Discrete Mathematics

$S_2(n, j)$, $p_j(n)$, principle of inclusion–exclusion, pigeonhole principle

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Summary of Lecture 12

$\{x_1+x_2+\cdots+x_n=r$
$0\le x_i\le r$

r-Combination of Set $A=\{a_1,a_2,\dots,a_n\}$ $\binom{n}{r}=\frac{n!}{r!(n-r)!}$

- Without repetition: an r-subset of A;
- With repetition: an r-multiset of the form $\{x_1\cdot a_1,\dots,x_n\cdot a_n\}$

r-Combination of Multiset $A=\{n_1\cdot a_1,n_2\cdot a_2,\dots,n_k\cdot a_k\}$

- an r-subset of A

乘序
第一项：a_s,
b_s,

$\{a_n\}_{n\ge s}$ $\xrightarrow{B.T.}$ $\{b_n\}_{n\ge s}$

Binomial Transform: $b_n=\sum_{k=s}^{n}\binom{n}{k}a_k$

Inverse Binomial Transform: $b_n=\sum_{k=s}^{n}(-1)^{n-k}\binom{n}{k}a_k$

- $a_n=\sum_{k=s}^{n}\binom{n}{k}b_k\Rightarrow b_n=\sum_{k=s}^{n}(-1)^{n-k}\binom{n}{k}a_k\ (n\ge s)$

Distributing Objects into Boxes: n objects, k boxes

- Labeled/unlabeled objects + labeled/unlabeled box

① Labeled 多重集全排 labeled ③ labeled Unlabeled
② unlabeled 可重复组合 Labeled ④ unlabeled unlabeled

用法：
化数问题中找到
$\{a_n\}$, $\{b_n\}$
且 $b_n=$ B.T.(a_n)
$\Rightarrow$ $b_n=$ I.B.T.(a_n)

(-- labeled objects, -- unlabeled boxes

(n label objects, j

$S_2(n,j)$ no empty box.)

THEOREM: $S_2(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n$ when $n \geq j \geq 1$.

$n<j$ default $S_2(n,j)=0$

Simpler case:

- $T(n,j)$: the number of ways of distributing n labeled objects into j labeled boxes such that no box is empty
  - $T(n,j) = j! \cdot S_2(n,j)$ labelling boxes.

Simpler-er case: box labelled

  - $T(n,j) = ?$
- X: the set of ways of distributing n labeled objects into j labeled boxes.
  - By the product rule, $|X| = j^n$ (不要求不空)
- $X_i \subseteq X$: the set of ways where exactly i boxes are used, $i = 1,2,\ldots,j$
  - 将X的方案分成j类
  - $\{X_1, X_2, \ldots, X_j\}$ is a partition of X and $|X_i| = \boxed{\binom{j}{i}} T(n,i)$ choose i boxes.
  - (实际)占1个 占2个 B.T.
  - 方案数 $j^n = |X| = \sum_{i=1}^{j}|X_i| = \sum_{i=1}^{j}\binom{j}{i}T(n,i)$

target: find $T(n,j) = \sum_{i=1}^{j}(-1)^{j-i}\binom{j}{i}i^n = \sum_{i=0}^{j-1}(-1)^t\binom{j}{i}(j-t)^n$ //inversion

下标替换 $j-i=t$, i(右边i) $1\leq i\leq j$, $0\leq j-i\leq j-1$, $0\leq t\leq j-1$

- $S_2(n,j) = \frac{1}{j!}\cdot T(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$

here: $a_j = j^n$ $a_j' = B.T.(b_j)$ $\Leftarrow \{ b_i = T(n,i)$ $\Rightarrow b_j = I.B.T.(a_j)$ 二项式反演! t再替换回i

$a_n = \sum_{k=0} \binom{n}{k} b_k$

$b_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k.$

指标 n, k, s

$a_k \quad b_k.$

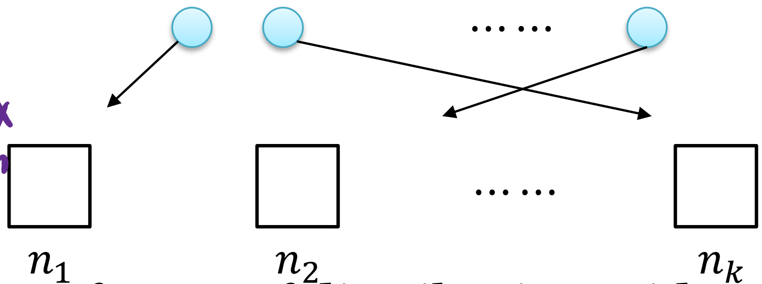
Type 4

n unbabled obj k unlabled box.

