Module 4: Linear Algebra

Contents

- Matrices and Vectors,
- Solving System of Linear Equations, Eigenvalues and Eigenvectors,
- Inverse of Matrix Decomposition,
- Implementation of Eigenvalues and Eigenvectors in Python,
- Built-in function in Python. Diagonalization of square matrices

Introduction:

Linear algebra is the study of vectors and linear functions. It comprises of the theory and application of linear system of equations, linear transformations, Eigen values and Eigen vectors, problems.

Elementary transformations of a matrix:

The following are the elementary row transformations of a matrix. The transformations can be applied for columns.

- The interchange of any two rows (columns).
- The multiplication of any row (column) by a non-zero constant.
- The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Example: A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

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SI. No.	Elementary row transformation	Notation	Resultant of a matrix A
1	Interchange of first and second row	$R_1 \leftrightarrow R_2$	$\begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$
2	Multiplication of third row by a constant k	kR_3	$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ kc_1 & kc_2 & kc_3 \end{bmatrix}$
3	Addition to second row k times the first row	$R_2 \rightarrow kR_1 + R_2$	$\begin{bmatrix} a_1 & a_2 & a_3 \\ (ka_1 + b_1) & (ka_2 + b_2) & (ka_3 + b_3) \\ c_1 & c_2 & c_3 \end{bmatrix}$

Equivalent matrices:

Two matrices A and B of the same order are said to be *equivalent* if one matrix can be obtained from the other by a finite number of successive elementary row or column transformations. We use the notation ‘A ~ B’

Echelon form of a matrix:

A non – zero matrix **A** is said to be in *row echelon form* if the following conditions prevail:

i) All the zero rows are below non zero rows.

ii) The first non zero entry in any non zero row is 1 [and the entries below 1 in the same column are zero].

Ex : $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank of a matrix:

The rank of a matrix A in its echelon form is equal to the number of non zero rows. It is denoted by $\rho(A)$.

Steps to find the rank of a matrix

Step 1. In order to reduce the given matrix to a row echelon form we must prefer to have the leading entry (first entry in the first row) non zero, much preferably 1.

Step 2. In case this entry is zero, we can interchange with any suitable row to meet the requirement.

Step 3. We then focus on the leading non zero entry (starting from the first row) to make all the elements in that column zero. However the transformation has to be performed for the entire row.

Step 4. Row echelon form will be achieved first and we can instantly write down the rank, which being the number of zero rows.

NOTE:

- The **rank** of a matrix A in its **echelon form** is equal to the number of **non zero rows**.

- Elementary transformations do not change either the order or rank of a matrix.

- It is advisable to avoid fraction as far as possible during the process of elementary transformation.

- Further we can say that if r is the rank of a matrix A of order $m \times n \ (r \leq m),\ r$ number of rows of the matrix are linearly independent.

Problems:

1. Find the rank of the following matrix by reducing it to the row echelon form. $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Solution: $R_2 = R_2 - 2R_1$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

Rank of the matrix is 2

2. Find the rank of the following matrix by reducing it to the row echelon form.

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

Solution:

Performing $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to {}^{R_1}/_2$ and $R_2 \to {}^{R_2}/_2$

$$A \sim \begin{bmatrix} 1 & 3/2 & 5/2 & 2 \\ 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2$$

Therefore Matrix A is in the row echelon form having **two** non zero rows.

3. Find the rank of the following matrix by reducing it to the row echelon form.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution:

Performing $R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$R_4 \to R_4 - (R_1 + R_2 + R_3)$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to {R_2}/_5$ and $R_3 \to {R_3}/_4$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 1 & 9/4 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & \frac{33}{20} & \frac{11}{10} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \to R_3 \times \frac{20}{33}$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

4. Find the rank of the following matrix by reducing it to the row echelon form.

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Solution:

Performing $R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 3 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$ and $R_4 \to R_4 - 3R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3/2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

5. **Find the rank of the following matrix by reducing it to the row echelon form.**

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

Solution:

Performing $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to (-R_2)$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2$$

Solving System of Linear Equations

Consider a system of '*m*' linear equations in '*n*' unknowns as follows

$$a_{11}\,x_1 + a_{12}\,x_2 + \dots\dots\dots + a_{1n}\,x_n = b_1$$

$$a_{21}\,x_1 + a_{22}\,x_2 + \dots\dots\dots + a_{2n}\,x_n = b_2$$

$$\dots\dots\dots \qquad\qquad \dots\dots\dots$$

$$\dots\dots\dots \qquad\qquad \dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}\,x_2 + \dots\dots\dots\dots + a_{mn}\,x_n = b_m$$

Where $a_{ij}'s$ and $b_i's$ are constants.

- If $b_1, b_2, \dots\dots\dots, b_m$ are all zero, the system is said to be homogeneous. Otherwise, it is said to be Non-homogeneous.
- The set of values $x_1, x_2, \dots, x_n$ which satisfy all the equations simultaneously is called a solution of the system of equations.
- A system of linear equations is said to be **consistent** if it possess a solution. Otherwise it is said to be **inconsistent**.

The above system of equations can be written in the matrix equation A X = B, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_m \end{bmatrix}$$

- $x_1 = x_2 = x_3 = \dots\dots\dots. = x_n = 0$ is a solution of the homogeneous system of equations and is called a **trivial solution.**
- If at least one x_i , (i = 1, 2, …, n) is not equal to zero then it is called a **non trivial solution.**
- The concept of the rank of a matrix helps us to conclude

  (i). Whether the system is consistent or not.

  (ii). Whether the system possess unique solution or many solution.

