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MODULE 5:

Ordinary Differential Equations

CONTENTS

- Solution of system of linear differential equations by diagonalization method and discuss the stability of the system
- Higher order linear differential equations with constant Coefficients
- Differential Equations using Python

Solve the system of linear differential equations by diagonalization method and stability of the system of linear differential equations

Solution to the system of homogenous differential equation by diagonalization method

Let us consider the linear first order homogenous differential equation of the form.

$$X' = AX$$

$$\begin{pmatrix} x_1' \\ x_2' \\ .. \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{pmatrix} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(1)}$$

In which each x_i' is expressed as a linear combination of $x_1, x_2, x_3, \dots\dots\dots, x_n$

Let $X = PY$ …….(2), be the solution for the equation (1)

Put equation (2) in (1), we have

$$PY' = APY \Rightarrow Y' = P^{-1}APY \Rightarrow Y' = DY$$

where D is a diagonal matrix.

$$\begin{pmatrix} y_1' \\ y_2' \\ .. \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ .. \\ y_n \end{pmatrix}$$

Therefore each differential equation in the system is of the form

$$y_i' = \lambda_i y_i,\ i = 1,2,..........,n\ .$$

The solution of each of these linear equations is $y_i = c_i e^{\lambda_i t},\ i = 1,2,..........,n$.

Hence the general solution is

$$Y = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \\ c_n e^{\lambda_n t} \end{pmatrix}$$

Since the matrix P can be constructed from the eigen vectors of A. The general solution of the original system $X' = AX$ is obtained from $X = PY$.

Solution to the system of non-homogenous differential equation by diagonalization method

Let us consider non-homogenous differential equation $X^I = AX + F(t)$

$$\text{i.e., } \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} F_1(t) \\ F_{12}(t) \\ \vdots \\ F_n(t) \end{bmatrix} \text{...........(1)}$$

Let $X = PY.$ ….(2), be the solution for the non-homogenous equation

Substitute equation (2) in (1), we have

$$(PY)' = APY + F$$
$$PY' = APY + F$$
$$Y' = (P^{-1}AP)Y + P^{-1}F$$
$$Y' = DY + P^{-1}F \text{....................(3)}$$

by solving equation (3), we get $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Hence the solution for the non - homogenous differential equation $X^I = AX + F$ is $X = PY$,

Where P = modal matrix

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Eigenvalue and Stability of the system

The table below gives a complete overview of the stability corresponding to each type of eigenvalue.

Eigen values	Stability
All real & positive	Unstable
All real & negative	Stable
Mixed positive & negative real	Unstable
a+bi	Unstable
-a+bi	Stable
0+bi	Unstable

Problems:

1. **Solve** $X' = \begin{pmatrix} -2 & -1 & 8 \\ 0 & -3 & 8 \\ 0 & -4 & 9 \end{pmatrix} X$ **by diagonalization.**

Solution: Let $X=PY$, be the solution for above equation, so we need to find P and Y

$$|A-\lambda I| = 0 \Rightarrow -(\lambda+2)(\lambda-1)(\lambda-5) = 0 \Rightarrow \lambda = -2,1,5.$$

Now construct the homogenous differential equation, i.e.,

$$\begin{aligned} &(-2-\lambda)x - y + 8z = 0 \\ &0x + (-3-\lambda)y + 8z = 0 \\ &0x - 4y + (9-\lambda)z = 0 \end{aligned}$$

Since eigen values are distinct then eigen vectors are linearly independent.

$$\text{Solving } |A-\lambda_i I| = 0, \quad \text{for } i = 1,2,3$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus a modal matrix P, that diagonalizes the coefficient matrix is $P = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

The entries on the main diagonal of D are the eigen values of A.

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

and $$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_3 t} \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^{t} \\ c_3 e^{5t} \end{bmatrix}$$

Hence the solution of the given system is $X' = AX$ is,

$$X = PY = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{t} \\ c_3 e^{5t} \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} + 2c_2 e^{t} + c_3 e^{5t} \\ 2c_2 e^{t} + c_3 e^{5t} \\ c_2 e^{t} + c_3 e^{5t} \end{pmatrix}$$

This can also be written in usual manner as

$$X = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{5t}$$

Stability: Since at least one eigen value is real & positive, hence the system is unstable.

2. **Solve** $X' = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} X + \begin{pmatrix} 3e^t \\ e^t \end{pmatrix}$ **by diagonalization and hence discuss the stability of the system.**

Solution: Let $X = PY$ be the solution of above non-homogenous differential equation.

Here

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \qquad F(t) = \begin{bmatrix} 3e^t \\ e^t \end{bmatrix}$$

Now, $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda_1 = 0$ and $\lambda_2 = 5$

- The eigen vector corresponding to eigen value $\lambda_1 = 0$ is $X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
- The eigen vector corresponding to eigen value $\lambda_2 = 5$ is $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The modal matrix

$$P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \text{ and } P^{-1} = \frac{1}{|P|} adj(P) = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\text{and } P^{-1}F = \begin{bmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{bmatrix}\begin{bmatrix} 3e^t \\ e^t \end{bmatrix} = \begin{bmatrix} 1/5\, e^t \\ 7/5\, e^t \end{bmatrix}$$

$$Y' = DY + P^{-1}F$$

Now to compute Y: consider $Y' = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} Y + \begin{bmatrix} \frac{1}{5} e^t \\ \frac{7}{5} e^t \end{bmatrix}$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{5} e^t \\ \frac{7}{5} e^t \end{bmatrix}$$

i.e., $y_1' = \frac{1}{5} e^t$ & $y_2' = 5y_2 + \frac{7}{5} e^t$

Integrating with respect to t,

$$y_1 = \frac{e^t}{5} + c_1$$

and

$$y_2' - 5y_2 = \frac{7}{5} e^t$$

This is of the form

$\frac{dy}{dx} + P(x)y = Q(x)$ is called leibnitz linear equation and $ye^{\int pdx} = \int Qe^{\int Pdx}$

Which is the required solution.

Here, $P(t) = -5$ and $Q(t) = \frac{7}{5} e^t$

$$e^{\int pdx} = e^{\int -5dx} = e^{-5t}$$

$$y_2 e^{-5t} = \int \frac{7}{5} e^t e^{-5t} dt + c$$

$$y_2 e^{-5t} = \frac{7}{5} \int e^{-4t} dt + c$$

$$y_2 = \frac{7}{5} \left(\frac{e^{-4t}}{-4} \right) \frac{1}{e^{-5t}} + c \frac{1}{e^{-5t}}$$

$$y_2 = -\frac{7}{20} e^t + ce^{5t}$$

Hence the solution of the original system is

$$X = PY = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5}e^{t} + c_1 \\ -\frac{7}{20}e^{t} + c_2 e^{5t} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}e^{t} + c_1 + 2c_2 e^{5t} \\ -\frac{3}{4}e^{t} - 2c_1 + c_2 e^{5t} \end{pmatrix}$$

Writing in usual manner

$$X = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} - \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix} e^{t}$$

Stability: Since all the eigen values are real, distinct & positive, hence the system is unstable.

