Reminders

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TSIA201

TSIA201 : Representations of signals

- This course introduces various digital signal processing tools :
  - Reminders about filtering, Z-transform, and recursive filters
  - Filter synthesis methods
  - Conversion of the sampling rate
  - Short-time Fourier transform
  - Filter banks
  - Wavelets
- These various notions will be used in the TSIA study track :
  - multirate processing and filter banks allow us to pre-process signals before applying machine learning methods
  - they find many applications, including denoising (SD-TSIA205)
  - time-frequency representations are essential tools in speech, audio, and multimedia signal processing (TSIA206, TSIA207)

Program of the course unit

- Lesson *Reminders*
- Lesson + Tutorial *Filter synthesis*
- Lesson *Frequency conversion*
- Lesson *Short time Fourier transform*
- Practical work *Frequency conversion, STFT*
- Lesson + Tutorial *Filter banks*
- Lesson *Wavelets*
- Practical work *Wavelets*
- Tutorial *Revision*
- Written examination

Educational resources

- Website `https://ecampus.paris-saclay.fr/` (English/French)
  - Links to course handout + slides
  - Subjects of tutorials and practical works + data
  - Solutions available online after every tutorial
  - Online submission of the practical works reports
- Grading (20 points)
  - Practical works reports (3 points for each practical work)
  - Examination (14 points)

Part I

Discrete filtering

Causality and stability

- Convolution product : $y(n) = \sum_{m \in \mathbb{Z}} h(m)x(n-m)$
- Causality :
  - $y(n)$ only depends on $x(k)$, $k \leq n$
  - Necessary and sufficient condition : $h(m) = 0$ if $m < 0$
  - Property : causal input $\Rightarrow$ causal output
  - Remark : compulsory for real-time processing
- Stability :
  - Definition : bounded input $\Rightarrow$ bounded output
  - Necessary and sufficient condition : $\sum_{m=-\infty}^{+\infty} |h(m)| < +\infty$ (h in $l^1(\mathbb{Z})$)
  - Property : continuous frequency response
  - Remark : numerically compulsory
    - If $x_q = x + e$, then $y_q = y + h * e$.
    - If $h \in l^1(\mathbb{Z})$, $\|h * e\|_\infty \leq \|h\|_1 \|e\|_\infty$, otherwise $h * e$ may diverge.

Ideal filters

- Low-pass filter (periodic spectrum)
  - Frequency response : $H(e^{2i\pi\nu}) = \begin{cases} 1 & \text{if } |\nu| < \nu_c \\ 0 & \text{if } |\nu| > \nu_c \end{cases}$
  - Impulse response : $h(n) = 2\nu_c \,\mathrm{sinc}(2\pi\nu_c n)$
- Exercise : band-pass filter
  - Frequency response : $H(e^{2i\pi\nu}) = \begin{cases} 1 & \text{if } ||\nu| - |\nu_0|| < \nu_c \\ 0 & \text{if } ||\nu| - |\nu_0|| > \nu_c \end{cases}$
  - Impulse response : $h(n) = 4\nu_c \,\mathrm{sinc}(2\pi\nu_c n) \cos(2\pi\nu_0 n)$
- Causality ? Stability ?

Transient and steady states

- Example :
  - $x(n)$ is a unit step signal : $x(n) = 1_{[0,+\infty[}(n)$
  - $h(n)$ is an averaging filter : $h(n) = \frac{1}{N} 1_{[0,N-1]}(n)$
  - From $n = 0$ to $N-2$: transient state (ramp)
  - From $n = N-1$ to $+\infty$: steady state (constant signal)

Phase and group delays

- Frequency response : $H(e^{i2\pi\nu}) = H_R(\nu)e^{i\phi(\nu)}$
- Phase and group delays

$$\begin{cases} \tau_p(\nu_0) &= -\frac{1}{2\pi}\frac{\phi(\nu_0)}{\nu_0} \\ \tau_g(\nu_0) &= -\frac{1}{2\pi}\frac{d\phi}{d\nu}(\nu_0) \end{cases}$$

- Frequency response in the neighborhood of ν_0

$$H(e^{i2\pi\nu}) \simeq H_R(\nu_0)\, e^{-i2\pi(\nu_0\tau_p(\nu_0)+(\nu-\nu_0)\tau_g(\nu_0))}$$