Condition for consistency and types of solution

Consider a system of m equations in n unknowns represented in the matrix form $AX = B$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_m \end{bmatrix}$$

Here A is called the coefficient matrix.

The matrix formed by appending to A an extra column consistent of the elements of B is called the **augmented matrix** denoted by **[*A*: *B*]**

That is, $$[A:B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ .. & .. & \dots & .. & \vdots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & b_m \end{bmatrix}$$

The system of equations represented by the matrix equation $AX = B$ is **consistent** if $\rho(A) = \rho(A:B)$

Suppose $\rho(A) = \rho(A:B)$ $=r$, then the condition for two types of solution are as follows.

1. **Unique Solution:** $\rho(A) = \rho(A:B) = r = n$, (where n is the number of unknowns).

2. **Infinite Solution:** $\rho(A) = \rho(A:B) = r < n$,

   In case $(n - r)$ unknowns can take arbitrary value, Obviously $\rho[A] \neq \rho[A:B]$ implies that the system in **inconsistent** (does not possess a solution).

Working procedure for problems

Step 1: We first form the augmented matrix [A: B] and we can clearly identify the portion of the coefficient matrix A in it.

Step 2: We reduce the matrix [A: B] to an echelon form by elementary row transformations. This will enable us to immediately write down the rank of A and also [A: B], with the result we can decide the consistency aspect of the system of equations.

Step 3: The echelon form of [A: B] is converted back to the equation form and the solution will emerge easily.

Problems

1. Test for consistency and solve

$$x + y + z = 6$$
$$x - y + 2z = 5$$
$$3x + y + z = 8$$

Solution:

$$[A:B] = \begin{bmatrix} 1 & 1 & 1: & 6 \\ 1 & -1 & 2: & 5 \\ 3 & 1 & 1: & 8 \end{bmatrix}$$ is the augmented matrix.

Perform $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 3R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1: & 6 \\ 0 & -2 & 1: & -1 \\ 0 & -2 & -2: & -10 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1: & 6 \\ 0 & -2 & 1: & -1 \\ 0 & 0 & -3: & -9 \end{bmatrix}$$

[**Note:** *We need not make the leading non zero entry in every row 1 as we can decide on the rank of the matrices A and [A: B] at this stage.*]

Both A and [A : B] matrices have all the three rows non zero.

Therefore $\rho[A] = 3$ and $\rho[A : B] = 3$ that is, $r = 3$.

Also, the number of independent variables $n = 3$.

Since $\rho[A] = \rho[A : B] = 3$ ($r = n = 3$) the given system of equations is **consistent**

and

will have **unique solution**.

Let us now convert the prevailing form of [A : B] into a set of equations as follows,

$$x + y + z = 6 ....(1)$$
$$-2y + z = -1 ....(2)$$
$$-3z = -9 ....(3)$$

From (3), $z = 3$,

Substitute this value in (2), we get $y = 2$.

Finally substituting these values in (1), we get $x = 1$.

Thus $x = 1, y = 2, z = 3$ is the unique solution.

2. Test for consistency and solve

$$x+2y+3z=14$$
$$4x+5y+7z=35$$
$$3x+3y+4z=21$$

Solution:

$$[A:B]=\begin{bmatrix}1 & 2 & 3 & : & 14\\ 4 & 5 & 7 & : & 35\\ 3 & 3 & 4 & : & 21\end{bmatrix}$$ is the augmented matrix.

$R_2 \to R_2-4R_1$ and $R_3 \to R_3-3R_1$

$$[A:B]\sim\begin{bmatrix}1 & 2 & 3 & : & 14\\ 0 & -3 & -5 & : & -21\\ 0 & -3 & -5 & : & -21\end{bmatrix}$$

$R_3 \to R_3-R_2$

$$[A:B]\sim\begin{bmatrix}1 & 2 & 3 & : & 14\\ 0 & -3 & -5 & : & -21\\ 0 & 0 & 0 & : & 0\end{bmatrix}$$

Therefore $\rho[A]=2$ and $\rho[A:B]=2$ that is, $r = 2$.

Also, the number of independent variables $n = 3$.

Since $\rho[A]=\rho[A:B]=2<3$ $(r<n)$ the given system of equations is **consistent** and will have **infinite solution**.

Here $(n - r) = 1$ and hence one of the variables can take arbitrary values.

We now have, $x+2y+3z=14\ldots\ldots(1)$

$$-3y-5z=-21\ldots\ldots(2)$$

Let $z = k$ be arbitrary,

Therefore, from (2),

$$-3y-5k=-21 \text{ or } y=\frac{-21+5k}{-3}=7-\frac{5k}{3}$$

Substitute this value in (1), we get

$$x+2\left[7-\frac{5k}{3}\right]+3k=14$$

$$\Rightarrow x+14-\frac{10k}{3}+3k-14=0 \Rightarrow x=k/3.$$

Thus

$$x=\frac{k}{3},\ y=7-\frac{5k}{3},\ z=k$$

represents infinite solutions, since k is arbitrary.

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3. Test for consistency and solve

$$x-4y+7z=14$$
$$3x+8y-2z=13$$
$$7x-8y+26z=5$$

Solution:

$$[A:B]=\begin{bmatrix}1 & -4 & 7 & : & 14\\ 3 & 8 & -2 & : & 13\\ 7 & -8 & 26 & : & 5\end{bmatrix}$$ is the augmented matrix.

$R_2 \to R_2-3R_1$ and $R_3 \to R_3-7R_1$

$$[A:B]\sim\begin{bmatrix}1 & -4 & 7 & : & 14\\ 0 & 20 & -23 & : & -29\\ 0 & 20 & -23 & : & -93\end{bmatrix}$$

$R_3 \to R_3-R_2$

$$[A:B]\sim\begin{bmatrix}1 & -4 & 7 & : & 14\\ 0 & 20 & -23 & : & -29\\ 0 & 0 & 0 & : & -64\end{bmatrix}$$

Therefore $\rho[A]=2$ and $\rho[A:B]=3$.