Higher order linear differential equations with constant Coefficients

Introduction

In this unit, we're going to learn how to solve second and higher order linear ordinary differential equations with constant coefficients. Differential equations of higher order arise very often in physical problems.

A differential equation in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together is called a **linear differential equation**.

Thus, the general linear differential equations with constant co-efficient of the n^{th} order is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots\dots + a_n y = r(x) \qquad \dots\dots\dots\dots\dots\dots (1)$$

Where, $a_1, a_2, a_3, \dots\dots a_n$ are constants and $r(x)$ are functions of x only.

If $r(x) = 0$, then the equation (1) is called a **homogeneous equation**. Otherwise, it is a **non-homogeneous equation**.

Operator D:

Let us denote $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$,…………, $\frac{d^n}{dx^n} = D^n$

$$\therefore \frac{dy}{dx} = Dy, \ \frac{d^2 y}{dx^2} = D^2 y, \ \frac{d^3 y}{dx^3} = D^3 y, \dots\dots\dots\dots$$

The above equation (1) can be written as, $(D^n + a_1 D^{n-1} + \dots\dots + a_n) y = r(x)$

i.e. $f(D) y = r(x)$ where, $f(D) = D^n + a_1 D^{n-1} + \dots\dots + a_n$, is a polynomial in D.

Here the symbol D is called the **differential operator** and it stands for the operation of differentiation.

General solution of a homogeneous differential equation with constant co-efficient

Let us consider a second order differential equation to explain the method of solving a homogeneous linear differential equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0 \quad \text{, where } a_1, a_2 \text{ are constants} \qquad \ldots\ldots\ldots\ldots (2)$$

i.e. $(D^2 + a_1 D + a_2)y = 0$

Or $f(D)y = 0$ where, $f(D) = D^2 + a_1 D + a_2$ ……………(3)

Theorem

If y_1 and y_2 are linearly independent solutions (one cannot be expressed in terms of the other) of (3) then $c_1y_1 + c_2y_2$ is also a solution of (3), where c_1 and c_2 are arbitrary constants.

Proof : Since y_1 and y_2 are solutions of (3) we have

$$f(D)y_1 = 0 \quad \text{and} \quad f(D)y_2 = 0 \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

$$\therefore \quad f(D)(c_1y_1 + c_2y_2) = c_1 f(D)y_1 + c_2 f(D)y_2 \qquad \text{(Using 4)}$$
$$= c_1.0 + c_2.0 = 0$$

Thus $f(D)(c_1y_1 + c_2y_2) = 0$

This shows that $c_1y_1 + c_2y_2$ is a solution of (3)

Therefore, $y = c_1y_1 + c_2y_2 = y_c$

which contains two arbitrary constants c_1 and c_2 , is called the **General Solution** of (3).

y_c is called as the **Complimentary Function** (C.F)

Note:

For nth order homogeneous differential equation,

If $y_1, y_2, y_3, \ldots\ldots, y_n$ are the solutions of this differential equation,

general solution of this equation is

$$y(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x) + c_4y_4(x) + \ldots + c_ny_n(x)$$

Methods of finding Complimentary function

Consider a second order homogeneous differential equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0 \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

Using operator D, the above equation can be written as

$$(D^2 + a_1 D + a_2)y = 0 \quad \Rightarrow f(D)y = 0 \qquad \ldots\ldots\ldots\ldots (6)$$

Let $y = e^{mx}$

$\therefore\ Dy = me^{mx}$ and $D^2 y = m^2 e^{mx}$

Using this in (5), we get $(m^2 + a_1 m + a_2)e^{mx} = 0$

i.e. $m^2 + a_1 m + a_2 = 0 \qquad \ldots\ldots\ldots\ldots (7)$

This is called **Auxiliary Equation (A.E) or characteristic equation.**

Equation (7) is a quadratic which has two roots that may be

(i) real and distinct
(ii) real and repeated
(iii) complex

Case (i) Suppose the roots, m_1 and m_2 are real and distinct then the two independent solution of (5) are $y = e^{m_1 x}$ and $y = e^{m_2 x}$

By theorem, $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ is the general solution of (5).

In general, the solution of the n^{th} order differential equation is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots\dots\dots\dots\dots + c_n e^{m_n x}$$

Where, $c_1, c_2, c_3, \dots\dots\dots\dots, c_n$ are arbitrary constants

Case (ii) Suppose the roots are real and repeated i.e. $m_1 = m_2$ then the differential equation is $(D - m_1)^2 y = 0$

Let $(D - m_1)y = p$ $\therefore$ The D.E becomes $\dfrac{dp}{dx} - m_1 p = 0$, This is a linear D.E in p.

The Integrating factor is $e^{\int -m_1\, dx} = e^{-m_1 x}$

The solution of linear D.E is $p\, e^{-m_1 x} = \int e^{-m_1 x}(0)dx + c_2 \Rightarrow p = c_2\, e^{m_1 x}$

But $p = (D - m_1)y \Rightarrow \dfrac{dy}{dx} - m_1 y = c_2\, e^{m_1 x}$

Integrating factor for the above equation is $e^{\int -m_1\, dx} = e^{-m_1 x}$ and the solution is

$$y\, e^{-m_1 x} = \int c_2\, e^{m_1 x} e^{-m_1 x}\, dx + c_1 \Rightarrow y\, e^{-m_1 x} = c_2 x + c_1$$

Therefore, $y = (c_1 + c_2 x)e^{m_1 x}$ is the general solution when the roots are repeated.

If three roots are repeated then the general solution is $y = (c_1 + c_2 x + c_3 x^2)e^{m_1 x}$

If four roots are repeated then the complete solution is $y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^{m_1 x}$

Case(iii) $m = \alpha \pm i\beta$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

i.e. $y = e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$

This can be written as

$$y = e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right] = e^{\alpha x}\left[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x\right]$$

$\therefore\ y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$, where $A = c_1 + c_2$ and $B = i(c_1 - c_2)$ are arbitra

If the complex root $\alpha \pm i\beta$ is repeated n times then the general

$$y = e^{\alpha x}\left[(a_1 + a_2 x + a_3 x^2 + \ldots\ldots\ldots\ldots + a_{n-1}x^{n-1})\cos\beta x + (b_1 + b_2 x + b_3 x^2 + \ldots\ldots + b_{n-1}x^{n-1})\sin\beta x\right]$$

	Roots of the auxiliary equation	Corresponding complementary function
1	One real root α_1	$C_1 e^{\alpha_1 x}$
2.	Two real and differential root α_1 and α_2	$C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x}$
3.	Two real and equal roots α_1 and α_2	$(C_1 + C_2 x)\ e^{\alpha_1 x}$
4.	Three real and equal roots $\alpha_1, \alpha_2, \alpha_3$	$(C_1 + C_2 x + C_3 x^2)\ e^{\alpha_1 x}$
5.	One pair of imaginary roots $\alpha \pm i\beta$	$(C_1 \cos\beta x + C_2 \sin\beta x)\ e^{\alpha x}$
6.	Two Pair of equal imaginary roots $\alpha \pm i\beta$ and $\alpha \pm i\beta$	$[\ (C_1 + C_2 x) \cos\beta\ + (C_1 + C_2 x)\sin\beta]\ e^{\alpha x}$

Problems:

1. **Find the General solution for the differential equation** $4y'' + 4y' - 3y = 0$.

