

Market Making with Fads, Informed, and Uninformed Traders

EMILIO BARUCCI[§], ADRIEN MATHIEU^{†,‡}, AND LEANDRO SÁNCHEZ-BETANCOURT^{†,‡}

[§]Department of Mathematics, Politecnico di Milano

[†]Mathematical Institute, University of Oxford

[‡]Oxford-Man Institute of Quantitative Finance, University of Oxford

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Abstract

We characterise the solutions to a continuous-time optimal liquidity provision problem in a market populated by informed and uninformed traders. In our model, the asset price exhibits fads —these are short-term deviations from the fundamental value of the asset. Conditional on the value of the fad, we model how informed traders and uninformed traders arrive in the market. The market maker is aware of the existence of the two groups of traders but only observes the anonymous order arrivals. We study both, the complete and the partial information versions of the control problem faced by the market maker. In such frameworks, we characterise the value of information, and we find the price of liquidity as a function of the proportion of informed traders in the market. Lastly, for the partial information setup, we explore how to go beyond the Kalman-Bucy filter to extract information about the fad from the market arrivals.

Keywords: market making, signals, informed traders, noise trading, stochastic filtering, fad

1 Introduction

The market making problem has been studied in a large body of literature dealing mainly with two issues: asymmetric information and inventory management; see the seminal contribution of [29] and [2, 35], respectively.¹ Asymmetric information is concerned with the financial losses from trading with agents who have privileged information about the fundamental value of the asset (adverse selection). Inventory management focuses on controlling the risk associated with an open inventory, that is, the risk that the price goes up when the inventory is negative (short position) or that the price goes down when the inventory is positive (long position).

¹See also the books [27, 32, 15] for a comprehensive account.

In what follows, we approach the problem through an angle that combines asymmetric information and inventory management in a new way: we study the market making problem when the market is affected by “fads”. By fads we mean short-term deviations from the fundamental price —a component that is associated with market inefficiencies and that may be related to cognitive limits of market participants (e.g. over-optimism, short-termism) or other behavioural explanations (e.g. herding); see [28, 40]. The fad affects the market in two different ways. First, it affects the price of the asset. In addition to the classical arrival of information, which is modelled as a Brownian motion, there is a mean reverting process (the fad) that drives mid-prices —the mean-reversion well describes the fad’s long run bursting properties. Second, the fad also affects the arrival of market orders. This is because a fraction of agents populating the market (informed traders) *distinguish* prices with fads from the fundamental value of the asset. The problem of the market maker is then to manage inventory while facing both (i) uninformed traders, who are sensitive to the displacements of bid-ask quotes from the mid-price as in [2], and (ii) informed traders, who identify the true fundamental value of the asset (disentangling it from the fad component) and arrive in the market according to the displacements of the quotes from the fundamental.

The above market making problem crucially depends on the information of the market maker about the fad. In this paper, we work out optimal bid-ask pricing strategies under both, a perfect information setup and an incomplete information setup. This allows us to obtain insights into the value of information depending on the proportion of informed traders in the market.

This paper is closely related to the strand of the literature that studies how brokers deal with toxic order flow. Toxic flow is the trading activity of informed traders who have access to signals about the short-term trend of the asset and exploit it to their advantage. For example, [16] analyse a model where informed and uninformed traders carry out their trading with a broker.² Similar to them, in this paper, the observed value of the asset shows temporal deviations (fad) from the fundamental and, conditional on the fad, we model how informed traders and uninformed traders arrive in the market in a framework similar to [2].³ To the best of our knowledge, this is the first paper where the order arrival from informed traders and uninformed traders is incorporated in the Avellaneda-Stoikov framework.

Since the reference paper by [1], several articles have incorporated predictive signals into a stochastic control framework of execution cost minimisation. Predictive signals mostly refer to measurable processes built from order book dynamics, e.g., the order flow or the order book imbalance, which affects the asset price in the short term; see e.g., [21, 14]. There are a number of contributions in the optimal execution literature that address the problem in a full information setting, see e.g., [14, 38, 39, 4], other papers that consider a noisy signal of future stock prices, see e.g., [5], and others that deal with the incomplete information setting, see e.g., [18]. Within the market making problem there are fewer examples. One of the examples is [17] who study the problem of posting at-the-touch when the market maker observes the short-term drift of the asset. Another example is [22] who studies market making when prices are mean-reverting (here the signal is whether the price is above or below the long run level). In

²See [13, 23] for methods to detect toxic order flow.

³An interesting example of a mean-field game of strategic players interacting in an Avellaneda-Stoikov framework is [6].

what follows, we consider a setting in which both the arrival rate and the asset price contain information about the fundamental value. We consider both a full information and a partial information setting. In the partial information setting, the key issue is to filter information about the fad.

We assess the value of information for the market maker by comparing the performance of trading strategies with increasing levels of knowledge about the market. We show that the optimal strategy under full information outperforms the partial information strategy which in turn outperforms the strategy obtained by a market maker who ignores the fad in the market. The optimal displacements in the full information setup are affected by the fad component asymmetrically: as the fad component increases, the market maker decreases the price of liquidity in the ask side and increases it in the bid side to manage the inventory optimally. This behaviour arises because the market maker aims to balance the arrival of trades by informed traders (who consider the fundamental value rather than the mid-price as reference price) and uninformed traders (who consider the mid-price as reference price). We also study the sensitivity of the optimal strategies to changes in the values of model parameters. When the trades sent by informed traders become “sharper” or more “toxic”, i.e., trades carried out when the fundamental value of the asset is far from the mid-price, the market maker’s performance worsens and the bid-ask spread increases.

The remainder of the paper is organized as follows. In Section 2 we introduce the model. In Section 3 we address the full information problem. In Section 4 we study the incomplete information problem. Section 5 carries out a sensitivity analysis on the optimal strategies through simulations, and Section 6 presents directions for future research where one goes beyond the Kalman-Bucy filtering framework. In particular, we discuss how the market maker would filter order arrivals in search of the unobserved value of the fad. In Section 7 we provide our conclusions. We collect proofs and further robustness checks in the appendix.