- Filtering of a narrowband signal ($y = h * x$)

$$x(n) = a(n)e^{i2\pi\nu_0 n} \Rightarrow y(n) \simeq H_R(\nu_0)a(n-\tau_g(\nu_0))e^{i2\pi\nu_0(n-\tau_p(\nu_0))}$$

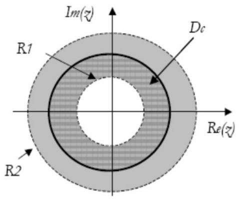
- Linear phase filters (constant delay)

Part II

Z-transform

Z-transform

- Definition : $H(z) = \sum_{n=-\infty}^{+\infty} h(n)z^{-n}$, called Transfer function
- Domain of convergence : $\mathscr{D} = \{z / \sum_{n=-\infty}^{+\infty} |h(n)||z|^{-n} < +\infty\}$
- Causal case : $R = \inf\{|z|, z \in \mathbb{C} / \sum h(n)z^{-n} < +\infty\}$, $\in \mathbb{R} \cup \{+\infty\}$



$z = 1$ at $\nu = 0$
$z = i$ at $\nu = 1/4$

- FIR filters : $D = \mathbb{C}\backslash 0$ or ∞
- Anti-causality : $\mathscr{D}$ is a disk
- Causality : $\mathscr{D}$ is the complement of a disk
- General case : $\mathscr{D}$ is a ring (or $\emptyset$)

Basic properties

- Stability : the ring $\mathscr{D}$ contains the unit circle
  - The ZT matches the DTFT on the unit circle
- Linearity : $a_1h_1 + a_2h_2 \to a_1H_1 + a_2H_2$ $(\mathscr{D} \supset \mathscr{D}_1 \cup \mathscr{D}_2)$
- Delay : $y(n) = x(n-k) \Leftrightarrow Y(z) = z^{-k}X(z)$
- Convolution product
  - If $y = h * x$, then $Y(z) = H(z)X(z)$, $\mathscr{D}_y \supset \mathscr{D}_h \cap \mathscr{D}_x$
- Insertion of zeros : $Y(z) = X(z^L) \Leftrightarrow y(nL) = x(n)$, $y = 0$ elsewhere
- Inverse filter
  - If $h * h_i = \delta$, then $H(z)H_i(z) = 1$ for $z \in \mathscr{D}_h \cap \mathscr{D}_{h_i}$

Examples of ZT

- $h(n) = \delta(n) \Rightarrow H(z) = 1\ \forall z \in \mathbb{C}$
- $h(n) = \mathbf{1}_{[0,+\infty[}(n) \Rightarrow H(z) = \frac{1}{1-z^{-1}}\ \forall |z| > 1$
- $h(n) = \mathbf{1}_{[0\ldots N-1]}(n) \Rightarrow H(z) = \frac{1-z^{-N}}{1-z^{-1}}\ \forall z \neq 0$
- AR1 filter :
  - $h(n) = \begin{cases} a^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases} \Rightarrow H(z) = \frac{1}{1-az^{-1}}$, $\mathscr{D} = \{z \in \mathbb{C}/|z| > |a|\}$
  - $h(n) = \begin{cases} -a^n & \text{if } n < 0 \\ 0 & \text{if } n \geq 0 \end{cases} \Rightarrow H(z) = \frac{1}{1-az^{-1}}$, $\mathscr{D} = \{z \in \mathbb{C}/|z| < |a|\}$

Autoregressive filter of order 1

I/O relationship	$y(n) - ay(n-1) = x(n)$
Transfer function	$H(z) = \frac{1}{1-az^{-1}}$

Implementation	$y(n)=ay(n-1)+x(n)$	$y(n) = \frac{y(n+1)-x(n+1)}{a}$
IR ($x(n)=\delta_0(n)$)	$h(n) = a^n 1_{\{n\geq 0\}}$	$h(n) = -a^n 1_{\{n<0\}}$
Domain $\mathscr{D}$	$\{z \in \mathbb{C}/|z| > |a|\}$	$\{z \in \mathbb{C}/|z| < |a|\}$
Properties	causal, stable if $|a|<1$	anti-causal, stable if $|a|>1$

Part III

Recursive filters

Recursive filters

- Input-output relationship : $\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$
- Computation of the output (causal implementation)

$$y(n) = \frac{1}{a_0}\left[\sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k)\right]$$