Solution: Auxiliary equation is $4m^2 + 4m - 3 = 0$

Roots are $m_1 = \frac{1}{2}, m_2 = -\frac{3}{2}$

$\therefore$ The complementary function is $y = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{3}{2}x}$

2. **Solve:** $2y'' - gy' = 0$.

Solution: Auxiliary equation is $2m^2 - gm = 0$

$m_1 =$ m (2m-g) =0

Roots are, $m_1 = 0, m_2 = \frac{g}{2}$

The general solution is $y = c_1 e^{0x} + c_2 e^{\frac{g}{2}x}$.

3. **Solve:** $y'' + 2ky' + k^2 y = 0$

Solution: $(m+k)^2 = 0$

Roots are, $m = -k, -k$

Therefore the roots are repeated.

Hence the general solution is y= $(c_1 + c_2 x)e^{-kx}$

4. Solve the initial value problem $\left(D^3 - D^2 - 4D + 4\right)y = 0$, $y(0) = 1, y'(0) = 2$ *and* $y''(0) = 1$

Solution: Auxiliary equation is $m^3 - m^2 - 4m + 4 = 0$

$$\Rightarrow m^2(m-1) - 4(m-1) = 0$$

$$\Rightarrow (m-1)\left(m^2 - 4\right) = 0$$

$$\Rightarrow m = 1, m = \pm 2$$

General solution is $y = c_1 e^x + c_2 e^{2x} + c_3{}^{-2x}$ ...... (1)

Differentiating (1) w.r.t x

$y' = c_1 e^x + 2c_2 e^{2x} - 2c_3 e^{-2x}$ ...... (2)

Differentiating (2) w.r.t x

$y'' = c_1 e^x + 4c_2 e^{2x} + 4c_3 e^{-2x}$ .... (3)

Using the given conditions in (1), (2) and (3), we obtain

$$\left.\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_1 + 4c_2 + 4c_3 &= 1 \\ c_1 + 2c_2 - 2c_3 &= 2 \end{aligned}\right\} \quad \text{.......... (4)}$$

Solving above system of equations, we obtain $c_1 = 1, c_2 = \frac{1}{4}$ *and* $c_3 = -\frac{1}{4}$

Hence the general solution is $y = e^x + \frac{e^{2x}}{4} - \frac{e^{-2x}}{4}$

5. Solve: $(D^2 - 2D + 4)y = 0$ **[Practice]**

x

6. Solve $y''' - 5y'' + 7y' - 3y = 0$

Solution: The auxiliary equation is $m^3 - 5m^2 + 7m - 3 = 0$

$m = 1$ is a root by inspection

$$
\begin{array}{c|cccc}
1 & 1 & -5 & 7 & -3 \\
 & 0 & 1 & -4 & 3 \\
\hline
 & 1 & -4 & 3 & \underline{0}
\end{array}
$$

Now we have $(m-1)(m^2 - 4m + 3) = 0$

i.e. $(m-1) = 0, (m^2 - 4m + 3) = 0 \Rightarrow m = 1,\ m = 1, 3$

The roots are $m = 1, 1, 3$

Therefore the general solution is $y = (c_1 + c_2 x)e^x + c_3 e^{3x}$

7. Solve $\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}-y=0$

Solution: The auxiliary equation is $m^3-3m^2+3m-1=0$

$m=1$ is a root by inspection

By Synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -3 & 3 & -1 \\ & 0 & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

Now we have $(m-1)(m^2-2m+1)=0$

i.e. $(m-1)=0,\ (m^2-2m+1)=0 \Rightarrow m=1,\ (m-1)^2=0$

The roots are $m=1,1,1$

Roots are real and repeated. Therefore the general solution is

$$y=\left(c_1+c_2x+c_3x^2\right)e^x$$

8. Solve $\left(D^4-2D^3+2D^2-2D+1\right)y=0$

Solution: The auxiliary equation is $m^4-2m^3+2m^2-2m+1=0$

$m=1$ is a root by inspection

By Synthetic division

$$\begin{array}{c|ccccc} 1 & 1 & -2 & 2 & -2 & 1 \\ & 0 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 0 \end{array}$$

Thus we obtain $m^3-m^2+m-1=0$

$m=1$ is a root by inspection

Again by Synthetic division,

$$\begin{array}{c|cccc} 1 & 1 & -1 & 1 & -1 \\ & 0 & 1 & 0 & 1 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

Now we have, $m^2+1=0 \Rightarrow m=\pm i$

The roots are $m=1,1,\pm i$

Thus the General solution is $y=(c_1+c_2x)e^x+c_3\cos x+c_4\sin x$

A differential equation of the form $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots\dots + a_n y = r(x)$ is called as the Non-Homogeneous Linear Differential Equation of nth order with constant coefficients, where $a_1, a_2, a_3, \dots\dots a_n$ are real constants. Let us denote $\frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2, \frac{d^3}{dx^3} = D^3$ etc, then above equation becomes $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots\dots\dots\dots + a_n) y = r(x)$

This is in the form of $f(D) y = r(x)$, where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots\dots\dots\dots + a_n$

The **General Solution** of the above equation is $y = C.F + P.I$, where, C.F is Complimentary function and P.I is particular integral.

i.e $\quad y = y_c + y_p$

Particular Integral

The evaluation of $\frac{1}{f(D)} r(x)$ is called as Particular Integral and it is denoted by y_p or φ_p

i.e. $y_p = \frac{1}{f(D)} r(x)$

$\frac{1}{f(D)} r(x)$ satisfies the equation $f(D) y = r(x)$, i.e. $f(D) \left\{ \frac{1}{f(D)} r(x) \right\} = r(x)$

Therefore, it is the particular integral.

Hence, $f(D)$ and $\frac{1}{f(D)}$ are **inverse operators.**

Let us prove the following two results:

(a) $\frac{1}{D}r(x) = \int r(x)dx$

Proof: Let $\frac{1}{D}r(x) = y$ ……… (1)

Operating (1) by D, $D\frac{1}{D}r(x) = Dy \Rightarrow r(x) = \frac{dy}{dx}$

i.e. $\int r(x)dx = \int \frac{dy}{dx}dx \Rightarrow y = \int r(x)\, dx$

[Here no constant is added as the equation (1) does not contain any constant]

Using (1), $\frac{1}{D}r(x) = \int r(x)\, dx$

(b) $\frac{1}{D-a}r(x) = e^{ax}\int r(x)e^{-ax}dx$

Proof: Let $\frac{1}{D-a}r(x) = y$ ……… (2)

Operating by $D-a$, $(D-a)\frac{1}{D-a}r(x) = (D-a)y$.

