Filter banks

Roland Badeau,
roland.badeau@telecom-paris.fr

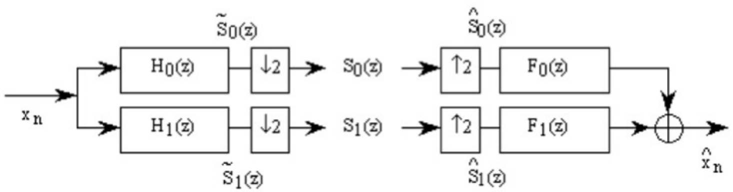
TSIA201

Part I

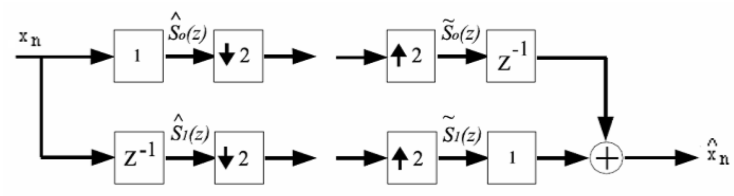
Two-channel filter banks

Ideal filter bank

- H_0 and F_0 are ideal low-pass half-band filters
- H_1 and F_1 are ideal high-pass half-band filters

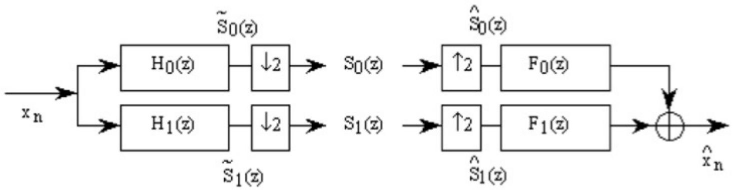


Example with aliasing



- H_0, F_0, H_1 and F_1 are all-pass filters
- But perfect reconstruction at output

General case of 2-channel filter bank



- Input-output relationship :
  $\widehat{X}(z) = T(z)X(z) + A(z)X(-z)$
  where $\begin{cases} T(z) = \frac{1}{2}(H_0(z)F_0(z) + H_1(z)F_1(z)) \\ A(z) = \frac{1}{2}(H_0(-z)F_0(z) + H_1(-z)F_1(z)) \end{cases}$
- Aliasing cancellation : $A(z) = 0$
- Perfect reconstruction : $T(z) = cz^{-n_0}$

Perfect reconstruction (PR)

- Exact solution :
  $\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix} = \begin{bmatrix} 2cz^{-n_0} \\ 0 \end{bmatrix}$
  $\Rightarrow \begin{cases} F_0(z) = \frac{2cz^{-n_0}}{D(z)} H_1(-z) \\ F_1(z) = -\frac{2cz^{-n_0}}{D(z)} H_0(-z) \end{cases}$
  where $D(z) = H_0(z)H_1(-z) - H_0(-z)H_1(z)$
- Solution with FIR filters H_k and F_k :
  - AC condition : $\begin{cases} F_0(z) = H_1(-z) \\ F_1(z) = -H_0(-z) \end{cases}$
  - PR condition : $T(z) = \frac{1}{2}D(z) = cz^{-n_0}$

