# Discrete Mathematics

recurrence relation

Liangfeng Zhang
School of Information Science and Technology
ShanghaiTech University

# Summary of Lecture 13

**THEOREM:** $S_2(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n$ when $n \geq j \geq 1$.

**Partition of Integers** $p_j(n)$: the number of ways of writing $n$ as the sum of $j$ positive integers no Exmplict

- solution to the type 4 problem$=\sum_{j=1}^{k} p_j(n)$ and $\begin{cases} P_1(n)=1 \\ P_n(n)=1 \end{cases}$

**THEOREM:** For $n \in \mathbb{Z}^+, j \in [n]$, $p_j(n+j) = \sum_{k=1}^{j} p_k(n)$ $\Rightarrow$ find all $P_j(n)$

**Principle of Inclusion–Exclusion:**

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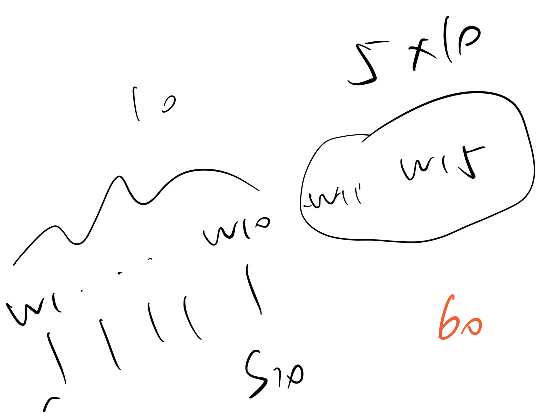
- $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{t=1}^{n}(-1)^{t-1}\sum_{1\leq i_1<\cdots<i_t\leq n}|A_{i_1}\cap\cdots\cap A_{i_t}|$
- $\left|\bigcap_{i=1}^{n} A_i\right| = \sum_{t=1}^{n}(-1)^{t-1}\sum_{1\leq i_1<\cdots<i_t\leq n}|A_{i_1}\cup\cdots\cup A_{i_t}|$

**Pigeonhole Principle** (general form): $\{A_1, A_2, \ldots, A_n\}$ is a cover of $A$ and $|A| \geq N \Rightarrow \exists k \in [n], |A_k| \geq \lceil N/n \rceil$. 忘掉下 n

$A \subseteq \bigcup_{i=1}^{n} A_i$, $A_i \subseteq A$

- $N = n + 1$: $|A_k| \geq 2$ (simple form)

向上取 $|A| \leq |A_n|$

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# Pigeonhole Principle

**EXAMPLE**: Connect 15 workstations $W_1, \ldots, W_{15}$ to 10 servers $S_1, \ldots, S_{10}$ such that any $\geq 10$ workstations have access to all servers. How many cables are needed?

- **Solution 2**: $S_i$ is connected to $W_i$ for every $i \in [10]$; and each of $W_{11}, W_{12}, W_{13}, W_{14}, W_{15}$ is connected to all servers. // 60 lines, optimal?
- Consider an optimal scheme $\Pi$. $< 60?$
  - not connected
  - Let $A = \{(W_i, S_j): i \in [15], j \in [10], W_i \text{ is not connected to } S_j\}$ in $\Pi$
  - $A_t = \{(W_i, S_j) \in A: j = t\}$ for $t = 1,2,\ldots,10$ $|A| > 90$
    - actually - partition
    - $\{A_1, A_2, \ldots, A_{10}\}$ is a cover of $A$ $15 \times 10$
- If there are $< 60$ lines in $\Pi$, then $|A| > 150 - 60 = 90$.
  - pigeon-hole
  - $\exists k \in [10]$ such that $|A_k| \geq \lceil 91/10 \rceil = 10$
    - There are 10 workstations not connected to $S_k$ , contradict!