$r(x) = \frac{dy}{dx} - ay \Rightarrow \frac{dy}{dx} - ay = r(x)$ is a linear equation.

$\therefore$ I.F= $e^{\int -a\, dx} = e^{-ax}$ and the solution of linear equation is

$y\,(I.F) = \int r(x)\,(I.F)dx \Rightarrow y\,e^{-ax} = \int r(x)\,e^{-ax}\,dx$

$\therefore \quad y = e^{ax}\int r(x)e^{-ax}dx$

Using (2), $\frac{1}{D-a}r(x) = e^{ax}\int r(x)e^{-ax}dx$

Methods to find Particular Integral

Method 1: Method to find P.I of f (D) y=r(x) where r(x) =e^{ax}, where a is a constant.

We know that $y = \dfrac{1}{f(D)} r(x) = \dfrac{1}{f(D)} e^{ax}$

$\therefore y = \dfrac{1}{f(a)} e^{ax}$ if $f(a) \neq 0$ [Directly substitute a in place of D]

If $f(a) = 0$, then the above rule fails and we proceed further as

i.e. $\dfrac{1}{f(D)} e^{ax} = x \dfrac{1}{f'(a)} e^{ax}$

If $f'(a)$=0, we get , $\dfrac{1}{f(D)} e^{ax} = x^2 \dfrac{1}{f''(a)} e^{ax}$, provided $f'(a) \neq 0$ and so on.

1. **Find the** y_p **of** $(D^2 + 5D + 6)y = e^{3x}$

Solution: y_p= $\dfrac{1}{D^2 + 5D + 6} e^{3x}$

Replace D by a i.e. 3,

$$y_p = \frac{1}{3^2 + 5(3) + 6} = \frac{e^x}{30} \Rightarrow \phi_p = \frac{e^x}{30}$$

2. Find the y_p of $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Solution: $y_p = \frac{1}{(D+2)(D-1)^2}\{e^{-2x} + 2\sinh x\} = \frac{1}{(D+2)(D-1)^2}\left[e^{-2x} + 2\left(\frac{e^{x}-e^{-x}}{2}\right)\right]$

$$y_p = \frac{1}{(D+2)(D-1)^2}e^{-2x} + \frac{1}{(D+2)(D-1)^2}e^{x} - \frac{1}{(D+2)(D-1)^2}e^{-x}$$

Let us evaluate three terms separately

Consider $\frac{1}{(D+2)(D-1)^2}e^{-2x} = \frac{1}{(D+2)}\left[\frac{1}{(D-1)^2}e^{-2x}\right]$

Replace D by -2, we have

$$= \frac{1}{(D+2)}\frac{1}{(-2-1)^2}e^{-2x} = \frac{1}{9}\frac{1}{(D+2)}e^{-2x}$$

By using the result (2) i.e. $\frac{1}{D-a}r(x) = e^{ax}\int r(x)e^{-ax}dx$,we get

$$= \frac{1}{9}e^{-2x}\int e^{-2x}e^{2x}dx = \frac{1}{9}e^{-2x}x$$

Now consider the second term ,

$$\frac{1}{(D+2)(D-1)^2}e^{x} = \frac{1}{(D-1)^2}\left[\frac{1}{D+2}e^{x}\right] = \frac{1}{3}\frac{1}{(D-1)^2}e^{x} \quad \text{[Replacing D by 1]}$$

$$= \frac{1}{3}x\frac{1}{2(D-1)}e^{x} = \frac{x^2}{6}e^{x}$$

And consider the last term $\frac{1}{(D+2)(D-1)^2}e^{-x} = \frac{1}{(-1+2)}\frac{1}{(-1-1)^2}e^{-x} = \frac{1}{4}e^{-x}$

Hence $y_p = x\frac{e^{-2x}}{9} + \frac{x^2}{6}e^{x} + \frac{1}{4}e^{-x}$

Method 2: Method to find P.I of f (D) y = r(x) where r(x) = sin (ax+b) (or) cos (ax+b), a is constant.

ere $y_p = \frac{1}{f(D^2)} \sin(ax+b)$

In this case replace D^2 by $-a^2$

$$y_p = \frac{1}{f(-a^2)} \sin(ax+b), \text{ provided } f(-a^2) \neq 0$$

- If $f(-a^2) = 0$, the above rules fails and we proceed further as

$$y_p = \frac{1}{f(D^2)} \sin(ax+b) = x\frac{1}{f'(-a^2)} \sin(ax+b) \text{ , provided } f'(-a^2) \neq 0 \text{ and so on.}$$

- If $f'(-a^2) = 0, \frac{1}{f(D^2)} \sin(ax+b) = x^2 \frac{1}{f''(-a^2)} \sin(ax+b)$ provided $f''(-a^2) \neq 0$

Note: Same rule applies even when r(x) =cos (ax+b) or cos(ax)

1. Find y_p of $(D_2 - 2D + 2)y = \cos(x-1)$

Solution: The particular integral is $y_p = \frac{1}{(D^2 - 2D + 2)} \cos(x-1)$

Replace D^2 by -1^2, we get

$$y_p = \frac{1}{(-1-2D+2)} \cos(x-1) = \frac{1}{(1-2D)} \cos(x-1)$$

$$= \frac{(1+2D)}{(1-2D)(1+2D)} \cos(x-1) = \frac{(1+2D)}{(1-4D^2)} \cos(x-1)$$

Now replace D^2 by -1^2, we get

$$= \frac{(1+2D)}{(1-4(-1))} \cos(x-1) = \frac{1}{5}(1+2D)\cos(x-1)$$

$$\therefore y_p = \frac{1}{5}[\cos(x-1) - 2\sin(x-1)]$$

Method 3: Method to find P.I of f (D) y = r(x) where r(x) $= x^m$, $m \in Z^+$.

Here $y_p = \dfrac{1}{f(D)} x^m = [f(D)]^{-1} x^m$.

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since $(m+1)^{th}$ and the higher derivatives of x^m are zero, we need not consider terms by D^m.

