Discrete Mathematics: Lecture 21

predicate logic, WFFs, from NL to WFFs, logic equivalence, tautological implication

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Well-Formed Formulas

DEFINIITON: well-formed formulas合式公式/formulas

1) propositional constants, propositional variables, and propositional functions without connectives are WFFs
2) If A is a WFF, then $\neg A$ is also a WFF
3) If A, B are WFFs and there is no individual variable x which is bound in one of A, B but free in the other, then $(A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B)$ are WFFs.
4) If A is a WFF with a free individual variable x, then $\forall x\, A, \exists x\, A$ are WFFs.
5) WFFs can be constructed with 1)-4).
   - Example: $\forall x\, F(x) \vee G(x), \forall x P(x)$ are not WFFs
   - Example: $\exists x\, (A(x) \rightarrow \forall y\, B(x, y))$ is a WFF

must have

Precedence: $\forall, \exists$ have higher precedence than $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

   - $\forall x\, P(x) \rightarrow Q(y)$ means $(\forall x P(x)) \rightarrow Q(y)$, not $\forall x (P(x) \rightarrow Q(y))$

Should !

From Natural Language to WFFs

The Method of Translation:

- Introduce symbols to represent propositional constants, propositional variables, individual constants, individual variables, predicate constants, predicate variables, functions of individuals
- Construct WFFs with 1)-4) such that WFFs reflect the real meaning of the natural language

EXAMPLE: All irrational numbers are real numbers.

- Every irrational number is a real number.
- For every x, if x is an irrational number, then x is a real number.
  - $I(x) =$ "x is an irrational number"
  - $R(x) =$ "x is a real number"
    - Translation: $\forall x (I(x) \rightarrow R(x))$

$$\forall \qquad \rightarrow \qquad fixed$$
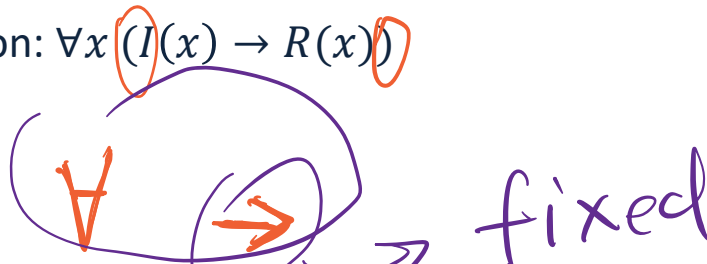
From Natural Language to WFFs

EXAMPLE: Some real numbers are irrational numbers.
- There is a real number which is also an irrational number.
- There is an x such that x is a real number and also an irrational number.
  - $I(x) =$ "x is an irrational number"
  - $R(x) =$ "x is a real number"
    - Translation: $\exists x \, (R(x) \wedge I(x))$

$\exists \, \bigcirc \; \textit{fixed}$

EXAMPLE: There is a symbol that can not be understood by any person's brain.
- There is a symbol such that any person's brain can not understand it.
- There is an x such that x is a symbol and any person's brain can not understand x.
  - $S(x)$: "x is a symbol"
  - Translation: $\exists x \, (S(x) \wedge (\cdots))$

From Natural Language to WFFs

Do not change origin sentence, do not do equal

EXAMPLE: There is a symbol that can not be understood by any person's brain.

- Any person's brain can not understand x.
- For any y, if y is a person, then y's brain cannot understand x.
  - $P(y)$: "y is a person"
  - Translation: $\forall y \ (P(y) \rightarrow (\cdots))$
- y's brain cannot understand x
  - $U(z, x)$: "z can understand x"
  - $b(y) =$ the brain of y
    - Translation: $\neg U(b(y), x)$
- Translation: $\exists x \ \big(S(x) \wedge \forall y \ (P(y) \rightarrow \neg U(b(y), x)) \big)$

no exist $\cdots$

Interpretation

給 variable 賦值

DEFINITION: an **interpretation**解释 requires one to (remove all uncertainty)
- assign a concrete proposition to every **proposition variable**
- assign a concrete predicate to every **predicate variable**
- restrict the domain of every **bound individual variable**
- assign a concrete individual to every **free individual variable**
- choose a concrete **function**, if there is any

EXAMPLE: $\exists x P(x) \rightarrow q$
- Domain of x={Alice, Bob, Eve}
- $P(x) =$ "x gets A+"
- $q =$ "I get A+"
- If at least one of Alice, Bob, and Eve gets A+, then I get A+.

