Institut Mines-Télécom

Solution of the tutorial on filter synthesis

Roland Badeau

1 Eigenvalue method

1. We are interested here in the synthesis of linear phase FIR filters. We consider the particular case of a type I filter, of odd length N and symmetrical impulse response, whose transfer function is denoted $H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$. Let $M = \frac{N-1}{2}$. Verify that we can write

$$H\left(e^{i2\pi\nu}\right) = e^{-i2\pi\nu M}\, H_R\left(e^{i2\pi\nu}\right) \tag{1}$$

where $H_R\left(e^{i2\pi\nu}\right)$ is a real-valued function, called the amplitude response of filter H, defined by the equality $H_R\left(e^{i2\pi\nu}\right) = \boldsymbol{a}^T \boldsymbol{c}(\nu)$, where $\boldsymbol{c}(\nu) = [1, \cos(2\pi\nu), \ldots, \cos(2\pi M\nu)]^T$, and where the coefficients of vector $\boldsymbol{a} = [a_0, a_1, \ldots, a_M]^T$ are to be expressed in terms of $h(n)$.

We have

$$\begin{aligned} H(z) &= z^{-M}\left(\sum_{n=0}^{M-1} h(n)\, z^{-n+M} + h(M) + \sum_{n=M+1}^{N-1} h(n)\, z^{-n+M}\right) \\ &= z^{-M}\left(\sum_{n=0}^{M-1} h(n)\, z^{-n+M} + h(M) + \sum_{m=0}^{M-1} h(N-1-m)\, z^{m-M}\right) \\ &= z^{-M}\left(h(M) + \sum_{n=0}^{M-1} h(n)\, (z^{-n+M} + z^{n-M})\right). \end{aligned}$$

At $z = e^{i2\pi\nu}$, we get

$$H\left(e^{i2\pi\nu}\right) = e^{-i2\pi\nu M}\left(H(M) + 2\sum_{n=0}^{M-1} h(n)\cos(2\pi\nu(M-n))\right).$$

We thus retrieve (1) *with $a_0 = h(M)$ and $a_m = 2h(M-m)$ $\forall m \in [[1, M]]$.*

2. We wish to synthesize a low-pass filter with cutoff frequency $\nu_c \in \left]0, \frac{1}{2}\right[$ and whose stop-band starts at $\nu_a \in \left]\nu_c, \frac{1}{2}\right[$. The energy in the stop-band is $E_a = 2\int_{\nu_a}^{\frac{1}{2}} \left(H_R(e^{i2\pi\nu})\right)^2 \mathrm{d}\nu$. Show that we can write $E_a = \boldsymbol{a}^T \boldsymbol{P}\, \boldsymbol{a}$, where $\boldsymbol{P}$ is a positive semidefinite matrix, whose coefficients $\{\boldsymbol{P}_{(m,n)}\}_{(m,n)\in[[0,M]]^2}$ are to be determined in function of ν_a.

We have $E_a = 2\int_{\nu_a}^{\frac{1}{2}} \left(\boldsymbol{a}^T \boldsymbol{c}(\nu)\right)^2 \mathrm{d}\nu = \boldsymbol{a}^T \boldsymbol{P}\, \boldsymbol{a}$ with $\boldsymbol{P} = 2\int_{\nu_a}^{\frac{1}{2}} \boldsymbol{c}(\nu)\boldsymbol{c}(\nu)^T \mathrm{d}\nu$. Matrix $\boldsymbol{P}$ is positive semidefinite, as a sum of rank-1 positive semidefinite matrices. Moreover,

$$\begin{aligned} \boldsymbol{P}_{(m,n)} &= 2\int_{\nu_a}^{\frac{1}{2}} \cos(2\pi\nu m)\cos(2\pi\nu n)\mathrm{d}\nu \\ &= \int_{\nu_a}^{\frac{1}{2}} \cos(2\pi\nu(m+n)) + \cos(2\pi\nu(m-n))\mathrm{d}\nu \\ &= \left[\frac{\sin(2\pi\nu(m+n))}{2\pi(m+n)} + \frac{\sin(2\pi\nu(m-n))}{2\pi(m-n)}\right]_{\nu_a}^{\frac{1}{2}} \\ &= \frac{\delta(m+n)+\delta(m-n)}{2} - \nu_a\left(\mathrm{sinc}(2\pi\nu_a(m+n)) + \mathrm{sinc}(2\pi\nu_a(m-n))\right). \end{aligned}$$

3. Ideally, the amplitude response $H_R\left(e^{i2\pi\nu}\right)$ is equal to $H_R(1)$ in the bandwidth $[0, \nu_c]$. We therefore define the square error in the bandwidth as follows:

$$E_c = 2\int_0^{\nu_c} \left(H_R(e^{i2\pi\nu}) - H_R(1)\right)^2 \mathrm{d}\nu$$

Show that we can write $E_c = \boldsymbol{a}^T \boldsymbol{Q}\, \boldsymbol{a}$, where $\boldsymbol{Q}$ is a positive semidefinite matrix, whose coefficients $\left\{\boldsymbol{Q}_{(m,n)}\right\}_{(m,n)\in[[0,M]]^2}$ are to be determined in function of ν_c.

