Filter synthesis

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TSIA201

Part I

Linear phase FIR filters

Linear phase FIR filters

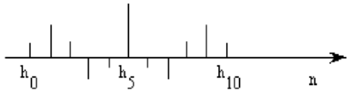
- Impulse response : $h(n) \in \mathbb{R}$, $n = 0 \dots N-1$
- Frequency response : $H(e^{i2\pi\nu}) = e^{i2\pi(\beta - \alpha\nu)} H_R(\nu)$
- Constant and equal group and phase delays (if $\beta = 0$)
- Advantages : always causal and stable, preserve the waveform of a narrow-band signal (if $\beta = 0$)
- Drawback : high computational complexity

Characterization

- 1 - periodicity of $H(e^{2i\pi\nu}) \Rightarrow \alpha = p/2$, $p \in \mathbb{Z}$ and H_R is 2-periodic
- Hermitian symmetry $\Rightarrow d = e^{2i\pi\beta} = 1$ or i and H_R is even or odd
- As $H_R(\nu)$ is 2-periodic, we can define $G(e^{2i\pi\nu}) = d\, H_R(2\nu)$ where $g(n)$ is real, even or odd
- We can thus write $H(z^2) = z^{-p} G(z)$ (filter of length $2N-1$)
- We choose $p = N-1$ for a causal filter
- 4 possibilities depending on d and the parity of N :
  - $d = 1$, N even or odd : $h(n) = h(N-1-n)$
  - $d = i$, N even or odd : $h(n) = -h(N-1-n)$

Types of filters

- Type 1 : N odd, symmetric ($d = 1$)

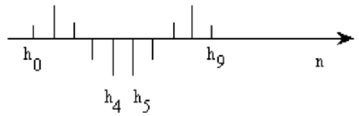


$$H(e^{i2\pi\nu}) = e^{-i2\pi\nu\frac{N-1}{2}} \sum_{n=0}^{\frac{N-1}{2}} a_n \cos(2\pi\nu n)$$

- Use : low-pass, high-pass, band-pass

Types of filters

- Type 2 : N even, symmetric ($d = 1$)

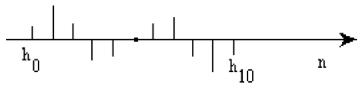


$$H(e^{i2\pi\nu}) = e^{-i2\pi\nu\frac{N-1}{2}} \sum_{n=1}^{\frac{N-1}{2}} b_n \cos\left(2\pi\nu\left(n - \tfrac{1}{2}\right)\right)$$

- Property : $H(-1) = 0$ ($\nu = \frac{1}{2}$)
- Use : low-pass, band-pass

Types of filters

- Type 3 : N odd, antisymmetric ($d = i$)

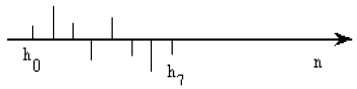


$$H(e^{i2\pi\nu}) = i\, e^{-i2\pi\nu\frac{N-1}{2}} \sum_{n=1}^{\frac{N-1}{2}} c_n \sin(2\pi\nu n)$$

- Property : $H(1) = H(-1) = 0$ ($\nu = 0$ or $\frac{1}{2}$)
- Use : differentiator ($H(f) = i2\pi f$), Hilbert transform ($H(f) = -i\,\mathrm{sign}(f)$) in band-pass form

Types of filters

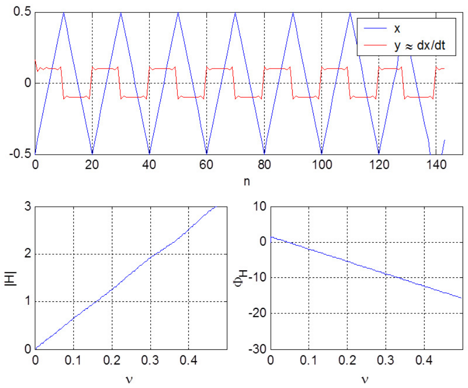
- Type 4 : N even, antisymmetric ($d = i$)