2 The model

We consider a market maker who trades an asset over a finite time horizon $\mathfrak{T} := [0, T]$. Over this period, she provides liquidity by setting the bid and the ask price at which she is willing to buy and sell one unit of the asset, respectively.⁴

The model builds on [2, 33, 34]. Similar to [2], we take the mid-price of the asset to be exogenous and assume that it evolves as

$$dS_t = \mu dt + \sigma dW_t, \quad S_0 \in \mathbb{R}^+, \quad (1)$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.⁵ Unlike [2], and similar to [5, 11, 31], we split the noise $(W_t)_{t \in \mathfrak{T}}$ into a persistent and a temporary component, respectively $(U_t)_{t \in \mathfrak{T}}$ and $(Z_t)_{t \in \mathfrak{T}}$. More precisely,

$$W_t = \mathfrak{p} Z_t + \mathfrak{q} U_t, \quad (2)$$

⁴We use one unit without loss of generality. For example, one unit can be thought as the average trade size in the market in which the market maker trades.

⁵In [2] the authors take $\mu = 0$.

with $\mathfrak{p}, \mathfrak{q} > 0$ satisfying $\mathfrak{p}^2 + \mathfrak{q}^2 = 1$, and $(U_t)_{t \in \mathfrak{T}}$ follows a mean reverting Ornstein-Uhlenbeck (OU) process

$$dU_t = -\eta U_t dt + dB_t, \quad U_0 = 0, \quad (3)$$

where $(B_t)_{t \in \mathfrak{T}}$ and $(Z_t)_{t \in \mathfrak{T}}$ are independent $(\mathcal{F}_t)_{t \in \mathfrak{T}}$ -Brownian motions; below, we provide a rigorous characterisation of the probability space. The two components $(Z_t)_{t \in \mathfrak{T}}$ and $(U_t)_{t \in \mathfrak{T}}$ aim to capture different features: Z_t is the classical martingale contribution that describes the arrival of information in the market affecting the fundamental value, U_t represents the fad or bubble in the market, a component affecting the price that after the hype tends to burst and therefore to dampen;⁶ this can be due to cognitive limits, irrational exuberance or herding effects. This component reverts to zero in the long run, which is why it has been proposed in several classical papers that deal with temporary deviations from the fundamental price, see [11, 26, 41].⁷ Within our model, we can interpret $S_t - \sigma q U_t$ as the fundamental price, *i.e.*, the asset price without the fad component. The values of the model parameters \mathfrak{p} and \mathfrak{q} control the relevance of the martingale/fundamental component and of the fad component, respectively.

As in [34], the inventory of the market maker evolves according to the following law of motion

$$dQ_t = dN_t^b - dN_t^a, \quad Q_0 \in \mathcal{Q}, \quad (4)$$

where Q_0 denotes the initial amount of the asset held by the market maker at $t = 0$. The processes $(N_t^b)_{t \in \mathfrak{T}}$ and $(N_t^a)_{t \in \mathfrak{T}}$ count the number of shares the market maker bought and sold, respectively; each transaction is of constant size equal to unity. We assume that the inventory is bounded from above and below, *i.e.*, $\underline{q} \leq Q_t \leq \bar{q}$ for $\underline{q} < 0 < \bar{q}$ with $\underline{q}, \bar{q} \in \mathbb{Z}$, and therefore the market maker does not quote on a given side if a trade on that side would breach these boundaries. The set $\mathcal{Q} = \{\underline{q}, \underline{q} + 1, \dots, \bar{q} - 1, \bar{q}\}$ is the set of possible values for the inventory. The market maker controls the intensity of the counting processes $(N_t^{a,b})_{t \in \mathfrak{T}}$ by defining the bid and the ask price, but arrival of orders is not fully in his hands as the fad component affects it. To model both components, we write the intensities of the two processes as follows

$$\lambda^a(\delta_t^a) = \left(\underbrace{\varphi e^{-k\delta_t^a}}_{\text{uninformed}} + \underbrace{\psi e^{-k\delta_t^a - \gamma(\sigma \mathfrak{q} U_t \vee S^-)}}_{\text{informed}} \right) \mathbf{1}_{\{Q_{t-} > \underline{q}\}}, \quad (5)$$

$$\lambda^b(\delta_t^b) = \left(\underbrace{\varphi e^{-k\delta_t^b}}_{\text{uninformed}} + \underbrace{\psi e^{-k\delta_t^b + \gamma(\sigma \mathfrak{q} U_t \wedge S^+)}}_{\text{informed}} \right) \mathbf{1}_{\{Q_{t-} < \bar{q}\}}, \quad (6)$$

for the ask quote $S_t^a = S_t + \delta_t^a$ and the bid quote $S_t^b = S_t - \delta_t^b$, respectively, and where $k, \gamma > 0$ and $S^- \leq 0 \leq S^+$. In what follows, we refer to δ_t^a and δ_t^b as the ask and bid displacements, respectively. The parameters φ and ψ control the proportion of uninformed and informed traders in the market. They modulate the baseline intensity of order arrivals from informed and uninformed traders, respectively. Thus, their ratio is linked to the fraction of informed-uninformed traders populating the market. We

⁶In our model $(S_t)_{t \in \mathfrak{T}}$ satisfies $dS_t = (\mu - \eta \sigma \mathfrak{q} U_t) dt + \sigma d\bar{W}_t$, where $\bar{W}_t = \mathfrak{p} Z_t + \mathfrak{q} B_t$ is a Brownian Motion and $(U_t)_{t \in \mathfrak{T}}$ is the OU process in (3) with $\langle \bar{W}, B \rangle_t = \mathfrak{q} t$.

⁷The Ornstein-Uhlenbeck process is widely used to model the dynamic of order flow imbalance/liquidity, see [14, 38].

return to this point below.

Remark 1. *The well-known model in [2] is recovered by setting $q = \psi = 0$ (the fad does not affect the asset price nor the arrival rate of informed traders).*

The stochastic intensities are the result of the activity of two groups of traders: uninformed and informed traders. (a) Uninformed traders do not know about the fad and therefore cannot disentangle the fad component from the mid-price to determine the fundamental price. Thus, their arrivals depend solely on how generous the quotes $S_t^{a,b}$ are with respect to the mid-price S_t in the market (fundamental plus fad), i.e., δ_t^a, δ_t^b . On the other hand, (b) informed traders observe the fad (as smart traders in [11]), and therefore they observe the fundamental value of the asset and their arrivals depend on how generous the quotes are with respect to that level. The formulation of the arrival rates in (5)-(6) can be derived as follows. Let $F_t = S_t - \sigma q U_t$ be the fundamental at time t (mid-price minus fad), then the ‘generosity’ of the ask quote with respect to fundamental is $S_t^a - F_t = S_t + \delta_t^a - S_t + \sigma q U_t = \delta_t^a + \sigma q U_t$. Similarly, for the bid quote (at which the informed trader may sell one share), the ‘generosity’ is $F_t - S_t^b = S_t - \sigma q U_t - S_t + \delta_t^b = \delta_t^b - \sigma q U_t$. The minus factor in the exponent of the arrival rate of informed traders in (5) is because larger values of $\delta_t^a + \sigma q U_t$ corresponds to less generosity of the quote and therefore a lower arrival rate of informed traders, the same argument holds true for (6).

