Lagrangian duality 2/2: strong duality

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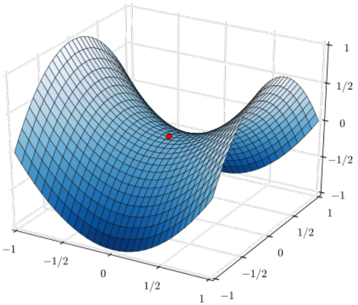
Saddle point



Definition: Saddle point

(x^*, ϕ^*) is a saddle point of L if for all x and all ϕ,

$$L(x^*, \phi) \leq L(x^*, \phi^*) \leq L(x, \phi^*)$$

Proposition:

L has a saddle point (x^*, ϕ^*) if and only if

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- As $\sup_{\phi} \inf_{x} L(x, \phi) = \inf_{x} \sup_{\phi} L(x, \phi)$, we say that strong duality holds

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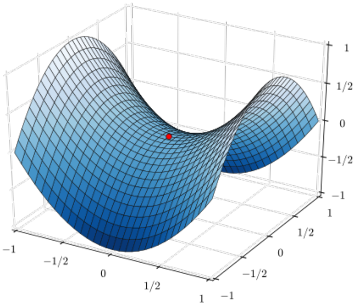

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- x^* is a minimizer of the primal problem

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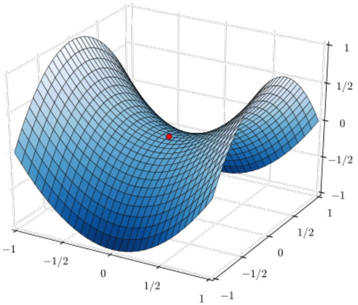

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- x^* is a minimizer of the primal problem
- ϕ^* is a maximizer of the dual problem

Karush-Kuhn-Tucker conditions

$$L(x, \phi_E, \phi_I) = f(x) + \langle \phi_E, A(x) \rangle + \langle \phi_I, g(x) \rangle - \iota_{\mathbb{R}^p_+}(\phi_I)$$

A saddle point verifies $0 \in \partial_x L(x^*, \phi^*)$ and $0 \in \partial_\phi(-L)(x^*, \phi^*)$

This gives the Karush-Kuhn-Tucker conditions

Theorem

$(x^*, \phi_E^*, \phi_I^*)$ is a saddle point of L if and only if

$$\begin{aligned}
&0 \in \partial f(x^*) + \textstyle\sum_{i=1}^n \phi_{E,i}^* \nabla A(x^*) + \sum_{j=1}^p \phi_{I,j}^* \partial g_j(x^*) \\
&A(x^*) = 0 \\
&g(x^*) \leq 0 \\
&\phi_I^* \geq 0 \\
&\forall j \in \{1, \dots, p\},\ g_j(x^*) \phi_{I,j}^* = 0
\end{aligned}$$

Strong duality theorem (Equality case)

How can we guarantee strong duality without computing a saddle point?

Theorem:
Consider the problem

$$\min_x f(x)$$
$$A(x) = 0$$

Suppose that

- f is convex and A is affine
- $0 \in \operatorname{relint}(A(\operatorname{dom} f))$ (constraint qualification)

Then there exists a dual solution ϕ_E^* and

$$\sup_{\phi_E} \inf_x L(x, \phi_E) = \inf_x \sup_{\phi_E} L(x, \phi_E)$$

Strong duality theorem (Inequality case)

How can we guarantee strong duality without computing a saddle point?

Theorem:

Consider the problem

$$\min_x f(x)$$

$$g(x) \leq 0$$

Suppose that

- f is convex and g_j is convex for all j
- $\exists x_0 \in \operatorname{dom} f$ such that $g_j(x_0) < 0$ for all j (Slater's constraint qualification)

Then there exists a dual solution ϕ_I^* and

$$\sup_{\phi_I} \inf_x L(x, \phi_I) = \inf_x \sup_{\phi_I} L(x, \phi_I)$$