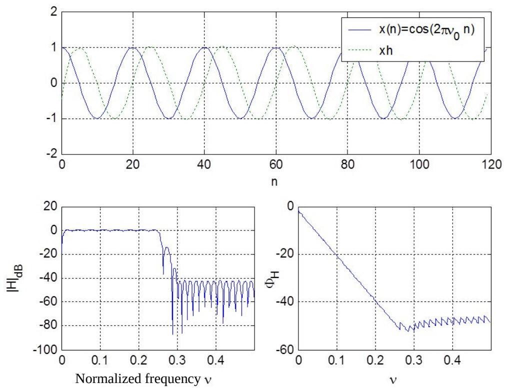
$$H(e^{i2\pi\nu}) = i\, e^{-i2\pi\nu\frac{N-1}{2}} \sum_{n=1}^{\frac{N-1}{2}} d_n \sin\left(2\pi\nu\left(n - \tfrac{1}{2}\right)\right)$$

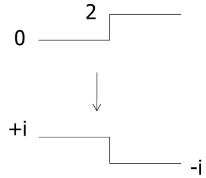
- Property : $H(1) = 0$ ($\nu = 0$)
- Use : differentiator ($H(f) = i2\pi f$), Hilbert transform ($H(f) = -i\,\mathrm{sign}(f)$) in high-pass form

Differentiators



Hilbert transform

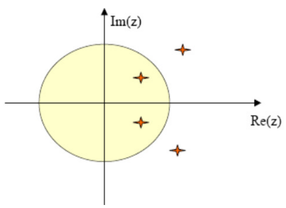




Position of the zeros

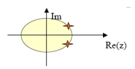
- Complex zero outside the unit circle :

$$(1-\rho e^{i\theta}z^{-1})(1-\rho e^{-i\theta}z^{-1})(1-\frac{1}{\rho}e^{i\theta}z^{-1})(1-\frac{1}{\rho}e^{-i\theta}z^{-1})$$

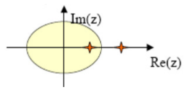


Position of the zeros

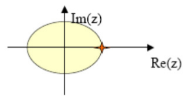
- Zero on the unit circle



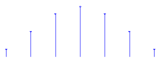
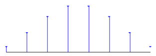
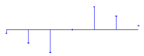
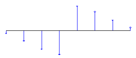
- Real zero



- Real zero on the unit circle



Summary

Type I N odd symmetric		-	Low-pass High-pass Band-pass
Type II N even symmetric		$H(-1)=0$	Low-pass, Band-pass
Type III N odd antisym.		$H(1)=$ $H(-1)=0$	Differentiator, Hilbert Transform, Band-pass
Type IV N even antisym.		$H(1)=0$	Differentiator, Hilbert Transform, High-pass

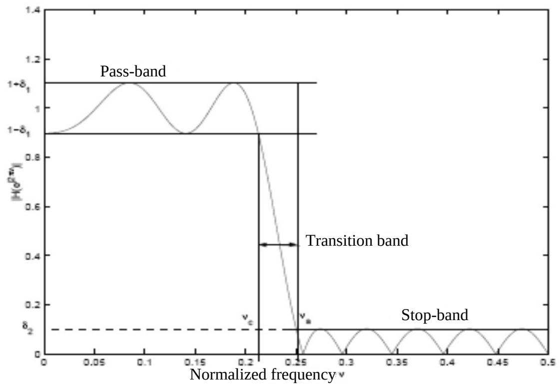
Part II

FIR filters: iterative methods

Iterative methods

- Advantages
  - Optimal design
  - Flexible method
  - Constant amplitude ripples
  - Minimum order for a given template
- Drawbacks
  - Computationally demanding synthesis
  - Not suitable for real-time processing
  - Not suitable for long filters (numerical stability issues)

