SLR 206 – Fondements des Algorithmes Répartis

Randomized Consensus and Other Agreement Protocols

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So far...

- Consensus is impossible to solve deterministically in a wait-free manner using read-write registers if at least one process may crash.

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- Consensus is impossible to solve deterministically in an asynchronous message passing system if at least one process may crash.

Next

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- Consensus is impossible to solve deterministically in an asynchronous message passing system if at least one process may crash.

These constructions have *Consensus number 1*

Consensus Number	Object
1	read/write registers
2	test&set, swap, fetch&add, queue, stack
⋮	⋮
$2n-2$	n-register assignment
⋮	⋮
∞	memory-to-memory move and swap, augmented queue, compare&swap, fetch&cons, sticky byte

Figure: Maurice Herlihy – Wait-free synchronization.

Next

- Consensus is impossible to solve deterministically in an asynchronous message passing system if at least one process may crash.

Consensus can be deterministically solved in *partial-synchronous* and *synchronous* networks.

- Asynchronous – No bound on message delays.
- Partial-synchronous – There exists a delay, but it is unknown.
- Synchronous – There exists a known delay.

Next

- Consensus is impossible to solve deterministically in a wait-free manner using read-write registers if at least one process may crash.
- Consensus is impossible to solve deterministically in an asynchronous message passing system if at least one process may crash.

In this lecture: There are other agreement protocols.

- Approximate Agreement
- Commit-adopt

Next

- Consensus is impossible to solve deterministically in a wait-free manner using read-write registers if at least one process may crash.

In this lecture: Consensus can be deterministically solved in an *obstruction-free* run.

- Lock-free – If processes try infinitely many times to take steps, they eventually finish.
- Obstruction-free – If a process takes steps in isolation, then it finishes in finitely many steps.
- Wait-free – Every process finishes in finitely many steps.

Next

- Consensus is impossible to solve deterministically in a wait-free manner using read-write registers if at least one process may crash.
- Consensus is impossible to solve deterministically in an asynchronous message passing system if at least one process may crash.

In this lecture: Consensus can be solved by randomized protocols with probabilistic guarantees.

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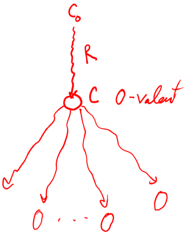
Review – FLP



C0
R
C
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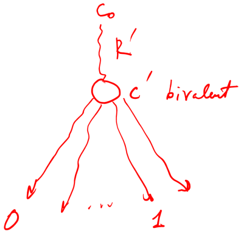

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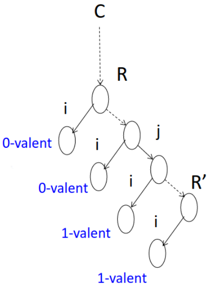

C
R
i
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0-valent
i
0-valent
i
R'
1-valent
i
1-valent

The iterated memory model

Algorithm 1 Two processes iterated memory – Code for process i

Infinite array of n atomic registers $R_k[1\ldots]$

$k \leftarrow 0$

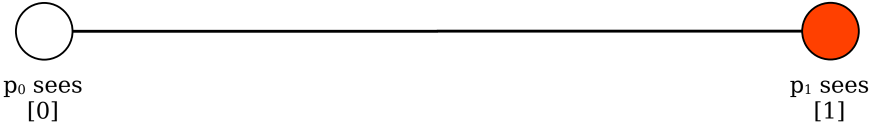
$v_i \leftarrow$ input

while decision not reached **do**

  $k \leftarrow k + 1$

  $R_k[i]$.write(v_i)

  $v_i \leftarrow [v_i, R_k[i-1]$.read()$]$

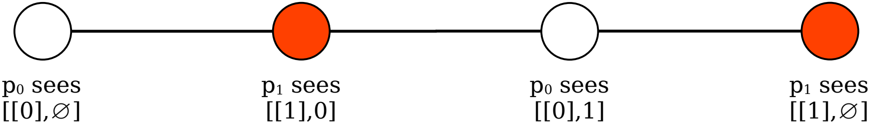


The iterated memory model

Algorithm 2 Two processes iterated memory – Code for process i

Infinite array of n atomic registers $R_k[1\ldots]$

$k \leftarrow 0$
$v_i \leftarrow$ input
while decision not reached **do**
    $k \leftarrow k+1$ $//k=1$
    $R_k[i]$.write(v_i)
    $v_i \leftarrow [v_i, R_k[i-1]$.read$()]$



