Lagrangian duality 1/2: weak duality

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Problem at stake

Convex optimization with constaints

$$\min_{x \in \mathbb{R}^n} f(x)$$
$$A(x) = 0$$
$$g(x) \leq 0$$

- $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex, lower continuous
- $A : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function
- For all i, $g_i : \mathbb{R}^n \to \mathbb{R}$ is convex
- $g(x) \leq 0$ means $g_i(x) \leq 0$ for all $i \in \{1, \ldots, p\}$

m equality constraints and p inequality constraints

Problem at stake

Convex optimization with constaints

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \; & f(x) \\ & A(x) = 0 \\ & g(x) \leq 0 \end{aligned}$$

Equivalent formulation using convex indicator functions

$$\min_{x \in \mathbb{R}^n} f(x) + \iota_{\{0\}}(A(x)) + \iota_{\mathbb{R}^p_-}(g(x))$$

$$\iota_C(y) = \begin{cases} 0 & \text{if } y \in C \\ +\infty & \text{if } y \notin C \end{cases}$$

m equality constraints and p inequality constraints

Lagrangian function

$$\min_{x\in\mathbb{R}^n} f(x)$$
$$A(x) = 0$$
$$g(x) \leq 0$$

Definition: Lagrangian function

$$L(x,\phi_E,\phi_I) = f(x) + \langle \phi_E, A(x)\rangle + \langle \phi_I, g(x)\rangle - \iota_{\mathbb{R}^p_+}(\phi_I)$$

$$L(x,\phi_E,\phi_I) = \begin{cases} f(x) + \sum_{i=1}^m \phi_{E,i} A_i(x) + \sum_{j=1}^p \phi_{I,j} g_j(x) & \text{if } \forall j, \phi_{I,j} \geq 0 \\ -\infty & \text{if } \exists j, \phi_{I,j} < 0 \end{cases}$$

Proposition

$$\sup_{\phi_E,\phi_I} L(x,\phi_E,\phi_I) = f(x) + \iota_{\{0\}}(A(x)) + \iota_{\mathbb{R}^p_-}(g(x))$$

Proof in the case $m = 0$

$$L(x, \phi_I) = f(x) + \langle \phi_I, g(x) \rangle - \iota_{\mathbb{R}^p_+}(\phi_I)$$

$$\sup_{\phi_I} L(x, \phi_I) = f(x) + \iota_{\mathbb{R}^p_-}(g(x))$$

Dual problem

We call primal problem the optimization problem with constraints

$$\inf_x f(x) + \iota_{\{0\}}(A(x)) + \iota_{\mathbb{R}^p_-}(g(x)) = \inf_x \sup_{\phi_E, \phi_I} L(x, \phi_E, \phi_I)$$

Definition: dual problem

$$\max_{\phi_E, \phi_I} \inf_x L(x, \phi_E, \phi_I)$$

Definition: dual function

$$D(\phi) = D(\phi_E, \phi_I) = \inf_x L(x, \phi_E, \phi_I)$$

Proposition:
The dual function D is concave

Theorem: Weak duality

$$\inf_x \sup_{\phi_E, \phi_I} L(x, \phi_E, \phi_I) \geq \sup_{\phi_E, \phi_I} \inf_x L(x, \phi_E, \phi_I)$$

Proof of the weak duality theorem

$$\inf_x \sup_{\phi_E, \phi_I} L(x, \phi_E, \phi_I) \geq \sup_{\phi_E, \phi_I} \inf_x L(x, \phi_E, \phi_I)$$