Also, the number of independent variables $n=3$.

Since $\rho[A]\neq\rho[A:B]$, the given system of equations is **inconsistent.**

4. Test for consistency and solve

$$5x_1 + x_2 + 3x_3 = 20$$
$$2x_1 + 5x_2 + 2x_3 = 18$$
$$3x_1 + 2x_2 + x_3 = 14$$

Solution:

$$[A:B] = \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 2 & 5 & 2 & : & 18 \\ 3 & 2 & 1 & : & 14 \end{bmatrix} \text{ is the augmented matrix.}$$

$R_2 \to 5R_2 - 2R_1$ and $R_3 \to 5R_3 - 3R_1$

$$[A:B] \sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 7 & 4 & : & 10 \end{bmatrix}$$

$R_3 \to 23R_3 + 7R_2$

$$[A:B] \sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 0 & -120 & : & -120 \end{bmatrix}$$

Therefore $\rho[A] = 3$ and $\rho[A : B] = 3$ that is, $r = 3$.

Also, the number of independent variables $n = 3$.

Since $\rho[A] = \rho[A : B] = 3$ $(r = n = 3)$ the given system of equations is **consistent** and will have **unique solution**.

Let us now convert the prevailing form of [A : B] into a set of equations as follows,

$$5x_1 + x_2 + 3x_3 = 20 \ldots\ldots (1)$$
$$23x_2 + 4x_3 = 50 \ldots\ldots (2)$$
$$-120x_3 = -120 \ldots\ldots\ldots (3)$$

From (3), $x_3 = 1$,

Substitute this value in (2), we get

$x_2 = 2$.

Finally substituting these values in (1), we get

$x_1 = 3$.

Thus $x_1 = 3,\ x_2 = 2,\ x_3 = 1$ is the unique solution.

5. Show that the following system of equation does not possess any solution

$$5x+3y+7z=5$$
$$3x+26y+2z=9$$
$$7x+2y+10z=5$$

Solution:

$$[A:B]=\begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$ is the augmented matrix.

$$\mathrm{R}_2 \to 5R_2 - 3R_1 \quad \text{and} \quad \mathrm{R}_3 \to 5R_3 - 7R_1$$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & -11 & 1 & : & -10 \end{bmatrix}$$

$$\mathrm{R}_3 \to R_2 + 11R_3$$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & 0 & 0 & : & -80 \end{bmatrix}$$

Therefore $\rho[A] = 2$ and $\rho[A : B] = 3$.

Also, the number of independent variables $n = 3$.

Since $\rho[A] \neq \rho[A : B]$, the given system of equations is **inconsistent.**

7. Test for consistency and solve

$$x+2y+2z=1$$
$$2x+y+z=2$$
$$3x+2y+2z=3$$
$$y+z=0$$

Solution:

$$[A:B]=\begin{bmatrix}1 & 2 & 2 & : & 1\\ 2 & 1 & 1 & : & 2\\ 3 & 2 & 2 & : & 3\\ 0 & 1 & 1 & : & 0\end{bmatrix}$$ is the augmented matrix.

$R_2 \to R_2-2R_1$ and $R_3 \to R_3-3R_1$

$$[A:B]\sim\begin{bmatrix}1 & 2 & 2 & : & 1\\ 0 & -3 & -3 & : & 0\\ 0 & -4 & -4 & : & 0\\ 0 & 1 & 1 & : & 0\end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$[A:B]\sim\begin{bmatrix}1 & 2 & 2 & : & 1\\ 0 & 1 & 1 & : & 0\\ 0 & -4 & -4 & : & 0\\ 0 & -3 & -3 & : & 0\end{bmatrix}$$

$R_3 \to 4R_2+R_3$ and $R_4 \to 3R_2+R_4$

$$[A:B]\sim\begin{bmatrix}1 & 2 & 2 & : & 1\\ 0 & 1 & 1 & : & 0\\ 0 & 0 & 0 & : & 0\\ 0 & 0 & 0 & : & 0\end{bmatrix}$$

Therefore $\rho[A]=2$ and $\rho[A:B]=2$ that is, $r=2$.

Also, the number of independent variables $n=3$.

Since $\rho[A]=\rho[A:B]=2<3$ ($r<n$) the given system of equations is **consistent** and will have **infinite solution**.

Here $(n - r) = 1$ and hence one of the variables can take arbitrary values.

We now have,
$$x+2y+2z=1.......(1)$$
$$y+z=0 \qquad ....(2)$$

Let $z=k$ be arbitrary,

Therefore, from (2),

$$y+k=0 \text{ or } y=-k$$

Substitute this value in (1), we get

$$x=1.$$

Thus $x=1, y=-k, z=k$ represents infinite solutions, since k is arbitrary.

Eigen values and Eigen vectors

Let ‘A’ be a given square matrix of order $n \times n$. Suppose there exists a non-zero column matrix ‘X’ of order $1 \times n$ and a real or complex number λ such that $AX = \lambda X$ then X is called an **Eigen vector** of A and λ is called the corresponding **Eigen value** of A.