- Transfer function : $H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$

Examples of recursive filters

- Auto-Regressive filters (if $M=0$, AR of order N)

$$y(n)=\frac{b_0}{a_0}x(n)-\sum_{k=1}^{N}\frac{a_k}{a_0}y(n-k)$$

- Finite Impulse Response filters
  - If $N=0$, FIR filter of length M
    $h(n)=\begin{cases}\frac{b_n}{a_0} & \text{if } n=0\dots M\\ 0 & \text{sinon}\end{cases}$
  - Causal filters, unconditionally stable
  - If the IR is symmetric or anti-symmetric, the phase is linear

Recursive filters

- Transfer function of a recursive filter

$$H(z)=\frac{\sum_{k=0}^{M}b_k z^{-k}}{\sum_{k=0}^{N}a_k z^{-k}}=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

  - Definition of the poles and zeros
  - Domain of convergence : $\mathscr{D}$ is a ring bounded by 2 poles, with no pole inside
  - Stable filters : $\mathscr{D}$ is the largest ring containing the unit circle and no pole
  - Stable and causal filters : whose poles are all strictly inside the unit circle
  - If the a_k and b_k are real, the poles and zeros are either real, or form conjugate pairs

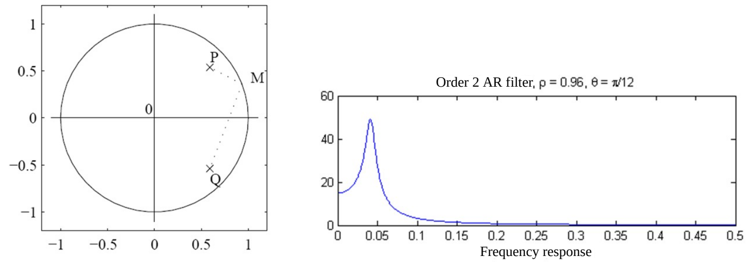
Computation of the IR from the ZT

1. If necessary, extract from $H(z)$ a term in z^n, so that the numerator and denominator of $H(z)$ only contain negative powers of z plus a constant
2. Factorize the numerator and denominator as products of monomials of z^{-1}
3. Partial fraction decomposition of $H(z)$ as a function of z^{-1}
4. Expansion in power series :
   - of z^{-1} if we compute the causal IR, or if we compute the stable IR and if the pole is inside the unit circle ;
   - of z if we compute the anti-causal IR, or if we compute the stable IR and if the pole is outside the unit circle.
5. Identify the resulting expression with $H(z)=\sum_{n\in\mathbb{Z}}h(n)z^{-n}$

Example : $H(z)=\frac{1}{1-\frac{5}{2}z^{-1}+z^{-2}}$

Geometric interpretation of the FR

Example (AR2 filter) : $H(z)=\frac{1}{1+a_1z^{-1}+a_2z^{-2}}=\frac{1}{(1-pz^{-1})(1-p^*z^{-1})}$



$$|H(z)|=\frac{1}{PM\times QM}$$
$$\arg H(z)=2\arg(OM)-\arg(PM)-\arg(QM)$$

Spectral factorization

- Factorization of the transfer function

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1-c_k z^{-1})}{\prod_{k=1}^{N} (1-d_k z^{-1})}$$

- Amplitude response ($z = e^{2i\pi\nu}$)

$$|H(e^{2i\pi\nu})|^2 = H(z)H(1/z^*)^* = \left|\frac{b_0}{a_0}\right|^2 \frac{\prod_{k=1}^{M} (1-c_k z^{-1})(1-c_k^* z)}{\prod_{k=1}^{N} (1-d_k z^{-1})(1-d_k^* z)}$$

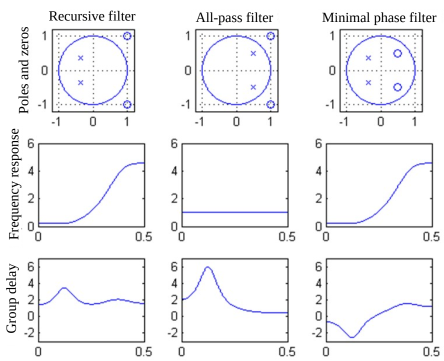
- Poles of $|| < 1$: stability + causality
- Zeros of $|| < 1$: minimal phase

