Review

Generating Functions

Ex1. Let a_r be the number of ways of distributing r identical books to 4 persons such that person 1 receives an even number (including 0) of books, person 2 receives at most 2 books, person 3 receives 2 or 3 books, and person 4 receives an odd number of books and at least 3 books. Determine a_{24}.

Ex2. Let a_r be the number of elements in $A_r = \{s : s \in \{0, 1, 2\}^r$, s has even number (including 0) of 1 s, odd number of 2 s and no more than two 0 s. $\}$. Determine a_{14}.

Ex3. Let $a_r = |\{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}, x_1 + x_2 + 2x_3 + 3x_4 = r\}|$. Determine a_{11}.

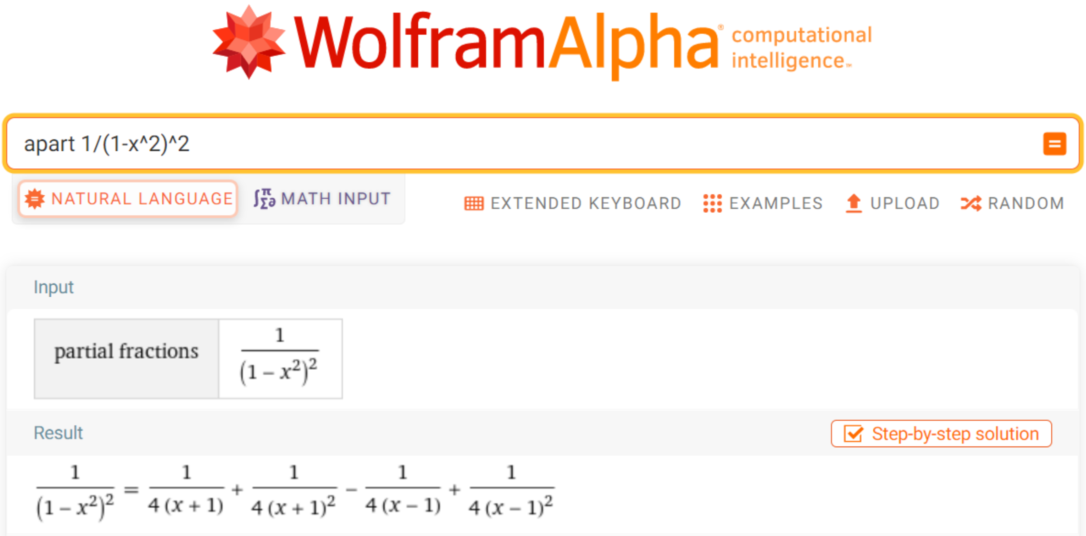
Solutions

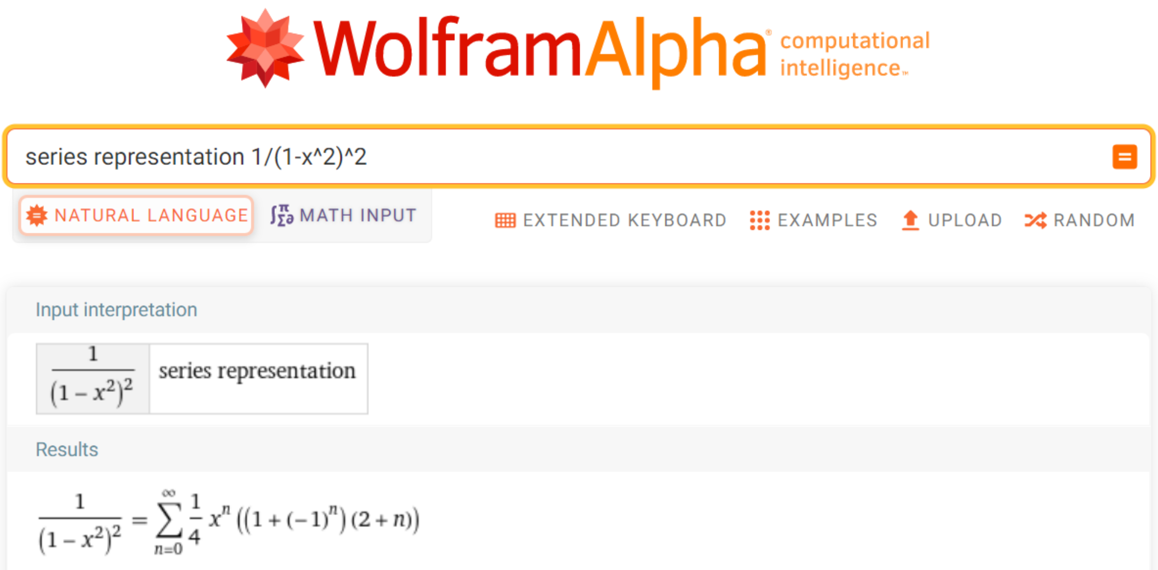
1. $R_1 = \{0, 2, 4, \cdots\}$ $R_2 = \{0, 1, 2\}$ $R_3 = \{2, 3\}$ $R_4 = \{3, 5, 7, \cdots\}$