Filter template



Parametrization of H_R

- Factorization : $H_R(\nu) = P(\nu)Q(\nu)$

$H_R(\nu)$	$P(\nu)$	$Q(\nu)$
$\sum_{n=0}^{\frac{N-1}{2}} a_n \cos(2\pi\nu n)$	$\sum_{n=0}^{\frac{N-1}{2}} a_n \cos(2\pi\nu n)$	1
$\sum_{n=1}^{\frac{N}{2}} b_n \cos\left(2\pi\nu\left(n-\frac{1}{2}\right)\right)$	$\sum_{n=0}^{\frac{N}{2}-1} b'_n \cos(2\pi\nu n)$	$\cos(\pi\nu)$
$\sum_{n=1}^{\frac{N-1}{2}} c_n \sin(2\pi\nu n)$	$\sum_{n=0}^{\frac{N-3}{2}} c'_n \cos(2\pi\nu n)$	$\sin(2\pi\nu)$
$\sum_{n=1}^{\frac{N}{2}} d_n \sin\left(2\pi\nu\left(n-\frac{1}{2}\right)\right)$	$\sum_{n=0}^{\frac{N}{2}-1} d'_n \cos(2\pi\nu n)$	$\sin(\pi\nu)$

Remez method

- Problem formulation : minimization of the error

$$E(\nu) = W(\nu)|H_D(\nu) - H_R(\nu)|$$

with N, ν_c and ν_a fixed
- We set $W(\nu) = 1/\delta_1$ in passband, $W(\nu) = 1/\delta_2$ in stopband, and $W(\nu) = 0$ in transition band
- With $P(\nu) = \sum_{n=0}^{M} a_n \cos(2\pi\nu n)$, we get :

$$\begin{aligned} E(\nu) &= W(\nu)(H_D(\nu) - P(\nu)\,Q(\nu)) \\ &= W'(\nu)(H'_D(\nu) - P(\nu)) \end{aligned}$$

Alternance theorem

- Minimization in Chebyshev sense :

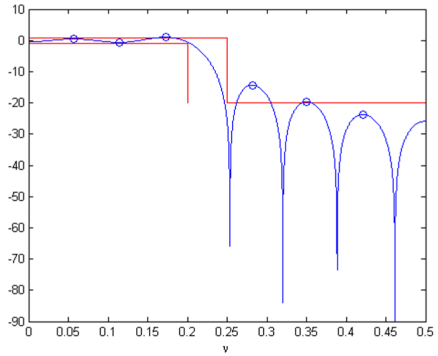
$$H(e^{i2\pi\nu}) = \min_H ||E(\nu)||_\infty$$

on a closed set of frequencies B
- Alternance theorem : the unique and best approximation is obtained when there exist $M+2$ frequencies $\nu_0 \dots \nu_{M+1}$ in B such that $E(\nu_k) = \pm(-1)^k\,\delta$ where $\delta = ||E(\nu)||_\infty$

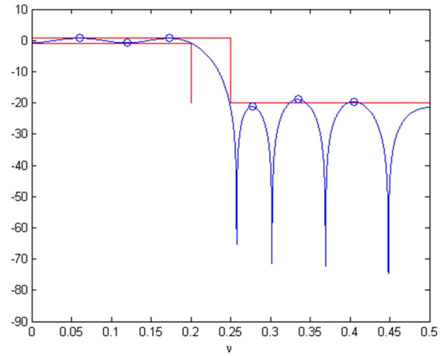
Remez algorithm

- Initialization : the $M+2$ alternances are set uniformly in B
- Iteration
  - Direct resolution of the linear system
    $\sum_{n=0}^{M} a_n \cos(2\pi\nu_k n) + \frac{(-1)^k\delta}{W(\nu_k)} = H_D(\nu_k)$
    (or solution by Lagrangian interpolation)
  - Search of the extrema of this polynomial
  - Choice of the new values of the ν_k
- Convergence in a few iterations

