Probability Distribution Function - Histogram

Idea 1) convolute samples w/ scaled Gaussian function

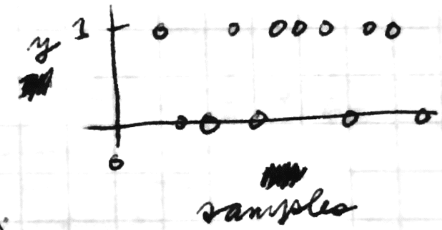
→ $\hat{f}(x) = \frac{1}{n} \sum_{y \in Y} f_{N, \sigma_y}(x-y)$

- for simplicity, use constant $\sigma_y = \hat{\sigma}\ \forall y$ → $\hat{Y} = Y * G_{0,\hat{\sigma}}$
    - $\hat{\sigma} \propto \frac{1}{n}$?
    - ~~$\hat{\sigma}^2 \propto \frac{1}{n}$?~~ (see next page)
- (dumb) - leave $\hat{\sigma}$ to be user specified ?
    - $\hat{\sigma} \propto \sigma_Y$?

let → $\hat{\sigma} \propto \frac{\sigma_Y}{n}$

Idea 2) same setup as I1

- let σ_y be some measure of compactness of surrounding points
    - ie., consider $f(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-1)$

→ 

compact output

→ no need to spread results w/ wide bell-curve

$\cancel{\text{let } Y = N,\ \hat{\sigma}}$

$\cancel{\rightarrow \hat{Y} \approx Y * G_{0,\hat{\sigma}} \text{ as } n \rightarrow \infty}$

$\cancel{\rightarrow E[\hat{Y}^2] = 1^2 +}$

let $X = G_{0,\sigma_X}$

→ $Y \approx X * G_{0,\hat{\sigma}}$ as $|Y| \rightarrow \infty$

→ $\sigma_Y^2 \approx \sigma_X^2 + \hat{\sigma}^2$

- let $\mathcal{Y} = \{0, 1, 2, \ldots, n\}$ w/ uniform prob

$\rightarrow \mu_{\mathcal{Y}} = \frac{1}{n+1}\sum_{i=1}^{n} i = \frac{1}{2}(n)$

$\sigma^2_{\mathcal{Y}} = \frac{1}{n+1}\sum_{i=1}^{n}\left(i - \frac{1}{2}(n)\right)^2 = \frac{2}{n+1}\sum_{i=0}^{\lfloor n/2 \rfloor}\left(i + \frac{n \bmod 2}{2}\right)^2$

$= \frac{2}{\cancel{n+1}}\left(\frac{1}{6}\frac{n}{\cancel{2}}\left(\frac{n}{2}+1\right)\cancel{(n+1)}\right) = \frac{1}{6}n\left(\frac{n}{2}+1\right)$ (for $2|n$)

$= \frac{1}{12}n^2 + \frac{1}{6}n$

samples of $\mathcal{Y}$

- let $Y = \{0, \ldots, n\} \longrightarrow \frac{1}{n+1}\sum_i \left(y - \mathrm{avg}(Y)\right)^2 =: \mathrm{var}(Y) = \frac{1}{12}n^2 + \frac{1}{6}n$

$\rightarrow \hat{\mathcal{Y}} = \sum_{y \in Y} G_{y,\hat{\sigma}^2}$,

~~$\hat{\sigma}^2 \propto \frac{1}{n}$~~

$\hat{\sigma}^2 \propto \mathrm{var}(Y) \rightarrow \hat{\sigma}^2 = \frac{1}{12}n + \frac{1}{6} \rightarrow$ ~~$\lim_{|Y|\to\infty} \hat{\sigma}^2 = \infty$~~

$\hat{\sigma}^* \propto \frac{1}{n} \rightarrow \hat{\sigma}^2 = \frac{1}{12} + \frac{1}{6n} \rightarrow \lim_{|Y|\to\infty} \hat{\sigma}^2 = \frac{1}{12} < \infty$ ✓