$$\begin{aligned}
\sum_{r=0}^{\infty} a_r x^r &= \left(1 + x^2 + x^4 + \cdots\right)\left(1 + x + x^2\right)\left(x^2 + x^3\right)\left(x^3 + x^5 + x^7 + \cdots\right) \\
&= \frac{1}{1 - x^2} \cdot \left(1 + x + x^2\right)\left(x^2 + x^3\right) \cdot \frac{x^3}{1 - x^2} \\
&= \frac{1}{4}\left(x^8 + 2x^7 + 2x^6 + x^5\right)\left[\frac{1}{(1+x)^2} + \frac{1}{(1-x)^2} + \frac{1}{1+x} + \frac{1}{1-x}\right] \\
&= \frac{1}{4}\left(x^8 + 2x^7 + 2x^6 + x^5\right)\sum_{r=0}^{\infty}(r+2)\left[1 + (-1)^r\right]x^r
\end{aligned}$$

$$a_{24} = \frac{1}{4}(18 \times 2 + 2 \times 20 \times 2) = 29$$

Use $Wolfram$ to check:





2. See 2021Mid-term Q23.

3. Take $2x_3, 3x_4$ as a whole.

$R_1 = \{0, 1, 2, \cdots\} \quad R_2 = \{0, 1, 2, \cdots\} \quad R_3 = \{0, 2, 4, \cdots\} \quad R_4 = \{0, 3, 6, \cdots\}$

$$\begin{aligned}
\sum_{r=0}^{\infty} a_r x^r &= \left(1 + x + x^2 + \cdots\right)\left(1 + x + x^2 + \cdots\right)\left(1 + x^2 + x^4 + \cdots\right)\left(1 + x^3 + x^6 + \cdots\right) \\
&= \frac{1}{1-x} \cdot \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \\
&= \frac{1}{1+x+x^2} \cdot \frac{1}{16}\left[\frac{8}{(1-x)^4} + \frac{4}{(1-x)^3} + \frac{2}{(1-x)^2} + \frac{1}{1-x} + \frac{1}{1+x}\right] \\
&= \frac{1}{1+x+x^2} \cdot \frac{1}{48} \cdot \sum_{r=0}^{\infty}\left[4r^3 + 30r^2 + 68r + 45 + 3(-1)^r\right]x^r \\
&= \frac{1}{1+x+x^2} \cdot \frac{1}{48} \cdot \sum_{r=0}^{\infty} b_r x^r
\end{aligned}$$