The iterated memory model

Algorithm 3 Two processes iterated memory – Code for process i

Infinite array of n atomic registers $R_k[1\ldots]$

$k \leftarrow 0$

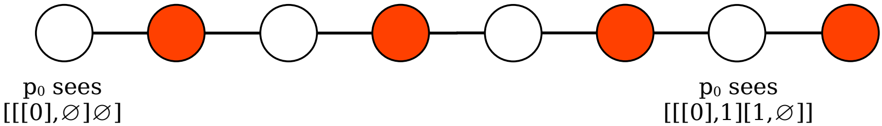
$v_i \leftarrow$ input

while decision not reached **do**

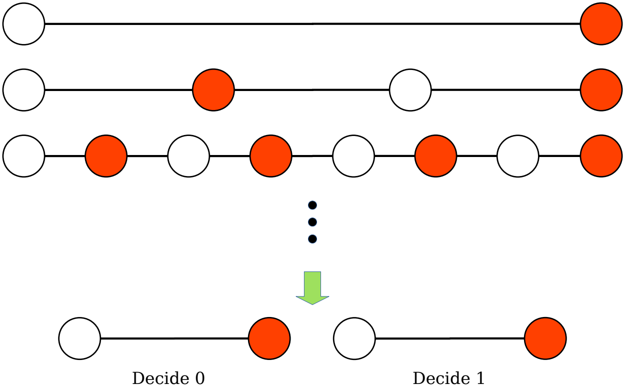
  $k \leftarrow k + 1$ $//k = 2$

  $R_k[i]$.write(v_i)

  $v_i \leftarrow [v_i, R_k[i-1]$.read$()]$



The topological argument for FLP



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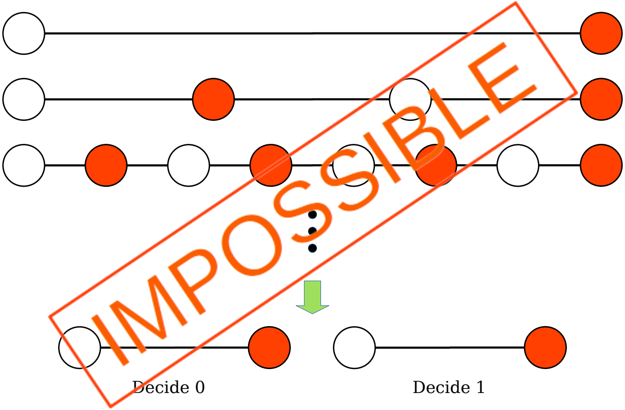

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Approximate Agreement

An approximate agreement protocol provides the following properties:

- Validity – Every output value is in the range of input values.

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Limitations? The problem must be "continuous" and admit disagreement.

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Limitations? The problem must be "continuous" and admit disagreement.

Bad examples	Good examples
input: $v = [1\ 1\ 1\ 1\ 1\ 1], \epsilon = 0.01$ **output**: $[1\ 1\ 0.99\ 1\ 1]$	

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Bad examples	Good examples
input: $v = [1\ 1\ 1\ 1\ 1\ 1], \epsilon = 0.01$ **output**: $[1\ 1\ 0.99\ 1\ 1]$	**input**: $v = [1\ 1\ 1\ 1\ 1\ 1], \epsilon = 10000$ **output**: $[1\ 1\ 1\ 1\ 1]$

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- Validity – Every output value is in the range of input values.
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Bad examples	Good examples
input: $v = [1\ 1\ 1\ 1\ 1\ 1], \epsilon = 0.01$ **output**: $[1\ 1\ 0.99\ 1\ 1]$	**input**: $v = [1\ 1\ 1\ 1\ 1\ 1], \epsilon = 10000$ **output**: $[1\ 1\ 1\ 1\ 1]$
input: $v = [0\ 2\ 3\ 9\ 7\ 3], \epsilon = 0.5$ **output**: $[9.3\ 9\ 8.9\ 8.95\ 1]$	

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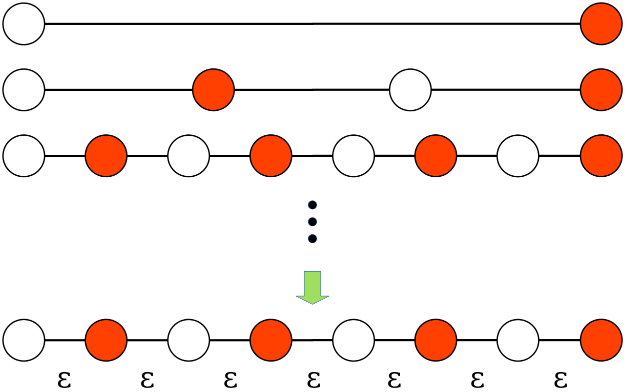
input: $v = [1\ 1\ 1\ 1\ 1\ 1], \epsilon = 0.01$ **output**: [1 1 0.99 1 1]	**input**: $v = [1\ 1\ 1\ 1\ 1\ 1], \epsilon = 10000$ **output**: [1 1 1 1 1]
input: $v = [0\ 2\ 3\ 9\ 7\ 3], \epsilon = 0.5$ **output**: [9.3 9 8.9 8.95 1]	**input**: $v = [0\ 2\ 3\ 9\ 7\ 3], \epsilon = 0.5$ **output**: [9 9 8.9 8.95 1]

The topology of AA



Solving AA

Algorithm 4 Two process wait-free ϵ-AA

Given two atomic registers $R[0], R[1]$

$v_i \leftarrow$ input

while true **do**

  // R[i] is originally $\perp$

  $R[i]$.write(v_i)