This argument holds for the baseline case $\gamma = k$. The parameters γ and k modulate how the order flow of informed and uninformed traders reacts to the fad and the displacements of the quotes from mid-price. The larger (resp. smaller) the values of these parameters, the more (resp. less) market order arrivals react to the fad and the displacements. The model has enough flexibility to allow informed traders to react more to the fad ($\gamma > k$) or less ($\gamma < k$) than in the baseline case.

The above arguments provide an explanation of the arrival rate of informed traders in (5)-(6). The peculiarity is that the effect of the fad is asymmetric. A positive (resp. negative) U_t signals an inflated (resp. deflated) price which in turn induces informed traders to sell (resp. buy) with a higher intensity than otherwise, because they know that, all else being equal, the asset is overvalued (resp. undervalued) with respect to the fundamental and is likely to come down (resp. go up) in the future.

For technical reasons we impose that the effect of the fad in the intensity of order arrivals is bounded. More precisely, we ask the baseline intensity $\psi \exp(\pm \gamma \sigma q U_t)$ of informed traders to be bounded by $\psi \exp(-\gamma \mathcal{S}^-)$ and $\psi \exp(\gamma \mathcal{S}^+)$ in the ask and bid price, respectively. This assumption is needed to show that the Radon-Nikodym derivative that we introduce is a true martingale and that the problem is well defined. From a financial point of view, the motivation of the assumption comes from the fact that the intensity of market orders cannot be arbitrarily large because exchanges have a maximum orders they can process per unit time.⁸

The cash of the market maker satisfies

$$dX_t = (S_t + \delta_t^a)dN_t^a - (S_t - \delta_t^b)dN_t^b, \quad X_0 = 0. \quad (7)$$

⁸One way to see this is to rewrite the stochastic intensities in (5)-(6) as $\psi(e^{\pm \gamma \sigma q U_t} \wedge \mathcal{Z})$ for \mathcal{Z} a function of \mathcal{S}^+ and \mathcal{S}^- and where \mathcal{Z} is the bound (per unit time) from the exchange. For example, 1000 orders per millisecond.

The market maker controls the bid and ask quotes and optimises the following criterion

$$X_T + Q_T S_T - \alpha Q_T^2 - \phi \int_0^T Q_u^2 du. \quad (8)$$

The interpretation is as follows. Over the horizon $[0, T]$, the market maker is interested in the final realized cash X_T , the market value of the inventory at time T , given by $Q_T S_T$, and inventory control (over time and at the end of the period), which is given by the remaining two terms. This performance criterion is widely used in the algorithmic trading literature, e.g. see [15]. The mark-to-market valuation of the inventory is at the mid-price in the market S_T (regardless of whether the fad is positive or negative). This assumption implies that the market maker is not interested in discerning *per se* the fundamental value. In Section 6 we discuss the role of this assumption and study the case where the market maker evaluates the inventory at the fundamental price.⁹

In what follows, we study the problem depending on the information available to the market maker. First, we address the full information scenario, *i.e.*, the case where the market maker observes the realization of the process U_t and defines the control problem accordingly.¹⁰ Second, we address the more realistic case in which the market maker filters the unobserved fad of the observed price process. We conclude by presenting alternative filters (filtering arrivals instead of prices) and other directions for future research.

3 Full information

3.1 Formal derivation of the probability space

Before pursuing the solution to the full information problem, it is important to provide a rigorous characterisation of the probability space we employ.

Consider Ω_d the set of increasing piecewise constant càdlàg functions from $[0, T]$ into \mathbb{N} with jumps equal to one and Ω_c the set of continuous functions from $[0, T]$ into \mathbb{R} . We define $\Omega = \Omega_c^2 \times \Omega_d^2$ as the sample space. We let $(B_t, Z_t, \bar{N}_t^a, \bar{N}_t^b)_{t \in \mathfrak{T}}$ be the first canonical process on Ω . The associated filtration is $\bar{\mathcal{F}} = (\mathcal{F}_t^c \otimes \mathcal{F}_t^c \otimes \bar{\mathcal{F}}_t^d \otimes \bar{\mathcal{F}}_t^d)_{t \in \mathfrak{T}}$ where $(\bar{\mathcal{F}}_t^d)_{t \in \mathfrak{T}}$ (*resp.* $(\mathcal{F}_t^c)_{t \in \mathfrak{T}}$) is the right continuous completed filtration associated with \bar{N}^a and \bar{N}^b (*resp.* B or Z). We denote by \mathbb{P}_0 the probability measure on (Ω, \mathcal{F}) such that $(\bar{M}_s^a = \bar{N}_s^a - s, \bar{M}_s^b = \bar{N}_s^b - s)_{s \in \mathfrak{T}}$ are martingales and $(Z_t)_{t \in \mathfrak{T}}, (B_t)_{t \in \mathfrak{T}}$ are Brownian motions. We also assume independence between the processes. We define the point processes $(N_t^a)_{t \in \mathfrak{T}}, (N_t^b)_{t \in \mathfrak{T}}$ as solutions of the coupled SDE

$$dN_t^b = \mathbb{1}_{\{N_{t-}^b - N_{t-}^a < \bar{q}\}} d\bar{N}_t^b, \quad dN_t^a = \mathbb{1}_{\{N_{t-}^b - N_{t-}^a > \underline{q}\}} d\bar{N}_t^a. \quad (9)$$

⁹Note that $Q_T S_T - \alpha Q_T^2 = Q_T(S_T - \alpha Q_T)$. This renders the interpretation that at T the market maker liquidates the open inventory and αQ_T captures the slippage of such a liquidation (the asset price is affected linearly by the amount to be liquidated). Here, liquidation is from S_T because this is the mid-price in the market. This is similar to [10] where their mark-to-market is done at the prices of the competing market makers.

¹⁰The full observation setting is relevant in its own right given that in some OTC markets the liquidity provider is able to profile their clients and is able to detect the short-term signal from the trading activity of their clients [13, 16].

