1. if $\log_5 7$ rational. $\log_5 7 = \frac{m}{n}$, m, n are relatively prime and both integer.

$\frac{\ln 7}{\ln 5} = \frac{m}{n}$

$5^{\frac{m}{n}} = 7$

$5^m = 7^n$

$5^m, 7^n$ are relative prime since $\gcd(5,7)=1$

$\Rightarrow 5^m \neq 7^m$. $\log_5 7$ rational impossible $\Rightarrow \log_5 7$ irrational

proof: $\gcd(a,b)=1$, $\gcd(a^m, b^n)=1$ $(m,n,a,b \in \mathbb{Z})$.

$a = p_1^{i_1} \cdots p_m^{i_m}$ $b = q_1^{j_1} \cdots q_m^{j_m}$ (FTA).

$\gcd(a,b)=1 \Rightarrow$ for any p_s, q_t (p_s in $p_1 \cdots p_m$, q_t in $q_1 \cdots q_m$) $p_s \neq q_t$.

$\Rightarrow a^m = p_1^{i_1 m} \cdots p_m^{i_m m}$, $b = q_1^{j_1 n} \cdots q_m^{j_m n}$ have no common prime factor, too.

$\gcd(a^m, b^n) = 1$

2. $\binom{p}{k} = \frac{p!}{k!(p-k)!}$ $\binom{p-1}{k-1} = \frac{(p-1)!}{(k-1)!(p-k)!}$

$\binom{p}{k} = \frac{p}{k}\binom{p-1}{k-1}$

$k\binom{p}{k} = p(p-1)$

$p \mid k\binom{p}{k}$

p prime, $0<k<p$, $\gcd(p,k)=1$, $p \nmid k$

$\Rightarrow p \mid \binom{p}{k}$

Discrete Mathematics: Homework 1

(Deadline: 8:00am, Feb 25, 2022)

1. (15 points) Show that $\log_5 7$ is an irrational number.

2. (20 points) Let p be a prime and let k be an integer such that $0 < k < p$. We know that the binomial coefficient
$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$
is an integer. Show that $\binom{p}{k}$ is a multiple of p.

3. (20 points) Let $a, b > 1$ be relatively prime integers. Show that if $a \mid n$ and $b \mid n$, then $ab \mid n$.

4. (25 points) Let $a, b, c \in \mathbb{Z}^+$. Show that $\gcd(a, bc) = 1$ if and only if $\gcd(a, b) = \gcd(a, c) = 1$.

5. (20 points) Let $\mathbb{R}$ be the set of real numbers. Let $S = (\mathbb{R} \times \mathbb{R}) \setminus \{(0,0)\}$. Let
$$R = \left\{((a,b),(c,d)) : (a,b),(c,d) \in S \text{ and } \exists \lambda \in \mathbb{R} \setminus \{0\} \text{ such that } (a,b) = (\lambda c, \lambda d)\right\}$$
Show that R is an equivalence relation.

3. $a \mid n$, $b \mid n$

$\Rightarrow$ def $as = bt = n$, $s \in \mathbb{Z}$, $t \in \mathbb{Z}$; $a, b > 1$

$\gcd(a,b)=1$ $ai + bj = 1$ for some $i, j \in \mathbb{Z}$

$nai + nbj = n$

$\Rightarrow abit + abjs = n$.

$i, t, j, s \in \mathbb{Z}$

$\Rightarrow ab \mid n$.

4.

if: $\gcd(a,b)=1$, $\gcd(a,c)=1$

$\Rightarrow ax + by = 1$ $az + cw = 1$

$x, y, z, w \in \mathbb{Z}$

$(ax+by)(az+cw) = 1$

$a(axz + xcw + byz) + bcwy = 1$

$(axz + xcw + byz)$, wy are $\in \mathbb{Z}$

$\Rightarrow ak + bcl = 1$, $k, l \in \mathbb{Z}$

$\Rightarrow \gcd(a, bc) = 1$.

only if: $\gcd(a, bc) = 1$

$a \nmid bc$.

$\Rightarrow a \nmid b$, $a \nmid c$

$\Rightarrow \gcd(a,b)=1$ $\gcd(a,c)=1$

5. $S = (R \times R) \setminus \{(0,0)\}$

$(a,b), (c,d) \in S \Rightarrow (a,b) \neq (0,0)$ $(c,d) \neq (0,0)$.

① reflective. $(a,b) \in S$.

$(a,b) R (a,b)$: $(a,b) = (1 \times a, 1 \times b)$, $\lambda = 1$. satisfies.

②. Symmetric

suppose $R: \{((a,b),(c,d)) : (a,b)(c,d) \in S$ and $(a,b) = (\lambda c, \lambda d)$ $\lambda \neq 0\}$.

$\Rightarrow a = \lambda c$ $b = \lambda d$. $\lambda \neq 0 \Rightarrow c = \frac{1}{\lambda} a$ $d = \frac{1}{\lambda} b$.

$\Rightarrow R: \{((c,d)(a,b)) : (c,d)(a,b) \in S$. and $(c,d) = (\frac{a}{\lambda}, \frac{b}{\lambda})$ $\frac{1}{\lambda} \neq 0\}$

$(c,d) R (a,b)$ satisfies

suppose ③. Transitive.

$(a,b) R (c,d) \Rightarrow (a,b) = (\lambda_1 c, \lambda_1 d)$ $\lambda_1 \neq 0$

$(c,d) R (s,t) \Rightarrow (c,d) = (\lambda_2 s, \lambda_2 t)$ $\lambda_2 \neq 0$

$\Rightarrow a = \lambda_1 \lambda_2 s$.

$b = \lambda_1 \lambda_2 t$.

$\lambda_1 \neq 0$, $\lambda_2 \neq 0$ $\lambda_1 \lambda_2 \neq 0$.

$\Rightarrow (a,b) = (\lambda_1 \lambda_2 s, \lambda_1 \lambda_2 t)$

$\Rightarrow (a,b) R (s,t)$.