  $v_j \leftarrow R[1-i]$.read(v_i)

  if $v_j = \perp \vee |v_j - v_i| \leq \epsilon$ **then** Decide v_i

  else $v_i \leftarrow \frac{v_i + v_j}{2}$

Solving AA

Algorithm 5 Two process wait-free ϵ-AA

1: Given two atomic registers $R[0], R[1]$
2: $v_i \leftarrow$ input
3: **while** true **do**
4: // R[i] is originally $\perp$
5: $R[i].\mathrm{write}(v_i)$
6: $v_j \leftarrow R[1-i].\mathrm{read}(v_i)$
7: **if** $v_j = \perp \vee |v_j - v_i| \leq \epsilon$ **then** Decide v_i
8: **else** $v_i \leftarrow \frac{v_i+v_j}{2}$

Solving AA

Algorithm 6 Two process wait-free ϵ-AA

Given two atomic registers $R[0], R[1]$

$v_i \leftarrow$ input

while true **do**

  // R[i] is originally $\perp$

  $R[i]$.write(v_i)

  $v_j \leftarrow R[1-i]$.read(v_i)

  if $v_j = \perp \vee |v_j - v_i| \leq \epsilon$ **then** Decide v_i

  else $v_i \leftarrow \frac{v_i + v_j}{2}$

Solving AA

Algorithm 7 Two process wait-free ϵ-AA

```
Given two atomic registers R[0], R[1]
v_i ← input
while true do
    // R[i] is originally ⊥
    R[i].write(v_i)
    v_j ← R[1 − i].read(v_i)
    if v_j = ⊥ ∨ |v_j − v_i| ≤ ε then Decide v_i
    else v_i ← (v_i + v_j) / 2
```

Solving AA

Algorithm 8 Two process wait-free ϵ-AA

Given two atomic registers $R[0], R[1]$

$v_i \leftarrow$ input

while true **do**

    // R[i] is originally $\perp$

    $R[i]$.write(v_i)

    $v_j \leftarrow R[1-i]$.read(v_i)

    if $v_j = \perp \vee |v_j - v_i| \leq \epsilon$ **then** Decide v_i

    else $v_i \leftarrow \frac{v_i+v_j}{2}$

Solution Analysis

- Validity – The decided values are either v_i or some iteration of $\frac{v_i + v_j}{2}$.

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Solution Analysis

- Validity – The decided values are either v_i or some iteration of $\frac{v_i + v_j}{2}$.
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- Liveness – The algorithm is wait-free, a process does not need the other to take steps in order to make progress and always terminates.

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Solution Analysis

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- Liveness – The algorithm is wait-free, a process does not need the other to take steps in order to make progress and always terminates.

The algorithm terminates in at most $O(\log(\frac{|v_1-v_0|}{\epsilon}))$ steps.

AA with n processes

Simply extending the case for $n = 2$ by using snapshot objects results in a $O((n + \log n)\log(\frac{|v_1 - v_0|}{\epsilon}))$ complexity algorithm. In "Are wait-free algorithms fast?" by Attiya, Lynch and Shavit, the authors provide a wait-free solution with complexity $O(\log n)$.

In message passing:

1. Broadcast v
2. Wait for n-f values
3. Compute mean of received values
4. Repeat

The problem is more interesting when there are Byzantine failures.

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Commit Adopt Definition

In commit adopt, processes propose an input value from V and output a couple $(c, v) \in \{false, true\} \times V$. If a process outputs either $(false, v)$ or $(true, v)$, we say that it adopts v. If a process outputs $(true, v)$, then we say that it commits to v.

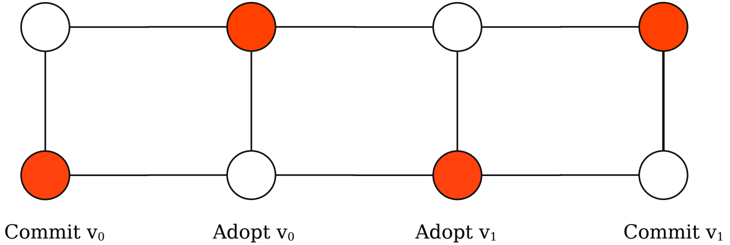
- Validity – Every adopted value is an input value.
- Termination – Every correct process eventually adopts a value.
- CA-Agreement
  - If a process commits a value v, then every correct process adopts v.
  - If every process inputs the same value, then every correct process commits to a value.

Graded Commit Adopt Definition

In graded commit adopt, processes propose an input value from V and output a couple $(c, v) \in \{0, 1, 2\} \times V$. If a process outputs either $(1, v)$ or $(2, v)$, we say that it adopts v. If a process outputs $(2, v)$, then we say that it commits to v, finally, if a process outputs $(0, v)$ we say that it sticks with its input.

- Validity – Every output value is an input value.
- Termination – Every correct process eventually outputs a value.
- CA-Agreement
  - If a process commits a value v, then every correct process adopts v.
  - If every process inputs the same value, then every correct process commits to a value.