Problem: distributing n unlabeled objects into k unlabled boxes

$123 = 321$

\# of obj the box contain



Classifications

$$n_1 + n_2 + \cdots + n_k = n$$
$$n_1, n_2, \ldots, n_k \in \mathbb{N}$$
$$n_1 \geq n_2 \geq \cdots \geq n_k$$

unlabled (order not matter)

EXAMPLE: # of ways of distributing 4 identical books into 3 identical boxes.

$$\begin{cases} n_1 + n_2 + n_3 = 4 \\ n_1 \geq n_2 \geq n_3 \geq 0. \end{cases}$$

- 4 0 0
- 3 1 0
- 2 2 0
- 2 1 1

partition method classifier

REMARK: The schemes are determined by $\{n_1, \ldots, n_k\}$

Partitions of Integers

DEFINITION: $n = a_1 + a_2 + \cdots + a_j$ is called an ***n*-partition** with exactly j parts if $a_1 \geq a_2 \geq \cdots \geq a_j$ are all positive integers.

- $p_j(n) = \overset{\#}{|}\{(a_1, \ldots, a_j): a_1 + \cdots + a_j = n, a_1 \geq a_2 \geq \cdots \geq a_j \geq 1 \text{ are integers}\}|$
  - $p_j(n)$: # of ways of writing n as the sum of j positive integers.

EXAMPLE: The integer 4 has four different partitions:

- $4 = 4$
- $4 = 3 + 1$
- $4 = 2 + 2$
- $4 = 2 + 1 + 1$

REMARK: solution to the type 4 problem$=\sum_{j=1}^{k} p_j(n)$

partition into #

$P_k(n) = P_k(n-k) + P_{k-1}(n-1)$

Partitions of Integers

Combinational proof example

THEOREM: For $n \in \mathbb{Z}^+, j \in [n]$, $p_j(n+j) = \sum_{k=1}^{j} p_k(n)$

- $p_1(n) = 1$, $p_n(n) = 1$
- Let $S_k = \{\text{partitions of } n \text{ into } k \text{ positive integers}\}, k \in [j]$
- Let $S = \bigcup_{k=1}^{j} S_k$. $\Sigma p_k(n)$
  - $|S| = |S_1| + \cdots + |S_j| = p_1(n) + \cdots + p_j(n)$
- Let $T = \{\text{partitions of } n+j \text{ into } j \text{ positive integers}\}$

proof1

bij 证

  - $|T| = p_j(n+j)$ $n_1 \geq n_2 \geq \cdots n_k \geq 1$, $n_1 + \cdots n_k = n$.
- $f: S \to T \quad (n_1, \ldots, n_k) \mapsto (\underbrace{n_1 + 1, \ldots, n_k + 1}_{n_1 + \cdots n_k = \text{sum} = n}, \underbrace{1, \ldots, 1}_{j-k})$

逆映射证

  - f is bijective
  - $|T| = |S|$

+1 → sum+k

sum $= n + k + j - k$
$= n + j$

EXAMPLE: determine $p_3(6)$ and $p_4(6)$ with the above theorem

- $p_3(6) = p_3(3+3) = p_1(3) + p_2(3) + p_3(3) = 1 + 1 + 1 = 3$
- $p_4(6) = p_4(2+4) = p_1(2) + p_2(2) + p_3(2) + p_4(2) = 1 + 1 + 0 + 0 = 2$

