Examination of the teaching unit
Représentations des signaux - TSIA201

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Duration: 2 hours
All documents are permitted. However electronic devices (including calculators) are forbidden.

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1 Rejector filter

Let us consider the transfer function $H(z) = \frac{1-2\cos(\theta)z^{-1}+z^{-2}}{1-2\rho\cos(\theta)z^{-1}+\rho^2 z^{-2}}$, with $0 < \rho < 1$.

a) Check that $H(z)$ can be factorized in the form $H(z) = \frac{(1-z_0 z^{-1})(1-z_0^* z^{-1})}{(1-\rho z_0 z^{-1})(1-\rho z_0^* z^{-1})}$, where z_0 is to be expressed as a function of θ.

b) What is the normalized frequency rejected by this filter?

c) What is the domain of convergence of its stable implementation?

d) Is this implementation causal?

e) Write the corresponding input/output relationship.

2 Downsampling

Let us consider a discrete time signal $x(n)$, that we wish to downsample by a factor 2. We recall the standard downsampling diagram:

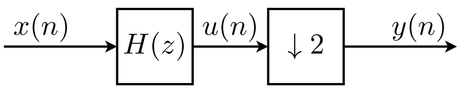


Figure 1: Downsampling diagram

where H is an ideal low-pass filter with cut-off frequency $\frac{1}{4}$: $\forall \nu \in [-\frac{1}{2}, +\frac{1}{2}]$, $H(e^{2i\pi\nu}) = 1$ if $\nu \in]-\frac{1}{4}, \frac{1}{4}[$, and $H(e^{2i\pi\nu}) = 0$ otherwise.

a) What is the role of filter H in Figure 1?

b) We define the filter of frequency response $G(e^{2i\pi\nu}) = e^{i\pi\nu}$ for all $\nu \in]-\frac{1}{2}, +\frac{1}{2}[$.

   1) Calculate its impulse response $g(n)$ $\forall n \in \mathbb{Z}$.

2) Is this filter stable? Is it causal? (justify). From now on, we will assume that we have synthesized a linear phase FIR filter that approximates $g(n)$.

c) We consider the diagram in Figure 2.

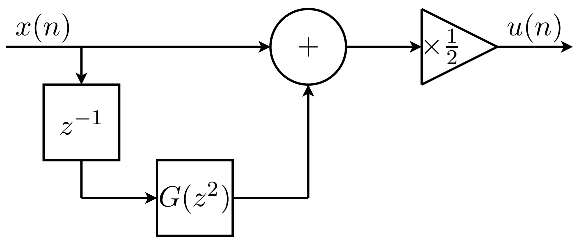


Figure 2: Equivalent implementation of filter H

1) Express $U(z)$ as a function of $G(z)$ and $X(z)$.
2) Evaluate $G(z^2)$ at $z = e^{2i\pi\nu}$, for $\nu \in [0, \frac{1}{4}[$ on the one hand, and $\nu \in]\frac{1}{4}, \frac{1}{2}]$ on the other hand.
3) Deduce the relationship between $U(e^{2i\pi\nu})$ and $X(e^{2i\pi\nu})$, and conclude that this diagram defines an equivalent implementation of filter H.

d) We consider the diagram in Figure 3.

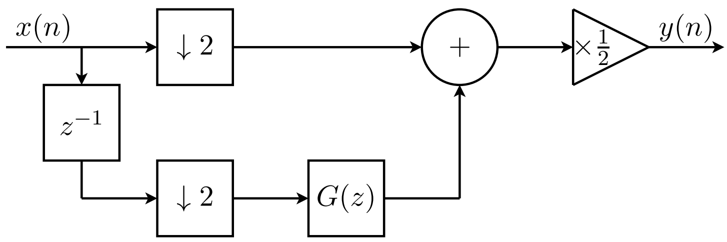
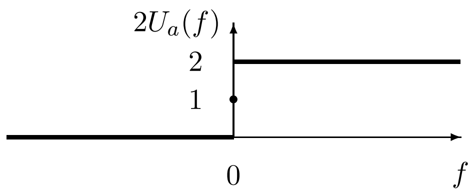


Figure 3: Efficient implementation

1) Check that this diagram is equivalent to the one in Figure 1.
2) What is the advantage of implementing the diagram in Figure 3 over that in Figure 1?

