

## **Tutorial on filter synthesis**

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## 1 Eigenvalue method

1. We are interested here in the synthesis of linear phase FIR filters. We consider the particular case of a type I filter, of odd length  $N$  and symmetrical impulse response, whose transfer function is denoted  $H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$ . Let  $M = \frac{N-1}{2}$ . Verify that we can write

$$H(e^{i2\pi\nu}) = e^{-i2\pi\nu M} H_R(e^{i2\pi\nu})$$

where  $H_R(e^{i2\pi\nu})$  is a real-valued function, called the amplitude response of filter  $H$ , defined by the equality  $H_R(e^{i2\pi\nu}) = \mathbf{a}^T \mathbf{c}(\nu)$ , where  $\mathbf{c}(\nu) = [1, \cos(2\pi\nu), \dots, \cos(2\pi M\nu)]^T$ , and where the coefficients of vector  $\mathbf{a} = [a_0, a_1, \dots, a_M]^T$  are to be expressed in terms of  $h(n)$ .

2. We wish to synthesize a low-pass filter with cutoff frequency  $\nu_c \in ]0, \frac{1}{2}[$  and whose stop-band starts at  $\nu_a \in ]\nu_c, \frac{1}{2}[$ . The energy in the stop-band is  $E_a = 2 \int_{\nu_a}^{\frac{1}{2}} (H_R(e^{i2\pi\nu}))^2 d\nu$ . Show that we can write  $E_a = \mathbf{a}^T \mathbf{P} \mathbf{a}$ , where  $\mathbf{P}$  is a positive semidefinite matrix, whose coefficients  $\{\mathbf{P}_{(m,n)}\}_{(m,n) \in [[0,M]]^2}$  are to be determined in function of  $\nu_a$ .
3. Ideally, the amplitude response  $H_R(e^{i2\pi\nu})$  is equal to  $H_R(1)$  in the bandwidth  $[0, \nu_c]$ . We therefore define the square error in the bandwidth as follows:

$$E_c = 2 \int_0^{\nu_c} (H_R(e^{i2\pi\nu}) - H_R(1))^2 d\nu$$

Show that we can write  $E_c = \mathbf{a}^T \mathbf{Q} \mathbf{a}$ , where  $\mathbf{Q}$  is a positive semidefinite matrix, whose coefficients  $\{\mathbf{Q}_{(m,n)}\}_{(m,n) \in [[0,M]]^2}$  are to be determined in function of  $\nu_c$ .

4. The FIR filter synthesis method called *eigenvalue method* consists in minimizing with respect to  $\mathbf{a}$  the cost function  $E(\mathbf{a}) = \alpha E_c + (1 - \alpha) E_a$ , where  $\alpha \in ]0, 1[$  is a trade-off parameter between pass-band and stop-band. We thus obtain  $E(\mathbf{a}) = \mathbf{a}^T \mathbf{R} \mathbf{a}$ , where  $\mathbf{R} = \alpha \mathbf{Q} + (1 - \alpha) \mathbf{P}$  is a positive semidefinite matrix. Show that vector  $\mathbf{a}$  minimizes function  $E$  under unit norm constraint if and only if it is an eigenvector of  $\mathbf{R}$ , associated to the lowest eigenvalue (*Rayleigh's principle*).

## 2 Synthesis of an integrator filter

We consider a digital signal  $x(n)$ , defined from an analog signal  $x^a(t)$  sampled at sampling rate  $T$ :  $x(n) = x^a(nT)$ . This exercise aims at synthesizing a digital filter which allows to obtain, from the discrete signal  $x(n)$ , a sampled version of the integrated signal  $y^a(t) = \int_{-\infty}^t x^a(u) du$ .

**Question 1** Show that the integrated signal  $y^a(t)$  can be written as the convolution product between the signal  $x^a(t)$  and the analog filter  $h^a(t) = 1$  if  $t \geq 0$  and  $h^a(t) = 0$  otherwise (Heaviside function). Is this filter causal? Is it stable? (reminder : the filter is stable if and only if  $\int_{-\infty}^{+\infty} |h^a(t)| dt < +\infty$ ). Compute the transfer function  $H^a(p) = \int_{-\infty}^{+\infty} h^a(t) e^{-pt} dt$  (Laplace transform of  $h^a$ , with  $p \in \mathbb{C}$ ), and specify its domain of definition.