Part II

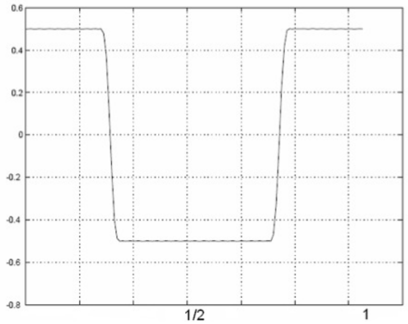
Half-band filters

Half-band filters

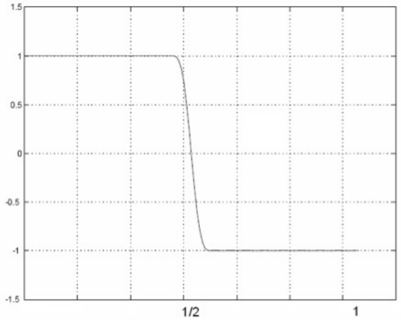
- Ideal low-pass half-band filter :
  $G_R(\nu) = \begin{cases} 1 & \text{for } 0 \leq |\nu| < 0.25 \\ 0 & \text{for } 0.25 \leq |\nu| < 0.5 \end{cases}$
- General definition : $G_R(\nu) + G_R(\nu + \frac{1}{2}) = 2c$ with $c > 0$
- Synthesis of a type I half-band filter $g(n)$
  - The length of $g(n)$ is $2N - 1$, where N is necessarily even
  - Since $g(n)$ is causal, $G(e^{2\imath\pi\nu}) = G_R(\nu)e^{-2\imath\pi\nu(N-1)}$ with $G_R(\nu) \in \mathbb{R}$
  - The half-band condition implies that $\exists V(z)$ such that
    $$G(z) = c(V(z^2) + z^{-(N-1)})$$
  - $v(n)$ is a type II filter of length N
  - $V(e^{2\imath\pi\nu})$ is nearly all-pass, but cuts frequency $1/2$

Half-band filters

- Synthesis of $V(z)$ by the Remez method



$\frac{1}{2c}G_R(\nu) - \frac{1}{2} = \frac{1}{2}V_R(2\nu)$



$V_R(\nu)$

Part III

Conjugate Quadrature Filters

CQF filters

- Reminder : input-output relationship
  $\widehat{X}(z) = T(z)X(z) + A(z)X(-z)$
  where $\begin{cases} T(z) = \frac{1}{2}(H_0(z)F_0(z) + H_1(z)F_1(z)) \\ A(z) = \frac{1}{2}(H_0(-z)F_0(z) + H_1(-z)F_1(z)) \end{cases}$
- Aliasing cancellation : $\begin{cases} F_0(z) = H_1(-z) \\ F_1(z) = -H_0(-z) \end{cases}$
- CQF constraint : N even, $H_1(z) = -z^{-(N-1)}\tilde{H}_0(-z)$ (analysis filters are *conjugate quadrature filters*), where $\tilde{H}_0(z) = H_0^*(\frac{1}{z})$
  - Analysis and synthesis filters are *paraconjugate* : $\forall k \in \{0,1\},\ F_k(z) = z^{-(N-1)}\widetilde{H}_k(z)$

CQF filters

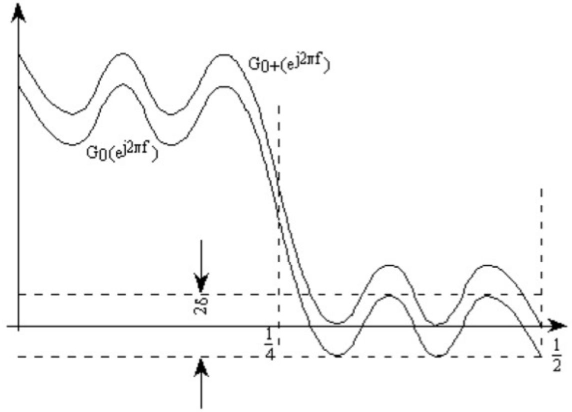
- Transfer function :
  $T(z) = \frac{z^{-(N-1)}}{2}\left(\tilde{H}_0(z)H_0(z) + \tilde{H}_0(-z)H_0(-z)\right)$
- *Symmetric power* constraint :
  $\widetilde{H}_0(z)H_0(z) + \widetilde{H}_0(-z)H_0(-z) = 2c$

$\Rightarrow$ Perfect reconstruction : $T(z) = c\, z^{-(N-1)}$

- Method : factorization of a half-band filter
  - Let $G(z)$ be a half-band filter of length $2N-1$
  - Function $G_R(\nu) = G(e^{2\imath\pi\nu})e^{2\imath\pi\nu(N-1)}$ is such that $G_R(\nu) + G_R(\nu + \frac{1}{2}) = 2c$

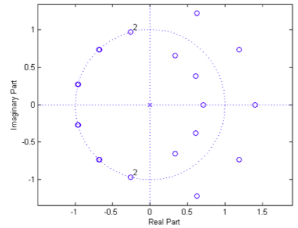
Raising of the half-band filter

- Let $G_R^+(\nu) = G_R(\nu) + \varepsilon$
  $\Rightarrow g^+(n) = g(n) + \varepsilon\,\delta_0(n-(N-1))$ is still a half-band filter