$\Longrightarrow \boxed{\text{let } \hat{\sigma}^* = \sqrt{\frac{\mathrm{var}(Y)}{n}}}$

let $\mathbb{X} = \begin{cases} 0 & \frac{1}{n+1} \\ \vdots & \\ n & \frac{1}{n+1} \end{cases} \longrightarrow \begin{array}{l} \mu_{\mathbb{X}} = \frac{1}{2} n \\ \sigma_{\mathbb{X}}^2 = \frac{1}{12} n^2 + \frac{1}{6} n \end{array}$

let $Y_m = \{ \underbrace{0, \dots, 0}_{m}, \underbrace{1, \dots, 1}_{m}, \dots, \underbrace{n, \dots, n}_{m} \}$ ie, $m(n+1)$ samples of the var, $\mathbb{X}$

$\rightarrow \text{avg}(Y) = \frac{1}{2} n$

$\rightarrow \text{var}(Y) = \frac{1}{12} n^2 + \frac{1}{6} n$

$\rightarrow \hat{\sigma}^2 \propto \frac{1}{m} \longrightarrow \hat{\sigma}^2 = \frac{1}{m(n+1)} \text{var}(Y) \Longrightarrow \frac{n}{12m}$ as $n \to \infty$

$\rightarrow \hat{\sigma}^2 \propto \frac{1}{m^2} \rightarrow \hat{\sigma}^2 = \_\_\_ \Longrightarrow \frac{1}{12m}$ as $m \to \infty$

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let $\mathbb{X} = \begin{cases} \frac{1}{m} & x \in [0, m] \\ 0 & \text{else} \end{cases} \longrightarrow \mu_{\mathbb{X}} = \frac{1}{2} m$

$\sigma_{\mathbb{X}}^2 = \frac{1}{m} \int_{-m/2}^{m/2} x^2 \, dx$

$= \frac{2}{m} \int_0^{m/2} x^2 \, dx$

$= \underbrace{\frac{2}{m} \left[\frac{x^3}{3} \right]_0^{m/2}}_{\frac{1}{3} \frac{m^3}{8}}$

$= \frac{1}{12} m^2$

let X_n be n samples of $\mathbb{X}$

$\rightarrow E[\text{avg}(X_n)] = E\left[\frac{1}{n} \sum_{i=1}^{n} \mathbb{X}\right] = \frac{1}{n} E\left[\sum^{n} \mathbb{X}\right] = \mu_{\mathbb{X}}$

$\rightarrow E[\text{var}(X_n)] = E\left[\frac{1}{n} \sum_{i=1}^{n} (\mathbb{X}_i - \mu_{\mathbb{X}})^2\right] = \frac{1}{n}(n \cdot \sigma_{\mathbb{X}}^2) = \sigma_{\mathbb{X}}^2$

$\xrightarrow{\sigma \propto \frac{1}{n}} E[\hat{\sigma}] = \left(\frac{\sigma_{\mathbb{X}}}{n}\right) = \frac{1}{\sqrt{12}} \frac{m}{n}$

$\xrightarrow{\sigma^2 \propto \frac{1}{n}} E[\hat{\sigma}] = \frac{\sigma_{\mathbb{X}}}{\sqrt{n}} = \frac{m}{\sqrt{12n}}$

assume X, Y independent

$$E[X+Y] := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f_X(x)\, f_Y\, dx\, dy$$

$$= \int \Big(\underbrace{\int x\, f_X(x)\, f_Y(y)\, dx}_{= f_Y(y) \int x\, f_X(x)\, dx \;=\; f_Y(y)\cdot \mu_X} + \underbrace{\int y\, f_X(x)\, f_Y(y)\, dx}_{= y\, f_Y(y) \cancel{\int f_X(x)\, dx}^{=1}} \Big)\, dy$$

$$= \int f_Y(y)\,(\mu_X + y)\, dy = \overset{=\mu_X}{\int \mu_X f_Y(y)\, dy} + \overset{=\mu_Y}{\int y\, f_Y(y)\, dy}$$

$$\boxed{= \mu_X + \mu_Y}$$

$$E[X \cdot Y] \overset{X,\,Y \text{ ind.}}{=} \iint xy\, f_X(x)\, f_Y(y)\, dx\, dy = \int y \Big(\underbrace{\int x\, f_X(x)\, dx}_{=\mu_X} \Big)\, dy$$

$$= \mu_X\, \mu_Y$$