Commit Adopt Topology



Graded Commit Adopt Algorithm

Algorithm 9 Graded Commit Adopt (GCA)

$A[1..n]$ n atomic registers
$B[1..n]$ n atomic registers
$v_i \leftarrow v$ // v is the original input
$A[i]$.write(v_i)
$U \leftarrow$ read $A[1..n]$
if All non-$\perp$ values in U are v_i **then** $B[i]$.write(($true$, v_i))
else $B[i]$.write(($false$, v_i))
$U \leftarrow$ read $B[1..n]$
if All non-$\perp$ values in U are $(true, v_i)$ **then** Decide $(2, v_i)$
else if $\exists(true, v') \in U$ **then** Decide $(1, v')$
else Decide$(0, v_i)$

Graded Commit Adopt Algorithm

Algorithm 10 Graded Commit Adopt (GCA)

```
A[1..n] n atomic registers
B[1..n] n atomic registers
v_i ← v    // v is the original input
A[i].write(v_i)
U ← read A[1..n]
if All non-⊥ values in U are v_i then B[i].write((true, v_i))
else B[i].write((false, v_i))
U ← read B[1..n]
if All non-⊥ values in U are (true, v_i) then Decide (2, v_i)
else if ∃(true, v′) ∈ U then Decide (1, v′)
else Decide(0, v_i)
```

Graded Commit Adopt Algorithm

Algorithm 11 Graded Commit Adopt (GCA)

$A[1..n]$ n atomic registers
$B[1..n]$ n atomic registers
$v_i \leftarrow v$ // v is the original input
$A[i]$.write(v_i)
$U \leftarrow$ read $A[1..n]$
if All non-$\perp$ values in U are v_i **then** $B[i]$.write(($true$, v_i))
else $B[i]$.write(($false$, v_i))
$U \leftarrow$ read $B[1..n]$
if All non-$\perp$ values in U are ($true$, v_i) **then** Decide (2, v_i)
else if $\exists$($true$, v') $\in U$ **then** Decide (1, v')
else Decide(0, v_i)

Graded Commit Adopt Algorithm

Algorithm 12 Graded Commit Adopt (GCA)

$A[1..n]$ n atomic registers
$B[1..n]$ n atomic registers
$v_i \leftarrow v$ // v is the original input
$A[i]$.write(v_i)
$U \leftarrow$ read $A[1..n]$
if All non-$\perp$ values in U are v_i **then** $B[i]$.write(($true$, v_i))
else $B[i]$.write(($false$, v_i))
$U \leftarrow$ read $B[1..n]$
if All non-$\perp$ values in U are $(true, v_i)$ **then** Decide $(2, v_i)$
else if $\exists(true, v') \in U$ **then** Decide $(1, v')$
else Decide$(0, v_i)$

Graded Commit Adopt Algorithm

Algorithm 13 Graded Commit Adopt (GCA)

$A[1..n]$ n atomic registers
$B[1..n]$ n atomic registers
$v_i \leftarrow v$ // v is the original input
$A[i]$.write(v_i)
$U \leftarrow$ read $A[1..n]$
if All non-$\perp$ values in U are v_i **then** $B[i]$.write(($true$, v_i))
else $B[i]$.write(($false$, v_i))
$U \leftarrow$ read $B[1..n]$
if All non-$\perp$ values in U are ($true$, v_i) **then** Decide (2, v_i)
else if $\exists(true, v') \in U$ **then** Decide (1, v')
else Decide(0, v_i)

Graded Commit Adopt Algorithm

Algorithm 14 Graded Commit Adopt (GCA)

$A[1..n]$ n atomic registers
$B[1..n]$ n atomic registers

$v_i \leftarrow v$ // v is the original input
$A[i]$.write(v_i)
$U \leftarrow$ read $A[1..n]$
if All non-$\perp$ values in U are v_i **then** $B[i]$.write(($true$, v_i))
else $B[i]$.write(($false$, v_i))

$U \leftarrow$ read $B[1..n]$
if All non-$\perp$ values in U are ($true$, v_i) **then** Decide (2, v_i)
else if $\exists(true, v') \in U$ **then** Decide (1, v')
else Decide(0, v_i)

Graded Commit Adopt Algorithm

Algorithm 15 Graded Commit Adopt (GCA)

$A[1..n]$ n atomic registers
$B[1..n]$ n atomic registers
$v_i \leftarrow v$ // v is the original input
$A[i]$.write(v_i)
$U \leftarrow$ read $A[1..n]$
if All non-$\perp$ values in U are v_i **then** $B[i]$.write(($true$, v_i))
else $B[i]$.write(($false$, v_i))
$U \leftarrow$ read $B[1..n]$
if All non-$\perp$ values in U are ($true$, v_i) **then** Decide (2, v_i)
else if $\exists$($true$, v') $\in U$ **then** Decide (1, v')
else Decide(0, v_i)

Graded Commit Adopt Algorithm

Algorithm 16 Graded Commit Adopt (GCA)