Factorization of the half-band filter

- The $2N-2$ roots of $G^+(z)$ form pairs :



  - The equation $G_R^+(\nu) = \widetilde{H}_0(e^{2\imath\pi\nu})H_0(e^{2\imath\pi\nu})$ admits several solutions H_0
  - We choose the one with minimal phase : $H_0(z)$ is the N sample-long filter whose roots are the $N-1$ roots of $G^+(z)$ located inside the unit circle

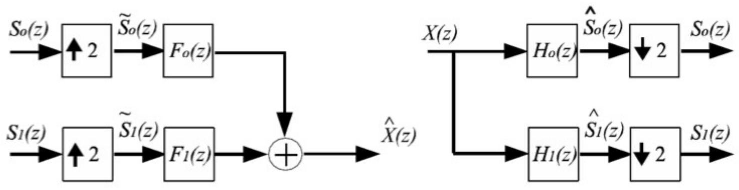
  $\Rightarrow$ Perfect reconstruction

Bi-orthogonal filters

- Reminder : input-output relationship
  $\widehat{X}(z) = T(z)X(z) + A(z)X(-z)$
  where $\begin{cases} T(z) = \frac{1}{2}(H_0(z)F_0(z) + H_1(z)F_1(z)) \\ A(z) = \frac{1}{2}(H_0(-z)F_0(z) + H_1(-z)F_1(z)) \end{cases}$
- Aliasing cancellation : $\begin{cases} F_0(z) = H_1(-z) \\ F_1(z) = -H_0(-z) \end{cases}$
- Let $G(z) = H_0(z)F_0(z)$; PR $\Rightarrow G(z) - G(-z) = 2cz^{-n_0}$
  - General solution : $G(z) = c(V(z^2) + z^{-n_0})$
- Synthesis in 2 steps :
  - Synthesis of $G(z)$, factorization as $H_0(z)F_0(z)$

Application : transmultiplexer

- Purpose : transmit several signals in a single channel

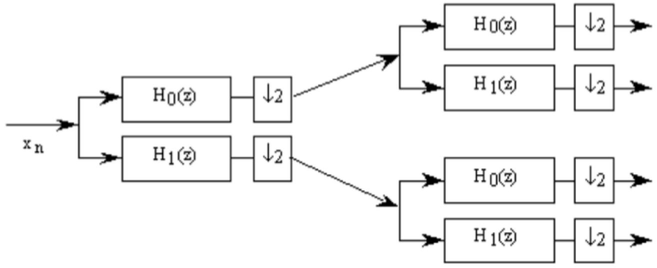


- Problem posed : channel equalization

Part IV

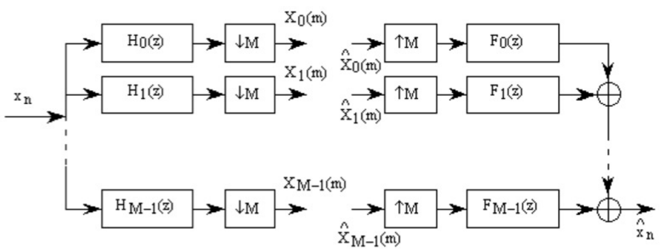
Filter banks: from 2 channels to M channels

Pyramidal structure



M-channel filter banks

General implementation of an M-channel filter bank



M-channel filter banks

Input-output relationship :

$$\widehat{X}(z) = T(z)X(z) + \sum_{l=1}^{M-1} A_l(z)X(zW_M^l)$$

where $W_M^l = e^{-2\imath\pi\frac{l}{M}}$ and $\begin{cases} T(z) = \frac{1}{M}\sum_{k=0}^{M-1} H_k(z)F_k(z) \\ A_l(z) = \frac{1}{M}\sum_{k=0}^{M-1} H_k(zW_M^l)F_k(z) \end{cases}$

- Aliasing cancellation : $\forall l,\ A_l(z) = 0$
- Perfect reconstruction : $T(z) = cz^{-n_0}$