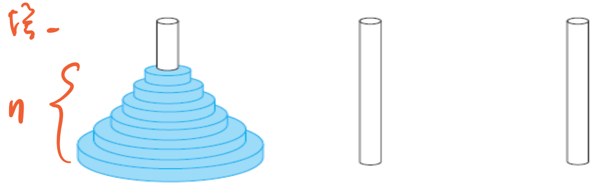
递推关系

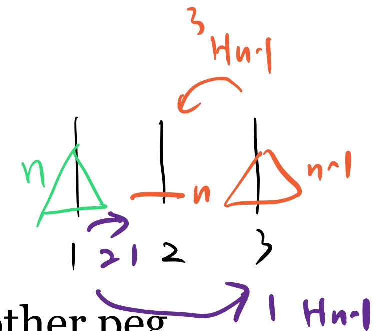
# Recurrence Relation (RR)

**Fibonacci Sequence:** The solution is a sequence $\{f_n\}_{n\geq 0}$ such that $f_0 = 1, f_1 = 1,\ \ f_n = f_{n-1} + f_{n-2}$ for every $n \geq 2$

**The Tower of Hanoi:**

汉诺塔




- Every time move only 1 disk from one peg to another peg
- Always place a smaller disk on top of a larger disk
- Move all the disks from peg 1 to peg 2.
  - $H_n$: the smallest number of moves ($n$ disks).
    - $H_1 = 1, H_2 = 3, H_n = 2H_{n-1} + 1$ for $n \geq 2$

**QUESTION**: $f_n = ?\ \ H_n = ?$ Find explicit formulas.

# Linear Homogeneous RR

k阶常系数线性关系

**DEFINITION:** A **linear homogeneous RR (LHRR) of degree *k* with constant coefficients** is an RR of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, where $n \geq k$, $\{c_i\}_{i=1}^{k}$ are constant real numbers, and $c_k \neq 0$.

degree k

- **degree *k***: every term depends on $k$ terms preceding it
- **constant coefficients:** $c_1, \ldots, c_k$ are independent of $n$
- **linear:** the right-hand side is a linear combination of $a_1, a_2, \ldots, a_{n-1}$.
- **homogeneous:** every term is a multiple of some $a_j$.
  - $f_n = f_{n-1} + f_{n-2}, n \geq 2$ LHRR of degree 2 with constant coefficients
  - $H_n = 2H_{n-1} + 1, n \geq 2$ not homomogenous
- $\{x_n\}_{n\geq 0}$ is a **solution** if $x_n = \sum_{i=1}^{k} c_i x_{n-i}$ for all $n \geq k$

Sequence

non-homo

$f_0$ $f_1$

eg 数: $n \geq 2$

$+C$ ×

$+n^2$ ✗

# Existence and Uniqueness

前k项确定，关系确定，解序列存在并唯一

**THEOREM:** For any $a_0, a_1, \ldots, a_{k-1}$, $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ has a unique solution $\{x_n\}_{n\geq 0}$ such that $x_i = a_i$ for every $0 \leq i < k$.

不是通解..

- **Existence**:
  - $x_n = a_n$ for all $0 \leq n < k$
  - $x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$ for all $n \geq k$
- **Uniqueness**:
  a) $x'_n = a_n$ for all $0 \leq n < k$
  b) $x'_n = c_1 x'_{n-1} + c_2 x'_{n-2} + \cdots + c_k x'_{n-k}$ $(n \geq k)$
  c) $x_n = a_n$ for all $0 \leq n < k$
  d) $x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$ $(n \geq k)$
  - a) + c) $\Rightarrow x'_n = x_n$ for all $0 \leq n < k$
  - b) + d) $\Rightarrow x'_n = x_n$ for all $n \geq k$

# Characteristic Roots

齐次关系的特征方程

**THEOREM:** $\{r^n\}_{n\geq 0}$ is a solution of the LHRR $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$.

- **characteristic equation**: $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$
- **characteristic roots**: solutions of the characteristic equation.

