proposition: a declarative sentence that is either true or false.

- simple, compound, propositional constant/variable

Logical Connectives: \neg (unary) $\wedge, \vee, \rightarrow, \leftrightarrow$ (binary)

- Truth table
- Example 14 (Textbook Page 11)

Well-Formed Formulas: formulas

- propositional constant, variables
- $\neg A, (A \wedge B), (A \vee B), (A \leftarrow B), (A \leftrightarrow B)$
- Finite

Precedence of Logical Operators

Precedence (priority): $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

- formulas inside $()$ are computed firstly
- different connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ (Decreasing Precedence)
- same connectives: from left to the right

• Example 1: $\neg p \wedge q$: $(\neg p) \wedge q$

• Example 2: $\neg(p \wedge q)$: First $(.)$, then \neg .

• Example 3: $p \vee q \wedge r$: $p \vee (q \wedge r)$

• Example 4:

$$\underline{\underline{((p \rightarrow q) \wedge (q \rightarrow r)) \leftrightarrow (p \rightarrow r)}}$$

From Natural Language to WFFs

The Method of Translation:

- Introduce symbols to represent simple propositions
- Connect the symbols with logical connectives to obtain WFFs

Example:

- "It is not the case that snow is black."
 - p : "Snow is black"
 - Translation: $\neg p$
 - **Remark:** it is better to choose the simple proposition to be affirmative sentence. *positive*
- " π and e are both irrational"
 - p : " π is irrational"; q : " e is irrational"
 - Translation: $p \wedge q$
- "If π is irrational, then 2π is irrational"
 - p : " π is irrational"; q : " 2π is irrational"
 - Translation: $p \rightarrow q$

Example:

- " $e^\pi > \pi^e$ if and only if $\pi > e \ln \pi$ "
 - $p : e^\pi > \pi^e$; $q : \pi > e \ln \pi$
 - Translation: $p \leftrightarrow q$
- “ $(\sqrt{2})^{\sqrt{2}}$ is rational or irrational.” (ambiguity in natural language)
 - $p = “(\sqrt{2})^{\sqrt{2}}$ is rational ”; $q = “(\sqrt{2})^{\sqrt{2}}$ is irrational”
 - Explanation 1: $(\sqrt{2})^{\sqrt{2}}$ cannot be neither rational nor irrational.
 - **Emphasis:** $(\sqrt{2})^{\sqrt{2}}$ is a real number, only two possibility
 - Translation 1: $p \vee q$ (by default, this is the translation of “or”)
 - Explanation 2: $(\sqrt{2})^{\sqrt{2}}$ cannot be both rational or irrational
 - It is obvious that $(\sqrt{2})^{\sqrt{2}}$ is real number. **Emphasis:** not both
 - Translation 2: $(p \wedge \neg q) \vee (\neg p \wedge q)$ (not both)
- The specific translations remove the ambiguity.