Computing $p_j(n)$ Recursively

$p_1(n)=1$

1

$p_1(1)$

1 $\quad p_1(2) \quad p_2(2) \quad$ 1

1 $\quad p_1(3) \quad$ 2 $\quad p_2(3) \quad p_3(3) \quad$ 1

1 $\quad p_1(4) \quad$ 2 $\quad p_2(4) \quad p_3(4) \quad p_4(4) \quad$ 1

$p_1(5) \quad p_2(5) \quad p_3(5) \quad p_4(5) \quad p_5(5)$

$p_1(6) \quad p_2(6) \quad p_3(6) \quad p_4(6) \quad p_5(6) \quad p_6(6)$

$p_1(7) \quad p_2(7) \quad p_3(7) \quad p_4(7) \quad p_5(7) \quad p_6(7) \quad p_7(7)$

$p_1(8) \quad p_2(8) \quad p_3(8) \quad p_4(8) \quad p_5(8) \quad p_6(8) \quad p_7(8) \quad p_8(8)$

$p_1(9) \quad p_2(9) \quad p_3(9) \quad p_4(9) \quad p_5(9) \quad p_6(9) \quad p_7(9) \quad p_8(9) \quad p_9(9)$

$p_n(n)=1$

$p_j(n+j)=\sum_{k=1}^{j} p_k(n)$

客厅

Principle of Inclusion–Exclusion

subset

Problem: S is a finite set and $A_1, A_2, \ldots, A_n \subseteq S$.

- $|\bigcup_{i=1}^{n} A_i| = ?$
- $|\bigcap_{i=1}^{n} A_i| = ?$

\#

∪求解 ⇄ ∩求解

n!个全排列

EXAMPLE: Let S be the set of permutations of $[n]$. Find $|A|$ for

第i位数≠i

$$A = \{x_1x_2 \ldots x_n : x_1x_2 \ldots x_n \in S;\ x_i \neq i \text{ for all } i \in [n]\}.$$

- $A_i = \{x_1x_2 \cdots x_n : x_1x_2 \cdots x_n \in S; x_i = i\}, i = 1,2, \ldots, n$
  - $A = S - \bigcup_{i=1}^{n} A_i$
  - $|S| = n!$
  - $|\bigcup_{i=1}^{n} A_i| = ?$

reverse: $x_i = i$

method.

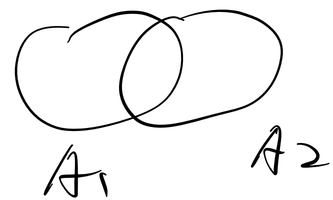
Principle of IE (Two Sets)

THEOREM: Let S be a finite set. Let A_1, A_2 be subsets of S. Then

- $|S - A_1| = |S| - |A_1|$; $|A_1 - A_2| = |A_1| - |A_1 \cap A_2|$
  - $S = A_1 \cup (S - A_1)$, $A_1 \cap (S - A_1) = \emptyset$;
    - $\{A_1, S - A_1\}$ is a partition of S
      - $|S| = |A_1| + |S - A_1|$
        - $|S - A_1| = |S| - |A_1|$
    - $A_1 - A_2 = A_1 - A_1 \cap A_2$
      - $|A_1 - A_2| = |A_1| - |A_1 \cap A_2|$

加法原则：

$|A_1 \cup A_2| = |A_1| + |A_2|$



- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
  - $A_1 \cup A_2 = (A_1 - A_2) \cup A_2, (A_1 - A_2) \cap A_2 = \emptyset$; partition !
    - $\{A_1 - A_2, A_2\}$ is a partition of $A_1 \cup A_2$
      - $|A_1 \cup A_2| = |A_1 - A_2| + |A_2| = |A_1| - |A_1 \cap A_2| + |A_2|$
- $|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2|$

Principle of IE (Three Sets)

THEOREM: Let S be a finite set. Let A_1, A_2, A_3 be subsets of S.