**EXAMPLE:** Solve the LHRR $f_n = f_{n-1} + f_{n-2}, n \geq 2$.

有很多解、

- characteristic equation: $r^2 - r - 1 = 0$

  $c_1 = c_2 = 1$

- characteristic roots: $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$
  - $\{r_1^n\}_{n\geq 0}$, $\{r_2^n\}_{n\geq 0}$ are solutions

特解，但不能找到所解

$f_0 = 1 \quad f_1 = 1$

if $f_n = r^n$

(特征方程)

$r^0=1 \quad r=r+1$ 无重根

# LHRR (no multiple roots)

事实上可能会有重根.

**THEOREM:** If $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ has $k$ distinct characteristic roots $r_1, r_2, \dots, r_k$, then $\{x_n\}_{n\geq 0}$ is a solution of the LHRR iff $x_n = \sum_{j=1}^{k} \alpha_j r_j^n$ for some constants $\alpha_1, \dots, \alpha_k$. $\in \mathbb{R}$

**EXAMPLE:** Solve $f_n = f_{n-1} + f_{n-2}$ with $f_0 = f_1 = 1$. ·· 初始条件

- Characteristic equation: $r^2 - r - 1 = 0$
- Characteristic roots: $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$
- $f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ 对任意初始条件都有效的形式.
  - $f_0 = 1 \Rightarrow \alpha_1 * r_1^0 + \alpha_2 * r_2^0 = 1$
  - $f_1 = 1 \Rightarrow \alpha_1 * r_1^1 + \alpha_2 * r_2^1 = 1$
    - $\alpha_1 = \frac{1}{\sqrt{5}} \cdot \frac{1+\sqrt{5}}{2}, \alpha_2 = -\frac{1}{\sqrt{5}} \cdot \frac{1-\sqrt{5}}{2}$
- $f_n = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \quad (n \geq 0)$

eg. $(r-1)^2(r-2)=0$

什么可能有重根

特征方程 有重根

# LHRR (multiple roots)

**THEOREM:** If $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ has distinct characteristic roots $r_1, r_2, \dots, r_t$ with multiplicities $m_1, m_2, \dots, m_t$, then $\{x_n\}_{n\geq 0}$ is a solution of the LHRR iff $x_n = \sum_{j=1}^{t} \left( \sum_{\ell=0}^{m_j-1} \alpha_{j,\ell}\, n^{\ell} \right) r_j^n$ for some constants $\{\alpha_{j,\ell} : j \in [t], 0 \leq \ell < m_j\}$.

重次

互异

n的函数

$m_1 + m_2 \cdots + m_t = k$

**EXAMPLE**: Solve $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$, $a_1 = 6$.

- Characteristic equation: $r^2 - 6r + 9 = 0$
- Characteristic roots: $r_1 = 3$ $m_1=2$
- $a_n = \alpha_{1,0} 3^n + \alpha_{1,1} n\, 3^n$ $t=1$ $m_j-1=2-1=1 \Rightarrow 0,1$
  - $a_0 = 1 \Rightarrow \alpha_{1,0} * 3^0 + \alpha_{1,1} * 0 * 3^0 = 1$
  - $a_1 = 6 \Rightarrow \alpha_{1,0} * 3^1 + \alpha_{1,1} * 1 * 3^1 = 6$
- $\alpha_{1,0} = 1, \alpha_{1,1} = 1$
- $a_n = 3^n + n3^n = 3^n(n+1)$

任意初始适用

多项式函数

$x_n = (\alpha_{1,1,0} \cdots\cdots) r_1^n$

$+ (\qquad) r_2^n$

$n$ 的多项式，次数小于……

$m_1$

$$\left(a_{1,1,0} + a_{1,1,1} n + a_{1,1,m_1-1} \cdot n^{m_1-1}\right) + \left(\cdots\right) r_E^n$$