If I is the unit matrix of the same order as that of A, we have $X = IX$ and hence

$$AX = \lambda X \text{ can be written as,}$$

$$AX = \lambda(IX)$$

$$AX - \lambda IX = 0$$

$$[A - I\lambda]X = [0], \quad \text{where } [0] \text{ is the null matrix.}$$

$$[A - I\lambda]X = [0] \text{ represents a set of homogeneous equations.}$$

Let us consider a square matrix of order 3, represented by

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \text{Also, } \lambda I = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore [A - \lambda I] = \begin{bmatrix} (a_1 - \lambda) & a_2 & a_3 \\ b_1 & (b_2 - \lambda) & b_3 \\ c_1 & c_2 & (c_3 - \lambda) \end{bmatrix} \qquad \text{Also, Let } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

It can be easily seen that [A- λI][X] = [0] represents a set of homogenous equation [RHS being zero] in 3 – unknowns.

$$(a_1 - \lambda)x + a_2 y + a_3 z = 0$$
$$b_1 x + (b_2 - \lambda)y + b_3 z = 0$$
$$c_1 x + c_2 y + (c_3 - \lambda)z = 0$$

A nontrivial solution [at least one of $x, y, z \neq 0$]for this system exists if the determinant of the co-efficient matrix is zero.

$$\begin{vmatrix} (a_1 - \lambda) & a_2 & a_3 \\ b_1 & (b_2 - \lambda) & b_3 \\ c_1 & c_2 & (c_3 - \lambda) \end{vmatrix} = 0$$

On expanding, we get a cubic equation λ which is called the characteristic equation of A. The roots of this equation are eigen values, which are also called eigen roots or characteristic roots or latent roots.

For each value of λ there will be an eigen vector $X \neq 0$, which is also called a characteristic vector.

Problems

1. Find all the Eigen values and the corresponding Eigen vectors of the matrix

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution:

The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

On expanding, we have $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

After solving, we get $\lambda = 0,\ 3,\ 15$ are the Eigen values.

Now the system of equations is

$$(8-\lambda)x - 6y + 2z = 0$$
$$-6x + (7-\lambda)y - 4z = 0 \ldots\ldots\ldots\ldots\ldots(1)$$
$$2x - 4y + (3-\lambda)z = 0$$

Case (i): Let $\lambda = 0$ and the system of equations becomes

$$8x - 6y + 2z = 0 \qquad -\ (i)$$
$$-6x + 7y - 4z = 0 \qquad -\ (ii)$$
$$2x - 4y + 3z = 0 \qquad -\ (iii)$$

Applying the rule of cross multiplication for (i) and (ii)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x}{10} = \frac{-y}{-20} = \frac{z}{20} \qquad or \qquad \frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

$\therefore (x, y, z)$ are proportional to $(1,\ 2,\ 2)$ and we can write

$x = k,\ \ y = 2k,\ \ z = 2k$ where k is arbitrary

$\therefore$ the Eigen vector X_1 for $\lambda = 0$ *is* $\begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix}$

$\therefore$ Eigen Vector $X_1 = [1\ \ 2\ \ 2\ ]$

Case (ii): Let $\lambda = 3$ and the system of equations becomes

$$5x - 6y + 2z = 0 \qquad - (iv)$$
$$-6x + 4y - 4z = 0 \qquad - (v)$$
$$2x - 4y + 0z = 0 \qquad - (vi)$$

Applying the rule of cross multiplication for (iv) and (v)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x}{16} = \frac{-y}{-8} = \frac{z}{-16} \qquad or \qquad \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

the Eigen vector X_2 for $\lambda = 3\ is\ [2 \quad 1 \quad -2]$

Case (iii): Let $\lambda = 15$ and the system of equations becomes

Applying the rule of cross multiplication for (vii) and (viii)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x}{40} = \frac{-y}{40} = \frac{z}{20} \qquad or \qquad \frac{x}{2} = \frac{-y}{2} = \frac{z}{1}$$

the Eigen vector X_3 for $\lambda = 15\, is\ [2 \quad -2 \quad 1]$

2. Find all the Eigen values and the corresponding Eigen vectors of the matrix

$$\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$$

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(4-\lambda) + 6 = 0$$
$$-4 - 4\lambda + \lambda + \lambda^2 + 6 = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$
$$\lambda = 1, 2$$

Now the system of equations is

$$(-1-\lambda)x + 3y = 0$$
$$-2x + (4-\lambda)y = 0$$

Case (i): Let $\lambda = 1$ and the system of equations becomes

$$-2x + 3y = 0$$
$$-2x + 3y = 0$$

The above sets of equations are all same as we have only one independent equation $-2x + 3y = 0$

$$2x = 3y$$
$$\frac{x}{3} = \frac{y}{2}$$

$$X_1 = [\,3 \quad 2]$$

Case (ii): Let $\lambda = 2$ and the system of equations becomes

$$-3x + 3y = 0$$
$$-2x + 2y = 0$$

The above sets of equations are all same as we have only one independent equation $-3x + 3y = 0$

$$3x = 3y$$
$$\frac{x}{1} = \frac{y}{1}$$

$$X_2 = [\,1 \quad 1]$$

3. Find all the Eigen values and the corresponding Eigen vectors of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

On expanding, we have $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

After solving, we get $\lambda = 2,\ 2,\ 8$ are the Eigen values.

Now the system of equations is

$$(6-\lambda)x - 2y + 2z = 0$$
$$-2x + (3-\lambda)y - z = 0$$
$$2x - y + (3-\lambda)z = 0$$

Case (i): Let $\lambda = 2$ and the system of equations becomes

$$4x - 2y + 2z = 0 \quad -\ (i)$$
$$-2x + y - z = 0 \quad -\ (ii)$$
$$2x - y + z = 0 \quad -\ (iii)$$

The above sets of equations are all same as we have only one independent equation $2x - y + z = 0$ and hence we can choose two variables arbitrarily.