$A[1..n]$ n atomic registers
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$U \leftarrow$ read $A[1..n]$
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else $B[i]$.write(($false$, v_i))
$U \leftarrow$ read $B[1..n]$
if All non-$\perp$ values in U are ($true$, v_i) **then** Decide (2, v_i)
else if $\exists(true, v') \in U$ **then** Decide $(1, v')$
else Decide(0, v_i)

Algorithm 17 Graded Commit Adopt (GCA)

$A[1..n]$ n atomic registers
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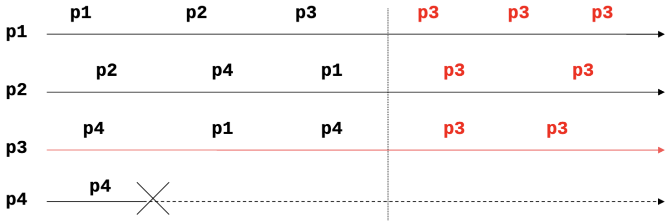
else if $\exists(true, v') \in U$ **then** Decide (1, v')
else Decide(0, v_i)

Ω failure detector

- Eventual Leader Detector.
- Outputs at every process a process identifier.
- Eventually, the same correct process is output at every correct process.



It can be implemented in partial-synchronous systems.

Consensus

Algorithm 18 GCA + Ω = Consensus

D a regular register
$R[1..n]$ n regular registers
$GCA_1, GCA_2, \ldots$ a series of graded commit adopt instances
Ω failure detector

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
$R[i] \leftarrow v$
while decision not reached **do**
    $k \leftarrow k + 1$
    if $v' \leftarrow D$.read() $\neq \perp$ **then** Decide v'
    $(c, v_i) \leftarrow GCA_k$.propose$(v_i)$
    if $c = 0 \wedge R[\Omega]$.read() $\neq \perp$ **then** $v_i \leftarrow R[\Omega]$.read()
        // $c = 1$ the process adopts the value from GCA
    else if $c = 2$ **then** D.write(v_i)

Consensus

Algorithm 19 GCA $+ \Omega =$ Consensus

D a regular register
$R[1..n]$ n regular registers
$GCA_1, GCA_2, \ldots$ a series of graded commit adopt instances
Ω failure detector

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
$R[i] \leftarrow v$
while decision not reached **do**
  $k \leftarrow k + 1$
  if $v' \leftarrow D.\text{read}() \neq \perp$ **then** Decide v'
  $(c, v_i) \leftarrow GCA_k.\text{propose}(v_i)$
  if $c = 0 \wedge R[\Omega].\text{read}() \neq \perp$ **then** $v_i \leftarrow R[\Omega].\text{read}()$
    // $c = 1$ the process adopts the value from GCA
  else if $c = 2$ **then** $D.\text{write}(v_i)$

Consensus

Algorithm 20 GCA $+ \Omega =$ Consensus

D a regular register
$R[1..n]$ n regular registers
$GCA_1, GCA_2, \ldots$ a series of graded commit adopt instances
Ω failure detector

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
$R[i] \leftarrow v$
while decision not reached **do**
    $k \leftarrow k + 1$
    if $v' \leftarrow D$.read() $\neq \perp$ **then** Decide v'
    $(c, v_i) \leftarrow GCA_k$.propose$(v_i)$
    if $c = 0 \wedge R[\Omega]$.read() $\neq \perp$ **then** $v_i \leftarrow R[\Omega]$.read()
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    else if $c = 2$ **then** D.write(v_i)

Consensus

Algorithm 21 GCA + Ω = Consensus

```
D a regular register
R[1..n] n regular registers
GCA_1, GCA_2, ... a series of graded commit adopt instances
Ω failure detector

v_i ← v   // v is the original input
k ← 0
R[i] ← v
while decision not reached do
    k ← k + 1
    if v' ← D.read() ≠ ⊥ then Decide v'
    (c, v_i) ← GCA_k.propose(v_i)
    if c = 0 ∧ R[Ω].read() ≠ ⊥ then v_i ← R[Ω].read()
        // c = 1 the process adopts the value from GCA
    else if c = 2 then D.write(v_i)
```

Consensus

Algorithm 22 GCA + Ω = Consensus

D a regular register
$R[1..n]$ n regular registers
$GCA_1, GCA_2, \ldots$ a series of graded commit adopt instances
Ω failure detector

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
$R[i] \leftarrow v$
while decision not reached **do**
  $k \leftarrow k + 1$
  if $v' \leftarrow D.\text{read}() \neq \perp$ **then** Decide v'
  $(c, v_i) \leftarrow GCA_k.\text{propose}(v_i)$
  if $c = 0 \wedge R[\Omega].\text{read}() \neq \perp$ **then** $v_i \leftarrow R[\Omega].\text{read}()$
    // $c = 1$ the process adopts the value from GCA
  else if $c = 2$ **then** $D.\text{write}(v_i)$

Consensus

Algorithm 23 GCA + Ω = Consensus

D a regular register
$R[1..n]$ n regular registers
$GCA_1, GCA_2, \ldots$ a series of graded commit adopt instances
Ω failure detector

