Bayesian tests

Exercise 1 (Bernoulli model – Simple hypotheses)**:**
Let θ be the fraction of electric cars in Paris. You want to test $H_0 = \{\theta_0\}$ with $\theta_0 = \frac{1}{4}$ against $H_1 = \{\theta_1\}$ with $\theta_1 = \frac{1}{3}$ using n i.i.d. observations. The prior information is $\pi(\theta_0) = \frac{2}{3}$, $\pi(\theta_1) = \frac{1}{3}$.

1. Give the Bayesian test.
2. Give the result of the test if you get 30 electric cars among $n = 100$.
3. Using a Gaussian approximation, give the risk of the test for $n = 100$.
4. What is the limiting risk when $n \to +\infty$?

Exercise 2 (Exponential model – One-tailed hypothesis)**:**
The delay to receive a shipment is supposed to be exponential with parameter $\theta > 0$; the prior on θ is itself exponential with parameter $\lambda > 0$. You want to test $H_0 = \{\theta > \frac{1}{2}\}$ against $H_1 = \{\theta \leq \frac{1}{2}\}$.

1. Give the Bayesian test based on n i.i.d. observations.
2. Give the result of the test for $\lambda = 1$ and the following observations: $1, 2, 5, 1, 2, 3$.
   Does this result depend on the value of λ?

Exercise 3 (Bernoulli model – Two-tailed hypothesis)**:**
Let θ be the winning probability of a game. You want to test $H_0 = \{\theta = \frac{1}{2}\}$ against $H_1 = \{\theta \neq \frac{1}{2}\}$ using n i.i.d. observations. The prior information is $\pi(\theta) = \frac{1}{2} 1_{[0,1]}(\theta)$ with respect to the measure $\mu = \delta_{\frac{1}{2}} + \lambda$ where λ refers to the Lebesgue measure.

1. Check that π is a probability density function with respect to μ
2. Give the Bayesian test for that problem.
3. Give the result of the test for all possible outcomes when $n = 2$.

χ^2 tests

Exercise 4 (Test for fit)**:**
A dice gives $1, \ldots, 6$ with respective probabilities $\theta_1, \ldots, \theta_6$, with $\theta_1 + \ldots + \theta_6 = 1$. You want to test $H_0 = \{\theta = (\frac{1}{6}, \ldots, \frac{1}{6})\}$ against $H_1 = \{\theta \neq (\frac{1}{6}, \ldots, \frac{1}{6})\}$ based on n i.i.d. observations.

1. Give the χ^2 test at level α.
2. You get the following numbers of $1, \ldots, 6$ over 100 samples: $20, 12, 18, 24, 11, 15$.
   Give the result of the test at level $\alpha = 5\%$.

Exercise 5 (Uniform model – Test for fit)**:**
You get the following statistics for the number of births declared in Poitiers in 2019[1]:

Period	Jan-Feb	Mar-May	Jun-Aug	Sep-Oct	Nov-Dec
Count	570	813	909	593	618

The null hypothesis is that births are uniformly distributed over the year.

1. Propose a χ^2 test at level α. For simplicity, we assume that all months have 30 days.
2. Give the result of this test for $\alpha = 1\%$.

Exercise 6 (Poisson model – Test for fit)**:**
The daily number of emails you receive is supposed to be Poisson distributed.
You collect the following statistics:

Range	$[0, 15)$	$[15, 20)$	$[20, 25)$	$[25, +\infty)$
Count	10	20	25	15

The empirical average is 20 emails per day.

1. Propose a χ^2-test at level α to test the Poisson distribution.
2. Give the result of the test for $\alpha = 1\%$.

Exercise 7 (Gaussian model – Test for fit)**:**
The daily power consumption of a company is supposed to be Gaussian.
You get the following statistics:

Range	< 80	$80 - 100$	$100 - 120$	> 120
Count	30	70	65	35

The empirical mean is 100 and the empirical standard deviation is 20.

1. Propose a χ^2-test at level α to test the Gaussian distribution.
2. Give the result of the test for $\alpha = 1\%$.

[1] See https://www.data.gouv.fr/fr/datasets/citoyennete-nombre-de-naissances-sur-poitiers/.

Exercise 8 (Test for independence)**:**
The following statistics were collected on patients with Covid-19 in France in April 2020[2].

1. Cumulated days of patients with Covid-19 in hospitals (in thousands):

	Total	Intensive cares
Men	1 400	281
Women	1 230	102

Test the independence between gender and severe forms of Covid-19 disease.

2. Status of patients with Covid-19:

Age	Back home	Death
< 20	2 189	7
$20 - 40$	10 682	133
$40 - 60$	24 678	1 297
$60 - 80$	38 023	7 644
> 80	29 128	13 332
Total	104 700	22 413

Test the independence between age and death by Covid-19.

[2] See https://www.data.gouv.fr/fr/datasets/donnees-hospitalieres-relatives-a-lepidemie-de-covid-19/