M-channel filter banks

- Exact solution : $\mathbf{f}(z) = \left(\mathbf{H}_M^\top(z)\right)^{-1}\mathbf{t}(z)$

$$\underbrace{\begin{bmatrix} H_0(z) & \dots & H_{M-1}(z) \\ H_0(zW_M^1) & \dots & H_{M-1}(zW_M^1) \\ \vdots & \dots & \vdots \\ H_0(zW_M^{M-1}) & \dots & H_{M-1}(zW_M^{M-1}) \end{bmatrix}}_{\mathbf{H}_M^\top(z)} \underbrace{\begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix}}_{\mathbf{f}(z)} = \underbrace{\begin{bmatrix} Mcz^{-n_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{t}(z)}$$

- A solution with FIR filters H_k and F_k :
  - AC condition : $\mathbf{f}(z) = \mathrm{Adj}(\mathbf{H}_M(z))\mathbf{e}_1$
  - PR condition $T(z) = \frac{1}{M}\det(\mathbf{H}_M(z)) = cz^{-n_0}$

Polyphase implementation

Type I polyphase components

$$H_k(z) = \sum_{l=0}^{M-1} E_{kl}(z^M)z^{-l}$$

$$\Downarrow \quad \underbrace{\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix}}_{\mathbf{h}(z)} =$$

$$\underbrace{\begin{bmatrix} E_{0,0}(z^M) & E_{0,1}(z^M) & \dots & E_{0,M-1}(z^M) \\ E_{1,0}(z^M) & E_{1,1}(z^M) & \dots & E_{1,M-1}(z^M) \\ \vdots & \vdots & \vdots & \vdots \\ E_{M-1,0}(z^M) & E_{M-1,1}(z^M) & \dots & E_{M-1,M-1}(z^M) \end{bmatrix}}_{\mathbf{E}(z^M)} \underbrace{\begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}}_{\mathbf{e}(z)}$$

Polyphase resolution

Type II polyphase components

$$F_k(z) = \sum_{l=0}^{M-1} R_{lk}(z^M)z^{-(M-1-l)}$$

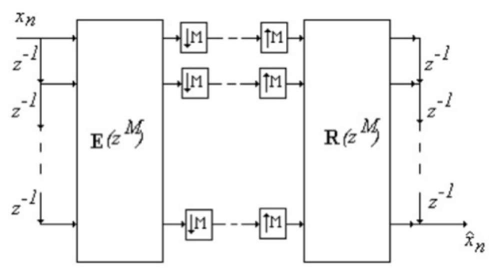
$$\Downarrow \quad \underbrace{\begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix}}_{\mathbf{f}(z)} =$$

$$\underbrace{\begin{bmatrix} R_{0,0}(z^M) & R_{1,0}(z^M) & \dots & R_{M-1,0}(z^M) \\ R_{0,1}(z^M) & R_{1,1}(z^M) & \dots & R_{M-1,1}(z^M) \\ \vdots & \vdots & \vdots & \vdots \\ R_{0,M-1}(z^M) & R_{1,M-1}(z^M) & \dots & R_{M-1,M-1}(z^M) \end{bmatrix}}_{\mathbf{R}^\top(z^M)} \underbrace{\begin{bmatrix} z^{-(M-1)} \\ z^{-(M-2)} \\ \vdots \\ 1 \end{bmatrix}}_{\tilde{\mathbf{e}}(z)}$$

Polyphase implementation

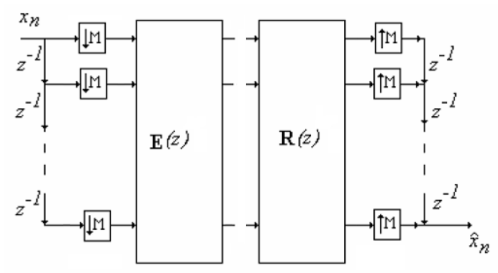
Transfer function :

$$T(z) = \frac{1}{M}\mathbf{f}(z)^\top\mathbf{h}(z) = \frac{1}{M}\tilde{e}(z)^\top \underbrace{\mathbf{R}(z^M)\mathbf{E}(z^M)}_{\mathbf{P}(z^M)} e(z)$$