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
$R[i] \leftarrow v$
while decision not reached **do**
    $k \leftarrow k + 1$
    if $v' \leftarrow D.\text{read}() \neq \bot$ **then** Decide v'
    $(c, v_i) \leftarrow GCA_k.\text{propose}(v_i)$
    if $c = 0 \wedge R[\Omega].\text{read}() \neq \bot$ **then** $v_i \leftarrow R[\Omega].\text{read}()$
        // $c = 1$ the process adopts the value from GCA
    else if $c = 2$ **then** $D.\text{write}(v_i)$

Consensus

Algorithm 24 GCA + Ω = Consensus

D a regular register
$R[1..n]$ n regular registers
$GCA_1, GCA_2, \ldots$ a series of graded commit adopt instances
Ω failure detector

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
$R[i] \leftarrow v$
while decision not reached **do**
    $k \leftarrow k + 1$
    if $v' \leftarrow D.\text{read}() \neq \bot$ **then** Decide v'
    $(c, v_i) \leftarrow GCA_k.\text{propose}(v_i)$
    if $c = 0 \wedge R[\Omega].\text{read}() \neq \bot$ **then** $v_i \leftarrow R[\Omega].\text{read}()$
        // $c = 1$ the process adopts the value from GCA
    else if $c = 2$ **then** $D.\text{write}(v_i)$

Consensus

Algorithm 25 GCA $+ \Omega =$ Consensus

D a regular register
$R[1..n]$ n regular registers
$GCA_1, GCA_2, \ldots$ a series of graded commit adopt instances
Ω failure detector

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
$R[i] \leftarrow v$
while decision not reached **do**
  $k \leftarrow k + 1$
  if $v' \leftarrow D.\text{read}() \neq \perp$ **then** Decide v'
  $(c, v_i) \leftarrow GCA_k.\text{propose}(v_i)$
  if $c = 0 \wedge R[\Omega].\text{read}() \neq \perp$ **then** $v_i \leftarrow R[\Omega].\text{read}()$
    // $c = 1$ the process adopts the value from GCA
  else if $c = 2$ **then** $D.\text{write}(v_i)$

Table of Contents

Modified Ben-Or Consensus Protocol in Shared Memory

Algorithm 26 Ben-Or in shared memory

D a regular register
$R[1..n]$ n regular registers
$GCA_1, GCA_2, \ldots$ a series of graded commit adopt instances
$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
$R[i] \leftarrow v$
while decision not reached **do**
    $k \leftarrow k + 1$
    if $v' \leftarrow D.\text{read}() \neq \perp$ **then** Decide v'
        $(c, v_i) \leftarrow GCA_k.\text{propose}(v_i)$
    if $c = 0$ **then** $v_i \leftarrow$ random bit
        // $c = 1$ the process adopts the value from GCA
    else if $c = 2$ **then** $D.\text{write}(v_i)$

Ben-Or Consensus

Algorithm 27 Ben-Or in message passing

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
while decision not reached **do**
  $k \leftarrow k + 1$
  Send $\langle m_1, k, v_i \rangle$ to all
  Wait for $n - f$ messages $\langle m_1, k, * \rangle$
  if All msgs are $\langle m_1, k, v \rangle$ **then** send $\langle m_2, k, v \rangle$
  else send $\langle m_2, k, \perp \rangle$
  Wait for $n - f$ messages $\langle m_2, k, * \rangle$
  // $n - f = f + 1$
  if All msgs are $\langle m_2, k, v \rangle$ where $v \neq \perp$ **then** send $\langle DECIDE, k, v \rangle$
  else if $\exists \langle m_2, k, v \rangle$ where $v \neq \perp$ **then** $v_i \leftarrow v$
  else $v_i \leftarrow$ Random bit

Algorithm 28 Ben-Or in message passing

```
v_i ← v    // v is the original input
k ← 0
while decision not reached do
    k ← k + 1
    Send ⟨m_1, k, v_i⟩ to all
    Wait for n − f messages ⟨m_1, k, ∗⟩
    if All msgs are ⟨m_1, k, v⟩ then  send ⟨m_2, k, v⟩
    else  send ⟨m_2, k, ⊥⟩

    Wait for n − f messages ⟨m_2, k, ∗⟩
    // n − f = f + 1
    if All msgs are ⟨m_2, k, v⟩ where v ≠ ⊥ then  send ⟨DECIDE, k, v⟩
    else if ∃⟨m_2, k, v⟩ where v ≠ ⊥ then  v_i ← v
    else  v_i ← Random bit
```

Ben-Or Consensus

Algorithm 29 Ben-Or in message passing

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
while decision not reached **do**
  $k \leftarrow k + 1$
  Send $\langle m_1, k, v_i \rangle$ to all
  Wait for $n - f$ messages $\langle m_1, k, * \rangle$
  if All msgs are $\langle m_1, k, v \rangle$ **then** send $\langle m_2, k, v \rangle$
  else send $\langle m_2, k, \perp \rangle$

