Examination of the teaching unit
Représentations des signaux - TSIA201

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Duration : 1 :30
All documents are permitted. However electronic devices (including calculators) are forbidden.

Exercise 1 (Filter synthesis : differentiator filter) *Consider a continuous time signal $x(t)$ at the input of a filter with transfer function $H(f)$ (where f is the frequency expressed in Hz). Let us note $y(t)$ the output signal. Given $H(f)$, we propose to determine a discrete time filter of transfer function $H_e(e^{i2\pi\nu})$ (where ν is the normalized frequency) which, from the input samples $x_e(n) = x(nT)$, produces the output samples $y_e(n) = y(nT)$ (see figure 1).*

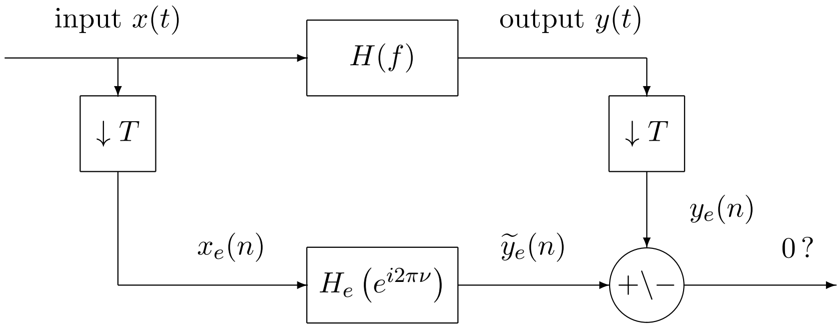

Figure 1 – Comparison of the outputs at sampling times

We remind the sampling formula : if $\forall n \in \mathbb{Z}$, $x_e(n) = x_a(nT)$ where $T \in \mathbb{R}_+^*$, then

$$X_e(e^{2i\pi\nu}) = \frac{1}{T}\sum_{k\in\mathbb{Z}} X_a\left(\frac{\nu+k}{T}\right) \tag{1}$$

1. *Using formula (1), express the DTFT $Y_e(e^{i2\pi\nu})$ and $\widetilde{Y}_e(e^{i2\pi\nu})$ of the discrete time signals $y_e(n)$ and $\widetilde{y}_e(n)$ as functions of $X(f)$.*
2. *Prove that the discrete filter, defined by the relation $H_e(e^{i2\pi\nu}) = H\left(\frac{\nu}{T}\right)$ for any $\nu \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$, is such that for any signal $x(t)$ whose spectrum has a support included in $\left[-\frac{1}{2T}, \frac{1}{2T}\right]$, $\widetilde{y}_e(n) = y_e(n)\ \forall n \in \mathbb{Z}$.*
3. *We now wish to synthesize a differentiator filter. We suppose that $x(t)$ is a summable function ($x \in L^1(\mathbb{R})$), continuously differentiable, whose derivative $x'(t)$ is also summable. Prove that $\lim\limits_{t\to\pm\infty} x(t) = 0$, and calculate the Fourier transform of $x'(t)$.*
4. *Deduce that the derivation can be seen as a filter with frequency response $H(f) = i2\pi f$, and express the complex gain $H_e(e^{i2\pi\nu})$ of the corresponding discrete-time linear filter.*
5. *Compute the impulse response $h_e(n)$. Deduce, by the window method, the coefficients of a type III linear phase FIR filter that performs the approximation of a derivative filter.*

6. *How could we obtain a type IV linear phase differentiator filter ? (hint : consider the filter of frequency response $H_e(e^{i2\pi\nu})\, e^{-i\pi\nu}$ and compute its impulse response). Deduce, by the window method, the coefficients of this type IV linear phase FIR filter.*

Exercise 2 (Conversion of sampling rate CD/DAT)

We want to perform the conversion of sampling rate from a Compact Disc (CD) player to a Digital Audio Tape (DAT) recorder. The standards of sampling frequency are the following ones : 44.1 kHz for the CD, 48 kHz for the DAT.

1. *At which intermediate sampling frequency is it necessary to interpolate the signals coming from a CD ? (note that $441 = 3 \times 147$ and that $480 = 3 \times 160$)*
2. *Draw the diagram (without efficient implementation) of the operations of this conversion. In particular, indicate the different sampling frequencies of the signals, and the characteristics of the filter, including its cut-off frequency (in absolute and normalized frequency) and its amplitude in pass-band.*
3. *Is there a loss of quality in performing this operation ? Justify your answer.*
4. *If we tried to perform the reverse conversion (DAT to CD), would the result be the same as in question 3 ? Justify your answer.*

Exercise 3 (Efficient implementations)

We want to implement a third order downsampling of a signal $x(n)$.