We have $E_c = 2\int_0^{\nu_c} \left(\boldsymbol{a}^T(\boldsymbol{c}(\nu) - \boldsymbol{c}(0)\right)^2 \mathrm{d}\nu = \boldsymbol{a}^T \boldsymbol{Q}\, \boldsymbol{a}$ *with* $\boldsymbol{Q} = 2\int_0^{\nu_c} (\boldsymbol{c}(\nu) - \boldsymbol{c}(0))(\boldsymbol{c}(\nu) - \boldsymbol{c}(0))^T \mathrm{d}\nu$. *Matrix* $\boldsymbol{Q}$ *is positive semidefinite, as a sum of rank-1 positive semidefinite matrices. Moreover,*

$$\begin{aligned}\boldsymbol{Q}_{(m,n)} &= 2\int_0^{\nu_c} (\cos(2\pi\nu m) - 1)(\cos(2\pi\nu n) - 1)\mathrm{d}\nu \\ &= \int_0^{\nu_c} \cos(2\pi\nu(m+n)) + \cos(2\pi\nu(m-n)) - 2\cos(2\pi\nu m) - 2\cos(2\pi\nu n) + 2\,\mathrm{d}\nu.\end{aligned}$$

The end of the calculation is left to the reader.

4. The FIR filter synthesis method called *eigenvalue method* consists in minimizing with respect to $\boldsymbol{a}$ the cost function $E(\boldsymbol{a}) = \alpha E_c + (1-\alpha)E_a$, where $\alpha \in]0, 1[$ is a trade-off parameter between pass-band and stop-band. We thus obtain $E(\boldsymbol{a}) = \boldsymbol{a}^T \boldsymbol{R}\, \boldsymbol{a}$, where $\boldsymbol{R} = \alpha\boldsymbol{Q} + (1-\alpha)\boldsymbol{P}$ is a positive semidefinite matrix. Show that vector $\boldsymbol{a}$ minimizes function E under unit norm constraint if and only if it is an eigenvector of $\boldsymbol{R}$, associated to the lowest eigenvalue (*Rayleigh's principle*).

   The Lagrangian of this optimization problem is

$$\mathcal{L}(\boldsymbol{a}, \lambda) = \boldsymbol{a}^T \boldsymbol{R}\, \boldsymbol{a} + \lambda(1 - \|\boldsymbol{a}\|^2).$$

   Its gradient w.r.t. $\boldsymbol{a}$ *is* $2\boldsymbol{R}\,\boldsymbol{a} - 2\lambda\boldsymbol{a}$. *This gradient is zero when* $\boldsymbol{R}\,\boldsymbol{a} = \lambda\boldsymbol{a}$ *(therefore* $\boldsymbol{a}$ *is an eigenvector of* $\boldsymbol{R}$*, associated to the eigenvalue* λ*), in which case* $E(\boldsymbol{a}) = \lambda$ *when* $\|\boldsymbol{a}\| = 1$. *Therefore vector* $\boldsymbol{a}$ *minimizes function* E *under unit norm constraint if and only if it is an eigenvector of* $\boldsymbol{R}$ *associated to the lowest eigenvalue* λ.

2 Synthesis of an integrator filter

We consider a digital signal $x(n)$, defined from an analog signal $x^a(t)$ sampled at sampling rate T: $x(n) = x^a(nT)$. This exercise aims at synthesizing a digital filter which allows to obtain, from the discrete signal $x(n)$, a sampled version of the integrated signal $y^a(t) = \int_{-\infty}^{t} x^a(u)\, du$.

Question 1 Show that the integrated signal $y^a(t)$ can be written as the convolution product between the signal $x^a(t)$ and the analog filter $h^a(t) = 1$ if $t \geq 0$ and $h^a(t) = 0$ otherwise (Heaviside function). Is this filter causal? Is it stable? (reminder : the filter is stable if and only if $\int_{-\infty}^{+\infty} |h^a(t)|\, dt < +\infty$). Compute the transfer function $H^a(p) = \int_{-\infty}^{+\infty} h^a(t)\, e^{-pt}\, dt$ (Laplace transform of h^a, with $p \in \mathbb{C}$), and specify its domain of definition.