Then, under \mathbb{P}_0 , $(N_t^a)_{t \in \mathfrak{T}}$ and $(N_t^b)_{t \in \mathfrak{T}}$ have intensities

$$\Lambda_t^a = \mathbb{1}_{\{N_{t-}^b - N_{t-}^a > q\}} \quad \Lambda_t^b = \mathbb{1}_{\{N_{t-}^b - N_{t-}^a < \bar{q}\}},$$

and they are independent of the Brownian motions. Now, we can define the second canonical process on Ω as $(B_t, Z_t, N_t^a, N_t^b)_{t \in \mathfrak{T}}$, with its associated filtration \mathcal{F} defined the same way.

We let \mathcal{U} be the set of admissible controls given by

$$\mathcal{U} := \left\{ \delta = (\delta_t^a, \delta_t^b)_{t \in \mathfrak{T}} \in L^2(\Omega \times [0, T]) : \delta \geq -\delta_\infty \text{ and } \mathcal{F} \text{ predictable} \right\},$$

where $\delta_\infty = (\delta_\infty^a, \delta_\infty^b)$ characterises the lower bound of the control. Next, for $\delta \in \mathcal{U}$, we define $\mathbb{X}_t = (X_t, Q_t, S_t, U_t)$ to be the state of the system, where the dynamics of $(\mathbb{X}_t)_{t \in \mathfrak{T}}$ are given by

$$\begin{cases} dX_t = (S_t + \delta_t^a) dN_t^a - (S_t - \delta_t^b) dN_t^b, & X_0 = 0, \\ dQ_t = dN_t^b - dN_t^a, & Q_0 \in \mathbb{R}, \\ dS_t = \mu dt + \sigma (\mathbf{q} dU_t + \mathbf{p} dZ_t), & S_0 \in \mathbb{R}^+, \\ dU_t = -\eta U_t dt + dB_t, & U_0 = 0. \end{cases} \quad (10)$$

Note that we do not include Z_t as state variable because knowledge of U_t and S_t implies knowledge of Z_t (given that $S_t = S_0 + \mu t + \sigma (\mathbf{p} Z_t + \mathbf{q} U_t)$).

Using these processes, we make an explicit change of measure associated with each control process. More precisely, we consider the functions

$$\lambda^a(U_t, Q_{t-}, \delta_t^a) := \lambda_t^a = (\varphi + \psi e^{-\gamma(\sigma \mathbf{q} U_t \vee \mathcal{S}^-)}) e^{-k \delta_t^a} \mathbb{1}_{\{Q_{t-} > q\}}, \quad (11)$$

$$\lambda^b(U_t, Q_{t-}, \delta_t^b) := \lambda_t^b = (\varphi + \psi e^{\gamma(\sigma \mathbf{q} U_t \wedge \mathcal{S}^+)}) e^{-k \delta_t^b} \mathbb{1}_{\{Q_{t-} < \bar{q}\}}, \quad (12)$$

which represent the ask and bid intensity for the control $\delta \in \mathcal{U}$. Then, for any $\delta \in \mathcal{U}$, we define \mathbb{P}^δ by

$$\frac{d\mathbb{P}^\delta}{d\mathbb{P}_0} = L_T^\delta, \quad (13)$$

where L_t^δ is the Doléans-Dade exponential of \mathcal{Z}_t , where

$$\mathcal{Z}_t = \int_0^t (\lambda_s^a - 1) dM_s^a + \int_0^t (\lambda_s^b - 1) dM_s^b.$$

Since $(\delta_t^a, \delta_t^b)_{t \in \mathfrak{T}}$ are bounded from below and the intensities are uniformly bounded, $(L_t^\delta)_{t \in \mathfrak{T}}$ is a \mathbb{P}_0 martingale by the Novikov criterion,¹¹ and under \mathbb{P}^δ , the processes

$$M_t^{a,\delta} := N_t^a - \int_0^t \lambda_u^a du \quad \text{and} \quad M_t^{b,\delta} := N_t^b - \int_0^t \lambda_u^b du,$$

¹¹See Section 3 in [25].

are true martingales. In the paper, we always refer to the expectation with respect to the probability measure \mathbb{P}^δ as \mathbb{E} instead of \mathbb{E}^δ . Finally, on the probability space $(\Omega, \mathcal{F}, \mathbb{P}^\delta)$, the processes $(Z_t)_{t \in \mathfrak{T}}$ and $(B_t)_{t \in \mathfrak{T}}$ are \mathcal{F} -Brownian motions, and the point processes $(N_t^a)_{t \in \mathfrak{T}}$ and $(N_t^b)_{t \in \mathfrak{T}}$ are \mathcal{F} -Poisson processes by the Watanabe characterisation.

3.2 Characterisation of the solution

The market maker aims to solve the following control problem

$$\sup_{\delta \in \mathcal{U}} \mathbb{E} \left[X_T + Q_T S_T - \alpha Q_T^2 - \phi \int_0^T Q_u^2 du \right]. \quad (14)$$

Using the Itô formula for $X_T + Q_T S_T$, we get

$$\mathbb{E}[X_T + Q_T S_T] = \mathbb{E}[X_0 + Q_0 S_0] + \mathbb{E} \left[\int_0^T (\delta_s^a \lambda_s^a + \delta_s^b \lambda_s^b + Q_s \mu - \eta \mathbf{q} \sigma Q_s U_s) ds \right].$$

The maximisation problem (14) is then equivalent to maximise

$$\mathbb{E} \left[-\alpha Q_T^2 + \int_0^T (\delta_s^a \lambda_s^a + \delta_s^b \lambda_s^b - \phi Q_s^2 + Q_s \mu - \eta \mathbf{q} \sigma Q_s U_s) ds \right], \quad (15)$$

from which we define the value function associated with the control problem

$$V(t, q, u) = \sup_{\delta \in \mathcal{U}} \mathbb{E}_{t, q, u} \left[-\alpha Q_T^2 + \int_t^T \left(-\phi Q_s^2 + Q_s \mu - \eta \mathbf{q} \sigma Q_s U_s + \delta_s^a \lambda^a(U_s, Q_{s-}, \delta_s^a) + \delta_s^b \lambda^b(U_s, Q_{s-}, \delta_s^b) \right) ds \right]. \quad (16)$$