$n$ 的多项式

对所有的 $a$ 作赋值

可以得到 $x_n$

$x_n$ 一定是该齐次关系的解

# Linear Nonhomogeneous RR

**DEFINITION:** A **linear nonhomogeneous RR (LNRR) of degree *k* with constant coefficients** is an RR of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$ where $c_1, c_2, \ldots, c_k$ are constants, $c_k \neq 0$, and $F(n) \neq 0$.

n可以有次数

伴随

- **Associated LHRR**: $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ (无F(n))
- $\{x_n\}_{n\geq 0}$ is a **solution** if $x_n = \sum_{i=1}^{k} c_i x_{n-i} + F(n)$ for all $n \geq k$.

c1=1 c2=1 F(n)=

**EXAMPLE:** $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$

- $c_1 = 1, c_2 = 1, F(n) = n^2 + n + 1$
- LNRR of degree 2 with constant coefficients
- associated LHRR: $a_n = a_{n-1} + a_{n-2}$

# Existence and Uniqueness

存在且唯一

**THEOREM:** For any $a_0, a_1, \ldots, a_{k-1}$, $a_n = \sum_{i=1}^{k} c_i a_{n-i} + F(n)$ has a unique solution $\{x_n\}_{n\geq 0}$ such that $x_n = a_n$ for all $0 \leq n < k$.

- **Existence**:
  - $x_n = a_n$ for all $0 \leq n < k$
  - $x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} + F(n)$ for all $n \geq k$
- **Uniqueness**:

  a) $x'_n = a_n$ for all $0 \leq n < k$

  b) $x'_n = c_1\, x'_{n-1} + c_2 x'_{n-2} + \cdots + c_k x'_{n-k} + F(n) \;\; (n \geq k)$

  c) $x_n = a_n$ for all $0 \leq n < k$

  d) $x_n = c_1\, x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} + F(n) \;\; (n \geq k)$

  - a) + c) $\Rightarrow x'_n = x_n$ for all $0 \leq n < k$
  - b) + d) $\Rightarrow x'_n = x_n$ for all $n \geq k$

# General Solutions

Xn是一个特解

**THEOREM:** If $\{x_n\}$ is a solution of $a_n = \sum_{i=1}^{k} c_i a_{n-i} + F(n)$, then $\{z_n\}$ is a solution iff $z_n = x_n + y_n$ for some solution $\{y_n\}$ of the associated LHRR $a_n = \sum_{i=1}^{k} c_i a_{n-i}$.

伴随的齐次的一个特解

- $\Leftarrow$: we prove that $z_n = x_n + y_n$ is a solution of the LNRR
  - $x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} + F(n)$
  - $y_n = c_1 y_{n-1} + \cdots + c_k y_{n-k}$
  - $x_n + y_n = c_1 \underbrace{(x_{n-1} + y_{n-1})}_{z_{n-1}} + \cdots + c_k (x_{n-k} + y_{n-k}) + F(n)$
  - $\{x_n + y_n\}$ is a solution of the LNRR
- $\Rightarrow$: we prove that a solution $\{z_n\}$ of the LNRR has the form $z_n = x_n + y_n$
  - $x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} + F(n)$
  - $z_n = c_1 z_{n-1} + \cdots + c_k z_{n-k} + F(n)$
  - Let $y_n = z_n - x_n$. Then $y_n = c_1 y_{n-1} + \cdots + c_k y_{n-k}$
    - $\{y_n\}$ is a solution of the associated LHRR

find {$x_n$

# Particular Solutions

如果F(n)是 多项式(n) × 指数(n)

**THEOREM:** Let $a_n = \sum_{i=1}^{k} c_i a_{n-i} + F(n)$ be an LNRR with $F(n) = (f_l n^l + \cdots + f_1 n + f_0)s^n = f(n)s^n$, where $c_i, f_j \in \mathbb{R}$. Suppose that $s$ is a root of $(r^k - c_1 r^{k-1} - \cdots - c_k)$ with multiplicity $m$, then the LNRR has a particular solution of the form $x_n = (p_l n^l + \cdots + p_1 n + p_0)s^n n^m$, where $\{p_j\}$ are undetermined coefficients.