Assume $(1 + x + x^2)^{-1} = \sum_{i=0}^{\infty} c_i x^i$,

$$\begin{cases} 1 = c_0 \\ 0 = c_0 + c_1 \\ 0 = c_i + c_{i+1} + c_{i+2} \end{cases} \Rightarrow \begin{cases} c_i = 1 \ , \ i \bmod 3 = 0 \\ c_i = -1, \ i \bmod 3 = 1 \\ c_i = 0 \ , \ i \bmod 3 = 2 \end{cases}$$

So $a_{11} = \frac{1}{48}\sum_{i=0}^{11} b_i c_{11-i} = 83.$

Homework7 Answers

1.

$$\begin{aligned} p_3(n) &= p_1(n-3) + p_2(n-3) + p_3(n-3) \\ &= 1 + p_2(n-3) + [p_1(n-6) + p_2(n-6) + p_3(n-6)] \\ &= 2 + p_2(n-3) + p_2(n-6) + p_3(n-6) \end{aligned}$$

$$(i)\ \ 2 \nmid n,\ p_2(n-3) = \frac{n-3}{2},\ p_2(n-6) = \frac{n-7}{2},\ p_3(n) = p_3(n-6) + 3$$

$$(ii)\ 2 \mid n,\ p_2(n-3) = \frac{n-4}{2},\ p_2(n-6) = \frac{n-6}{2},\ p_3(n) = p_3(n-6) + 3$$

2.

Define $A_i = \{x : x \in [n],\ p_i \mid x\},$ so $|A_i| = \dfrac{n}{p_i}$.

$$\begin{aligned} \varphi(n) = n - \left|\bigcup_{i=1}^{k} A_i\right| &= n - \sum_{t=1}^{k} (-1)^{t-1} \sum_{1 \le i_1 < \cdots < i_t \le k} |A_i \cap \cdots \cap A_{i_t}| \\ &= n - \sum_{i=1}^{k} |A_i| + \sum_{1 \le i_1 < i_2 \le k} |A_{i_1} \cap A_{i_2}| + \cdots + (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le k} |A_{i_1} \cap \cdots \cap A_{i_k}| \\ &= n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \cdots \frac{n}{p_k}\right) + \left(\frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \cdots + \frac{n}{p_{k-1} p_k}\right) + (-1)^{k-1} \frac{n}{p_1 p_2 \cdots p_k} \\ &= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \end{aligned}$$

3.

Define $A_i = [\frac{i-1}{n}, \frac{i}{n}),\ k_i = \{ia\} = ia - \lfloor ia \rfloor,\ i \in [n].$

$(i)\ \ a \in \mathbb{Z},\ q = pa$, done.

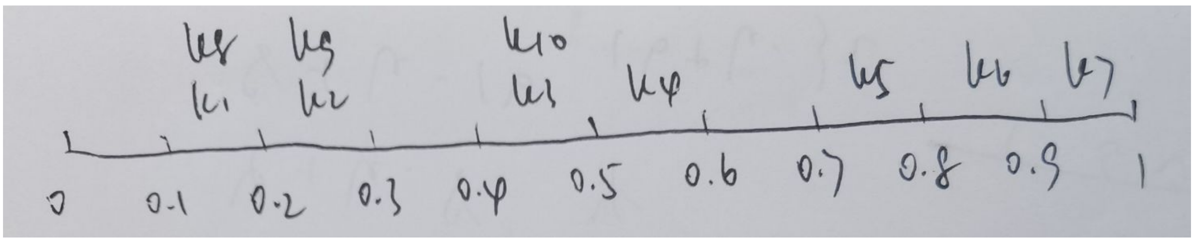
$(ii)\ a \notin \mathbb{Z},$

$(1)\ \exists t \in [n],\ k_t \in A_1$. Then $p = t,\ q = \lfloor ta \rfloor$, done.

$(2)\ \forall t \in [n],\ k_t \notin A_1$. By Pigeonhole Principle, $\exists t \in [n],\ |\{i : k_i \in A_t\}| \ge 2$.

Assume $k_r, k_s \in A_t$, then $p = |r - s|,\ q = |\lfloor ra \rfloor - \lfloor sa \rfloor|$.