Types of WFFs

DEFINITION: A WFF is **logically valid**普遍有效 if it is **T** in every interpretation

- $\forall x \, (P(x) \lor \neg P(x))$ is logically valid

DEFINITION: A WFF is **unsatisfiable**不可满足 if it is **F** in every interpretation

- $\exists x \, (P(x) \land \neg P(x))$ is unsatisfiable

DEFINITION: A WFF is **satisfiable**可满足 if it is **T** in some interpretation

- $\forall x \, (x^2 > 0)$
  - true when domain= nonzero real numbers

THEOREM: Let A be any WFF. A is logically valid iff $\neg A$ is unsatisfiable.

Rule of Substitution: Let A be a tautology in propositional logic. If we substitute any propositional variable in A with an arbitrary WFF from predicate logic, then we get a logically valid WFF.

- $p \lor \neg p$ is a tautology; hence, $P(x) \lor \neg P(x)$ is logically valid

Logical Equivalence

DEFINITION: Two WFFs A,B are **logically equivalent**等值 if they always have the same truth value in every interpretation.

- notation: $A \equiv B$; example: $\forall x \, P(x) \wedge \forall x \, Q(x) \equiv \forall x \, (P(x) \wedge Q(x))$

THEOREM: $A \equiv B$ iff $A \leftrightarrow B$ is logically valid.

- $A \equiv B$
- iff A, B have the same truth value in every interpretation I
- iff $A \leftrightarrow B$ is true in every interpretation I
- iff $A \leftrightarrow B$ is logically valid

THEOREM: $A \equiv B$ iff $A \rightarrow B$ and $B \rightarrow A$ are both logically valid.

- $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

Rule of Substitution

METHOD*: Applying the rule of substitution to the logical equivalences in propositional logic, we get logical equivalences in predicate logic.*

$$P \lor Q \equiv Q \lor P \quad A(x) \lor B(y) \equiv B(y) \lor A(x)$$

$$(P \land Q) \land R \equiv P \land (Q \land R) \quad (A(x) \land B(y)) \land c \equiv A(x) \land (B(y) \land c)$$

$$P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R) \quad A(x) \land (B(y) \lor c) \equiv (A(x) \land B(y)) \lor (A(x) \land c)$$

$$P \land (P \lor Q) \equiv P \quad A(x) \land (A(x) \lor B(y)) \equiv A(x)$$

$$\neg(P \land Q) \equiv \neg P \lor \neg Q \quad \neg(A(x) \land B(y)) \equiv \neg A(x) \lor \neg B(y)$$

$$P \rightarrow Q \equiv \neg P \lor Q \quad A(x) \rightarrow (\forall y\, B(y)) \equiv \neg A(x) \lor (\forall y\, B(y))$$

$$P \leftrightarrow Q \equiv (P \rightarrow Q) \land (Q \rightarrow P) \quad A(x) \leftrightarrow c \equiv (A(x) \rightarrow c) \land (c \rightarrow A(x))$$

var / constant both o/c

De Morgan's Laws for Quantifiers

THEOREM: $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

- Show that $\neg \forall x\, P(x) \rightarrow \exists x\, \neg P(x)$ is logically valid
  - Suppose that $\neg \forall x\, P(x)$ is **T** in an interpretation I
  - $\forall x\, P(x)$ is **F** in I
  - There is an x_0 such that $P(x_0)$ is **F** in I
  - There is an x_0 such that $\neg P(x_0)$ is **T** in I
  - $\exists x\, \neg P(x)$ is **T** in I

$\longrightarrow inner$

- Show that $\exists x\, \neg P(x) \rightarrow \neg \forall x\, P(x)$ is logically valid
  - Suppose that $\exists x \neg P(x)$ is **T** in an interpretation I
  - There is an x_0 such that $\neg P(x_0)$ is **T** in I
  - There is an x_0 such that $P(x_0)$ is **F** in I
  - $\forall x P(x)$ is **F** in I
  - $\neg \forall x P(x)$ is **T** in I

THEOREM: $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$.