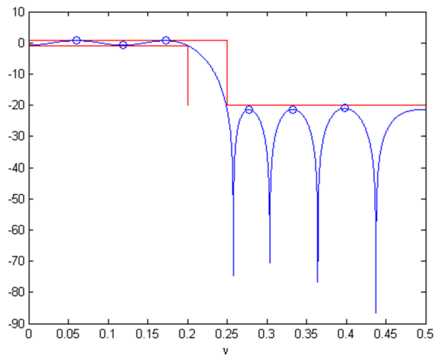
Remez method



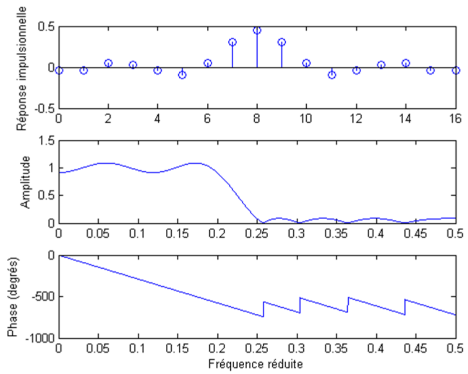
Remez method



Remez method



Remez method

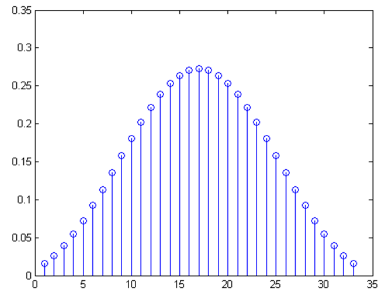


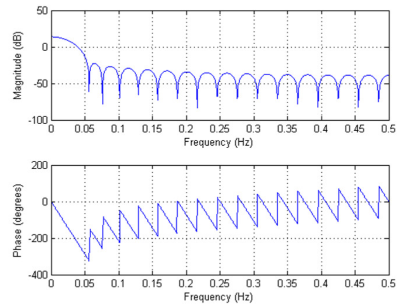
Part III

Eigenvalues methods

Prolate spheroidal sequences

- Let $h(n)$ be a low-pass FIR, real, causal filter
- Optimization under constraint : let us maximize $\int_{-\nu_c}^{\nu_c} |H(e^{i2\pi\nu})|^2 d\nu$ under the constraint $\int_{-\frac{1}{2}}^{\frac{1}{2}} |H(e^{i2\pi\nu})|^2 d\nu = \sum_{n=0}^{N-1} h(n)^2 = 1$





$$H = e(\nu)^H h, \; J = h^T [\int_{-\nu_c}^{+\nu_c} e(\nu) e(\nu)^H] h$$

Optimal eigen-filters

- Addition of a transition band $[\nu_C, \nu_A]$
- Let us minimize $E = \alpha E_c + (1-\alpha) E_a$ where $\alpha \in]0,1[$ where E_a is the stopband error :

$$E_a = 2 \int_{\nu_a}^{1/2} |H(e^{i2\pi\nu})|^2 d\nu$$

and E_c is the passband error :

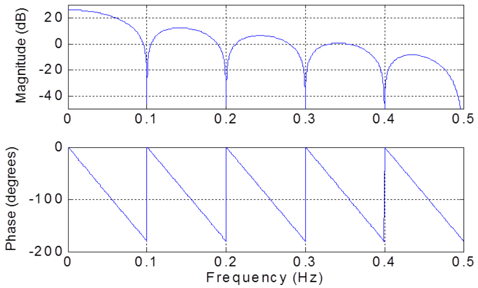
$$E_c = 2 \int_0^{\nu_c} |H(1) - H(e^{i2\pi\nu})|^2 d\nu$$

Part IV

Synthesis of recursive filters

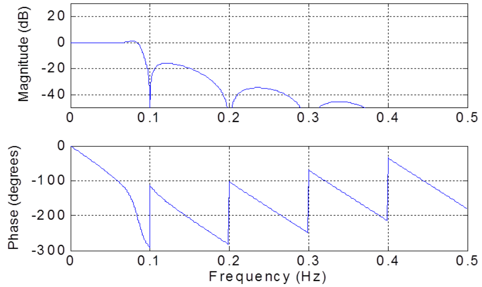
Generalities

- Example : $N(z) = \prod_k (1 - z_k z^{-1})$ with zeros at frequencies 0.1, 0.2, 0.3, 0.4, 0.5



