$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\frac{2\pi n x}{T} + \sum_{n=1}^{\infty} b_n \cdot \sin\frac{2\pi n x}{T}$$

$f(x)$'s period is $T$

① $$\int_0^T \sin\frac{2\pi x}{T}n \sin\frac{2\pi x}{T}m \, dx \quad (m \neq n)$$

$$= \int_0^T \frac{\cos\frac{2\pi x}{T}(m-n) - \cos\frac{2\pi x}{T}(m+n)}{2} dx$$

$$= \frac{1}{2} \frac{\sin\frac{2\pi x}{T}(m-n)}{\frac{2\pi}{T}(m-n)} \Bigg|_0^T - \frac{1}{2} \frac{\sin\frac{2\pi x}{T}(m+n)}{\frac{2\pi}{T}(m+n)} \Bigg|_0^T$$

$$= 0$$

② $$\int_0^T \sin\frac{2\pi x}{T} \cdot n \cos\frac{2\pi x}{T}m \, dx \quad (m \neq n)$$

$$= \int_0^T \frac{\sin\frac{2\pi x}{T}(m+n) + \sin\frac{2\pi x}{T}(n-m)}{2} dx$$

$$= -\frac{1}{2} \cdot \frac{\cos\frac{2\pi x}{T}(m+n)}{\frac{2\pi}{T}(m+n)} \Bigg|_0^T - \frac{1}{2} \cdot \frac{\cos\frac{2\pi x}{T}(n-m)}{\frac{2\pi}{T}(n-m)} \Bigg|_0^T$$

$$= 0$$

③ $$\int_0^T \cos\frac{2\pi x}{T}n \cos\frac{2\pi x}{T}m \, dx \quad (m \neq n)$$

$$= \int_0^T \frac{\cos\frac{2\pi x}{T}(m+n) + \cos\frac{2\pi x}{T}(m-n)}{2} dx$$

$$= \frac{1}{2} \cdot \frac{\sin\frac{2\pi x}{T}(m+n)}{\frac{2\pi}{T}(m+n)} \Bigg|_0^T + \frac{1}{2} \frac{\sin\frac{2\pi x}{T}(m-n)}{\frac{2\pi}{T}(m-n)} \Bigg|_0^T = 0$$

if $m=n$

① $\int_0^T \cos\frac{2\pi t}{T}n \cos\frac{2\pi t}{T}n\, dt$

$\overset{x=\frac{2\pi t}{T}}{=\!=} \frac{T}{2\pi}\int_0^{2\pi} \cos^2 nx\, dx$

$= \frac{T}{2\pi}\int_0^{2\pi} \frac{\cos 2nx+1}{2}\, dx$

$= \frac{T}{2\pi}\left[\frac{1}{2}\frac{\sin(2nx)}{2n} + \frac{1}{2}x\right]\Big|_0^{2\pi}$

$= \frac{T}{2}$

② $\int_0^T \sin\frac{2\pi t}{T}n \cos\frac{2\pi t}{T}n\, dt$

$\overset{x=\frac{2\pi t}{T}}{=\!=} \frac{T}{2\pi}\int_0^{2\pi} \sin nx \cos nx\, dx$

$= \frac{T}{4\pi}\int_0^{2\pi} \sin 2nx\, dx$

$= -\frac{T}{4\pi}\left[\frac{\cos 2nx}{2n}\right]\Big|_0^{2\pi}$

$= 0$

③ $\int_0^T \sin\frac{2\pi t}{T}n \sin\frac{2\pi t}{T}n\, dt$

$\overset{x=\frac{2\pi t}{T}}{=\!=} \frac{T}{2\pi}\int_0^{2\pi} \sin^2 nx\, dx$

$= \frac{T}{2\pi}\int_0^{2\pi} \frac{1-\sin 2nx}{2}\, dx$

$= \frac{T}{2\pi}\left[\frac{x}{2} + \frac{\cos 2nx}{4n}\right]\Big|_0^{2\pi}$

$= \frac{T}{2}$

$e^{ix} = \cos x + i\sin x$

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\frac{2\pi nx}{T} + \sum_{n=1}^{\infty} b_n \cdot \sin\frac{2\pi nx}{T}$

$\int_0^T f(x)\,dx = a_0 T \longrightarrow a_0 = \frac{1}{T}\int_0^T f(x)\,dx$

$\int_0^T f(x)\cos\frac{2\pi mx}{T}\,dx = \int_0^T a_0 \cdot \cos\frac{2\pi mx}{T} + \sum_{n=1}^{\infty} a_n \cdot \cos\frac{2\pi nx}{T}\cos\frac{2\pi mx}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nx}{T}\cdot\cos\frac{2\pi mx}{T}\,dx$