  Wait for $n - f$ messages $\langle m_2, k, * \rangle$
  // $n - f = f + 1$
  if All msgs are $\langle m_2, k, v \rangle$ where $v \neq \perp$ **then** send $\langle DECIDE, k, v \rangle$
  else if $\exists \langle m_2, k, v \rangle$ where $v \neq \perp$ **then** $v_i \leftarrow v$
  else $v_i \leftarrow$ Random bit

Ben-Or Consensus

Algorithm 30 Ben-Or in message passing

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
while decision not reached **do**
    $k \leftarrow k + 1$
    Send $\langle m_1, k, v_i \rangle$ to all
    Wait for $n - f$ messages $\langle m_1, k, * \rangle$
    if All msgs are $\langle m_1, k, v \rangle$ **then** send $\langle m_2, k, v \rangle$
    else send $\langle m_2, k, \perp \rangle$

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    // $n - f = f + 1$
    if All msgs are $\langle m_2, k, v \rangle$ where $v \neq \perp$ **then** send $\langle DECIDE, k, v \rangle$
    else if $\exists \langle m_2, k, v \rangle$ where $v \neq \perp$ **then** $v_i \leftarrow v$
    else $v_i \leftarrow$ Random bit

Ben-Or Consensus

Algorithm 31 Ben-Or in message passing

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
while decision not reached **do**
  $k \leftarrow k + 1$
  Send $\langle m_1, k, v_i \rangle$ to all
  Wait for $n - f$ messages $\langle m_1, k, * \rangle$
  if All msgs are $\langle m_1, k, v \rangle$ **then** send $\langle m_2, k, v \rangle$
  else send $\langle m_2, k, \perp \rangle$
  Wait for $n - f$ messages $\langle m_2, k, * \rangle$
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  if All msgs are $\langle m_2, k, v \rangle$ where $v \neq \perp$ **then** send $\langle DECIDE, k, v \rangle$
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  else $v_i \leftarrow$ Random bit

Ben-Or Consensus

Algorithm 32 Ben-Or in message passing

$v_i \leftarrow v$ // v is the original input
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while decision not reached **do**
    $k \leftarrow k + 1$
    Send $\langle m_1, k, v_i \rangle$ to all
    Wait for $n - f$ messages $\langle m_1, k, * \rangle$
    if All msgs are $\langle m_1, k, v \rangle$ **then** send $\langle m_2, k, v \rangle$
    else send $\langle m_2, k, \perp \rangle$

    Wait for $n - f$ messages $\langle m_2, k, * \rangle$
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    if All msgs are $\langle m_2, k, v \rangle$ where $v \neq \perp$ **then** send $\langle DECIDE, k, v \rangle$
    else if $\exists \langle m_2, k, v \rangle$ where $v \neq \perp$ **then** $v_i \leftarrow v$
    else $v_i \leftarrow$ Random bit

Ben-Or Consensus

Algorithm 33 Ben-Or in message passing

```
v_i ← v    // v is the original input
k ← 0
while decision not reached do
    k ← k + 1
    Send ⟨m_1, k, v_i⟩ to all
    Wait for n − f messages ⟨m_1, k, ∗⟩
    if All msgs are ⟨m_1, k, v⟩ then  send ⟨m_2, k, v⟩
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    Wait for n − f messages ⟨m_2, k, ∗⟩
    // n − f = f + 1
    if All msgs are ⟨m_2, k, v⟩ where v ≠ ⊥ then  send ⟨DECIDE, k, v⟩
    else if ∃⟨m_2, k, v⟩ where v ≠ ⊥ then  v_i ← v
    else  v_i ← Random bit
```

Ben-Or Consensus

Algorithm 34 Ben-Or in message passing

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
while decision not reached **do**
    $k \leftarrow k + 1$
    Send $\langle m_1, k, v_i \rangle$ to all
    Wait for $n - f$ messages $\langle m_1, k, * \rangle$
    if All msgs are $\langle m_1, k, v \rangle$ **then** send $\langle m_2, k, v \rangle$
    else send $\langle m_2, k, \perp \rangle$

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    else if $\exists \langle m_2, k, v \rangle$ where $v \neq \perp$ **then** $v_i \leftarrow v$
    else $v_i \leftarrow$ Random bit

Ben-Or Consensus

Algorithm 35 Ben-Or in message passing

$v_i \leftarrow v$ // v is the original input
$k \leftarrow 0$
while decision not reached **do**
  $k \leftarrow k + 1$
  Send $\langle m_1, k, v_i \rangle$ to all
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  if All msgs are $\langle m_1, k, v \rangle$ **then** send $\langle m_2, k, v \rangle$
  else send $\langle m_2, k, \perp \rangle$

  Wait for $n - f$ messages $\langle m_2, k, * \rangle$
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  else if $\exists \langle m_2, k, v \rangle$ where $v \neq \perp$ **then** $v_i \leftarrow v$
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Complexity

The expected number of rounds is $\mathbb{E}[R] = \sum_{r=1}^{\infty}(r \times P(\text{Decide at round r}))$.

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$P(\text{Decide at round r}) = (1-p)^{r-1}p$.