E.g.: $a = \pi, n = 10$.



$r = 1, 1 \cdot \pi = 3.1415, \quad s = 8, 8 \cdot \pi = 25.1327$
$|0.1415 - 0.1327| < 0.1$
$p = 8 - 1 = 7, \quad q = 25 - 3 = 22$
$|pa - q| = |7\pi - 22| = |(8-1)\pi - (25-3)| = |(8\pi - 25) - (\pi - 3)| = |0.1415 - 0.1327| < 0.1$

4.

$$r^4 - 8r^2 + 16 = 0 \Rightarrow r_1 = r_2 = 2 \quad r_3 = r_4 = -2$$

$$a_n = \alpha_{1,0} \cdot 2^n + \alpha_{1,1} \cdot n \cdot 2^n + \alpha_{2,0} \cdot (-2)^n + \alpha_{2,1} \cdot n(-2)^n$$

$$\begin{cases} a_0 = \alpha_{1,0} \cdot 1 + \alpha_{1,1} \cdot 0 \cdot 1 + \alpha_{2,0} \cdot 1 + \alpha_{2,1} \cdot 0 \cdot 1 = 3 \\ a_1 = \alpha_{1,0} \cdot 2 + \alpha_{1,1} \cdot 1 \cdot 2 + \alpha_{2,0} \cdot (-2) + \alpha_{2,1} \cdot 1 \cdot (-2) = 6 \\ a_2 = \alpha_{1,0} \cdot 4 + \alpha_{1,1} \cdot 2 \cdot 4 + \alpha_{2,0} \cdot 4 + \alpha_{2,1} \cdot 2 \cdot 4 = 44 \\ a_3 = \alpha_{1,0} \cdot 8 + \alpha_{1,1} \cdot 3 \cdot 8 + \alpha_{2,0} \cdot (-8) + \alpha_{2,1} \cdot 3 \cdot (-8) = 56 \end{cases} \Rightarrow \begin{cases} \alpha_{1,0} = 2 \\ \alpha_{1,1} = 3 \\ \alpha_{2,0} = 1 \\ \alpha_{2,1} = 1 \end{cases}$$

$$\Rightarrow a_n = 2 \cdot 2^n + 3 \cdot n \cdot 2^n + 1 \cdot (-2)^n + 1 \cdot n(-2)^n = (3n+2) \cdot 2^n + (n+1) \cdot (-2)^n$$

5.

$r^2 - 3r + 2 = 0 \Rightarrow r_1 = 1, r_2 = 2, \quad F(n) = n \cdot 2^n$ where $s = 2$ is a root with mutiplicity 1.
Particular solution: $x_n = (p_1 \cdot n + p_0) \cdot 2^n \cdot n^1$
General solution: $y_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n$

$$\Rightarrow \quad z_n = x_n + y_n = \left(p_1 \cdot n^2 + p_0 \cdot n + \alpha_2\right) 2^n + \alpha_1$$

$$\begin{cases} a_0 = \left(p_1 \cdot 0^2 + p_0 \cdot 0 + \alpha_2\right) 2^0 + \alpha_1 = 1 \\ a_1 = \left(p_1 \cdot 1^2 + p_0 \cdot 1 + \alpha_2\right) 2^1 + \alpha_1 = -1 \\ a_2 = \left(p_1 \cdot 2^2 + p_0 \cdot 2 + \alpha_2\right) 2^2 + \alpha_1 = 3 \\ a_3 = \left(p_1 \cdot 3^2 + p_0 \cdot 3 + \alpha_2\right) 2^3 + \alpha_1 = 35 \end{cases} \Rightarrow \begin{cases} p_0 = -1 \quad p_1 = 1 \\ \alpha_1 = 3 \quad \alpha_2 = -2 \end{cases}$$

$$\Rightarrow a_n = \left(n^2 - n - 2\right) \cdot 2^n + 3$$