Then $\left|\bigcup_{i=1}^{3} A_i\right| = \sum_{t=1}^{3}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le 3} |A_{i_1} \cap \cdots \cap A_{i_t}|$

$= (-1)^{1-1}\sum_{1\le i_1\le 3}|A_{i_1}| + (-1)^{2-1}\sum_{1\le i_1\le i_2\le 3}|A_{i_1}\cap A_{i_2}| + (-1)^{3-1}\sum_{1\le i_1\le i_2<i_3\le 3}|A_{i_1}\cap A_{i_2}\cap A_{i_3}|$

- $\left|\bigcup_{i=1}^{3} A_i\right| = |(A_1 \cup A_2) \cup A_3| = |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3|$

  $|A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|) + |A_1 \cap A_2 \cap A_3|$

  - $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
  - $|(A_1 \cup A_2) \cap A_3| = |(A_1 \cap A_3) \cup (A_2 \cap A_3)|$

$|A_1 \cup A_2 \cup A_3|$

$= |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3|$

$$= |A_1 \cap A_3| + |A_2 \cap A_3| - |(A_1 \cap A_3) \cap (A_2 \cap A_3)|$$

$$= |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

- $\left|\bigcup_{i=1}^{3} A_i\right| = \underbrace{|A_1| + |A_2| - |A_1 \cap A_2|}_{|A_1 \cup A_2|} + |A_3|$

  $-(|A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|)$

- $\left|\bigcap_{i=1}^{3} A_i\right| = \sum_{t=1}^{3}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le 3} |A_{i_1} \cup \cdots \cup A_{i_t}|$

Principle of IE (n Sets)

THEOREM: Let S be a finite set. Let $A_1, A_2, \dots, A_n$ be subsets of S.

Then $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{t=1}^{n} (-1)^{t-1} \sum_{1 \le i_1 < \cdots < i_t \le n} \left|A_{i_1} \cap \cdots \cap A_{i_t}\right|$

- $n = 1$: $|A_1| = |A_1|$
- $n = 2$: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- $n = 3$: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3|$
  $-|A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$
- **Induction hypothesis**: the identity holds for $n \le k$ ($k \ge 3$)
- Need to show the identity for $n = k + 1$
- $|\underbrace{A_1 \cup \cdots \cup}_{1} \underbrace{A_{k+1}}_{2}| = |A_1 \cup \cdots \cup A_k| + |A_{k+1}| - |(A_1 \cup \cdots \cup A_k) \cap A_{k+1}|$
  $= \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right|$

Principle of IE (n Sets)

- $\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{t=1}^{k}(-1)^{t-1}\sum_{1\le i_1<\cdots<i_t\le k}|A_{i_1}\cap\cdots\cap A_{i_t}|$
- $\left|\bigcup_{i=1}^{k}(A_i\cap A_{k+1})\right| = \sum_{t=1}^{k}(-1)^{t-1}\sum_{1\le i_1<\cdots<i_t\le k}|(A_{i_1}\cap A_{k+1})\cap\cdots\cap(A_{i_t}\cap A_{k+1})|$
- $\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{t=1}^{k}(-1)^{t-1}\sum_{1\le i_1<\cdots<i_t\le k}|A_{i_1}\cap\cdots\cap A_{i_t}| + |A_{k+1}| -$

归纳:

$$\sum_{t=1}^{k}(-1)^{t-1}\sum_{1\le i_1<\cdots<i_t\le k}|(A_{i_1}\cap A_{k+1})\cap\cdots\cap(A_{i_t}\cap A_{k+1})|$$

$t=1,\ t=2,\ \ldots$

左=右

$$= \sum_{t=1}^{k+1}(-1)^{t-1}\sum_{1\le i_1<\cdots<i_t\le k+1}|A_{i_1}\cap\cdots\cap A_{i_t}|$$

THEOREM: Let S be a finite set. Let $A_1, A_2, \ldots, A_n$ be subsets of S.

Then $|\bigcap_{i=1}^{n} A_i| = \sum_{t=1}^{n}(-1)^{t-1}\sum_{1\le i_1<\cdots<i_t\le n}|A_{i_1}\cup\cdots\cup A_{i_t}|$

$\cup$ $\cap$

TH:

$\cap$ $\cup$

Principle of Inclusion-Exclusion

EXAMPLE: Let S be the set of permutations of $[n]$. Find $|A|$ for

$$A = \{x_1 x_2 \dots x_n : x_1 x_2 \dots x_n \in S;\ x_i \neq i \text{ for all } i \in [n]\}.$$

