Probability Distribution Function - Histogram

Idea 1) Convolute samples w/ scaled Gaussian function

$\longrightarrow \hat{f}(x) = \frac{1}{n} \sum_{y \in Y} f_{N, \sigma_y}(x-y)$

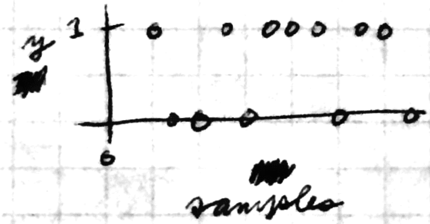
- for simplicity, use constant $\sigma_y = \hat{\sigma} \; \forall y \longrightarrow \hat{Y} = Y * G_{0,\hat{\sigma}}$
  - $\hat{\sigma} \propto \frac{1}{\sqrt{n}}$? (by Python experiment)
  - ~~$\hat{\sigma}^2 \propto \frac{1}{n}$? (see next page)~~

  let $\rightarrow \hat{\sigma} \propto \frac{\sigma_Y}{\sqrt{n}}$

(dumb) - leave $\hat{\sigma}$ to be user specified ?
  - $\hat{\sigma} \propto \sigma_Y$?

Idea 2) same setup as I1

- let σ_y be some measure of compactness of surrounding points
  - ie., consider $f(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-1)$

$\longrightarrow$ 

compact output

$\rightarrow$ no need to spread results w/ wide bell-curve

let $Y = \hat{Y}, \hat{\sigma}$

$\rightarrow \hat{Y} \approx Y * G_{0,\hat{\sigma}}$ as $n \rightarrow \infty$

$\rightarrow E[\hat{Y}^2] = 1^2 +$

let $X = G_{0,\sigma_X}$

$\rightarrow Y \approx X * G_{0,\hat{\sigma}}$ as $|Y| \rightarrow \infty$

$\rightarrow \sigma_Y^2 \approx \sigma_X^2 + \hat{\sigma}^2$

- let $\mathcal{Y} = \{0, 1, 2, \dots, n\}$ w/ uniform prob

$\longrightarrow \mu_{\mathcal{Y}} = \frac{1}{n+1}\sum_{i=1}^{n} i = \frac{1}{2}(n)$

$\sigma^2_{\mathcal{Y}} = \frac{1}{n+1}\sum_{i=1}^{n} \left(i - \frac{1}{2}(n)\right)^2 = \frac{2}{n+1}\sum_{i=0}^{\lfloor n/2 \rfloor} \left(i + \frac{n \bmod 2}{2}\right)^2$

$= \frac{\cancel{2}}{n+1}\left(\frac{1}{6}\frac{n}{\cancel{2}}\left(\frac{n}{2}+1\right)\cancel{(n+1)}\right) = \frac{1}{6}n\left(\frac{n}{2}+1\right)$ (for $2|n$)

$= \frac{1}{12}n^2 + \frac{1}{6}n$

samples of $\mathcal{Y}$

- let $Y = \{0, \dots, n\} \longrightarrow \frac{1}{n+1}\sum (y - avg(Y))^2 =: var(Y) = \frac{1}{12}n^2 + \frac{1}{6}n$

$\longrightarrow \hat{\mathcal{Y}} = \sum_{y \in Y} G_{y, \hat\sigma^2}$,

$\cancel{\hat\sigma^2 \propto \frac{1}{n}}$

$\hat\sigma^2 \propto var(Y) \longrightarrow \hat\sigma^2 = \frac{1}{12}n + \frac{1}{6} \longrightarrow \cancel{\lim_{|Y|\to\infty} \hat\sigma^2 = \infty}$

$\hat\sigma^2 \propto \frac{1}{n} \longrightarrow \hat\sigma^2 = \frac{1}{12} + \frac{1}{6n} \longrightarrow \lim_{|Y|\to\infty} \hat\sigma^2 = \frac{1}{12} < \infty$ ✓