All-pass filters

- Definition : $H(z) = \prod_{k=1}^{N} \frac{z^{-1}-c_k^*}{1-c_k z^{-1}}$ where $|c_k| < 1 \ \forall k$
- $H(z)$ is all-pass, **causal** and stable
- Properties (if $y = h * x$) :
  - $\sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=-\infty}^{+\infty} |y(n)|^2$
  - An all-pass filter is such that $\sum_{n=-\infty}^{N} |x(n)|^2 \geq \sum_{n=-\infty}^{N} |y(n)|^2$

    Proof : $x_N(n) = x(n) \times \mathbf{1}_{]-\infty,N]}(n)$

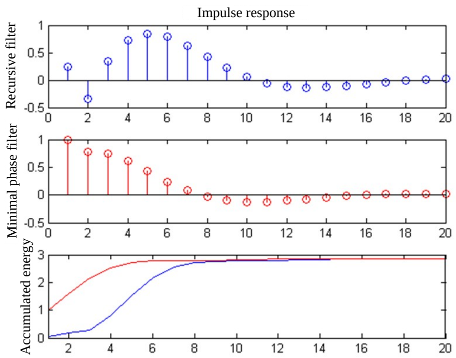
Spectral factorization



Minimal phase filters

- Definition : filter whose poles and zeros are all inside the unit circle
- Any causal and stable recursive filter is the product of an all-pass filter and a minimal phase filter
- Properties (if g is causal and stable, $y = g * x$ and $|G| = |G_m|$) :
  - The inverse filter is stable and causal
  - $\sum_{n=-\infty}^{N} |y_m(n)|^2 \geq \sum_{n=-\infty}^{N} |y(n)|^2$
  - g_m has the lowest group delay

Minimal phase filters

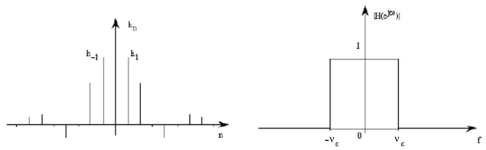


Part IV

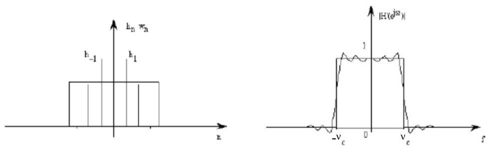
FIR filters: window method

Window method

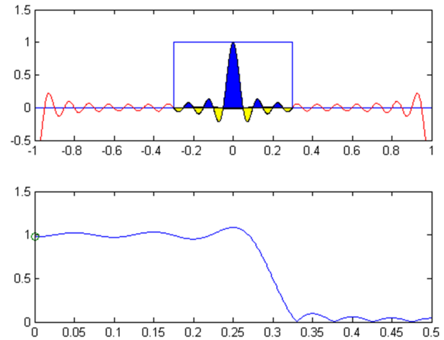
- Example : the low-pass filter
- Ideal low-pass filter : $h(n) = 2\nu_c \operatorname{sinc}(2\nu_c n)$
  - The response is IIR, non-causal, non-stable



- Synthesis of a type I causal FIR filter
  - Truncation and temporal shift



Gibbs phenomenon



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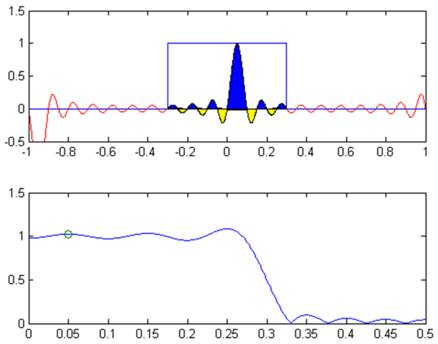

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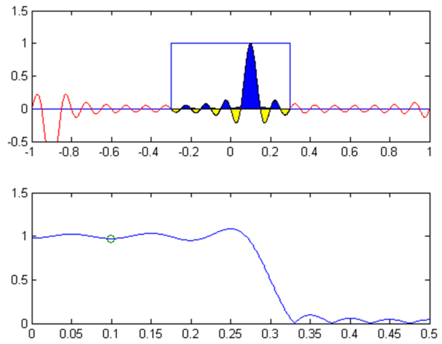