We have $(h^a * x^a)(t) = \int_{v\in\mathbb{R}} h^a(v)x^a(t-v)dv = \int_{v=0}^{+\infty} x^a(t-v)dv = \int_{u=-\infty}^{t} x^a(u)\, du$ *with the change of variable* $v = t - u$. *Filter* h^a *is causal by definition, and it is unstable because* $h^a \notin L^1(\mathbb{R})$. *We get* $H^a(p) = \int_0^{+\infty} e^{-pt}\, dt = \frac{1}{p}$, *which is defined for* $Re(p) > 0$.

2.1 Approximation by the rectangle method

We wish to approximate the integral of the signal $x^a(t)$ by the rectangle method (an example is given on Figure 1-(a)), which amounts to computing the integral of the interpolated signal

$$x_0^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\, f_0(t - mT)$$

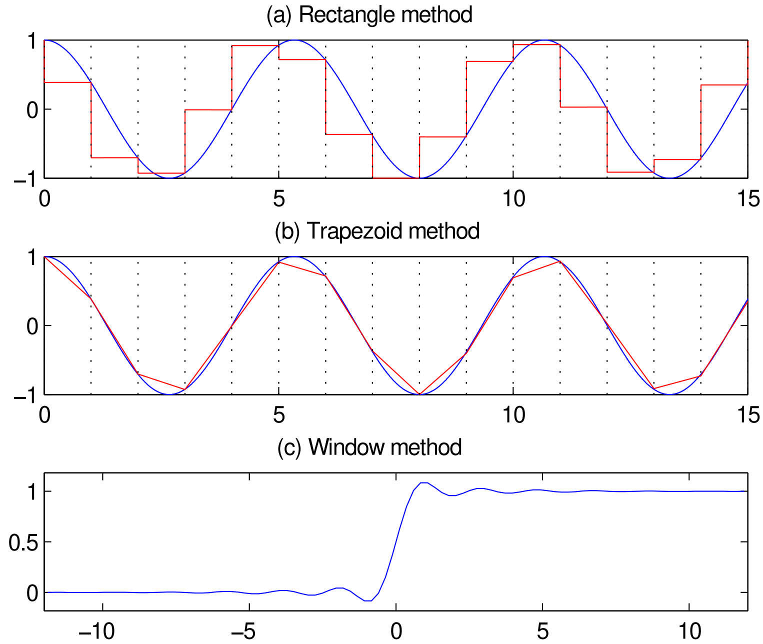

Figure 1: Three synthesis methods of an integrator filter

where $f_0(t) = 1$ if $t \in [-T, 0]$ and $f_0(t) = 0$ otherwise (rectangle function). We define the discrete-time integrated signal $y_0(n) = \int_{-\infty}^{nT} x_0^a(t)\,dt$.

Question 2 Show that $y_0(n)$ can be written as the convolution product between the signal $x(n)$ and a digital filter $h_0(n)$, and give the expression of its impulse response. Is this filter causal? Is it stable? Calculate the transfer function $H_0(z)$, and specify its domain of definition.

We have $y_0(n) = \sum\limits_{m=-\infty}^{+\infty} x(m) \int_{-\infty}^{nT} f_0(t-mT)\,dt = T \sum\limits_{m=-\infty}^{n} x(m) = \sum\limits_{m\in\mathbb{Z}} h_0(n-m)x(m)$ *with* $h_0(m) = T\, 1_{\mathbb{N}}(m)$. *Filter* h_0 *is causal by definition, and it is unstable because* $h_0 \notin l^1(\mathbb{Z})$. *We get* $H_0(z) = T\sum_{m\in\mathbb{N}} z^{-m} = \frac{T}{1-z^{-1}}$, *which is defined for* $|z| > 1$.

2.2 Approximation by the trapezoid method

We wish to approximate the integral of signal $x^a(t)$ by the trapezoid method (an example is given in Figure 1-(b)), which amounts to computing the integral of the interpolated signal

$$x_1^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\, f_1(t-mT)$$

where $f_1(t) = 1 - |t|/T$ if $t \in [-T, T]$ and $f_1(t) = 0$ elsewhere (triangle function). We define the discrete-time integrated signal $y_1(n) = \int_{-\infty}^{nT} x_1^a(t)\,dt$.

Question 3 Show that $y_1(n)$ can be written as the convolution product between the signal $x(n)$ and a digital filter $h_1(n)$, and give the expression of its impulse response. Is this filter causal? Is it stable?