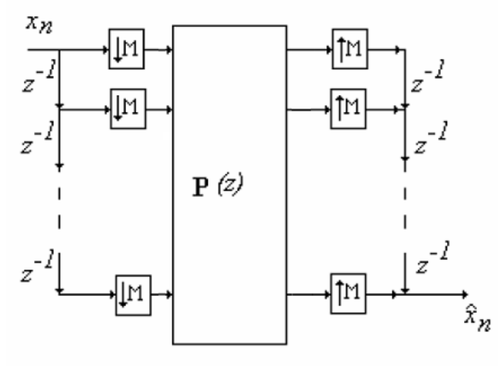
Polyphase implementation

Transfer function :
$T(z) = \frac{1}{M}\mathbf{f}(z)^\top \mathbf{h}(z) = \frac{1}{M}\widetilde{e}(z)^\top \underbrace{\mathbf{R}(z^M)\mathbf{E}(z^M)}_{\mathbf{P}(z^M)} e(z)$



Polyphase implementation

Transfer function :
$T(z) = \frac{1}{M}\mathbf{f}(z)^\top \mathbf{h}(z) = \frac{1}{M}\widetilde{e}(z)^\top \mathbf{P}(z^M) e(z)$



$\Rightarrow$ if $\mathbf{P}(z) = cz^{-n_0'}\mathbf{I}_M$, then $T(z) = cz^{-n_0}$ with $n_0 = Mn_0' + M - 1$

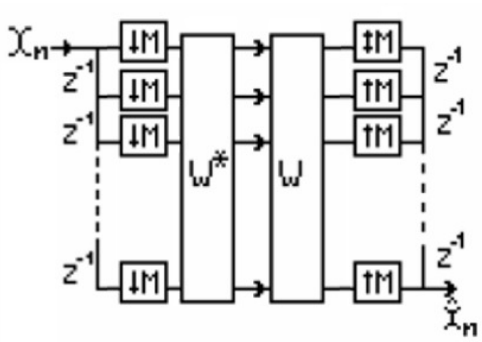
Paraunitary filter banks

- Para-conjugation : $\widetilde{\mathbf{E}}(z) = \mathbf{E}^H(z^{-1})$
- Paraunitary matrix : $\mathbf{E}(z)\widetilde{\mathbf{E}}(z) = c\,\mathbf{I}_M$
  - Let $\mathbf{R}(z) = z^{-(\frac{N}{M}-1)}\widetilde{\mathbf{E}}(z) \Rightarrow T(z) = cz^{-(N-1)}$
  - Consequence (CQF) : $F_k(z) = z^{-(N-1)}\widetilde{H}_k(z)$

    $T(z) = \frac{1}{M} z^{-(N-1)} \sum_{k=0}^{M-1} H_k(z)\widetilde{H}_k(z)$
- Synthesis of the analysis filters
  - Synthesis of a low-pass M-th band filter $G(z)$
  - Factorization as $G_R^+(zW_M^k) = H_k(z)\widetilde{H}_k(z)$

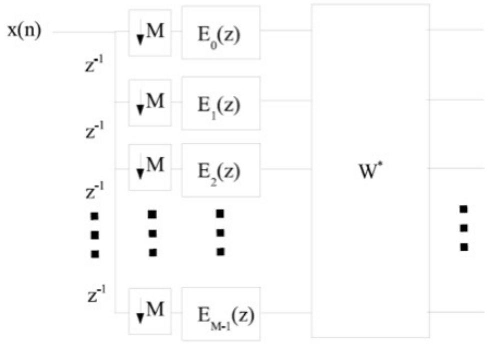
Example : DFT filter bank

- Let $\mathbf{E}(z) = \mathbf{W}^H$ (with $\mathbf{W}$ the DFT matrix) and $\mathbf{R}(z) = \widetilde{\mathbf{E}}(z) = \mathbf{W} \Rightarrow \mathbf{P}(z) = M\,\mathbf{I}_M$
- Rectangular window STFT



DFT filter bank

- General case : $H_k(z) = H_0(zW_N^k)$



- If the $E_k(z)$ are not constant, perfect reconstruction is not feasible

DFT filter bank