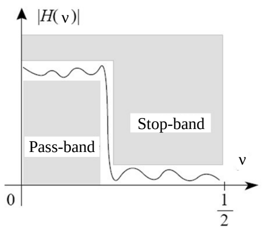
Generalities

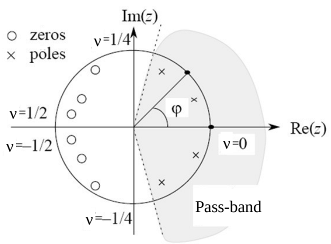
- Example : $H(z) = N(z)/D(z)$ where $D(z) = (1 - pz^{-1})(1 - p^* z^{-1})$, with $p = 0.95e^{i2\pi 0.085}$



Low-pass recursive filters

- Position of the poles and zeros

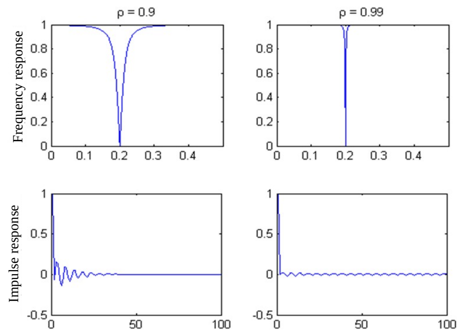




Rejector filter (audio feedback)

$$
\begin{aligned}
H(z) &= \frac{1 - e^{+i2\pi\nu_c} z^{-1}}{1 - \rho\, e^{+i2\pi\nu_c} z^{-1}} \frac{1 - e^{-i2\pi\nu_c} z^{-1}}{1 - \rho\, e^{-i2\pi\nu_c} z^{-1}} \\
&= \frac{1 - 2\cos(2\pi\nu_c) z^{-1} + z^{-2}}{1 - 2\rho\cos(2\pi\nu_c) z^{-1} + \rho^2 z^{-2}}
\end{aligned}
$$

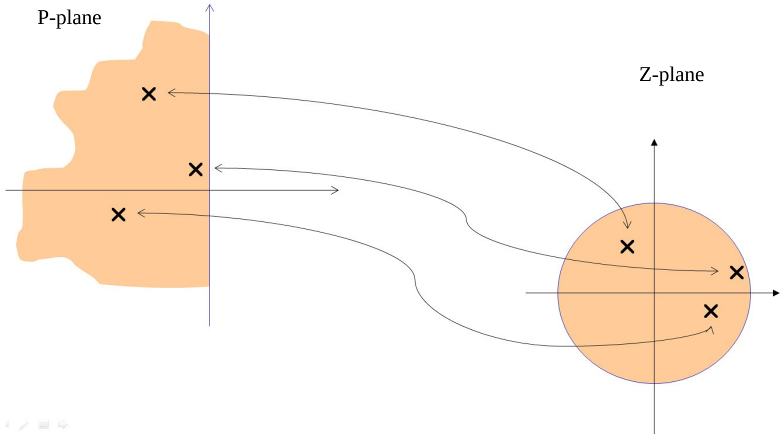
- Remark : the IR is long if $\rho \to 1$



Bilinear transform

- Analog filters : $H_a(s) = \int_{\mathbb{R}} h_a(t)e^{-st}dt$ where $s = 2i\pi f$
- Digital filters : $H(z) = \sum_{\mathbb{Z}} h(n)z^{-n}$ where $z = e^{2i\pi\nu}$
- Trapezoidal rule to approximate
  $$x(nT) = x((n-1)T) + \int_{(n-1)T}^{nT} x'(t)dt$$
- Example of the system $y(t) = x'(t)$ (differentiator filter : $H_a(s) = s$)
- Bilinear transform : $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$
- Frequency relationship : $f = \frac{1}{\pi T}\tan(\pi\nu)$

Bilinear transform



From continuous to discrete domain