$= \int_0^T a_m \cdot \cos^2\frac{2\pi mx}{T}\,dx$

$= a_m \cdot \frac{T}{2}$

$a_n = \frac{2}{T}\int_0^T f(x)\cos\frac{2\pi nx}{T}\,dx$

$$\int_0^T f(x)\sin\frac{2\pi mx}{T}dx = \int_0^T a_0 \sin\frac{2\pi mx}{T} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nx}{T}\sin\frac{2\pi mx}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nx}{T}\sin\frac{2\pi mx}{T}\, dx$$

$$= b_m \cdot \frac{T}{2}$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin\frac{2\pi nx}{T}dx$$

$$e^{ix} = \cos x + i\cdot \sin x$$

$$e^{-ix} = \cos x - i\sin x$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\sin x = \frac{ie^{-ix}-ie^{ix}}{2}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cdot\cos\frac{2\pi nx}{T} + \sum_{n=1}^{\infty} b_n\cdot\sin\frac{2\pi nx}{T}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \frac{e^{i\cdot\frac{2\pi nx}{T}}+e^{-i\cdot\frac{2\pi nx}{T}}}{2} + \sum_{n=1}^{\infty} b_n\cdot\frac{i\cdot e^{-i\cdot\frac{2\pi nx}{T}} - i\cdot e^{i\cdot\frac{2\pi nx}{T}}}{2}$$

$$= a_0 + \sum_{n=1}^{\infty} e^{i\cdot\frac{2\pi nx}{T}}\left(\frac{a_n - ib_n}{2}\right) + \sum_{n=1}^{\infty} e^{-i\cdot\frac{2\pi nx}{T}}\left(\frac{a_n + ib_n}{2}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} e^{i\cdot\frac{2\pi nx}{T}}\left(\frac{a_n - ib_n}{2}\right) + \sum_{n=-\infty}^{-1} e^{i\cdot\frac{2\pi nx}{T}}\left(\frac{a_{-n} + ib_{-n}}{2}\right)$$

$$= \sum_{n=-\infty}^{\infty} C_n\cdot e^{i\cdot\frac{2\pi nx}{T}}$$

$$C_n = \begin{cases} a_0, & n=0 \\ \frac{a_n - ib_n}{2}, & n>0 \\ \frac{a_{-n}+ib_{-n}}{2}, & n<0 \end{cases}$$

$$\frac{a_n - ib_n}{2} = \frac{\frac{2}{T}\int_0^T f(x)\cos\frac{2\pi nx}{T}dx - i\cdot\frac{2}{T}\int_0^T f(x)\sin\frac{2\pi nx}{T}dx}{2}$$

$$= \frac{1}{T}\int_0^T f(x)\left[\cos\frac{2\pi nx}{T} - i\cdot\sin\frac{2\pi nx}{T}\right]dx$$

$$= \frac{1}{T}\int_0^T f(x)\, e^{-\frac{2\pi nx}{T}} dx$$

$$a_0 = \frac{1}{T}\int_0^T f(x)\, e^{-\frac{2\pi nx i}{T}} dx \Big|_{n=0}$$

$$\frac{a_{-n} + ib_{-n}}{2} = \frac{\frac{2}{T}\int_0^T f(x)\cos\frac{2\pi nx}{T}dx - i\cdot\frac{2}{T}\int_0^T f(x)\sin\frac{2\pi nx}{T}dx}{2}$$

$$= \frac{1}{T}\int_0^T f(x)\, e^{-\frac{2\pi nx}{T}i} dx$$

$$\therefore \quad f(t) = \sum_{-\infty}^{+\infty} c_n\, e^{int\cdot\frac{2\pi}{T}}$$

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-in\cdot\frac{2\pi}{T}\cdot t} dt$$

$$f_T(t) = \sum_{-\infty}^{+\infty} \frac{\Delta w}{2\pi}\int_0^T f(t)\, e^{-inw_0 t} dt\; e^{inw_0 t}$$

$$T \to \infty$$

$$\int_0^T dt \to \int_{-\infty}^{+\infty} dt \qquad \longrightarrow f_T(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} f(t) e^{-iwt} dt}_{F(w)}\; e^{iwt}\, dw$$

$$nw_0 \to w$$

$$\sum_{n=-\infty}^{\infty} \Delta w \to \int_{-\infty}^{+\infty} dw$$

Fourier Transform $F(w) = \int_{-\infty}^{+\infty} f(t)\, e^{-iwt}\, dt$

Inverse Fourier Transform $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(w)\, e^{iwt}\, dw$

For every $w$, $f(t) = \frac{1}{2\pi} \left[ \int_{-\infty}^{+\infty} f(t)\, e^{-wt}\, dt \right] e^{iwt}$

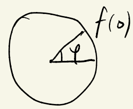
$$= \frac{1}{2\pi} (a + bi) \left[ \cos(wt) + \sin(wt) \cdot i \right]$$

$$= \frac{1}{2\pi} \left\{ \left[ a \cos(wt) - b \sin(wt) \right] + \left[ a \sin(wt) + b \cos(wt) \right] i \right.$$

$$|f(t)| = \frac{1}{2\pi} \sqrt{a^2 \cos^2(wt) + b^2 \sin^2(wt) - 2ab \sin(wt) \cos(wt) + a^2 \sin^2(wt) + b^2 \cos(wt) + 2ab \sin(wt) \cos(wt)}$$

$$= \frac{\sqrt{2}}{2\pi} \cdot \sqrt{a^2 + b^2} = \frac{\sqrt{2}}{2\pi} |F(w)|$$

$f(0) = \frac{1}{2\pi} (a + bi) \longrightarrow$




$$\varphi = \arctan \frac{b}{a}$$

$$\frac{\sqrt{2}}{2\pi} \left( \sqrt{a^2+b^2} \left[ \frac{a}{\sqrt{a^2+b^2}} \cos(wt) - \frac{b}{\sqrt{a^2+b^2}} \sin(wt) \right] \right)$$

$$f(w) = \underbrace{\frac{\sqrt{2}}{2\pi} \sqrt{a^2+b^2}}_{\text{amplitude}} \left[ \cos(wt + \varphi) + \sin(wt + \varphi) i \right], \quad \underbrace{\varphi = \arctan \frac{b}{a}}_{\text{phase}}$$