$\Longrightarrow \cancel{\boxed{\text{let } \hat\sigma^2 = \sqrt{\frac{var(Y)}{n}}}}$

let $\mathbb{X} = \begin{cases} 0 & \frac{1}{n+1} \\ \vdots & \\ n & \frac{1}{n+1} \end{cases} \longrightarrow \begin{aligned} \mu_{\mathbb{X}} &= \tfrac{1}{2} n \\ \sigma_{\mathbb{X}}^2 &= \tfrac{1}{12} n^2 + \tfrac{1}{6} n \end{aligned}$

let $Y_m = \{\underbrace{0, \dots, 0}_{m}, \underbrace{1, \dots, 1}_{m}, \dots, \underbrace{n, \dots, n}_{m}\}$ ie, $m(n+1)$ samples of the var, $\mathbb{X}$

$\rightarrow avg(Y) = \frac{1}{2} n$

$\rightarrow var(Y) = \frac{1}{12} n^2 + \frac{1}{6} n$

$\rightarrow \hat{\sigma}^2 \propto \frac{1}{n} \longrightarrow \hat{\sigma}^2 = \frac{1}{m(n+1)} var(Y) \Longrightarrow \frac{n}{12m}$ as $n \to \infty$

$\rightarrow \hat{\sigma}^2 \propto \frac{1}{n^2} \rightarrow \hat{\sigma}^2 = \_\_\_ \Longrightarrow \frac{1}{12m}$ as $m \to \infty$

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let $\mathbb{X} = \begin{cases} \frac{1}{m} & x \in [0, m] \\ 0 & \text{else} \end{cases} \longrightarrow \mu_{\mathbb{X}} = \frac{1}{2} m$

$$\sigma_{\mathbb{X}}^2 = \frac{1}{m} \int_{-m/2}^{m/2} x^2 \, dx = \frac{2}{m} \int_0^{m/2} x^2 \, dx = \frac{2}{m} \underbrace{\left[\frac{x^3}{3}\right]_0^{m/2}}_{\frac{1}{3}\frac{m^3}{8}} = \frac{1}{12} m^2$$

let X_n be n samples of $\mathbb{X}$

$\rightarrow E[avg(X_n)] = E\left[\frac{1}{n} \sum_{i=1}^{n} \mathbb{X}\right] = \frac{1}{n} E\left[\sum^{n} \mathbb{X}\right] = \mu_{\mathbb{X}}$

$\rightarrow E[var(X_n)] = E\left[\frac{1}{n} \sum_{i=1}^{n} (\mathbb{X}_i - \mu_{\mathbb{X}})^2\right] = \frac{1}{n}(n \cdot \sigma_{\mathbb{X}}^2) = \sigma_{\mathbb{X}}^2$

$\xrightarrow{\sigma \propto \frac{1}{n}} E[\hat{\sigma}] = \left(\frac{\sigma_{\mathbb{X}}}{n}\right) = \frac{1}{\sqrt{12}} \frac{m}{n}$

$\xrightarrow{\sigma^2 \propto \frac{1}{n}} E[\hat{\sigma}] = \frac{\sigma_{\mathbb{X}}}{\sqrt{n}} = \frac{m}{\sqrt{12 n}}$

assume X, Y independent

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int \Big(\underbrace{\int x\, f_X(x)\, f_Y(y)\, dx}_{= f_Y(y) \int x f_X(x)\, dx \;=\; f_Y(y)\cdot \mu_X} + \underbrace{\int y\, f_X(x)\, f_Y(y)\, dx}_{= y f_Y(y) \cancel{\int f_X(x)\, dx}^{\,=1}} \Big)\, dy$$

$$= \int f_Y(y)\,(\mu_X + y)\, dy = \overbrace{\int \mu_X f_Y(y)\, dy}^{=\mu_X} + \overbrace{\int y\, f_Y(y)\, dy}^{=\mu_Y}$$

$$\boxed{= \mu_X + \mu_Y}$$

X, Y ind.

$$E[X \cdot Y] = \iint xy\, f_X(x)\, f_Y(y)\, dx\, dy = \int y \Big(\underbrace{\int x\, f_X(x)\, dx}_{=\mu_X} \Big)\, dy$$

$$= \mu_X\, \mu_Y$$