- $A_i = \{x_1 x_2 \cdots x_n : x_1 x_2 \cdots x_n \in S; x_i = i\}, i = 1,2, \dots, n$
  - $A = S - \bigcup_{i=1}^{n} A_i$
    - $|S| = n!$
    - $|\bigcup_{i=1}^{n} A_i| = ?$
- $|\bigcup_{i=1}^{n} A_i| = \sum_{t=1}^{n} (-1)^{t-1} \underbrace{\sum_{1 \le i_1 < \cdots < i_t \le n}}_{\binom{n}{t}} \overbrace{|A_{i_1} \cap \cdots \cap A_{i_t}|}^{(n-t)!}$
  - $|A_{i_1} \cap \cdots \cap A_{i_t}| = (n-t)!$ for $t = 1,2, \dots, n$

$\{x_{i_1}=i_1,\ x_{i_2}=i_2,\ x_{i_t}=i_t \dots\}$ → 剩下(n-t)个数无动，全排列

- $|A| = |S| - |\bigcup_{i=1}^{n} A_i|$

$$= n! - \left(\binom{n}{1} * (n-1)! - \binom{n}{2} * (n-2)! + \cdots + (-1)^{n-1} * \binom{n}{n} * 1\right)$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^t \frac{1}{t!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

抽屉

Pigeonhole Principle

less cable

任意10个

EXAMPLE: Connect 15 workstations $W_1, \ldots, W_{15}$ to 10 servers $S_1, \ldots, S_{10}$ such that any ≥ 10 workstations have access to all servers. How many cables are needed?

- **Solution 1**: Connecting every workstation directly to every server. 150
- **Solution 2**: S_i is connected to W_i for every $i \in [10]$; and each of $W_{11}, W_{12}, W_{13}, W_{14}, W_{15}$ is connected to all servers.
  - This solution requires 60 lines.
  - Is this solution optimal?

5×10

W1 W2 ... W10 W11 ... W15

S1 S2 ... S10

10

覆盖

Cover

if any $A_i \cap A_j = \emptyset$ $\Rightarrow$ partition

DEFINITION: A **cover** of a finite set A is a family $\{A_1, A_2, \dots, A_n\}$ of subsets of A such that $\bigcup_{i=1}^{n} A_i = A$.

cover

P

LEMMA: Let $\{A_1, A_2, \dots, A_n\}$ be a cover of a finite set A. Then $|A| \le \sum_{i=1}^{n} |A_i|$. #

- $n = 1$: $|A| = |A_1|$
- $n = 2$: $|A| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \le |A_1| + |A_2|$
- Suppose true when $n \le k$ ($k \ge 2$).
- When $n = k + 1$, $|A| = \left|\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right|$

$$\le \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}|$$

$$\le \underline{\sum_{i=1}^{k} |A_i|} + |A_{k+1}|$$

$$= \sum_{i=1}^{k+1} |A_i|$$

pre

Th

Pigeonhole Principle

抽屉

元素个数≥1

THEOREM: (simple form) Let A be a set with $\geq n+1$ elements.

覆盖

Let $\{A_1, A_2, \ldots, A_n\}$ be a cover of A. Then $\exists k \in [n], |A_k| \geq 2$.

- Suppose that $|A_i| \leq 1$ for every $i \in [n]$. Then $n+1 \leq |A| \leq \sum_{i=1}^{n} |A_i| \leq n$.
  - If $\geq n + 1$ objects are distributed into n boxes, then there is at least one box containing ≥ 2 objects.

THEOREM: (general form) Let A be a set with $\geq N$ elements.

Let $\{A_1, A_2, \ldots, A_n\}$ be a cover of A. Then $\exists k \in [n], |A_k| \geq \lceil N/n \rceil$.

- If $|A_i| < \lceil N/n \rceil$ for all $i \in [n]$, then $N \leq |A| \leq \sum_{i=1}^{n} |A_i| < n \cdot N/n = N$
  - If we distribute $\geq N$ objects into n boxes, then there is at least one box that contains $\geq \lceil N/n \rceil$ objects.

$\cancel{\leq n\lceil \frac{A}{n} \rceil < n\frac{N}{n}}$

向上取整