Important Formulae:

1. $(1-D)^{-1} = 1 + D + D^2 + \ldots\ldots\ldots\ldots$
2. $(1+D)^{-1} = 1 - D + D^2 - \ldots\ldots\ldots\ldots$
3. $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 \ldots\ldots\ldots\ldots$
4. $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 \ldots\ldots\ldots\ldots$
5. $(1-D)^{-3} = 1 + 3D + 6D^2 + \ldots\ldots\ldots\ldots$
6. $(1+D)^{-3} = 1 - 3D + 6D^2 - \ldots\ldots\ldots\ldots$

1. **Find** y_p **of** $(D^2 + D)y = (x^2 + 2x + 25)$

Solution: $y_p = \dfrac{1}{D(D+1)}(x^2 + 2x + 25) = \dfrac{1}{D}(1+D)^{-1}(x^2 + 2x + 25)$

$$= \frac{1}{D}(1 - D + D^2 - \ldots\ldots\ldots)(x^2 + 2x + 25)$$

$$= \frac{1}{D}\{x^2 + 2x + 25 - (2x + 2) + 2\}$$

$$= \int (x^2 + 25)dx = \frac{x^3}{3} + 25x$$

$$\therefore y_p = \frac{x^3}{3} + 25x + C$$

Method 4: Method to find P.I of f (D) y=r(x), where r(x) =e^{ax} V, V being the function of x.

Here $y_p = \dfrac{1}{f(D)} e^{ax} V$

In such cases, first take e^{ax} term outside the operator, by substituting $(D+a)$ in place of D.

$$\Rightarrow y_p = e^{ax} \frac{1}{f(D+a)} V$$

Depending upon the nature of V we will solve further.

1. **Find** y_p **of** $(D^2 - 2D + 4)y = e^x \cos x.$

Solution: $y_p = \dfrac{1}{(D^2 - 2D + 4)} e^x \cos x.$

Here add 1 to D ,

$$y_p = e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^2 + 3} \cos x$$

Now replace D^2 by -1^2

$$= e^x \frac{1}{-1^2 + 3} \cos x = e^x \frac{1}{2} \cos x$$

Problems

1. Solve $(D^2+4)y = e^{4x}$

Solution: The auxiliary equation is $m^2+4=0$

The roots are $m=\pm 2i$

The complementary function is $y_c = c_1\cos 2x + c_2\sin 2x$

The particular integral is $y_p = \dfrac{1}{(D^2+4)}e^{4x}$

Replacing D by 4, we get

$$y_p = \frac{1}{(4^2+4)}e^{4x} = \frac{1}{20}e^{4x}$$

$\therefore$ The general solution is $y = y_c + y_p$

i.e. $y = c_1\cos 2x + c_2\sin 2x + \dfrac{1}{20}e^{4x}$

2. Solve $\dfrac{d^4x}{dt^4} + 4x = \sinh t$

Solution: We have $(D^4+4)x = \sinh t$

Auxiliary equation is $m^4+4=0$

i.e. $(m^2+2)^2 - 4m^2 = 0$

Or $[(m^2+2)-2m][(m^2+2)+2m]=0$

i.e. $m^2-2m+2=0$; $m^2+2m+2=0$

$$\therefore m = \frac{-2\pm 2i}{2} = -1\pm i \ ; \ m = \frac{2\pm 2i}{2} = 1\pm i$$

The complementary function is $x_c = e^t(c_1\cos t + c_2\sin t) + e^{-t}(c_3\cos t + c_4\sin t)$

The particular integral is

$$x_p = \frac{1}{D^4+4}\sinh t = \frac{1}{2}\left[\frac{e^t - e^{-t}}{D^4+4}\right] = \frac{1}{2}\left[\frac{e^t}{D^4+4} - \frac{e^{-t}}{D^4+4}\right]$$

$$= \frac{1}{2}\left[\frac{e^t}{1+4} - \frac{e^{-t}}{1+4}\right] = \frac{\sinh t}{5}$$

Thus the general solution is $y = y_c + y_p$

$$\therefore \quad y = e^t(c_1\cos t + c_2\sin t) + e^{-t}(c_3\cos t + c_4\sin t) + \sinh t/5$$

3. Solve $(D^4 - 18D^2 + 81)y = 36e^{3x} + 8^x$

Solution: Auxiliary equation is $m^4 - 18m^2 + 81 = 0$

i.e. $(m^2 - 9)^2 = 0$

$(m-3)^2 (m+3)^2 = 0$

The roots are $m = 3, 3, -3, -3$

$$\therefore y_c = (c_1 + c_2 x)e^{3x} + (c_3 + c_4 x)e^{-3x}$$

$$y_p = \frac{36}{D^4 - 18D^2 + 81} e^{3x} + \frac{1}{D^4 - 18D^2 + 81} 8^x$$

$$= 36\frac{1}{D^4 - 18D^2 + 81} e^{3x} + \frac{1}{(D^4 - 18D^2 + 81)} e^{(\log 8)x}$$

Replacing D by 3, we get

$$= 36\frac{1}{3^4 - 18(3^2) + 81} e^{3x} + \frac{1}{(\log 8)^4 - 18(\log 8)^2 + 81)} e^{(\log 8)x}$$

Denominator is zero in the first term, it follows that

$$\therefore y_p = 36x\frac{1}{4D^3 - 36D} e^{3x} + \frac{8^x}{(\log 8)^4 - 18(\log 8)^2 + 81}$$

Replacing D by 3 in the first term, we get

$$= 36x\frac{1}{4(3)^3 - 36(3)} e^{3x} + \frac{8^x}{(\log 8)^4 - 18(\log 8)^2 + 81)}$$

Again the denominator is zero in the first term, hence it follows that

$$y_p = 36x^2\frac{1}{12D^2 - 36} e^{3x} + \frac{8^x}{(\log 8)^4 - 18(\log 8)^2 + 81)}$$

Thus $y_p = 36x^2 \dfrac{e^{3x}}{72} + \dfrac{8^x}{(\log 8)^4 - 18(\log 8)^2 + 81)}$

The general solution is

$$y = (c_1 + c_2 x)e^{3x} + (c_3 + c_4 x)e^{-3x} + x^2\frac{e^{3x}}{2} + \frac{8^x}{(\log 8)^4 - 18(\log 8)^2 + 81}$$

4. Solve $(D^2+4)y = 4\sin 2x + \cos 5x$

Solution: Auxiliary equation is $m^2+4=0$

The roots are $m = \pm 2i$

The complementary function is $y_c = c_1\cos 2x + c_2 \sin 2x$

$$y_p = \frac{1}{D^2+4}(4\sin 2x + \cos 5x)$$

$$= 4\frac{1}{D^2+4}\sin 2x + \frac{1}{D^2+4}\cos 5x = P_1 + P_2$$

Consider $P_1 = 4\dfrac{1}{D^2+4}\sin 2x$

Replacing D^2 by -2^2, we get

$P_1 = 4\dfrac{1}{-(2)^2+4}\sin 2x$ [Denominator is zero]

It follows that $P_1 = 4x\dfrac{1}{2D}\sin 2x = 2x\int \sin 2x\, dx = -x\cos 2x$

Now consider $P_2 = \dfrac{1}{D^2+4}\cos 5x$

Replacing D^2 by -5^2, we get

$$P_2 = \frac{1}{-(5)^2+4}\cos 5x = \frac{-1}{21}\cos 5x$$

Thus $y_p = -x\cos 2x - \dfrac{1}{21}\cos 5x$

Hence, the general solution is $y = c_1\cos 2x + c_2\sin 2x - x\cos 2x - \dfrac{1}{21}\cos 5x$

