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Tutorial on filter synthesis

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1 Eigenvalue method

1. We are interested here in the synthesis of linear phase FIR filters. We consider the particular case of a type I filter, of odd length N and symmetrical impulse response, whose transfer function is denoted $H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$. Let $M = \frac{N-1}{2}$. Verify that we can write

$$H\left(e^{i2\pi\nu}\right) = e^{-i2\pi\nu M}\, H_R\left(e^{i2\pi\nu}\right)$$

where $H_R\left(e^{i2\pi\nu}\right)$ is a real-valued function, called the amplitude response of filter H, defined by the equality $H_R\left(e^{i2\pi\nu}\right) = \boldsymbol{a}^T \boldsymbol{c}(\nu)$, where $\boldsymbol{c}(\nu) = [1, \cos(2\pi\nu), \dots, \cos(2\pi M\nu)]^T$, and where the coefficients of vector $\boldsymbol{a} = [a_0, a_1, \dots, a_M]^T$ are to be expressed in terms of $h(n)$.

2. We wish to synthesize a low-pass filter with cutoff frequency $\nu_c \in \left]0, \frac{1}{2}\right[$ and whose stop-band starts at $\nu_a \in \left]\nu_c, \frac{1}{2}\right[$. The energy in the stop-band is $E_a = 2\int_{\nu_a}^{\frac{1}{2}} \left(H_R(e^{i2\pi\nu})\right)^2 \mathrm{d}\nu$. Show that we can write $E_a = \boldsymbol{a}^T \boldsymbol{P}\, \boldsymbol{a}$, where $\boldsymbol{P}$ is a positive semidefinite matrix, whose coefficients $\{\boldsymbol{P}_{(m,n)}\}_{(m,n)\in[[0,M]]^2}$ are to be determined in function of ν_a.

3. Ideally, the amplitude response $H_R\left(e^{i2\pi\nu}\right)$ is equal to $H_R(1)$ in the bandwidth $[0, \nu_c]$. We therefore define the square error in the bandwidth as follows:

$$E_c = 2\int_0^{\nu_c} \left(H_R(e^{i2\pi\nu}) - H_R(1)\right)^2 \mathrm{d}\nu$$

Show that we can write $E_c = \boldsymbol{a}^T \boldsymbol{Q}\, \boldsymbol{a}$, where $\boldsymbol{Q}$ is a positive semidefinite matrix, whose coefficients $\left\{\boldsymbol{Q}_{(m,n)}\right\}_{(m,n)\in[[0,M]]^2}$ are to be determined in function of ν_c.

4. The FIR filter synthesis method called *eigenvalue method* consists in minimizing with respect to $\boldsymbol{a}$ the cost function $E(\boldsymbol{a}) = \alpha E_c + (1-\alpha)E_a$, where $\alpha \in]0, 1[$ is a trade-off parameter between pass-band and stop-band. We thus obtain $E(\boldsymbol{a}) = \boldsymbol{a}^T \boldsymbol{R}\, \boldsymbol{a}$, where $\boldsymbol{R} = \alpha\boldsymbol{Q} + (1-\alpha)\boldsymbol{P}$ is a positive semidefinite matrix. Show that vector $\boldsymbol{a}$ minimizes function E under unit norm constraint if and only if it is an eigenvector of $\boldsymbol{R}$, associated to the lowest eigenvalue (*Rayleigh's principle*).

2 Synthesis of an integrator filter

We consider a digital signal $x(n)$, defined from an analog signal $x^a(t)$ sampled at sampling rate T: $x(n) = x^a(nT)$. This exercise aims at synthesizing a digital filter which allows to obtain, from the discrete signal $x(n)$, a sampled version of the integrated signal $y^a(t) = \int_{-\infty}^{t} x^a(u)\, du$.

Question 1 *Show that the integrated signal $y^a(t)$ can be written as the convolution product between the signal $x^a(t)$ and the analog filter $h^a(t) = 1$ if $t \geq 0$ and $h^a(t) = 0$ otherwise (Heaviside function). Is this filter causal? Is it stable? (reminder : the filter is stable if and only if $\int_{-\infty}^{+\infty} |h^a(t)|\, dt < +\infty$). Compute the transfer function $H^a(p) = \int_{-\infty}^{+\infty} h^a(t)\, e^{-pt}\, dt$ (Laplace transform of h^a, with $p \in \mathbb{C}$), and specify its domain of definition.*