3 Hilbert filter

Let $x_a(t)$ be a continuous time (analog) real signal. The *analytic* signal associated to $x_a(t)$ is the signal $z_a(t)$ whose the CTFT is expressed as $Z_a(f) = 2\,U_a(f)X_a(f)$, where $U_a(f)$ is the unit step function, whose value is 1 for $f > 0$, and 0 for $f < 0$. For continuity reasons, we assume that $U_a(0) = \frac{1}{2}$. The filter of frequency response $2U_a(f)$ is referred to as the *analytic filter*.



a) Which property does function $X_a(f)$ satisfy? Deduce the expression of $\frac{1}{2}(Z_a(f) + Z_a^*(-f))$ as a function of $X_a(f)$, and prove that the real part of $z_a(t)$ is equal to $x_a(t)$. We can then write $z_a(t) = x_a(t) + iy_a(t)$, where the real signal $y_a(t)$ is defined as the imaginary part of $z_a(t)$.

b) Prove that $y_a(t)$ can be obtained from $x_a(t)$ by linear filtering of frequency response $H_a(f) = -i\,\mathrm{sign}(f)$, where $\mathrm{sign}(f) = 1$ for $f > 0$, $\mathrm{sign}(f) = -1$ for $f < 0$, and $\mathrm{sign}(0) = 0$. Filter $H_a(f)$ is referred to as the *Hilbert filter*, and $y_a(t)$ is called the *Hilbert transform* of $x_a(t)$.

Let us assume that the signal $x_a(t)$ satisfies the assumptions of the sampling theorem: there exists a frequency F_s such that the support of $X_a(f)$ is included in $]-\frac{F_s}{2}, \frac{F_s}{2}[$. We then consider the sampled signals $x(n) = x_a(nT_s)$ and $y(n) = y_a(nT_s)$, where $T_s = 1/F_s$. We remind the relationship between the DTFT $X(e^{2i\pi\nu})$ and the CTFT $X_a(f)$:

$$X(e^{2i\pi\nu}) = \frac{1}{T_s} \sum_{k\in\mathbb{Z}} X_a\left(\frac{\nu+k}{T_s}\right) \tag{1}$$

c) Simplify the expression (1) when $\nu \in]-\frac{1}{2}, \frac{1}{2}[$. Check that $Y(e^{2i\pi\nu})$ satisfies a similar expression. Deduce that the signal $y(n)$ can also be expressed as the output of the discrete filter of frequency response $H(e^{2i\pi\nu}) = -i\,\mathrm{sign}(\nu)$ for $\nu \in]-\frac{1}{2}, \frac{1}{2}[$ (and $H(e^{2i\pi\nu}) = 0$ for $\nu = \pm\frac{1}{2}$), applied to the input signal $x(n)$.

Remark: the discrete filter $H(e^{2i\pi\nu})$ allows us to directly compute the samples $y(n)$ of the Hilbert transform from the samples $x(n)$, without having to perform a digital/analog conversion.

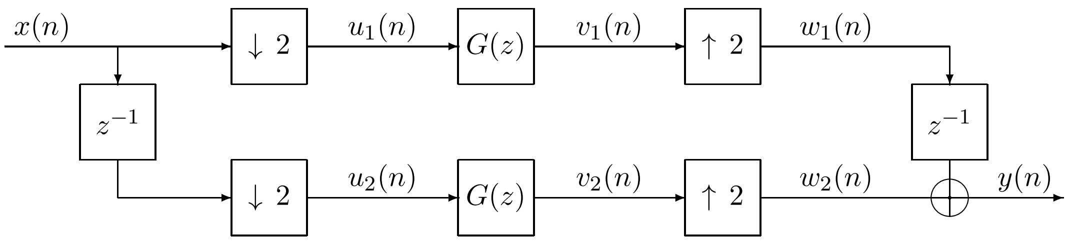
d) By applying the inverse DTFT, prove that the impulse response $h(n)$ satisfies $h(n) = \frac{2}{\pi n}$ if n is odd, and 0 if n is even.