5. Solve $\dfrac{d^4y}{dx^4}+8\dfrac{d^2y}{dx^2}+16y=4\sin^2 x$

Solution: Auxiliary equation is $m^4+8m^2+16=0$

Or $(m^2+4)^2=0 \Rightarrow m=\pm 2i, \pm 2i$

$\therefore\ y_c=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x$

$$y_p=\frac{1}{D^4+8D^2+16}4\sin^2 x=\frac{1}{D^4+8D^2+16}2(1-\cos 2x)$$

$$y_p=2\frac{1}{D^4+8D^2+16}-\frac{1}{D^4+8D^2+16}2\cos 2x=P_1-P_2$$

Consider $P_1=2\dfrac{1}{D^4+8D^2+16}e^{0x}=\dfrac{2}{0+0+16}e^{0x}=\dfrac{1}{8}$

Now consider $P_2=2\dfrac{1}{D^4+8D^2+16}\cos 2x$

Now replacing D^2 by $-(2)^2$ i.e. -4

$P_2=2\dfrac{1}{(-4)^2+8(-4)+16}\cos 2x=\dfrac{2}{32-32}\cos 2x$ (Denominator is zero)

$$\therefore P_2=2x\frac{1}{4D^3+16D}\cos 2x=\frac{2x}{4}\frac{1}{(D^2D+4D)}\cos 2x$$

Replacing D^2 by -4, we get

$=\dfrac{x}{2}\dfrac{1}{(-4D+4D)}\cos 2x$ (Again the denominator is zero)

$$\therefore P_2=\frac{x^2}{2}\frac{1}{(3D^2+4)}\cos 2x$$

Replacing D^2 by -4, we get $P_2=\dfrac{x^2}{2}\dfrac{1}{(-12+4)}\cos 2x=\dfrac{-x^2\cos 2x}{16}$

Substitute P_1 and P_2 in y_p,

Thus $y_p=\dfrac{1}{8}-\dfrac{(-x^2\cos 2x)}{16}=\dfrac{1}{8}+\dfrac{x^2\cos 2x}{16}$

$\therefore$ The general solution is $y=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x+\dfrac{1}{8}+\dfrac{x^2\cos 2x}{16}$

6. Solve $(D^2+5D-6)y=\cos 4x\cos x$

Solution: A. E is $m^2+5m-6=0$

Roots are $m=1,-6$

The Complementary function is $y_c=c_1e^x+c_2e^{-6x}$

The Particular Integral is $y_p=\dfrac{1}{D^2+5D-6}\cos 4x\cos x$

$$=\frac{1}{(D^2+5D-6)}\frac{1}{2}(\cos 5x+\cos 3x)$$

$$=\frac{1}{2}\left\{\frac{1}{(D^2+5D-6)}\cos 5x+\frac{1}{(D^2+5D-6)}\cos 3x\right\}$$

$$y_p=\frac{1}{2}(P_1+P_2)$$

Consider $P_1=\dfrac{1}{D^2+5D-6}\cos 5x$

Replacing D^2 by -5^2, we get

$$P_1=\frac{1}{-25+5D-6}\cos 5x=\frac{1}{5D-31}\cos 5x=\frac{1}{5D-31}\cos 5x$$

$$=\frac{(5D+31)}{(5D-31)(5D+31)}\cos 5x=\frac{(5D+31)}{(25D^2-961)}\cos 5x$$

Now replacing D^2 by -5^2, we get

$$P_1=\frac{(5D+31)}{(-625-961)}\cos 5x=-\frac{1}{1586}(5D+31)\cos 5x$$

$$=-\frac{1}{1586}(-25\sin 5x+31\cos 5x)=\frac{1}{1586}(25\sin 5x-31\cos 5x)$$

Now consider $P_2=\dfrac{1}{D^2+5D-6}\cos 3x$

Replacing D^2 by -3^2, we get

$$P_2=\frac{1}{(-9+5D-6)}\cos 3x=\frac{1}{5D-15}\cos 3x=\frac{1}{5(D-3)}\cos 3x$$

$$=\frac{(D+3)}{5(D-3)(D+3)}\cos 3x=\frac{1}{5}\frac{(D+3)}{(D^2-9)}\cos 3x$$

Replacing D^2 by -3^2, we get

$$P_2=\frac{1}{5}\frac{(D+3)}{(-9-9)}\cos 3x=\frac{-1}{90}(D+3)\cos 3x=\frac{3}{90}\sin 3x-\frac{3}{90}\cos 3x$$

$$\therefore P_2=\frac{1}{30}\sin 3x-\frac{1}{30}\cos 3x$$

Substitute P_1 and P_2 in y_p,

Thus $y_p=\dfrac{1}{2}\left[\dfrac{1}{1586}(25\sin 5x-31\cos 5x)+\dfrac{1}{30}(\sin 3x-\cos 3x)\right]$

Hence the general solution is $y=y_c+y_p$

i.e. $y=c_1e^x+c_2e^{-6x}+\dfrac{1}{2}\left[\dfrac{1}{1586}(25\sin 5x-31\cos 5x)+\dfrac{1}{30}(\sin 3x-\cos 3x)\right]$

7. Solve $(D^2 - D)y = x^2 - 2x - 32$

Solution: A. E is $m^2 - m = 0$

i.e. $m(m-1) = 0$

The roots are $m = 0, 1$

The complimentary function is $y_c = c_1 e^{0x} + c_2 e^{x}$

Particular Integral is $y_p = \dfrac{1}{D(D-1)}(x^2 - 2x - 32)$

$= \dfrac{1}{D}(D-1)^{-1}(x^2 - 2x - 32) = \dfrac{1}{D}\left(1 + D + D^2\right)\left(x^2 - 2x - 32\right)$

$= \dfrac{1}{D}\left[x^2 - 2x - 32 + (2x-2) + (2)\right] = \int \left(x^2 - 32\right)dx = \dfrac{x^3}{3} + 32x$

$\therefore$ The general solution is $y = c_1 + c_2 e^{-x} - \dfrac{x^3}{3} + 32x$

8. Solve $y' + 3y' + 2y = \sin x + e^x + 2x^2$

Solution: A. E is $m^2 + 3m + 2 = 0$

The roots are $m = -1, -2$

The complimentary function is $y_c = c_1 e^{-x} + c_2 e^{-2x}$

Particular Integral is $y_p = \frac{1}{(D^2+3D+2)}\left(\sin x + e^x + 2x^2\right)$

$$= \frac{1}{(D^2+3D+2)}\sin x + \frac{1}{(D^2+3D+2)}e^x + \frac{1}{(D^2+3D+2)}2x^2$$