De Morgan's Laws for Quantifiers

EXAMPLE: $R(x)$:"x is a real number"; $Q(x)$:"x is a rational number"

- $\neg \forall x \ (R(x) \rightarrow Q(x))$
  - Not all real numbers are rational numbers
- Negation: $\exists x \ \neg(R(x) \rightarrow Q(x)) \equiv \exists x \ (R(x) \wedge \neg Q(x))$
  - There is a real number which is not rational

EXAMPLE: Let the domain be the set of all real numbers. Let $Q(x)$:"x is a rational number" and $I(x)$:"x is an irrational number"

- $\neg \exists x \ (Q(x) \wedge I \ (x))$
  - No real number is both rational and irrational.
- Negation: $\forall x \ \neg(Q(x) \wedge I(x)) \equiv \forall x \ (\neg Q(x) \vee \neg I(x))$
  - Any real number is either not rational or not irrational.

Distributive Laws for Quantifiers

[handwritten: every x in I]

THEOREM: $\forall x \left(P(x) \land Q(x)\right) \equiv \forall x \, P(x) \land \forall x \, Q(x)$

- Show that $\forall x \left(P(x) \land Q(x)\right) \to \forall x \, P(x) \land \forall x \, Q(x)$ is logically valid
  - Suppose that $\forall x \left(P(x) \land Q(x)\right)$ is **T** in an interpretation I *[handwritten: proof]*
  - $P(x) \land Q(x)$ is **T** for every x in I *[handwritten: in this interpretation ⇐ Method!]*
  - $P(x)$ is **T** for every x in I and $Q(x)$ is **T** for every x in I
  - $\forall x \, P(x)$ is **T** in I and $\forall x \, Q(x)$ is **T** in I
  - $\forall x \, P(x) \land \forall x \, Q(x)$ is **T** in I

- Show that $\forall x \, P(x) \land \forall x \, Q(x) \to \forall x \left(P(x) \land Q(x)\right)$ is logically valid.
  - Suppose that $\forall x \, P(x) \land \forall x \, Q(x)$ is **T** in an interpretation I
  - $\forall x \, P(x)$ is **T** in I and $\forall x \, Q(x)$ is **T** in I
  - $P(x)$ is **T** for every x in I and $Q(x)$ is **T** for every x in I
  - $P(x) \land Q(x)$ is **T** for every x in I
  - $\forall x \left(P(x) \land Q(x)\right)$ is **T** in I

THEOREM: $\exists x \left(P(x) \lor Q(x)\right) \equiv \exists x \, P(x) \lor \exists x \, Q(x)$.

Tautological Implication

DEFINITION: Let A and B be WFFs in predicate logic. A **tautologically implies**(重言蘊涵) B if every interpretation that causes A to be true causes B to be true.

- notation: $A \Rightarrow B$, called a **tautological implication**(重言蘊涵)

THEOREM: $A \Rightarrow B$ iff $A \to B$ is logically valid.

- $A \Rightarrow B$
- iff every interpretation that causes A to be true causes B to be true
- iff there is no interpretation such that $(A, B) = (\mathbf{T}, \mathbf{F})$
- Iff $A \to B$ is true in every interpretation
- iff $A \to B$ is logically valid

THEOREM: $A \Rightarrow B$ iff $A \wedge \neg B$ is unsatisfiable.

- $A \to B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$

Rule of Substitution

Name	Tautological Implication	NO.
Conjunction(合取)	$(P) \wedge (Q) \Rightarrow P \wedge Q$	1
Simplification(化简)	$P \wedge Q \Rightarrow P$	2
Addition(附加)	$P \Rightarrow P \vee Q$	3
Modus ponens(假言推理)	$P \wedge (P \rightarrow Q) \Rightarrow Q$	4
Modus tollens(拒取)	$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$	5
Disjunctive syllogism(析取三段论)	$\neg P \wedge (P \vee Q) \Rightarrow Q$	6
Hypothetical syllogism(假言三段论)	$(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R)$	7
Resolution (归结)	$(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$	8

EXAMPLE: $P \wedge (P \rightarrow Q) \Rightarrow Q$ is a TI in propositional logic.