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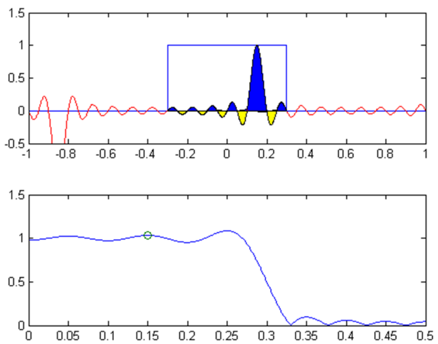
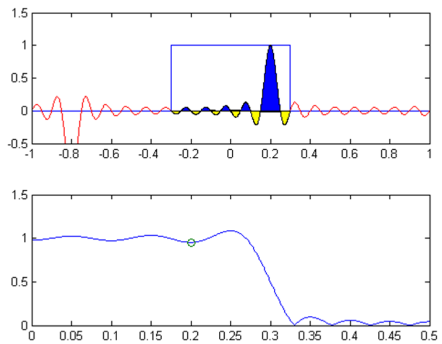


Gibbs phenomenon



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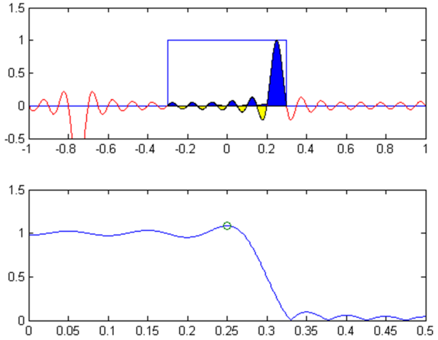
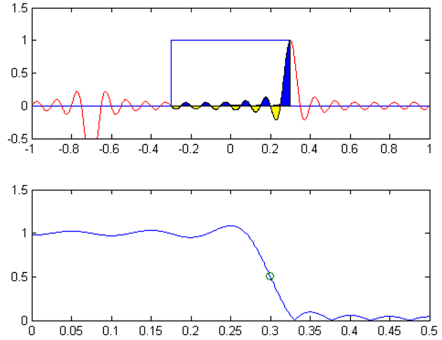


Gibbs phenomenon



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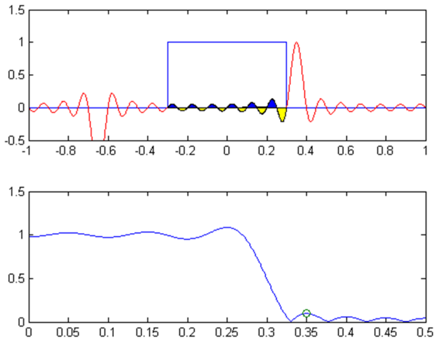
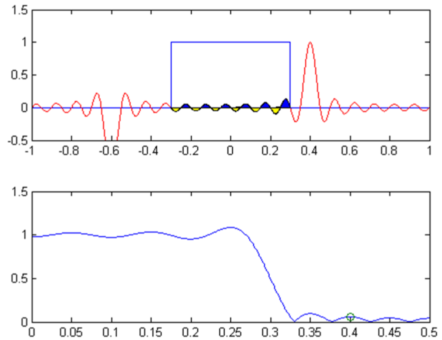
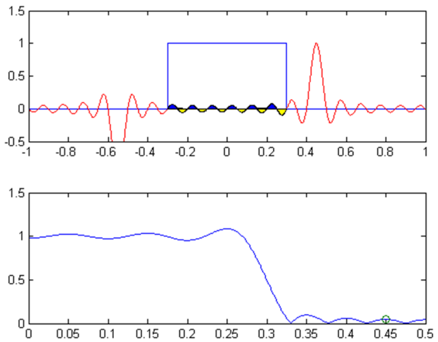