伴随的齐次特征方程的一个根

重数

待定

$F(n) = (1 \cdot n + 0)2^n$, $f_1$, $f_0$, $s=2$

**EXAMPLE**: Particular solution for $a_n = 4a_{n-1} - 4a_{n-2} + n2^n$.

- Characteristic equation of the associated LHRR: $r^2 - 4r + 4 = 0$
- Characteristic roots: $r_1 = 2$ (with multiplicity $m_1 = 2$)
    - Particular solution: $x_n = (p_1 n + p_0)2^n n^2$

不确定

靠初条件确定

# Solving LNRR

**EXAMPLE**: Solve $a_n = 4a_{n-1} - 4a_{n-2} + n2^n$ with $a_0 = 1, a_1 = 4$.

- Particular solution of the LNRR: $x_n = (p_1 n + p_0)2^n n^2$
- General solution of the associated LHRR: $y_n = (\alpha_{1,0} + \alpha_{1,1} n)2^n$
- General solution of the LNRR: 确定系数
  - $z_n = x_n + y_n = (\alpha_{1,0} + \alpha_{1,1} n + p_0 n^2 + p_1 n^3)2^n$
  - Initial conditions give an equation system:
    - $a_0 = 1$: $(\alpha_{1,0} + \alpha_{1,1} \cdot 0 + p_0 \cdot 0^2 + p_1 \cdot 0^3)2^0 = 1$
    - $a_1 = 4$: $(\alpha_{1,0} + \alpha_{1,1} \cdot 1 + p_0 \cdot 1^2 + p_1 \cdot 1^3)2^1 = 4$
    - $a_2 = 20$: $(\alpha_{1,0} + \alpha_{1,1} \cdot 2 + p_0 \cdot 2^2 + p_1 \cdot 2^3)2^2 = 20$
    - $a_3 = 88$: $(\alpha_{1,0} + \alpha_{1,1} \cdot 3 + p_0 \cdot 3^2 + p_1 \cdot 3^3)2^3 = 88$

4元

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,0} + \alpha_{1,1} + p_1 + p_0 & = 2 \\ \alpha_{1,0} + 2\alpha_{1,1} + 4p_1 + 8p_0 & = 5 \\ \alpha_{1,0} + 3\alpha_{1,1} + 9p_1 + 27p_0 & = 11 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_1 + p_0 & = 1 \\ 2\alpha_{1,1} + 4p_1 + 8p_0 & = 4 \\ 3\alpha_{1,1} + 9p_1 + 27p_0 & = 10 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_1 + p_0 & = 1 \\ 2p_1 + 6p_0 & = 2 \\ 6p_1 + 24p_0 & = 7 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_1 + p_0 & = 1 \\ 2p_1 + 6p_0 & = 2 \\ 6p_0 & = 1 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_1 + p_0 & = 1 \\ 2p_1 + 6p_0 & = 2 \\ p_0 & = \frac{1}{6} \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_1 & = \frac{5}{6} \\ 2p_1 & = 1 \\ p_0 & = \frac{1}{6} \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_1 & = \frac{5}{6} \\ p_1 & = \frac{1}{2} \\ p_0 & = \frac{1}{6} \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} & = \frac{1}{3} \\ p_1 & = \frac{1}{2} \\ p_0 & = \frac{1}{6} \end{cases}$$

- The solution $(\alpha_{1,0}, \alpha_{1,1}, p_0, p_1) = \left(1, \frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$ gives

$$a_n = \left(1 + \frac{1}{3}n + \frac{1}{2}n^2 + \frac{1}{6}n^3\right) 2^n$$