Let $z = k_1$ and $y = k_2$, $x = \dfrac{(k_2 - k_1)}{2}$

$$X_1 = \left(\frac{(k_2 - k_1)}{2}, k_2, k_1 \right)$$

is the eigen vector corresponding to $\lambda = 2$

Case (ii): Let $\lambda = 8$ and the system of equations becomes

$$-2x - 2y + 2z = 0 \quad -\ (iv)$$
$$-2x - 5y - z = 0 \quad -\ (v)$$
$$2x - y + -5z = 0 \quad -\ (vi)$$

Applying the rule of cross multiplication for (iv) and (v)

$$\frac{x}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\frac{x}{12} = \frac{-y}{6} = \frac{z}{6} \quad or \frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

The Eigen vector X_2 for $\lambda = 8 is$ $\begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$

4. Find all the Eigen values and the corresponding Eigen vectors of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution:

The characteristic equation of A is $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

On expanding, we have $\lambda^3 - 7\lambda^2 + 36 = 0$

After solving, we get $\lambda = -2,\ 6,\ 3$ are the Eigen values.

Now the system of equations is

$$(1-\lambda)x + y + 3z = 0$$
$$x + (5-\lambda)y + z = 0$$
$$3x + y + (1-\lambda)z = 0$$

Case (i): Let $\lambda = -2$ and the system of equations becomes

$$3x + y + 3z = 0 - \ (i)$$
$$x + 7y + z = 0 - \ (ii)$$
$$3x + y + 3z = 0 - (iii)$$

Applying the rule of cross multiplication for (i) and (ii)

$$\frac{x}{-20} = \frac{-y}{0} = \frac{z}{20} \qquad or \qquad \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

The Eigen vector X_1 for $\lambda = -2$ is $[-1 \quad 0 \quad 1]$

Case (ii): Let $\lambda = 3$ and the system of equations becomes

$$-2x + y + 3z = 0 - \ (iv)$$
$$x + 2y + z = 0 \ - \ (v)$$
$$3x + y - 2z = 0 \ - (vi)$$

Applying the rule of cross multiplication for (iv) and (v)

$$\frac{x}{-5} = \frac{-y}{-5} = \frac{z}{-5} \qquad or \qquad \frac{x}{-1} = \frac{y}{1} = \frac{z}{-1}$$

The Eigen vector X_2 for $\lambda = 3$ is $[-1 \quad 1 \quad -1]$

Case (iii): Let $\lambda = 6$ and the system of equations becomes

$$-5x + y + 3z = 0 - \ (vii)$$
$$x - y + z = 0 \ - \ (viii)$$
$$3x + y - 5z = 0 \ - (ix)$$

Applying the rule of cross multiplication for (vii) and (viii)

$$\frac{x}{4} = \frac{-y}{-8} = \frac{z}{4} \qquad or \qquad \frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

The Eigen vector X_3 for $\lambda = 6$ is $[1 \quad 2 \quad 1]$

5. Find all the Eigen values and the corresponding Eigen vectors of the matrix

$$\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

Solution:

The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{vmatrix}=0$$

On expanding, we have $\lambda^3-18\lambda^2+99\lambda-162=0$

After solving, we get $\lambda=3,\ 6,\ 9$ are the Eigen values.

Now the system of equations is

$$(7-\lambda)x-2y+0z=0$$
$$-2x+(6-\lambda)y-2z=0$$
$$0x-2y+(5-\lambda)z=0$$

Case (i): Let $\lambda=3$ and the system of equations becomes

$$4x-2y+0z=0\quad-\ (i)$$
$$-2x+3y-2z=0\ -\ (ii)$$
$$0x-2y+2z=0\quad-\ (iii)$$

Applying the rule of cross multiplication for (i) and (ii)

$$\frac{x}{\begin{vmatrix}-2 & 0\\ 3 & -2\end{vmatrix}}=\frac{-y}{\begin{vmatrix}4 & 0\\ -2 & -2\end{vmatrix}}=\frac{z}{\begin{vmatrix}4 & -2\\ -2 & -3\end{vmatrix}}$$

$$\frac{x}{4}=\frac{-y}{-8}=\frac{z}{-16}\quad or\ \frac{x}{1}=\frac{y}{2}=\frac{z}{-4}$$

The Eigen vector X_1 for $\lambda=3$ $is[1\quad 2\quad -4]$

Case (ii): Let $\lambda=6$ and the system of equations becomes

$$x-2y+0z=0-\ (iv)$$
$$-2x-0y-2z=0\ -\ (v)$$
$$0x-2y-z=0\quad-\ (vi)$$

Applying the rule of cross multiplication for (iv) and (v)

$$\frac{x}{4}=\frac{-y}{-2}=\frac{z}{-4}\quad or\quad \frac{x}{2}=\frac{y}{1}=\frac{z}{-2}$$

The Eigen vector X_2 for $\lambda=6$ is $[2\quad 1\quad -2]$

Case (iii): Let $\lambda=9$ and the system of equations becomes

$$-2x-2y+0z=0-\ (vii)$$
$$-2x-3y-2z=0\quad-\ (viii)$$
$$0x-2y-4z=0\quad-\ (ix)$$

Applying the rule of cross multiplication for (vii) and (viii)

$$\frac{x}{\begin{vmatrix}-2 & 0\\ -3 & -2\end{vmatrix}}=\frac{-y}{\begin{vmatrix}-2 & 0\\ -2 & -2\end{vmatrix}}=\frac{z}{\begin{vmatrix}-2 & -2\\ -2 & -3\end{vmatrix}}$$

$$\frac{x}{4}=\frac{-y}{4}=\frac{z}{2}\quad or\ \frac{x}{2}=\frac{y}{-2}=\frac{z}{1}$$

The Eigen vector X_3 for $\lambda=9$ is $[2\quad -2\quad 1]$

Similarity of Matrices and Diagonalization of Matrices.

