Statistics
MDI220
5. Part 1: Bayesian tests

Thomas Bonald
Institut Polytechnique de Paris

2023 – 2024

We present hypothesis testing in a Bayesian setting where the parameter of interest θ is considered as random, with some prior distribution π.

1 Bayes risk

Let $\delta(x) \in \{0,1\}$ be the decision, based on the observation x:

$$\delta(x) = \left\{ \begin{array}{ll} 0 & \rightarrow \text{ accept } H_0 \\ 1 & \rightarrow \text{ reject } H_0 \text{ in favor of } H_1. \end{array} \right.$$

Recall that in parametric models, the null and alternative hypotheses form subsets of the parameters:

$$\begin{array}{l} H_0 \rightarrow \Theta_0 \subset \Theta \\ H_1 \rightarrow \Theta_1 \subset \Theta \end{array} \qquad \Theta_0 \cap \Theta_1 = \emptyset$$

Given θ, the risk of the decision function δ is the probability of error:

$$R(\theta,\delta) = \left\{ \begin{array}{lll} P_\theta(\delta(X)=1) & \text{if } \theta \in \Theta_0 & \rightarrow \text{ type-I errors} \\ P_\theta(\delta(X)=0) & \text{if } \theta \in \Theta_1 & \rightarrow \text{ type-II errors} \end{array} \right.$$

Bayes risk is the expected risk. It is the overall error rate, including both types of errors (type-I and type-II).

The Bayes risk of the decision function δ is:

$$r(\delta) = \mathrm{E}(R(\theta,\delta)) = \int_{\Theta_0} \underbrace{P_\theta(\delta(X)=1)}_{\text{type-I}} \pi(\theta) \mathrm{d}\mu(\theta) + \int_{\Theta_1} \underbrace{P_\theta(\delta(X)=0)}_{\text{type-II}} \pi(\theta) \mathrm{d}\mu(\theta).$$

In the above expression, μ is the reference measure for the prior distribution of θ; it is typically either the Lebesgue measure or a counting measure (sum of Dirac measures).

Example. *Consider the Gaussian model $\mathcal{P} = \{P_\theta \sim \mathcal{N}(\theta,1), \theta \in \mathbb{R}\}$ with a single observation. The hypotheses are $H_0 = \{\theta = 0\}$ and $H_1 = \{\theta = 3\}$. The prior is $\pi(0) = \frac{2}{3}$ and $\pi(3) = \frac{1}{3}$. The Bayes risk of the decision function $\delta(x) = 1_{\{x>2\}}$ is:*

$$r(\delta) = \frac{2}{3} P_0(X > 2) + \frac{1}{3} P_3(X \leq 2) = \frac{2}{3} P(Z > 2) + \frac{1}{3} P(Z > 1)$$

with $Z \sim \mathcal{N}(0,1)$.

2 Bayesian test

Finding an optimal decision function for Bayes risk is easy. This is because the risk is integrated over all values of θ (weighted by the prior π). In particular, there is no need to control the type-I error rate as in Neyman-Pearson's approach. The null hypothesis is rejected whenever the *posterior* probability of the aternative hypothesis H_1 is higher than that of the null hypothesis H_0:

The Bayes risk is minimized for the decision function:

$$\delta^\star(x) = 1_{\{\pi(\Theta_1|x) > \pi(\Theta_0|x)\}}.$$

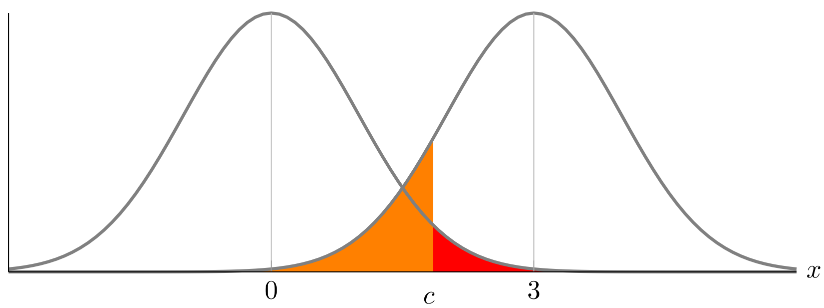
Example. *Consider the previous example with $\theta_0 = 0$ and $\theta_1 = 3$. The posterior distribution satisfies:*

$$\pi(\theta|x) \propto \begin{cases} \frac{2}{3} e^{-\frac{(x-\theta_0)^2}{2}} & \text{for } \theta = \theta_0, \\ \frac{1}{3} e^{-\frac{(x-\theta_1)^2}{2}} & \text{for } \theta = \theta_1. \end{cases}$$

In particular,

$$\pi(\theta_1|x) > \pi(\theta_0|x) \quad \Longleftrightarrow \quad e^{-\frac{(x-\theta_1)^2}{2}} > 2e^{-\frac{(x-\theta_0)^2}{2}} \quad \Longleftrightarrow \quad x > \frac{\theta_0+\theta_1}{2} + \frac{\log(2)}{\theta_1 - \theta_0}.$$

The optimal decision function is $\delta^\star(x) = 1_{\{x>c\}}$ with $c = \frac{\theta_0+\theta_1}{2} + \frac{\log(2)}{\theta_1-\theta_0} \approx 1.85$.



3 Simple hypotheses

For simple hypotheses, it is interesting to compare the test obtained with Neyman-Pearson approach:

$$\delta(x) = 1_{\{\frac{p_1(x)}{p_0(x)} > c\}},$$

with the Bayesian test:

$$\delta^\star(x) = 1_{\{\frac{\pi(\theta_1|x)}{\pi(\theta_0|x)} > 1\}}.$$

Since $\pi(\theta|x) \propto \pi(\theta)p(x|\theta)$, we get:

$$\delta^\star(x) = 1_{\{\frac{p_1(x)}{p_0(x)} > \frac{\pi(\theta_0)}{\pi(\theta_1)}\}}.$$

In particular, the Bayesian test has the same form as the UMP tests obtained with Neyman-Pearson approach.

Let α and β be its type-I and type-II error rates:

$$\alpha = P(\delta^\star(X) = 1 | \theta = \theta_0), \quad \beta = P(\delta^\star(X) = 0 | \theta = \theta_1).$$

The Bayesian test is UMP at level α. Its Bayes risk is:

$$r(\delta^\star) = \pi(\theta_0)\alpha + \pi(\theta_1)\beta.$$

Example. *In the previous example, we have:*

$$\alpha = P(\delta(X) = 1 | \theta = \theta_0) = P(X > c | \ \theta = 0) = P(Z > c)$$

and

$$\beta = P(\delta(X) = 0 | \theta = \theta_1) = P(X \leq c | \ \theta = 3) = P(Z \leq c - 3) = P(Z \geq 3 - c)$$

where $Z \sim \mathcal{N}(0, 1)$*. We obtain* $\alpha \approx 0.032$ *and* $\beta \approx 0.125$*, so that:*

$$r(\delta^\star) = \frac{2}{3}\alpha + \frac{1}{3}\beta \approx 0.063.$$