Bayesian statistics

Exercise 1 (Bernoulli model)**:**
You would like to estimate the probability θ of winning a lottery game through n i.i.d. observations. You assume a uniform prior distribution on θ.

1. Give the posterior distribution of θ.
2. Deduce the Bayes estimator of θ.
3. Compute the Bayes quadratic risk of this estimator.
4. Compare this estimator to the maximum likelihood estimator (MLE) in terms of Bayes quadratic risk and in terms of quadratic risk.

Exercise 2 (Poisson model)**:**
The daily number of admissions to a cinema is supposed to be Poisson with parameter θ. The prior distribution on θ is exponential with parameter $\lambda > 0$. You want to estimate θ using n i.i.d. samples.

1. Give the posterior distribution of θ.
2. Deduce the Bayes estimator of θ.
3. Compute the Bayes quadratic risk.

Exercise 3 (Gaussian model – mean)**:**
The daily power consumption of a company is supposed to have a gaussian distribution. You want to estimate the mean θ through n independent observations; the variance σ^2 is known. The prior distribution of θ is itself gaussian with mean μ and variance 1.

1. Give the posterior distribution of θ.
2. Deduce the Bayes estimator of θ.
3. Compute the Bayes quadratic risk. Which term dominates for large n, bias or variance?

Exercise 4 (Bernoulli model – discrete prior)**:**
You want to estimate the fraction θ of electric cars in Paris. The prior distribution on θ is uniform over $\{\frac{1}{4}, \frac{1}{3}\}$. You have n i.i.d. observations.

1. Give the posterior distribution of θ.
2. Deduce the Bayes estimator of θ.
3. Study the behavior of Bayes estimator for large n, depending on the true parameter $\theta \in (0, 1)$.

Exercise 5 (Translated exponential)**:**
The lifetimes of laptops are supposed to have a translated exponential distribution of the form:

$$P(X > x) = \exp(\theta - x), \quad \forall x \geq \theta,$$

where $\theta > 0$ is unknown. You have n observations. The prior distribution of θ is exponential with parameter $\lambda > 0$.

1. Give the posterior distribution of θ.
2. Deduce the Bayes estimator of θ.
3. Compare with the maximum likelihood estimator (MLE) for large n.
4. Compare the quadratic risk and the Bayes quadratic risk of the MLE.

Exercise 6 (Exponential model – mean)**:**
You want to estimate the mean $g(\theta) = \frac{1}{\theta}$ of an exponential distribution with parameter θ using n i.i.d. samples; the prior on θ is itself exponential with parameter λ.

1. Give the posterior distribution of θ.
2. Deduce the Bayes estimator of $g(\theta)$.
3. What is the Bayes risk?

Exercise 7 (Poisson model – Jeffreys prior)**:**
We would like to use a non-informative prior for the Poisson model.

1. Give the Jeffreys prior on θ. Observe that this prior is *improper*, in the sense that it is not a probability measure.
2. For that prior, give the posterior distribution of θ.
3. Deduce the Bayes estimator of θ and compare it to that of Exercise 2.
4. Compute the quadratic risk of this estimator.

Exercise 8 (Gaussian vector)**:**
Some signal is supposed to have a gaussian distribution with unknown mean $\theta \in \mathbb{R}^d$ and known covariance matrix Γ. You want to estimate $g(\theta) = w^T\theta$ for some $w \in \mathbb{R}^d$ through n independent observations $x_1, \ldots, x_n$. The prior of θ is itself Gaussian with zero mean and unit covariance matrix.

1. Give the posterior distribution of θ.
2. Deduce the Bayes estimator of $g(\theta)$.