e) Is this filter causal? Stable? Of finite (FIR) or infinite (IIR) impulse response?

f) For a discrete filter of impulse response $g(n)$ and of transfer function $G(z)$, what is the impulse response of the filter of transfer function $G(z^2)$? By using the fact that the even coefficients of $h(n)$ are zero, deduce that there exists a transfer function $G(z)$, such that $H(z) = z^{-1}G(z^2)$. What is the impulse response $g(n)$?

g) We want to approximate the ideal filter $G(z)$ by using the window method, in order to synthesize a linear phase FIR filter, of type 4 (even length N, antisymmetric impulse response $g(n)$). Quickly summarize the principle of the window method, its advantages and its drawbacks.

h) Now, we want to prove that the following diagram provides an efficient implementation of the discrete Hilbert filter $H(z)$:



We remind that $U_1(z) = \frac{1}{2}(X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}}))$. Express $U_2(z)$ as a function of $X(z)$, then $V_1(z)$ and $V_2(z)$ as a function of $U_1(z)$ and $U_2(z)$, then $W_1(z)$ and $W_2(z)$ as a function of $V_1(z)$ and $V_2(z)$, and finally $Y(z)$ as a function of $W_1(z)$ and $W_2(z)$. By substitution, retrieve the relationship $Y(z) = H(z)X(z)$.

4 DFT filter bank

We consider the signal processing system represented in Figure 4.

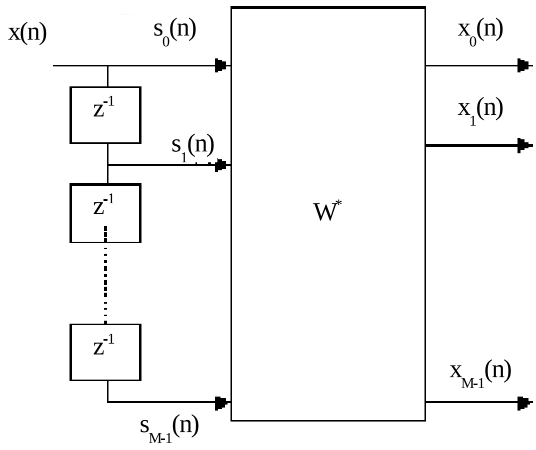


Figure 4: DFT filter bank

The matrix W^* implements the discrete Fourier transform (DFT) of length M. It is of dimension M, and each element $[W_{km}]$ is written in the form :

$$[W_{km}] = e^{2i\pi \frac{km}{M}}.$$

The input signal $x(n)$ is decomposed into M discrete signals by simply passing through a delay line. We thus define:

$$s_m(n) = x(n-m).$$

a) Give the expression of the Z-transform $S_m(z)$ as a function of $X(z)$.

b) Express the subband signals $x_k(n)$ as functions of the signals $s_m(n)$.

c) Calculate $X_k(z)$ as a function of the Z-transforms $S_m(z)$.

d) Express $X_k(z)$ in the form: $X_k(z) = H_k(z)X(z)$ and give the expression of $H_k(z)$.

e) Check that $H_k(z) = H_0(zW_{k1})$. Explain intuitively the spectral content of the subband signals $x_k(n)$.

f) Determine the type I polyphase components $E_{km}(z)$ of filter $H_k(z)$ at order M. Conclude that Figure 4 actually represents the polyphase implementation of filters h_k.

g) Draw the diagram of a signal processing system that perfectly reconstructs the original signal $x(n)$ given the subband signals $x_k(n)$.