The expression above is finite since all processes belong to $L^2(\Omega \times [0, T])$. Below, we also prove that V is locally bounded. Standard techniques from the theory of stochastic control show that the Hamilton Jacobi Bellman (HJB) equation associated with such a problem is

$$\begin{aligned} 0 &= \partial_t V - \phi q^2 - \eta u \partial_u V + \frac{1}{2} \partial_{u,u}^2 V + (\mu - \eta \sigma \mathbf{q} u) q \\ &\quad + \sup_{\delta^a} \left\{ e^{-k \delta^a} (\varphi + \psi e^{-\gamma(\sigma \mathbf{q} u \vee \mathcal{S}^-)} \times \right. \\ &\quad \left. (\delta^a + V(t, q-1, u) - V(t, q, u)) \right\} \mathbb{1}_{q > \underline{q}} \\ &\quad + \sup_{\delta^b} \left\{ e^{-k \delta^b} (\varphi + \psi e^{\gamma(\sigma \mathbf{q} u \wedge \mathcal{S}^+)}) \times \right. \\ &\quad \left. (\delta^b + V(t, q+1, u) - V(t, q, u)) \right\} \mathbb{1}_{q < \bar{q}}, \end{aligned} \quad (17)$$

with terminal condition $V(T, q, u) = -\alpha q^2$. We compute the maximisers of the above expression explic-

itly obtaining that

$$\delta^{a*} = \left(\frac{1}{k} - V(t, q-1, u) + V(t, q, u) \right) \vee -\delta_\infty^a, \quad q \neq \underline{q}, \quad (18)$$

$$\delta^{b*} = \left(\frac{1}{k} - V(t, q+1, u) + V(t, q, u) \right) \vee -\delta_\infty^b, \quad q \neq \bar{q}, \quad (19)$$

which transforms the HJB into the following partial differential equation (PDE)

$$\begin{aligned} 0 = & \partial_t V - \eta u \partial_u V - \phi q^2 + \frac{1}{2} \partial_{u,u}^2 V + (\mu - \eta \sigma \mathbf{q} u) q \\ & + \frac{1}{k} \exp(-1 + k[V(t, q-1, u) - V(t, q, u)]) (\varphi + \psi e^{-\gamma(\sigma \mathbf{q} u \vee \mathcal{S}^-)}) \mathbb{1}_{\{q > \underline{q}\}} \mathbb{1}_{\{\delta^{a,*} = \delta^{a,**}\}} \\ & + \frac{1}{k} \exp(-1 + k[V(t, q+1, u) - V(t, q, u)]) (\varphi + \psi e^{\gamma(\sigma \mathbf{q} u \wedge \mathcal{S}^+)}) \mathbb{1}_{\{q < \bar{q}\}} \mathbb{1}_{\{\delta^{a,*} = \delta^{b,**}\}} \\ & + e^{k\delta_\infty^a} (\varphi + \psi e^{-\gamma(\sigma \mathbf{q} u \vee \mathcal{S}^-)}) (-\delta_\infty^a + V(t, q-1, u) - V(t, q, u)) \mathbb{1}_{\{q > \underline{q}\}} \mathbb{1}_{\{\delta^{a,*} > \delta^{a,**}\}} \\ & + e^{k\delta_\infty^b} (\varphi + \psi e^{\gamma(\sigma \mathbf{q} u \wedge \mathcal{S}^+)}) (-\delta_\infty^b + V(t, q+1, u) - V(t, q, u)) \mathbb{1}_{\{q < \bar{q}\}} \mathbb{1}_{\{\delta^{a,*} > \delta^{b,**}\}}, \end{aligned} \quad (20)$$

where

$$\delta^{a,**} = \frac{1}{k} - V(t, q-1, u) + V(t, q, u), \quad \delta^{b,**} = \frac{1}{k} - V(t, q+1, u) + V(t, q, u).$$

3.2.1 Viscosity solution: complete information problem

In this section, we prove that the value function V in (16) is a viscosity solution of the HJB equation in (17). We recall that \mathcal{U} is the set of predictable processes $(\delta_t)_{t \in \mathfrak{T}} = (\delta_t^a, \delta_t^b)_{t \in \mathfrak{T}}$, bounded from below and taking values in $U = U^a \times U^b = (-\delta_\infty^a, +\infty) \times (-\delta_\infty^b, +\infty)$ and where the state processes are defined in (10). We rewrite the price process as

$$dS_t = (\mu - \eta \sigma \mathbf{q} U_t) dt + \sigma d\bar{W}_t, \quad (21)$$

where $(\bar{W}_t)_{t \in \mathfrak{T}}$ is an \mathcal{F} -Brownian motion, with correlation $\langle B, \bar{W} \rangle_t = \mathbf{q} t$. The stochastic intensities of the Poisson processes are given by (11) and (12).

We define the restriction of our control space \mathcal{U}_t as

$$\mathcal{U}_t := \{(\delta_t)_{t \in \mathfrak{T}} \in \mathcal{U} : (\delta_s)_{s \geq t} \text{ is independent of } \mathcal{F}_t\},$$

and we recall an important result. Let \tilde{V} given by

$$\tilde{V}(t, q, u) = \sup_{\delta \in \mathcal{U}_t} \mathbb{E}_{t,q,u} \left[-\alpha Q_T^2 + \int_t^T (\delta_s^a \lambda_s^a + \delta_s^b \lambda_s^b - \phi Q_s^2 + Q_s \mu - \eta \mathbf{q} \sigma Q_s U_s) ds \right],$$

where the supremum is taken over \mathcal{U}_t instead of \mathcal{U} . It follows from the Remark 2.2 (iv) in [43] that $V = \tilde{V}$. To simplify notations, we denote by $\mathcal{E} := [0, T] \times \mathcal{Q} \times \mathbb{R}$ the space in which (t, q, u) takes values.

Theorem 1 (Weak dynamic programming principle). Define the lower and upper semicontinuous envelopes

$$V_*(t, q, u) = \liminf_{(t', q', u') \rightarrow (t, q, u)} V(t', q', u'),$$

and

$$V^*(t, q, u) = \limsup_{(t', q', u') \rightarrow (t, q, u)} V(t', q', u'),$$

where V is locally bounded. Consider $(t, q, u) \in \mathcal{E}$ fixed and let $\{\theta^\alpha, \alpha \in \mathcal{U}_t\}$ be a family of finite stopping with values in $[t, T]$. Then

$$V(t, q, u) \geq \sup_{\alpha \in \mathcal{U}_t} \mathbb{E}_{t, q, u} \left[\int_t^{\theta^\alpha} \left(\delta_s^a \lambda_s^a + \delta_t^b \lambda_s^b - \phi Q_s^2 + (\mu - \eta \mathbf{q} \sigma U_s) Q_s \right) ds + V_*(\theta^\alpha, U_{\theta^\alpha}, Q_{\theta^\alpha}) \right], \quad (22)$$

and if for any α , $\left((U_s^{t,u}, Q_s^{t,q})^T \mathbf{1}_{s \in [t, \theta^\alpha]} \right)_{t \in \mathfrak{T}}$ is L^∞ bounded, then

$$V(t, q, u) \leq \sup_{\alpha \in \mathcal{U}_t} \mathbb{E}_{t, q, u} \left[\int_t^{\theta^\alpha} \left(\delta_s^a \lambda_s^a + \delta_t^b \lambda_s^b - \phi Q_s^2 + (\mu - \eta \mathbf{q} \sigma U_s) Q_s \right) ds + V^*(\theta^\alpha, U_{\theta^\alpha}, Q_{\theta^\alpha}) \right]. \quad (23)$$

We do not present the proof since it follows the one in [36] because the stochastic intensities are uniformly bounded. Before dealing with the viscosity solution, let us define the Hamiltonian of our problem.