$y_p = P_1 + P_2 + P_3$

Consider $P_3 = \frac{1}{(D^2+3D+2)}2x^2 = 2\frac{1}{2\left[1+\left(\frac{D^2+3D}{2}\right)\right]}x^2$

$$= \left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1}x^2 = \left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2\right]x^2$$

$$= x^2 - \left(\frac{2}{2}+\frac{6x}{2}\right) + \left(\frac{D^4+9D^2+6D^3}{4}\right)x^2$$

$$\therefore P_3 = x^2 - (1+3x) + \frac{18}{4} = x^2 - 3x + \frac{7}{2}$$

Consider $P_1 = \frac{1}{(D^2+3D+2)}\sin x$

Here replacing D^2 by -1^2,we get

$$P_1 = \frac{1}{(-1+3D+2)}\sin x = \frac{1}{3D+1}\sin x = \frac{(3D-1)}{(9D^2-1)}\sin x$$

Now replacing D^2 by -1^2,we get

$$P_1 = -\frac{1}{10}(3D-1)\sin x = -\frac{3\cos x}{10} + \frac{\sin x}{10}$$

Now consider $P_2 = \frac{1}{(D^2+3D+2)}e^x$

Replacing D by1,we get $P_2 = \frac{1}{(1+3+2)}e^x = \frac{e^x}{6}$

Thus $y_p = P_1 + P_2 + P_3 = \frac{\sin x}{10} - \frac{3\cos x}{10} + \frac{e^x}{6} + x^2 - 3x + \frac{7}{2}$

Hence the general solution is $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{\sin x}{10} - \frac{3\cos x}{10} + \frac{e^x}{6} + x^2 - 3x + \frac{7}{2}$

9. Solve $(D^2+1)^2 y = x^4 + 2\sin x\cos 3x$

Solution: A. E is $(m^2+1)^2 = 0 \Rightarrow (m^2+1)(m^2+1) = 0$

The roots are $m = \pm i, \pm i$

The complementary function is $y_c = e^{0x}\left[(c_1+c_2x)\cos x + (c_3+c_4x)\sin x\right]$

Particular Integral is $y_p = \dfrac{1}{(D^2+1)^2}\left(x^4 + 2\sin x\cos 3x\right)$

$$y_p = \frac{1}{(D^2+1)^2}x^4 + \frac{2}{(D^2+1)^2}\frac{1}{2}(\sin 4x - \sin 2x)$$

$$= (1+D^2)^{-2}x^4 + \frac{1}{(D^2+1)^2}\sin 4x - \frac{1}{(D^2+1)^2}\sin 2x$$

$$= \left[1 - 2D^2 + 3D^4 - 4D^6 + \ldots\ldots\ldots\right]x^4 + \frac{1}{(-4^2+1)^2}\sin 4x - \frac{1}{(-2^2+1)^2}\sin 2x$$

$$y_p = \left[x^4 - 2(12x^2) + 3(24)\right] + \frac{1}{225}\sin 4x - \frac{1}{9}\sin 2x$$

$$y = \left[(c_1+c_2x)\cos x + (c_3+c_4x)\sin x\right] + x^4 - 24x^2 + 72 + \frac{\sin 4x}{225} - \frac{\sin 2x}{9}$$

10. Solve $(D^2+4D+5)y = e^{-2x}\cos x$

Solution: A. E is $m^2+4m+5=0$

i.e. $m = \dfrac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

The roots are $m = -2 \pm i$

The complementary function is $y_c = e^{-2x}[c_1 \cos x + c_2 \sin x]$

Particular Integral is $y_p = \dfrac{1}{(D^2+4D+5)} e^{-2x} \cos x$

Add -2 to D , we get

$$y_p = e^{-2x} \frac{1}{((D-2)^2+4(D-2)+5)} \cos x = e^{-2x} \frac{1}{(D^2+1)} \cos x$$

Replacing D^2 by -1^2, denominator is zero

$$\therefore\ y_p = e^{-2x} x \frac{1}{2D} \cos x = \frac{x}{2} e^{-2x} \int \cos x \, dx$$

$$y_p = \frac{x}{2} e^{-2x} \sin x$$

Hence, the general solution is

$$y = e^{-2x}[c_1 \cos x + c_2 \sin x] + \frac{x}{2} e^{-2x} \sin x$$

11. Solve $(D^3 - 7D - 6)y = (1+x)e^{2x}$

Solution: A. E is $m^3 - 7m - 6 = 0$

$m = -1$ is a root by inspection.

By Synthetic Division,

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & 0 & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

Thus we obtain $(m+1)(m^2 - m - 6) = 0 \Rightarrow (m+1)(m-3)(m+2) = 0$

The roots are $m = -1, 3, -2$

The Complementary function is $y_c = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{-2x}$

The Particular Integral is $y_p = \dfrac{1}{(D^3 - 7D - 6)}\left(e^{2x} + xe^{2x}\right)$

$$y_p = \frac{1}{(D^3 - 7D - 6)}e^{2x} + \frac{1}{(D^3 - 7D - 6)}xe^{2x} = P_1 + P_2$$

Consider $P_1 = \dfrac{1}{(D^3 - 7D - 6)}e^{2x}$

Replace D by a i.e. 2,

$$\therefore P_1 = -\frac{e^{2x}}{12}$$

Now let us consider $P_2 = \dfrac{1}{(D^3 - 7D - 6)}xe^{2x}$

Add 2 to D, we get $P_2 = e^{2x}\dfrac{1}{(D+2)^3 - 7(D+2) - 6}x$

$$= e^{2x}\frac{1}{(D^3 + 6D^2 + 5D - 12)}x = e^{2x}\frac{1}{(-12)}\frac{1}{\left[1 - \left(\dfrac{D^3 + 6D^2 + 5D}{12}\right)\right]}x$$

$$= -\frac{e^{2x}}{12}\left[1 - \left(\frac{D^3 + 6D^2 + 5D}{12}\right)\right]^{-1}x$$

$$= -\frac{e^{2x}}{12}\left[1 + \left(\frac{D^3 + 6D^2 + 5D}{12}\right)\right]x = -\frac{e^{2x}}{12}\left[x + \frac{5}{12}\right]$$

$$\therefore y_p = -\frac{e^{2x}}{12} - \frac{e^{2x}}{12}\left(x + \frac{5}{12}\right) = -\frac{e^{2x}}{12}\left(x + \frac{7}{12}\right)$$

Hence the general solution is $y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{-2x} - \dfrac{e^{2x}}{12}\left(x + \dfrac{7}{12}\right)$

Differential Equations in Python

Introduction

Ordinary Differential Equation (ODE) can be used to describe a dynamic system. More formally, if we define a set of variables, like the temperatures in a day, or the amount of molecule X in a certain time point, and it changes with the independent variable (in a dynamic system, usually it will be time t).

ODE offers us a way to mathematically depict the dynamic changes of defined variables. More formally, if we define a set of variables, like the temperatures in a day, or the amount of molecule X in a certain time point, and it changes with the independent variable (in a dynamic system, usually it will be time t). ODE offers us a way to mathematically depict the dynamic changes of defined variables.