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$P(\text{Decide at round r}) = (1-p)^{r-1}p$.

$\mathbb{E}[R] = \sum_{r=1}^{\infty}(r(1-p)^{r-1}p) = \frac{1}{p}$.

The probability of a decision being reached at every round as a result of all flips matching is: $p = 2^{-n}$. Hence, the number of expected rounds is $O(2^n)$.

Shared Memory Common Coin

Algorithm 36 Bracha-Rachman common coin

done – shared atomic register
$R[1..n]$ – n atomic registers

$count \leftarrow 0$
$sum \leftarrow 0$
while not done **do**
  $count \leftarrow count + 1$
  $sum \leftarrow sum + \text{flip}()$ // The flip result is -1 or +1
  $R[i].\text{write}((count, sum))$
  if $count \mod n = 0$ **then**
    $U \leftarrow \text{Read } R[1..n]$
    if $\sum_{i=1}^{n} U[i].count \geq n^2$ **then**
      $done \leftarrow true$
      Decide $sgn(\sum_{i=1}^{n} U[i].sum)$

Shared Memory Common Coin

Algorithm 37 Bracha-Rachman common coin

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$R[1..n]$ – n atomic registers
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Shared Memory Common Coin

Algorithm 38 Bracha-Rachman common coin

done – shared atomic register
$R[1..n]$ – n atomic registers
$count \leftarrow 0$
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Shared Memory Common Coin

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$count \leftarrow 0$
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    if $\sum_{i=1}^{n} U[i].count \geq n^2$ **then**
      $done \leftarrow true$
      Decide $sgn(\sum_{i=1}^{n} U[i].sum)$

Shared Memory Common Coin

Algorithm 40 Bracha-Rachman common coin

done – shared atomic register
$R[1..n]$ – n atomic registers

$count \leftarrow 0$
$sum \leftarrow 0$
while not done **do**
    $count \leftarrow count + 1$
    $sum \leftarrow sum + \text{flip}()$ // The flip result is -1 or +1
    $R[i]\text{.write}((count, sum))$
    if $count \mod n = 0$ **then**
        $U \leftarrow \text{Read } R[1..n]$
        if $\sum_{i=1}^{n} U[i].count \geq n^2$ **then**
            $done \leftarrow true$
            Decide $sgn(\sum_{i=1}^{n} U[i].sum)$

Shared Memory Common Coin

Algorithm 41 Bracha-Rachman common coin

done – shared atomic register
$R[1..n]$ – n atomic registers

$count \leftarrow 0$
$sum \leftarrow 0$
while not done **do**
    $count \leftarrow count + 1$
    $sum \leftarrow sum +$ flip()   // The flip result is -1 or +1
    $R[i]$.write($(count, sum)$)
    if $count \mod n = 0$ **then**
        $U \leftarrow$ Read $R[1..n]$
        if $\sum_{i=1}^{n} U[i].count \geq n^2$ **then**
            $done \leftarrow true$
            Decide $sgn(\sum_{i=1}^{n} U[i].sum)$

Shared Memory Common Coin

Algorithm 42 Bracha-Rachman common coin

$done$ – shared atomic register
$R[1..n]$ – n atomic registers
$count \leftarrow 0$
$sum \leftarrow 0$
while not done **do**
    $count \leftarrow count + 1$
    $sum \leftarrow sum + \text{flip}()$ // The flip result is -1 or +1
    $R[i].\text{write}((count, sum))$
    if $count \mod n = 0$ **then**
        $U \leftarrow \text{Read } R[1..n]$
        if $\sum_{i=1}^{n} U[i].count \geq n^2$ **then**
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            Decide $sgn(\sum_{i=1}^{n} U[i].sum)$

Shared Memory Common Coin – Analysis

The summation of n^2 can be approximated to a normal random function.

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The probability that $count$ exceeds n^2 by more than $O(n)$ can be bounded by a constant.

Shared Memory Common Coin – Analysis

The summation of n^2 can be approximated to a normal random function.

The probability that $sum \geq cn$ can be lowerbounded by a constant.

The probability that $count$ exceeds n^2 by more than $O(n)$ can be bounded by a constant.

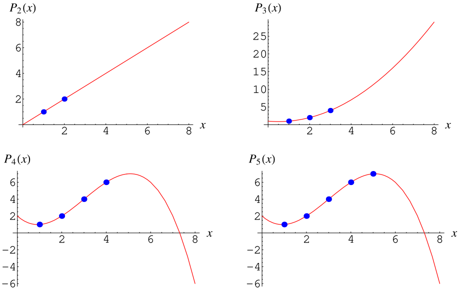
The probability that processes agree is therefore constant.

Message Passing Common Coin

Digital Signatures:

- Unforgeability – Processes with not enough keys cannot produce a signature.
- Uniqueness – Processes with enough keys always generate the same signature for the same message.

How to achieve it? – Polynomial interpolation.



These coins have constant success probability $\rightarrow$ randomized consensus takes a constant amount of times in expectation.

Good Luck with the Exam!