Two square matrices A and B of the same order are said to be similar if there exists a non-singular matrix P, such that $B = P^{-1}AP$. Here B is said to be similar to A.

Diagonalization of a square matrix

Property: If A is a square matrix of order n linearly independent eigen vectors then there exists an n^{th} order square matrix P such that $P^{-1}AP$ is a diagonal matrix. Here "P" is called as the *modal matrix*.

We shall establish this result by considering a square matrix A of order 3, to make an important and interesting observation.

- ❖ We find eigen values $\lambda_1, \lambda_2, \lambda_3$ and the corresponding eigen vectors,

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$ for the square matrix of order 3.

- ❖ We form the **Modal matrix P,**

  Let the square matrix P be equal to $[X_1 \; X_2 \; X_3]$, i.e.,

$$P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Now

or $$AP = A[X_1 \;\; X_2 \;\; X_3] = [AX_1 \;\; AX_2 \;\; AX_3] = [\lambda_1 X_1 \;\; \lambda_2 X_2 \;\; \lambda_3 X_3]$$

$$AP = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

i.e., $AP = PD$ where D is the diagonal matrix represented by

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Consider $AP = PD$

Pre multiplying by P^{-1} we have

$$P^{-1}AP = P^{-1}PD = (P^{-1}P)D = ID = D$$
$$P^{-1}AP = D$$

It is important that $P^{-1}AP$ is a diagonal matrix having the eigen values of $A(\lambda_1 \quad \lambda_2 \quad \lambda_3)$ in its principal diagonal. We say that the matrix P diagonalizes A, where P is constituted by the eigen vectors of A.

Computation of powers of square matrix

Diagonalization of a square matrix A also helps us to find the powers of $A : A^2 \quad A^3 \quad A^4$,........ etc.

We have $$D = P^{-1}AP$$

$$\therefore D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A[AP] = P^{-1}A^2P$$

$$D^2 = P^{-1}A^2P$$

Pre multiplying by P and post multiplying by P^{-1} we have

$$PD^2P^{-1} = (PP^{-1}) = A^2(PP^{-1}) = IA^2I = A^2$$

i.e., $A^2 = PD^2P^{-1}$.

Thus in general, $A^n = PD^nP^{-1}$

where $$D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

Working procedure for diagonalization of a square matrix A of order 3

- We find Eigen values $\lambda_1 \quad \lambda_2 \quad \lambda_3$.
- We find the Eigen vectors $X_1 \quad X_2 \quad X_3$ corresponding to the eigen values $\lambda_1 \quad \lambda_2 \quad \lambda_3$.
- We form the modal matrix $P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$
- We compute $P^{-1} = \frac{1}{|P|}(AdjP)$.

The diagonalization of A is given by $D = P^{-1}AP$ where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

Problems

1. Reduce the matrix $A=\begin{bmatrix}-1 & 3\\ -2 & 4\end{bmatrix}$ **to the diagonal form and hence find** A^4**.**

Solution:

The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix}(-1-\lambda) & 3\\ -2 & 4-\lambda\end{vmatrix}=0$$

$$(-1-\lambda)(4-\lambda)+6=0$$

$$\lambda^2-3\lambda+2=0$$

$\therefore \lambda=1,\ 2$ are the eigen values of A.

Now consider $[A-\lambda I][X]=[0]$

$$\begin{bmatrix}(-1-\lambda) & 3\\ -2 & (4-\lambda)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$(-1-\lambda)x+3y=0$$

$$-2x+(4-\lambda)y=0$$

Case-(i): Let $\lambda=1$,

We get $-2x+3y=0$ or $2x=3y$ or $\frac{x}{3}=\frac{y}{2}$.

$\therefore X_1=(3\quad 2)'$ is the eigen vector corresponding to $\lambda=1$.

Case-(ii): Let $\lambda=2$,

We get $-3x+3y=0$ or $x=y$ or $\frac{x}{1}=\frac{y}{1}$

$\therefore X_2=(1\quad 1)'$ is the eigen vector corresponding to $\lambda=2$.

Modal matrix: $P=[X_1\quad X_2]=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}$

We have $|P|=1$ and $P^{-1}=\frac{1}{|P|}(AdjP)$

$$P^{-1}=\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}$$

$$Now\ P^{-1}AP=\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}\begin{bmatrix}-1 & 3\\ -2 & 4\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}=\begin{bmatrix}1 & -1\\ -4 & 3\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$$

Thus $P^{-1}AP=D$

$=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ is the diagonal matrix.

$$P^{-1}AP=Diag(1\quad 2)$$

Now to find out A^4, consider the following

we have $A^n=PD^nP^{-1}$

$A^4=PD^4P^{-1}$ where $D^4=\begin{bmatrix}1^4 & 0\\ 0 & 2^4\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 16\end{bmatrix}$

$$A^4=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 16\end{bmatrix}\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}=\begin{bmatrix}-29 & 45\\ -30 & 46\end{bmatrix}$$

Thus $A^4=\begin{bmatrix}-29 & 45\\ -30 & 46\end{bmatrix}$.