- $A(x) \wedge \big(A(x) \rightarrow B(y)\big) \Rightarrow B(y)$ must be a TI in predicate logic.
  - Rule of substitution: let $P = A(x)$ and $Q = B(y)$

Tautological Implications

- $\forall x P(x) \lor \forall x\, Q(x) \Rightarrow \forall x\, (P(x) \lor Q(x))$
- $\exists x \big(P(x) \land Q(x)\big) \Rightarrow \exists x\, P(x) \land \exists x Q(x)$
- $\forall x\, \big(P(x) \to Q(x)\big) \Rightarrow \forall x P(x) \to \forall x\, Q(x)$
- $\forall x\, \big(P(x) \to Q(x)\big) \Rightarrow \exists x\, P(x) \to \exists x\, Q(x)$
- $\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \forall x\, P(x) \leftrightarrow \forall x\, Q(x)$
- $\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \exists x\, P(x) \leftrightarrow \exists x\, Q(x)$
- $\forall x\, \big(P(x) \to Q(x)\big) \land \forall x\, \big(Q(x) \to R(x)\big) \Rightarrow \forall x\, (P(x) \to R(x))$
- $\forall x\, \big(P(x) \to Q(x)\big) \land P(a) \Rightarrow Q(a)$

Examples

EXAMPLE: $\forall x \left(P(x) \to Q(x) \right) \wedge P(a) \Rightarrow Q(a)$
- Suppose that the left hand side is true in an interpretation I (domain=D)
  - $\forall x \left(P(x) \to Q(x) \right)$ is **T** and $P(a)$ is **T**
    - $P(a) \to Q(a)$ is **T** and $P(a)$ is **T**
      - $Q(a)$ is **T** in I.

EXAMPLE: Tautological implication in the following proof?
- All rational numbers are real numbers $\boxed{\forall x(P(x) \to Q(x))}$
- 1/3 is a rational number $\boxed{P(1/3)}$
- 1/3 is a real number $\boxed{Q(1/3)}$
  - $P(x) =$"x is a rational number"
  - $Q(x) =$"x is a real number"
  - rule of inference: $\forall x \left(P(x) \to Q(x) \right) \wedge P(1/3) \Rightarrow Q(1/3)$

Examples

EXAMPLE:$\forall x \left(P(x) \rightarrow Q(x)\right) \wedge \forall x \left(Q(x) \rightarrow R(x)\right) \Rightarrow \forall x \left(P(x) \rightarrow R(x)\right)$

- Suppose that the left hand side is **T** in an interpretation I (domain=D)
  - $\forall x \left(P(x) \rightarrow Q(x)\right)$ is **T** and $\forall x \left(Q(x) \rightarrow R(x)\right)$ is **T**
  - $P(x) \rightarrow Q(x)$ is **T** for all $x \in D$ and $Q(x) \rightarrow R(x)$ is **T** for all $x \in D$
    - $P(x) \rightarrow R(x)$ is **T** for all $x \in D$
    - $\forall x \left(P(x) \rightarrow R(x)\right)$ is **T** in I.

EXAMPLE: Tautological implication in the following proof?
- All integers are rational numbers. $\boxed{\forall x (P(x) \rightarrow Q(x))}$
- All rational numbers are real numbers. $\boxed{\forall x (Q(x) \rightarrow R(x))}$
- All integers are real numbers. $\boxed{\forall x (P(x) \rightarrow R(x))}$
  - $P(x) =$ "x is an integer"
  - $Q(x) =$ "x is a rational number"
  - $R(x) =$ "x is a real number"
  - rule of inference:$\forall x \left(P(x) \rightarrow Q(x)\right) \wedge \forall x \left(Q(x) \rightarrow R(x)\right) \Rightarrow \forall x \left(P(x) \rightarrow R(x)\right)$