Differential equations in Sympy

SymPy is a symbolic mathematics Python package. Its goal is to develop into a completely featured computer algebra system while keeping the code as basic as possible to make it understandable and extendable. The package is entirely written in python language. Anaconda distribution and Google colab both contains pre-installed SymPy.

Syntax for solving ODE

Mathematical expression	Type	SymPy function	Syntax
x	independent variable	`Symbol`	`x = Symbol('x')`
y	variable depending on x	`Function`	`y = Function('y')(x)`
$\frac{dy}{dx}$	Derivative	`diff`	`dydx = y.diff(x)`
$\frac{dy}{dx} = x$	Differential equation	`Eq`	`deq = Eq(dydx, x)`
General Solution of ODE		`dsovle`	`dsolve(deq)`

Solving second order ODE

Consider a second order ODE

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

It is convenient to use function Derivative of SymPy to write higher order derivatives.

Derivative(y,x) gives $\frac{dy}{dx}$

Derivative(y,x,x) gives $\frac{d^2y}{dx^2}$

Code:

```
from sympy import Symbol, Function, Derivative, Eq, dsolve
x=Symbol('x')
y = Function('y')(x)
deq = Eq(Derivative(y,x,x) + Derivative(y,x) - 2*y, 0)
solution = dsolve(deq)
Solution
```

Output:

$y(x) = C_1 e^{-2x} + C_2 e^{x}$

Solving second order ODE with IVP

Consider a second order IVP

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 6y = 0,\ y(0) = 1, y'(0) = -1$$

For second order IVP, there are two initial conditions.

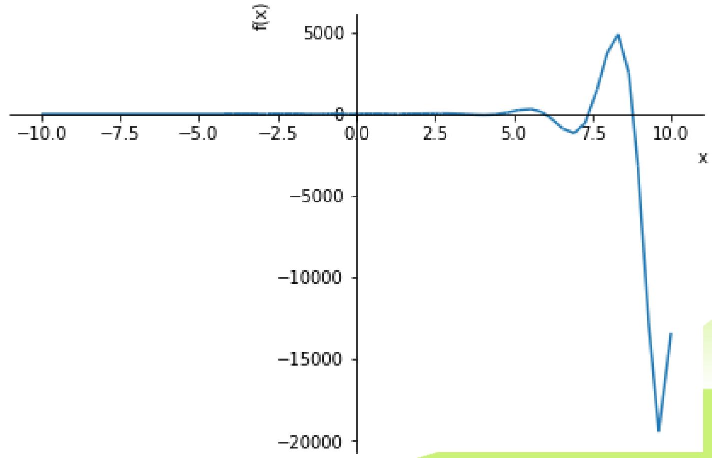
We use these conditions in the general solution, which will given two equations in two unknowns.

Two arbitrary constants can be obtained by solving these two equations.

Code:

```
from sympy import Symbol, Function, Derivative, Eq, dsolve, solve
from sympy.plotting import plot
x=Symbol('x')
y = Function('y')(x)
deq = Eq(Derivative(y,x,x) -2* Derivative(y,x) +6*y, 0)
s = dsolve(deq)
eqn1 = s.rhs.subs(x,0) -1
eqn2 = s.rhs.diff(x).subs(x,0) + 1
constants = solve([eqn1, eqn2])
solution = s.subs(constants)
plot(solution.rhs)
```

Output:



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Example-1

Solve the following differential equation by using Python

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$$

Code:

```
from sympy import Symbol, Function, Derivative, Eq, dsolve, solve
from sympy.plotting import plot
x=Symbol('x')
y = Function('y')(x)
deq = Eq(Derivative(y,x,x,x)-3*Derivative(y,x,x) +3* Derivative(y,x) -y, 0)
s = dsolve(deq)
S
```

Output:

$y(x) = (C_1 + x(C_2 + C_3x))e^x$

Example-2

Solve the following differential equation by using Python

$$(D^4 - 2D^3 + 2D^2 - 2D + 1)y = 0$$

Code:

```
from sympy import Symbol, Function, Derivative, Eq, dsolve, solve
from sympy.plotting import plot
x=Symbol('x')
y = Function('y')(x)
deq = Eq(Derivative(y,x,x,x,x)-2*Derivative(y,x,x,x)+2*Derivative(y,x,x)-
2*Derivative(y,x)+y, 0)
s = dsolve(deq)
s
```

Output:

$$y(x) = C_3 \sin(x) + C_4 \cos(x) + (C_1 + C_2 x)e^x$$

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Example-3

Solve the following IVP by using Python

$$(D^3 - D^2 - 4D + 4)y = 0,\ y(0) = 1, y'(0) = 2 \text{ and } y''(0) = 1$$

Code:

```
from sympy import Symbol, Function, Derivative, Eq, dsolve, solve
from sympy.plotting import plot
x=Symbol('x')
y = Function('y')(x)
deq = Eq(Derivative(y,x,x,x)-Derivative(y,x,x) -4* Derivative(y,x) +4*y, 0)
s = dsolve(deq)
eqn1 = s.rhs.subs(x,0) -1
eqn2 = s.rhs.diff(x).subs(x,0) - 2
eqn3 = s.rhs.diff(x,x).subs(x,0) - 1
constants = solve([eqn1, eqn2, eqn3])
solution = s.subs(constants)
Solution
```

Output:

$$y(x) = \frac{e^{2x}}{4} + e^x - \frac{e^{-2x}}{4}$$

Example-4

Solve the following non-homogenous differential equation by using Python

$$(D^2 - 2D + 4)y = e^x \cos(x),$$

Code:

```
from sympy import Symbol, Function, Derivative, Eq, dsolve, solve, exp, cos
from sympy.plotting import plot
x=Symbol('x')
y = Function('y')(x)
deq = Eq(Derivative(y,x,x) -2* Derivative(y,x)+4*y, exp(x)*cos(x))
s = dsolve(deq)
s
```

Output:

$$y(x) = \left(C_1 \sin(\sqrt{3}x) + C_2 \cos(\sqrt{3}x) + \frac{\cos(x)}{2}\right) e^x$$

Exercise question

1. Solve the following differential equation by using Python

(i) $(D^3 - 5D^2 + 7D - 3)y = 0$

(ii) $(D^3 - 3D^2 + 3D - 1)y = 0$

(iii) $(D^2 + D)y = x^2 + 2x + 25$

(iv) $(D^2 + 4)y = e^{4x}$

(v) $(D^4 - 18D^2 + 18)y = 36e^{3x} + 8^x$

2. Solve the following IVP by using Python

(i) $(D^2 - D)y = x^2 - 2x - 32, y(0) = 1, y'(0) = -1$

(ii) $(D^2 + 5D + 6)y = e^x, y(0) = 1, y'(o) = 2$

Thank You