Definition 2 (Hamiltonian). We define the Hamiltonian $\mathcal{H} : \mathcal{Q} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{aligned} \mathcal{H}(q, u, V_1, V_2) &= \sup_{\delta^a \in U^a} (\varphi + \psi e^{-\gamma(\sigma \mathbf{q} u \vee \mathcal{S}^-)} e^{-k \delta^a} (\delta^a + V_1) \mathbf{1}_{\{q > \underline{q}\}} \\ &\quad + \sup_{\delta^b \in U^b} (\varphi + \psi e^{\gamma(\sigma \mathbf{q} u \wedge \mathcal{S}^+)}) e^{-k \delta^b} (\delta^b + V_2) \mathbf{1}_{\{q < \bar{q}\}}). \end{aligned} \quad (24)$$

Finally, we define the HJB equation associated with the control problem.

Definition 3 (Hamilton-Jacobi-Bellman equation). Let $V \in C^{1,2}(\mathcal{E})$, then

$$\begin{aligned} 0 &= \partial_t V(t, q, u) - \phi q^2 + q \mu - \eta \mathbf{q} \sigma u - \eta u \partial_u V(t, q, u) + \frac{1}{2} \partial_{u,u}^2 V(t, q, u) \\ &\quad + \mathcal{H}(q, u, V(t, q-1, u) - V(t, q, u), V(t, q+1, u) - V(t, q, u)). \end{aligned} \quad (25)$$

To shorten the notation, we write the equation above as

$$0 = F(t, q, u, V(t, q, u), \partial_t V(t, q, u), \partial_u V(t, q, u), \partial_{u,u}^2 V(t, q, u)).$$

Remark 4. By a ball around a point $(t, q, u) \in \mathcal{E}$ with radius $r > 0$ we mean

$$\{(s, j, v) \in \mathbb{R}^3 : \|(s, j, v) - (t, q, u)\| < r\} \cap \mathcal{E},$$

where $\|\cdot\|$ is the Euclidean norm.

Now we can define the notions of viscosity sub-solution and viscosity super-solution

Definition 5 (Viscosity sub-solution). *We say that V is a viscosity sub-solution if, for all $\Phi \in \mathcal{C}^{1,2}(\mathcal{E})$ such that $V^*(t_0, q_0, u_0) - \Phi(t_0, q_0, u_0)$ is a local maximum, we have*

$$F(t_0, q_0, u_0, \Phi(t_0, q_0, u_0), \partial_t \Phi(t_0, q_0, u_0), \partial_u \Phi(t_0, q_0, u_0), \partial_{u,u}^2 \Phi(t_0, q_0, u_0)) \geq 0. \quad (26)$$

Definition 6 (Viscosity super-solution). *Let V be locally bounded on \mathcal{E} . We say that V is a viscosity super-solution if, for all $\Phi \in \mathcal{C}^{1,2}(\mathcal{E})$ such that $V_*(t_0, q_0, u_0) - \Phi(t_0, q_0, u_0)$ is a local minimum, we have*

$$F(t_0, q_0, u_0, \Phi(t_0, q_0, u_0), \partial_t \Phi(t_0, q_0, u_0), \partial_u \Phi(t_0, q_0, u_0), \partial_{u,u}^2 \Phi(t_0, q_0, u_0)) \leq 0. \quad (27)$$

Definition 7 (Viscosity solution). *We say that V is a viscosity solution, if it is a viscosity sub-solution and a viscosity super-solution.*

Before we state the main theorem of the section, we prove two important lemmas.

Lemma 8. *The value function V is locally bounded.*

Proof. We define $\hat{V} : (t, q, u, \delta) \in \mathcal{E} \times \mathcal{U} \rightarrow \mathbb{R}$ by

$$\hat{V}(t, q, u, \delta) = \mathbb{E}_{t,q,u} \left[-\alpha Q_T^2 + \int_t^T (\delta_s^a \lambda_s^a + \delta_s^b \lambda_s^b - \phi Q_s^2 + Q_s \mu - \eta \mathbf{q} \sigma Q_s U_s) ds \right].$$

Consider a neighbourhood of $(t, q, u) \in \mathcal{E}$, and let $\delta \in \mathcal{U}$. We bound \hat{V} as

$$\begin{aligned} |\hat{V}(t, q, u, \delta)| &\lesssim \mathbb{E}[Q_T^2] + \int_t^T \mathbb{E}[\delta_s^a \lambda_s^a + \delta_s^b \lambda_s^b] ds + \int_t^T \mathbb{E}[Q_s^2] ds + \int_t^T \mathbb{E}[U_s^2] ds \\ &\lesssim q^2 + u^2 + \int_t^T \mathbb{E}[(\lambda_s^a)^2 + (\lambda_s^b)^2] ds + \int_t^T \mathbb{E}[e^{-\gamma(\sigma \mathbf{q} U_s \vee S^-)} + e^{\gamma(\sigma \mathbf{q} U_s \wedge S^+)}] ds \\ &\quad + \int_t^T \int_t^s \mathbb{E}[(\lambda_u^a)^2 + (\lambda_u^b)^2] du ds \\ &\lesssim 1 + q^2 + u^2, \end{aligned}$$

because δ is bounded from below (so $\delta e^{-k\delta}$ and $e^{-k\delta}$ are bounded), and $(U_t)_{t \in \mathfrak{T}}$ is an OU process (so all its moments are bounded uniformly in t). The symbol $a \lesssim b$ means that there exists a constant $C > 0$ (independent of a and b) such that $a \leq C b$. Finally, we obtain

$$|\hat{V}(t, q, u, \delta)| \lesssim 1 + q^2 + u^2,$$

and then $(t, q, u) \mapsto \hat{V}(t, q, u, \delta)$ is locally bounded. Taking the supremum of the above expression implies that the value function is locally bounded. \square