2 **Reduce the matrix** $A=\begin{bmatrix}11 & -4 & -7\\ 7 & -2 & -5\\ 10 & -4 & -6\end{bmatrix}$ **into a diagonal matrix. Also find** A^5**.**

Solution: The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{bmatrix}11-\lambda & -4 & -7\\ 7 & (-2-\lambda) & -5\\ 10 & -4 & (-6-\lambda)\end{bmatrix}=0$$

$$\Rightarrow (11-\lambda)\,[(-2-\lambda)(-6-\lambda)-20]+4[7(-6-\lambda)+50]-7[-28-10(-2-\lambda)]=0$$

$$\Rightarrow (11-\lambda)[\lambda^2+8\lambda-8]+4[8-7\lambda]-7[10\lambda-8]=0$$

$$\Rightarrow 11\lambda^2+88\lambda-88-\lambda^3-8\lambda^2+8\lambda+32-28\lambda-70\lambda+56=0$$

$$\Rightarrow \lambda^3-3\lambda^2-2\lambda=0$$

$$\lambda=0,\quad 1,\quad 2.$$

Now consider $[A-\lambda I]\ [X]=[0]$.

$$(11-\lambda)x-4y-7z=0$$
$$7x+(-2-\lambda)y-5z=0$$
$$10x-4y+(-6-\lambda)z=0$$

Case (i): Let $\lambda=0$ and the corresponding equations are

$$11x-4y-7z=0$$
$$7x-2y-5z=0$$
$$10x-4y-6z=0$$

$$\frac{x}{6}=\frac{-y}{-6}=\frac{z}{6}\quad\text{or}\quad\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$$

$X_1=(1\quad 1\quad 1)'$ is the eigen vector corresponding to $\lambda=0$.

Case (ii): Let $\lambda=1$ and the corresponding equations are

$$10x-4y-7z=0$$
$$7x-3y-5z=0$$
$$10x-4y-7z=0$$

$$\frac{x}{-1}=\frac{-y}{-1}=\frac{z}{-2}\quad\text{or}\quad\frac{x}{1}=\frac{y}{-1}=\frac{z}{2}$$

$X_2=(1\quad -1\quad 2)'$ is the eigen vector corresponding to $\lambda=1$.

Case (iii): Let $\lambda=2$ and the corresponding equations are

$$9x-4y-7z=0$$
$$7x-4y-5z=0$$
$$10x-4y-8z=0$$

$$\frac{x}{-8}=\frac{-y}{4}=\frac{z}{-8}\quad\text{or}\quad\frac{x}{2}=\frac{y}{1}=\frac{z}{2}$$

$X_3=(2\quad 1\quad 2)'$ is the Eigen vector corresponding to $\lambda=2$.

Hence the modal matrix $P = \lfloor X_1 \quad X_2 \quad X_3 \rfloor = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

We have $|P| = 1(-2-2) - 1(2-1) + 2(2+1) = 1$

$$AdjP = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}(AdjP) = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

<u>Diagonalization of A is given by $P^{-1}AP$:</u>

Now $P^{-1}AP = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

$P^{-1}AP = D = Diag(0 \quad 1 \quad 2)$.

Now to find out A^5, consider the following

we have $A^n = PD^nP^{-1}$.

$A^5 = PD^5P^{-1}$ and $D^5 = Diag(0^5 \quad 1^5 \quad 2^5) = Diag(0 \quad 1 \quad 32)$.

Hence $A^5 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & 64 \\ 0 & -1 & 32 \\ 0 & 2 & 64 \end{bmatrix} \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 191 & -64 & -127 \\ 97 & -32 & -65 \\ 190 & -64 & -126 \end{bmatrix}.$$

3. Reduce the given matrix in diagonal form $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$

Solution: The Characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix}(-2-\lambda) & 2 & -3\\ 2 & (1-\lambda) & -6\\ -1 & -2 & (-\lambda)\end{vmatrix}=0$$

$$\Rightarrow (-2-\lambda)[-\lambda(1-\lambda)-12]-2[-2\lambda-6]-3[-4+1-\lambda]=0$$

$$\Rightarrow (-2-\lambda)[-\lambda+\lambda^2-12)+(4\lambda+12)+(9+3\lambda)=0$$

$$\Rightarrow (-2-\lambda)(\lambda+3)(\lambda-4)+4(\lambda+3)+3(\lambda+3)=0$$

$$\Rightarrow (\lambda+3)\left[(-2-\lambda)(\lambda-4)+4+3\right]=0$$

$$\Rightarrow (\lambda+3)(-\lambda^2+2\lambda+15)=0$$

$$\Rightarrow (\lambda+3)(\lambda+3)\left(\lambda-5\right)=0$$

$$\lambda=-3 \quad -3 \quad 5.$$

We now form the system of equations.

$$(-2-\lambda)+2y-3z=0$$

$$2x+(1-\lambda)y-6z=0$$

$$-1x-2y-\lambda z=0.$$

Case (i): Let $\lambda=-3$ and the corresponding equations are

$$x+2y-3z=0$$

$$2x+4y-6z=0$$

$$-x-2y+3z=0.$$

It should be observed that the equations are all same and we have only one independent equation $x+2y-3z=0$

(In case the rule of cross multiplication is applied, we get $x=y=z=0$ which is a trivial solution.)

Two variables can be arbitrary.

Let $z=k_1,\ y=k_2 \quad \therefore\ x=3k_1-2k_2$

The eigen vector corresponding to the coincident eigen value $\lambda=-3$ be denoted by $X_{1,2}$ and we have $X_{1,2}=(3k_1-2K_2 \quad k_2 \quad k_1)'$

where $k_1,\ k_2$ are arbitray. We choose convenient values for k_1 and k_2 to obtain two distinct eigen vectors.