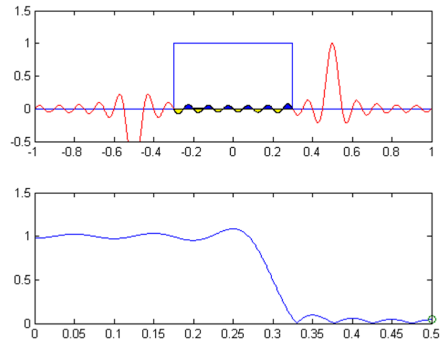
Gibbs phenomenon



Gibbs phenomenon



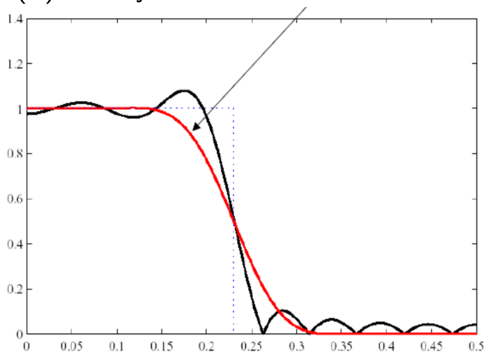
Gibbs phenomenon



Choice of a suitable window

$$h(n) = 2\nu_c \,\mathrm{sinc}(2\nu_c n)\, w(n)$$

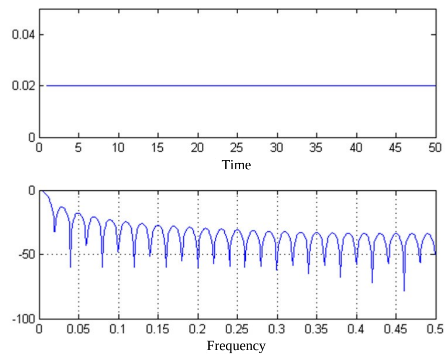
where $w(n)$ is a symmetric window of finite support



Rectangular window

$$w(n) = \mathbf{1}_{[0\ldots P-1]}(n)$$

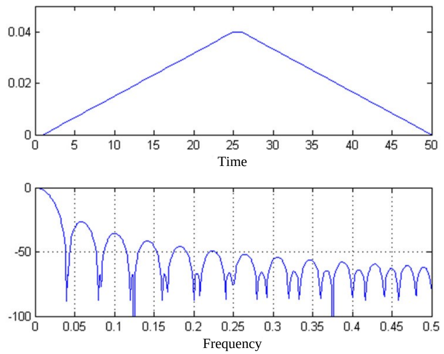
- Width : $2/P$, second lobe : -13 dB, decrease rate : -6 dB / octave



Bartlett window

$$w(n) = 1 - \left|\frac{2n}{P-1} - 1\right|$$

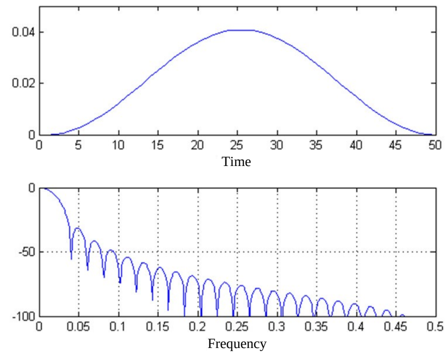
- Width : $4/P$, second lobe : -26 dB, decrease rate : -12 dB / octave



Hann window

$$w(n) = 0.5 - 0.5\cos(2\pi n/(P-1))$$

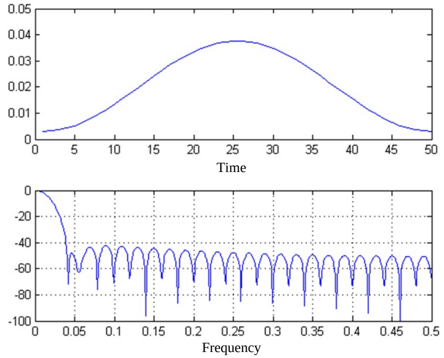
- Width : $4/P$, second lobe : -31 dB, decrease rate : -18 dB / octave



Hamming window

$$w(n) = 0.54 - 0.46\cos(2\pi n/(P-1))$$

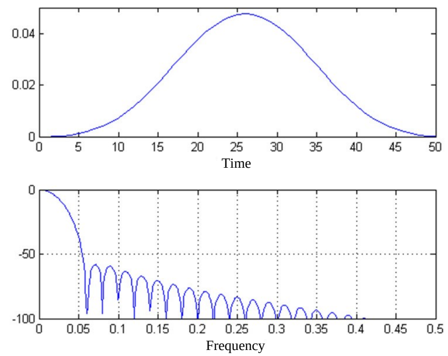
- Width : $4/P$, second lobe : -41 dB, decrease rate : -6 dB / octave



Blackman window

$$w(n) = 0.4266 - 0.4965\cos(2\pi n/(P-1)) + 0.076\cos(4\pi n/(P-1))$$

- Width : $6/P$, second lobe : -57 dB, decrease rate : -18 dB / octave



Window method

- Advantages :
  - Stability, causality
  - Linear phase filter if symmetric window
- Drawbacks :
  - the transition bands are widened
  - the spurious ripples
    - are due to side lobes
    - do not have a constant amplitude
    - are the same in passband and stopband