1. *Draw the diagram (without efficient implementation) of the operations to be performed. In particular, you will specify the characteristics of the filter.*
2. *Then draw the diagram of an efficient implementation of this downsampling, using the polyphase components of the filter.*

Exercise 4 (Multiresolution Analysis)

Reminders *A Multiresolution Analysis (MRA) is a sequence of closed vector subspaces V_j such that the following conditions hold :*

$$\forall j \in \mathbb{Z}, V_j \subset V_{j+1} \tag{2}$$
$$\cap_j V_j = \{0\} \tag{3}$$
$$\overline{\cup_j V_j} = \mathcal{L}^2(\mathbb{R}) \tag{4}$$
$$f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1} \tag{5}$$
$$f(t) \in V_0 \Leftrightarrow \forall k \in \mathbb{Z}, f(t-k) \in V_0 \tag{6}$$
$$\exists \phi \in \mathcal{L}^2(\mathbb{R}) : \{\phi(t-k)\}_k \text{ is an orthonormal basis of } V_0 \tag{7}$$

It can be shown that, if $\phi(t) = \mathbb{1}_{(0,1)}(t)$, then, the sequence of spaces $V_j = \overline{\{\phi_{j,k}(t)\}_k}$, with $\phi_{j,k}(t) = 2^{\frac{j}{2}}\phi(2^j t - k)$, is an MRA. In this case the mother wavelet is $\psi(t) = \phi(2t) - \phi(2t-1)$.

The mother and the father wavelet are related to the filter coefficients c and d by the dilation equation and the wavelet equation, respectively :

$$\phi(t) = \sqrt{2}\sum_k c_k \phi(2t-k) \tag{8}$$
$$\psi(t) = \sqrt{2}\sum_k d_k \phi(2t-k) \tag{9}$$

The projection of a function $f \in \mathcal{L}^2(\mathbb{R})$ onto V_j can be written as $f_j(t) = Proj(f|V_j) = \sum_k a_{j,k}\phi_{j,k}(t)$, with $a_{j,k} = \langle f(t), \phi_{j,k}(t)\rangle$.

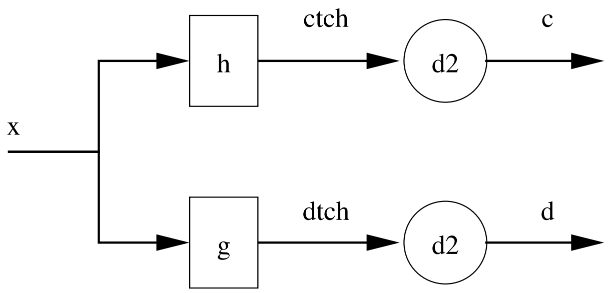

FIGURE 2 – The Haar analysis filterbank can be used to compute the lower-resolution representation of a signal

Questions

1. *Verify that, for* $\phi(t) = \mathbb{1}_{(0,1)}(t)$*,*

$$c_k = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } k \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} \qquad d_k = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } k = 0 \\ -\frac{1}{\sqrt{2}} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

2. *Verify that, with the current MRA (i.e.,* $\phi(t) = \mathbb{1}_{(0,1)}(t)$*),* $a_{j,k} = \langle f(t), \phi_{j,k}(t) \rangle$ *is* $2^{-\frac{j}{2}}$ *times the average value of f(t) in* $\left(2^{-j}k, 2^{-j}(k+1)\right)$
3. *Find a similar interpretation for* $b_{j,k} = \langle f(t), \psi_{j,k}(t) \rangle$
4. *Based on the previous interpretations of* $a_{j,k}$ *and* $b_{j,k}$*, compute the values of* $a_{0,k}$*,* $b_{0,k}$ *and* $a_{1,k}$*, for* $k = 0, 1, \ldots, 13$ *for the following function :*

$$f(t) = \begin{cases} t & \text{if } t \in (0,2) \\ 2 & \text{if } t \in (2,6) \\ 14 - 2t & \text{if } t \in (6,7) \end{cases}$$

5. *Use the previous results to sketch* $f_0 = Proj(f|V_0)$ *and* $f_1 = Proj(f|V_1)$
6. *It is possible to compute* $a_0[k] = a_{0,k}$ *and* $b_0[k] = b_{0,k}$ *from* $a_1[k] = a_{1,k}$ *by using the Haar filterbank, as shown in Fig. 2 ? Remember that* c^T *and* d^T *are just the time-reversed versions of* c_k *and* d_k*.*

$$c^T_k = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } k \in \{-1,0\} \\ 0 & \text{otherwise} \end{cases} \qquad d^T_k = \begin{cases} -\frac{1}{\sqrt{2}} & \text{if } k = -1 \\ \frac{1}{\sqrt{2}} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, start by the following : If x *is an* ℓ^1 *sequence, and* $y = c^T * x$ *(convolution), write the expression of* y_n *as a function of* x_n *and* x_{n-1} *Repeat the same as before but for* $y = d^T * x$

7. *Using the values of* $a_1[k]$ *computed in question 4, and the filter equations found in question 6, compute* $a_0[k]$ *and* $b_0[k]$*.*
8. *Verify that the representation of* f_1 *with* $a_0[k]$ *and* $b_0[k]$ *is sparser than the representation with* $a_1[k]$ *(meaning that there are more null coefficients).*