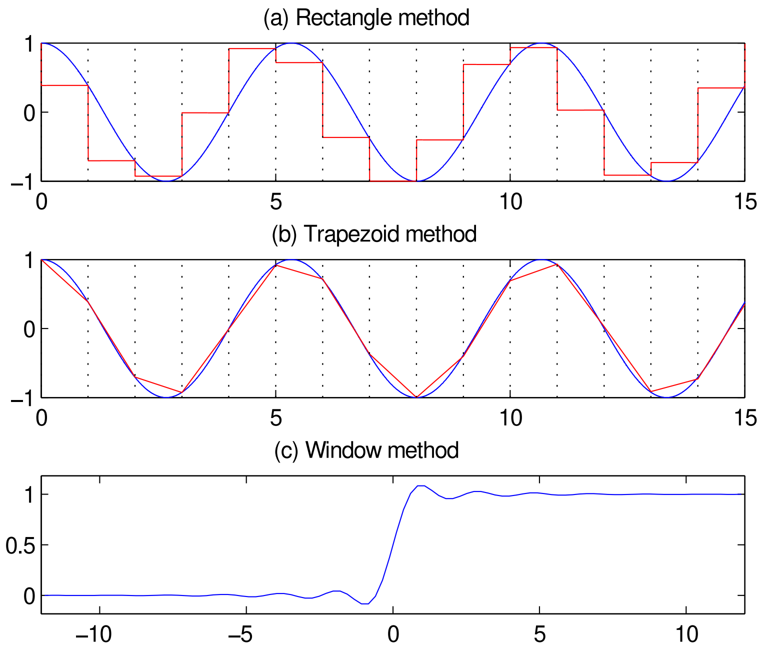

Figure 1: Three synthesis methods of an integrator filter

2.1 Approximation by the rectangle method

We wish to approximate the integral of the signal $x^a(t)$ by the rectangle method (an example is given on Figure 1-(a)), which amounts to computing the integral of the interpolated signal

$$x_0^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\, f_0(t - mT)$$

where $f_0(t) = 1$ if $t \in [-T, 0]$ and $f_0(t) = 0$ otherwise (rectangle function). We define the discrete-time integrated signal $y_0(n) = \int_{-\infty}^{nT} x_0^a(t)\, dt$.

Question 2 *Show that $y_0(n)$ can be written as the convolution product between the signal $x(n)$ and a digital filter $h_0(n)$, and give the expression of its impulse response. Is this filter causal? Is it stable? Calculate the transfer function $H_0(z)$, and specify its domain of definition.*

2.2 Approximation by the trapezoid method

We wish to approximate the integral of signal $x^a(t)$ by the trapezoid method (an example is given in Figure 1-(b)), which amounts to computing the integral of the interpolated signal

$$x_1^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\, f_1(t - mT)$$

where $f_1(t) = 1 - |t|/T$ if $t \in [-T, T]$ and $f_1(t) = 0$ elsewhere (triangle function). We define the discrete-time integrated signal $y_1(n) = \int_{-\infty}^{nT} x_1^a(t)\, dt$.

Question 3 *Show that $y_1(n)$ can be written as the convolution product between the signal $x(n)$ and a digital filter $h_1(n)$, and give the expression of its impulse response. Is this filter causal? Is it stable?*

Question 4 *Show that this method is equivalent to determining the digital filter from the analog filter of Question 1 by using the bilinear transformation (hint: we can identify the two transfer functions).*

2.3 Synthesis by the window method

We now wish to determine the integral of the signal $x^a(t)$ exactly. To do this, we assume that $x^a(t)$ satisfies the assumptions of the Shannon-Nyquist's theorem. It can then be reconstructed exactly from its samples:

$$x^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\, f(t - mT)$$

where $f(t) = \operatorname{sinc}\left(\frac{t}{T}\right)$ avec $\operatorname{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$. We define the discrete time integrated signal $y(n) = \int_{-\infty}^{nT} x^a(t)\, dt$.

Question 5 *Show that $y(n)$ can be written as the convolution product between the signal $x(n)$ and the digital filter $h(n) = T \int_{-\infty}^{n} \operatorname{sinc}(u)\, du$ (hint: we will assume that $x^a(t)$ satisfies strong enough assumptions to be able to switch $\int$ and $\sum$).*

Question 6 *The impulse response of filter h is represented in Figure 1-(c) (for $T = 1$). What phenomenon can be observed compared to the impulse responses calculated previously? Is this filter causal? Is it stable ? (hint: $h(n) \underset{n\to+\infty}{\longrightarrow} T$)*

Since $h(n) \underset{n\to+\infty}{\longrightarrow} 1$, it does not seem reasonable to synthesize filter h by directly applying the window method, which consists in truncating the impulse response. Instead, we define filter $G(z) = (1 - z^{-1})\, H(z)$, whose impulse response decreases towards 0 at infinity. This filter $G(z)$ can be synthesized by the window method. We can then deduce an integrating filter $H(z) = \frac{G(z)}{1-z^{-1}}$.

Question 7 *Show that the impulse response of the filter $g(n)$ is symmetrical with respect to $\frac{1}{2}$, and is upper bounded in absolute value by $O\left(\frac{1}{n}\right)$ (the proof is simple but it may be useful to make a drawing).*

Question 8 *Since the impulse response of filter g tends to 0 at infinity, it seems reasonable to synthesize this filter by the window method. In order for the resulting filter to be linear phase, should we choose an even or odd filter length N? What type of filter does this correspond to? (I, II, III or IV)*

Question 9 *Is the resulting filter $H(z)$ stable? What would you suggest to remedy this?*