Lemma 9. *The Hamiltonian \mathcal{H} defined in (24) is continuous.*

Proof. First, we derive explicitly the value of the Hamiltonian. We have that

$$\begin{aligned}
\mathcal{H}(q, u, V_1, V_2) &= \sup_{\delta^a \in U^a} (\varphi + \psi e^{-\gamma(\sigma q u \vee \mathcal{S}^-)} e^{-k\delta^a} (\delta^a + V_1) \mathbb{1}_{\{q > \underline{q}\}} \\
&\quad + \sup_{\delta^b \in U^b} (\varphi + \psi e^{\gamma(\sigma q u \wedge \mathcal{S}^+)} e^{-k\delta^b} (\delta^b + V_2) \mathbb{1}_{\{q < \bar{q}\}} \\
&= (\varphi + \psi e^{-\gamma(\sigma q u \vee \mathcal{S}^-)} e^{-k\delta^{a,*}} (\delta^{a,*} + V_1) \mathbb{1}_{\{q > \underline{q}\}} \\
&\quad + (\varphi + \psi e^{\gamma(\sigma q u \wedge \mathcal{S}^+)} e^{-k\delta^{b,*}} (\delta^{b,*} + V_2) \mathbb{1}_{\{q < \bar{q}\}} ,
\end{aligned}$$

where the optimal displacements $\delta^* = (\delta^{a,*}, \delta^{b,*})$ are given by

$$\delta^{a,*}(V_1) = \left(\frac{1}{k} - V_1 \right) \vee -\delta_a^\infty, \quad \delta^{b,*}(V_2) = \left(\frac{1}{k} - V_2 \right) \vee -\delta_b^\infty, \quad (28)$$

from which it follows that \mathcal{H} is continuous. Indeed, take $(q, u, V_1, V_2) \in \mathcal{Q} \times \mathbb{R}^3$ and consider $(q^n, u^n, V_1^n, V_2^n)_{n \in \mathbb{N}}$ that converges toward (q, u, V_1, V_2) . Since \mathcal{Q} is discrete, there exists $n_0 \in \mathbb{N}$, such that, for all $n \geq n_0$, $q^n = q$. Then, for $n \geq n_0$, we have $\mathcal{H}(q^n, u^n, V_1^n, V_2^n) = \mathcal{H}(q, u^n, V_1^n, V_2^n)$, and convergence of $\mathcal{H}(q, u^n, V_1^n, V_2^n)$ toward $\mathcal{H}(q, u, V_1, V_2)$ follows by continuity of composition of continuous functions. \square

Now, we are able to state the main theorem of the section.

Theorem 2. *V is a viscosity solution of the HJB equation (25).*

Proof. We first prove that V is a viscosity super-solution of the equation. Consider (t, q, u) and $\Phi \in C^{1,2}$ such that

$$0 = (V_* - \Phi)(t, q, u) = \min(V_* - \Phi),$$

and for all $(s, j, v) \neq (t, q, u)$, $(V - \Phi)(s, j, v) > (V_* - \Phi)(t, q, u)$. Let assume by contradiction that

$$h(t, q, u) := F(t, q, u, \Phi(t, q, u), \partial_t \Phi(t, q, u), \partial_u \Phi(t, q, u), \partial_{u,u}^2 \Phi(t, q, u)) > 0.$$

By continuity of the function Φ and the Hamiltonian, there exists an open neighbourhood B_r of (t, q, u) such that

$$h(t, q, u) := F(t, q, u, \Phi(t, q, u), \partial_t \Phi(t, q, u), \partial_u \Phi(t, q, u), \partial_{u,u}^2 \Phi(t, q, u)) > 0 \quad \text{on } B_r.$$

We can define the set $J(t, q, u)$ as the set of attainable values if a jump happens in B_r . We define

$$2\eta := \min_{\partial B_r \cup J(t, q, u)} (V_* - \Phi) > 0.$$

Now, we consider (t_n, q_n, u_n) a sequence such that

$$(t_n, q_n, u_n) \rightarrow (t, q, u) \text{ and } V(t_n, q_n, u_n) \rightarrow V^*(t, q, u).$$

Since $(V - \Phi)(t_n, q_n, u_n) \rightarrow 0$, we can assume that the sequence (t_n, q_n, u_n) also satisfies

$$|(V - \Phi)|(t_n, q_n, u_n) \leq \eta \text{ for all } n \geq 1. \quad (29)$$

Then, we consider the control $\nu \in \mathcal{U}_{t_n}$ defined by

$$\nu_t^a = \delta^{a,*} \left(\Phi(t, Q_{t^-}^{t_n, q_n, \nu} - 1, U_t^{t_n, u_n, \nu}) - \Phi(t, Q_{t^-}^{t_n, q_n, \nu}, U_t^{t_n, u_n, \nu}) \right), \quad (30)$$

$$\nu_t^b = \delta^{b,*} \left(\Phi(t, Q_{t^-}^{t_n, q_n, \nu}, U_t^{t_n, u_n, \nu}) - \Phi(t, Q_{t^-}^{t_n, q_n, \nu}, U_t^{t_n, u_n, \nu}) \right), \quad (31)$$

we define the stopping time

$$\theta_n^\nu := \inf\{t > t_n : (t, Q_t^{t_n, q_n, \nu}, U_t^{t_n, u_n, \nu}) \notin B_r\},$$

and we observe that $(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) \in \partial B_r \cup J(t, q, u)$. Then, we have

$$\Phi(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) \leq -2\eta + V^*(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}). \quad (32)$$