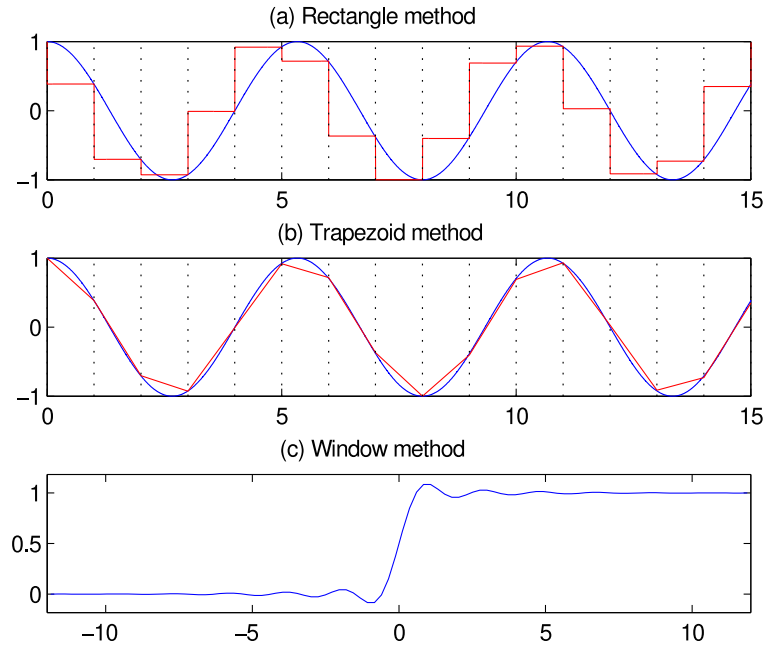


Figure 1: Three synthesis methods of an integrator filter

## 2.1 Approximation by the rectangle method

We wish to approximate the integral of the signal  $x^a(t)$  by the rectangle method (an example is given on Figure 1-(a)), which amounts to computing the integral of the interpolated signal

$$x_0^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT) f_0(t - mT)$$

where  $f_0(t) = 1$  if  $t \in [-T, 0]$  and  $f_0(t) = 0$  otherwise (rectangle function). We define the discrete-time integrated signal  $y_0(n) = \int_{-\infty}^{nT} x_0^a(t) dt$ .

**Question 2** Show that  $y_0(n)$  can be written as the convolution product between the signal  $x(n)$  and a digital filter  $h_0(n)$ , and give the expression of its impulse response. Is this filter causal? Is it stable? Calculate the transfer function  $H_0(z)$ , and specify its domain of definition.

## 2.2 Approximation by the trapezoid method

We wish to approximate the integral of signal  $x^a(t)$  by the trapezoid method (an example is given in Figure 1-(b)), which amounts to computing the integral of the interpolated signal

$$x_1^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT) f_1(t - mT)$$

where  $f_1(t) = 1 - |t|/T$  if  $t \in [-T, T]$  and  $f_1(t) = 0$  elsewhere (triangle function). We define the discrete-time integrated signal  $y_1(n) = \int_{-\infty}^{nT} x_1^a(t) dt$ .

**Question 3** Show that  $y_1(n)$  can be written as the convolution product between the signal  $x(n)$  and a digital filter  $h_1(n)$ , and give the expression of its impulse response. Is this filter causal? Is it stable?

**Question 4** Show that this method is equivalent to determining the digital filter from the analog filter of Question 1 by using the bilinear transformation (hint: we can identify the two transfer functions).

## 2.3 Synthesis by the window method

We now wish to determine the integral of the signal  $x^a(t)$  exactly. To do this, we assume that  $x^a(t)$  satisfies the assumptions of the Shannon-Nyquist's theorem. It can then be reconstructed exactly from its samples:

$$x^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT) f(t - mT)$$

where  $f(t) = \text{sinc}\left(\frac{t}{T}\right)$  avec  $\text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$ . We define the discrete time integrated signal  $y(n) = \int_{-\infty}^{nT} x^a(t) dt$ .

**Question 5** Show that  $y(n)$  can be written as the convolution product between the signal  $x(n)$  and the digital filter  $h(n) = T \int_{-\infty}^n \text{sinc}(u) du$  (hint: we will assume that  $x^a(t)$  satisfies strong enough assumptions to be able to switch  $\int$  and  $\sum$ ).

**Question 6** The impulse response of filter  $h$  is represented in Figure 1-(c) (for  $T = 1$ ). What phenomenon can be observed compared to the impulse responses calculated previously? Is this filter causal? Is it stable? (hint:  $h(n) \xrightarrow{n \rightarrow +\infty} T$ )

Since  $h(n) \xrightarrow{n \rightarrow +\infty} 1$ , it does not seem reasonable to synthesize filter  $h$  by directly applying the window method, which consists in truncating the impulse response. Instead, we define filter  $G(z) = (1 - z^{-1}) H(z)$ , whose impulse response decreases towards 0 at infinity. This filter  $G(z)$  can be synthesized by the window method. We can then deduce an integrating filter  $H(z) = \frac{G(z)}{1 - z^{-1}}$ .

**Question 7** Show that the impulse response of the filter  $g(n)$  is symmetrical with respect to  $\frac{1}{2}$ , and is upper bounded in absolute value by  $O\left(\frac{1}{n}\right)$  (the proof is simple but it may be useful to make a drawing).

**Question 8** Since the impulse response of filter  $g$  tends to 0 at infinity, it seems reasonable to synthesize this filter by the window method. In order for the resulting filter to be linear phase, should we choose an even or odd filter length  $N$ ? What type of filter does this correspond to? (I, II, III or IV)

**Question 9** Is the resulting filter  $H(z)$  stable? What would you suggest to remedy this?





Contexte académique } **sans modifications**

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