(i) Let $k_1=1,\ k_2=1 \ \therefore\ X_1=(1 \quad 1 \quad 1)'$

(ii) Let $k_1=1,\ k_2=0 \ \therefore\ X_2=(3 \quad 0 \quad 1)'$

Case (ii): Let $\lambda = 5$ and the corresponding equations are

$$-7x + 2y - 3z = 0 \quad \ldots\ldots (1)$$
$$2x - 4y - 6z = 0 \quad \ldots\ldots. (2)$$
$$-x - 2y - 5z = 0$$

Solving (1) and (2) , $\dfrac{x}{-12-12} = \dfrac{-y}{42+6} = \dfrac{z}{28-4}$

$$\frac{x}{-24} = \frac{-y}{48} = \frac{z}{24} \quad \text{or} \quad \frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$X_3 = (1 \quad 2 \quad -1)'$ is the eigen vector corresponding to $\lambda = 5$.

<u>We have modal matrix</u>

$$P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|P| = 1(-2) - 3(-3) + 1(1) = 8.$$

$$AdjP = \begin{bmatrix} +(0-2) & -(-3-1) & +(6-0) \\ -(-1-2) & +(-1-1) & -(2-1) \\ +(1-0) & -(1-3) & +(0-3) \end{bmatrix} = \begin{bmatrix} -2 & 4 & 6 \\ 3 & -2 & -1 \\ 1 & 2 & -3 \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$,

$$= \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 3 & -2 & -1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 6 & -12 & -18 \\ -9 & 6 & 3 \\ 5 & 10 & -15 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -24 & -0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$$

Thus $P^{-1}AP = D = Diag(-3 \quad -3 \quad 5)$.

4. Show that the following matrix is not diagnosable $A=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$

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Solution: The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{bmatrix}2-\lambda&1&0\\0&2-\lambda&1\\0&0&2-\lambda\end{bmatrix}=0$$

$$(2-\lambda)^3=0$$

$$\lambda=2,\quad 2,\quad 2..$$

The eigen vector corresponding to $\lambda=2$ has to be obtained by solving the system of equations.

$$(2-2)x+1y+0Z=0$$

$$0x+(2-2)y+1z=0$$

$$0x+0y+(2-2)z=0$$

$y=0,\ \ z=0;\ x$ can be arbitrary.

$\therefore\ x=k,\ \ y=0,\ \ z=0$ is the eigen vector corresponding to the

coincident eigen value $\lambda=2$.

It is evident that we cannot obtain three linearly independent eigen vectors.

Thus we conclude that the matrix A is not diagnozable.

Eigenvalues and Eigenvectors in Python

Propositional Logic

Let A be a square matrix. In Linear Algebra, a scalar λ is called an eigenvalue of matrix A if there exists a column vector v such that Av=λv and v is non-zero. Any vector satisfying the above relation is known as eigenvector of the matrix A corresponding to the eigen value λ.

Logical Expressions in Sympy

NumPy is a Python library used for working with arrays. It also has functions for working in domain of linear algebra, Fourier transform, and matrices. NumPy was created in 2005 by Travis Oliphant. It is an open source project and you can use it freely. NumPy stands for Numerical Python.

NumPy has the numpy.linalg.eig() function to deduce the eigenvalues and normalized eigenvectors of a given square matrix. Note the two variables w and v assigned to the output of numpy.linalg.eig(). The first variable w is assigned an array of computed eigenvalues and the second variable v is assigned the matrix whose columns are the normalized eigenvectors corresponding to the eigenvalues in that order.

Example-1: (2×2 matrix)

Code:

```
import numpy as np
a = np.array([[3, 1], [2, 2]])
w, v = np.linalg.eig(a)
print(w)
print(v)
```

Output:

```
 [4. 1.]
[[ 0.70710678 -0.4472136 ]
 [ 0.70710678  0.89442719]]
```

Example-2: **(3×3 matrix)**

Code:

```
import numpy as np
from numpy.linalg import eig
a = np.array([[2, 2, 4],[1, 3, 5],[2, 3, 4]])
w,v=eig(a)
print('Eigen value:', w)
print('Eigen vector:', v)
```

Output:

```
Eigen value: [ 8.80916362  0.92620912 -0.73537273]
Eigen vector: [[-0.52799324 -0.77557092 -0.36272811]
 [-0.604391    0.62277013 -0.7103262 ]
 [-0.59660259 -0.10318482  0.60321224]]
```

<u>Example-3:</u> Find all the Eigen values and the corresponding Eigen vectors of the matrix by using python

$$\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$$

Solution:

Code:

```
import numpy as np
from numpy.linalg import eig
a = np.array([[-1,3],[-2,4]])
w,v=eig(a)
print('Eigen value:', w)
print('Eigen vector:', v)
```

Output:

```
Eigen value: [1. 2.]
Eigen vector: [[-0.83205029 -0.70710678]
 [-0.5547002  -0.70710678]]
```

Example-4: **Find all the Eigen values and the corresponding Eigen vectors of the matrix by using python**

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution:

Code:

```
import numpy as np
from numpy.linalg import eig
a = np.array([[6,-2,2],[-2,3,-1],[2,-1,3]])
w,v=eig(a)
print('Eigen value:', w)
print('Eigen vector:', v)
```

Output:

```
Eigen value: [8. 2. 2.]
Eigen vector: [[ 0.81649658 -0.57735027 -0.11547005]
 [-0.40824829 -0.57735027  0.57735027]
 [ 0.40824829  0.57735027  0.80829038]]
```

Exercise question

1. Find all the eigenvalues and the corresponding eigenvectors of the matrix by using python

$$\begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}.$$

2. Find all the eigenvalues and the corresponding eigenvectors of the matrix by using python

$$\begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

3. Find all the eigenvalues and the corresponding eigenvectors of the matrix by using python

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

Thank You