We have $y_1(n) = \sum\limits_{m=-\infty}^{+\infty} x(m) \int_{-\infty}^{nT} f_1(t-mT)\,dt = T(\frac{1}{2}x(n) + \sum\limits_{m=-\infty}^{n-1} x(m)) = \sum\limits_{m\in\mathbb{Z}} h_1(n-m)x(m)$ *with* $h_1(m) = T\,(\frac{1}{2}\delta_0(m) + 1_{\mathbb{N}^*}(m))$*. Filter* h_1 *is causal by definition, and it is unstable because* $h_1 \notin l^1(\mathbb{Z})$*.*

Question 4 Show that this method is equivalent to determining the digital filter from the analog filter of Question 1 by using the bilinear transformation (hint: we can identify the two transfer functions).
We get $H_1(z) = \frac{1}{2} + \sum_{m\in\mathbb{N}^*} z^{-m} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$*, which is defined for* $|z| > 1$*. With the bilinear transform, we retrieve* $H_1(z) = H^a(p) = \frac{1}{p}$ *with* $p = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$*.*

2.3 Synthesis by the window method

We now wish to determine the integral of the signal $x^a(t)$ exactly. To do this, we assume that $x^a(t)$ satisfies the assumptions of the Shannon-Nyquist's theorem. It can then be reconstructed exactly from its samples:

$$x^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\, f(t-mT)$$

where $f(t) = \text{sinc}\left(\frac{t}{T}\right)$ avec $\text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$. We define the discrete time integrated signal $y(n) = \int_{-\infty}^{nT} x^a(t)\,dt$.

Question 5 Show that $y(n)$ can be written as the convolution product between the signal $x(n)$ and the digital filter $h(n) = T\int_{-\infty}^{n} \text{sinc}(u)\,du$ (hint: we will assume that $x^a(t)$ satisfies strong enough assumptions to be able to switch $\int$ and $\sum$).
We have $y(n) = \sum\limits_{m=-\infty}^{+\infty} x(m)\int_{-\infty}^{nT} f(t-mT)\,dt = \sum\limits_{m=-\infty}^{+\infty} x(m)\int_{-\infty}^{(n-m)T} f(t)\,dt = \sum\limits_{m\in\mathbb{Z}} h(n-m)x(m)$ *with* $h(m) = \int_{-\infty}^{mT} f(t)\,dt = \int_{-\infty}^{mT} \text{sinc}\left(\frac{t}{T}\right) dt = T\int_{-\infty}^{m} \text{sinc}(u)\,du$ *with the change of variable* $t = uT$*.*

Question 6 The impulse response of filter h is represented in Figure 1-(c) (for $T = 1$). What phenomenon can be observed compared to the impulse responses calculated previously? Is this filter causal? Is it stable ? (hint: $h(n) \underset{n\to+\infty}{\longrightarrow} T$)
In Figure 1-(c), we observe a Gibbs phenomenon in the time domain. This filter is non longer causal, nor stable because $h \notin l^1(\mathbb{Z})$*.*
Since $h(n) \underset{n\to+\infty}{\longrightarrow} T$, it does not seem reasonable to synthesize filter h by directly applying the window method, which consists in truncating the impulse response. Instead, we define filter $G(z) = (1-z^{-1})\,H(z)$, whose impulse response decreases towards 0 at infinity. This filter $G(z)$ can be synthesized by the window method. We can then deduce an integrating filter $H(z) = \frac{G(z)}{1-z^{-1}}$.

Question 7 Show that the impulse response of the filter $g(n)$ is symmetrical with respect to $\frac{1}{2}$, and is upper bounded in absolute value by $O\left(\frac{1}{n}\right)$ (the proof is simple but it may be useful to make a drawing).
We have $G(z) = (1-z^{-1})H(z)$*, therefore* $g(n) = h(n) - h(n-1) = T\int_{n-1}^{n} \text{sinc}(u)\,du = O(\frac{1}{n})$*. Moreover,* $g(1-n) = T\int_{-n}^{1-n} \text{sinc}(u)\,du = g(n)$*, thus the impulse response* $g(n)$ *is symmetrical with respect to* $\frac{1}{2}$*.*

Question 8 Since the impulse response of filter g tends to 0 at infinity, it seems reasonable to synthesize this filter by the window method. In order for the resulting filter to be linear phase, should we choose an even or odd filter length N? What type of filter does this correspond to? (I, II, III or IV)
The symmetry of $g(n)$ w.r.t. $\frac{1}{2}$ imposes a type II filter with even filter length N.

Question 9 Is the resulting filter $H(z)$ stable? What would you suggest to remedy this?
$H(z)$ is unstable because of its pole at $z = 1$. Stability can be enforced by moving this pole inside the unit circle (e.g. at $z = 1 - \varepsilon$ with a small $\varepsilon > 0$).