Now, using equation (29) and Itô formula, we obtain

$$\begin{aligned} V(t_n, q_n, u_n) &\leq \eta + \Phi(t_n, q_n, u_n) \\ &= \eta + \mathbb{E} \left[\Phi(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) \right] \\ &\quad - \mathbb{E} \left[\int_{t_n}^{\theta_n^\nu} \left(\partial_t \Phi(s, Q_s^{t_n, q_n, \nu}, U_s^{t_n, u_n, \nu}) - \eta U_s^{t_n, u_n, \nu} \partial_u \Phi(s, Q_s^{t_n, q_n, \nu}, U_s^{t_n, u_n, \nu}) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \partial_{u,u}^2 \Phi(s, Q_s^{t_n, q_n, \nu}, U_s^{t_n, u_n, \nu}) + \lambda_s^a \left(\Phi(s, Q_s^{t_n, q_n, \nu} - 1, U_s^{t_n, u_n, \nu}) \right. \right. \right. \\ &\quad \left. \left. \left. - \Phi(s, Q_s^{t_n, q_n, \nu}, U_s^{t_n, u_n, \nu}) \right) + \lambda_s^b \left(\Phi(s, Q_s^{t_n, q_n, \nu} + 1, U_s^{t_n, u_n, \nu}) \right. \right. \\ &\quad \left. \left. \left. - \Phi(s, Q_s^{t_n, q_n, \nu}, U_s^{t_n, u_n, \nu}) \right) \right) \right] \mathrm{d}s \\ &\leq \eta + \mathbb{E} \left[\Phi(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) + \int_{t_n}^{\theta_n^\nu} \left(\nu_s^a \lambda_s^a + \nu_s^b \lambda_s^b \right. \right. \\ &\quad \left. \left. - \phi(Q_s^{t_n, q_n, \nu})^2 + Q_s^{t_n, q_n, \nu} \mu - \eta \mathbf{q} \sigma Q_s^{t_n, q_n, \nu} U_s^{t_n, u_n, \nu} \right) \mathrm{d}s \right], \\ &= \eta + \mathbb{E} \left[\Phi(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) + \int_{t_n}^{\theta_n^\nu} \left(\nu_s^a \lambda_s^a + \nu_s^b \lambda_s^b \right. \right. \\ &\quad \left. \left. - \phi(Q_s^{t_n, q_n, \nu})^2 + Q_s^{t_n, q_n, \nu} \mu - \eta \mathbf{q} \sigma Q_s^{t_n, q_n, \nu} U_s^{t_n, u_n, \nu} \right) \mathrm{d}s \right], \end{aligned}$$

and using equation (32) we obtain

$$\begin{aligned} V(t_n, q_n, u_n) &\leq -\eta + \mathbb{E} \left[V_*(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) + \int_{t_n}^{\theta_n} (\nu_s^a \lambda_s^a + \nu_s^b \lambda_s^b \right. \\ &\quad \left. - \phi(Q_s^{t_n, q_n, \nu})^2 + Q_s^{t_n, q_n, \nu} \mu - \eta \mathfrak{q} \sigma Q_s^{t_n, q_n, \nu} U_s^{t_n, u_n, \nu}) ds \right]. \end{aligned}$$

Since $\eta > 0$ is independent of ν , it follows that the latter inequality is in contradiction with the second inequality of the dynamic programming principle (23). Then V is a viscosity super-solution. Now, let us prove that V is a viscosity sub-solution.

Consider (t, q, u) and $\Phi \in C^{1,2}$ such that

$$0 = (V^* - \Phi)(t, q, u) = \max(V^* - \Phi),$$

and such that for all $(s, j, v) \neq (t, q, u)$, $(V - \Phi)(s, j, v) < (V^* - \Phi)(t, q, u)$. Let assume by contradiction that there exists $(t, q, u) \in \mathcal{E}$ such that

$$h(t, q, u) := F(t, q, u, \Phi(t, q, u), \partial_u \Phi(t, q, u), \partial_{u,u}^2 \Phi(t, q, u)) < 0.$$

By continuity of the function Φ and the Hamiltonian, there exists an open neighbourhood B_r of (t, q, u) such that

$$h(t, q, u) := F(t, q, u, \Phi(t, q, u), \partial_u \Phi(t, q, u), \partial_{u,u}^2 \Phi(t, q, u)) < 0 \quad \text{on } B_r.$$

We define

$$-2\eta := \max_{\partial B_r \cup J(t, q, u)} (V^* - \Phi) < 0.$$

Now, we consider (t_n, q_n, u_n) a sequence such that

$$(t_n, q_n, u_n) \rightarrow (t, q, u) \text{ and } V(t_n, q_n, u_n) \rightarrow V^*(t, q, u).$$

Since $(V - \Phi)(t_n, q_n, u_n) \rightarrow 0$, we can assume that the sequence (t_n, q_n, u_n) also satisfies

$$|(V - \Phi)|(t_n, q_n, u_n) \leq \eta \text{ for all } n \geq 1. \quad (33)$$

Then, for an arbitrary control $\nu \in \mathcal{U}_{t_n}$, we define the stopping time

$$\theta_n^\nu := \inf\{t > t_n : (t, Q_t^{t_n, q_n, \nu}, U_t^{t_n, u_n, \nu}) \notin B_r\},$$

and we observe that $(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) \in \partial B_r \cup J(t, q, u)$. Then, we have

$$\Phi(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) \geq 2\eta + V^*(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}). \quad (34)$$

Now, using equation (33) and Itô formula, we obtain

$$\begin{aligned}
V(t_n, q_n, u_n) &\geq -\eta + \Phi(t_n, q_n, u_n) \\
&= -\eta + \mathbb{E} \left[\Phi(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) \right] \\
&\quad - \mathbb{E} \left[\int_{t_n}^{\theta_n^\nu} \left(\partial_t \Phi(s, Q_s^{t_n, q_n, \nu}, U_s^{t_n, u_n, \nu}) - \eta U_s^{t_n, u_n, \nu} \partial_u \Phi(s, Q_s^{t_n, q_n, \nu}, U_s^{t_n, u_n, \nu}) \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \partial_{u,u}^2 \Phi(s, Q_s^{t_n, q_n, \nu}, U_s^{t_n, u_n, \nu}) + \lambda_s^a (\Phi(s, Q_s^{t_n, q_n, \nu} - 1, U_s^{t_n, u_n, \nu}) \right. \right. \\
&\quad \left. \left. - \Phi(s, Q_s^{t_n, q_n, \nu}, U_s^{t_n, u_n, \nu})) + \lambda_s^b (\Phi(s, Q_s^{t_n, q_n, \nu} + 1, U_s^{t_n, u_n, \nu}) \right. \right. \\
&\quad \left. \left. - \Phi(s, Q_s^{t_n, q_n, \nu}, U_s^{t_n, u_n, \nu})) \right) ds \right] \\
&\geq -\eta + \mathbb{E} \left[\Phi(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) \right. \\
&\quad \left. + \int_{t_n}^{\theta_n^\nu} \left(\nu_s^a \lambda_s^a + \nu_s^b \lambda_s^b - \phi(Q_s^{t_n, q_n, \nu})^2 \right. \right. \\
&\quad \left. \left. + Q_s^{t_n, q_n, \nu} \mu - \eta \mathfrak{q} \sigma Q_s^{t_n, q_n, \nu} U_s^{t_n, u_n, \nu} \right) ds \right], \\
&= -\eta + \mathbb{E} \left[\Phi(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) \right. \\
&\quad \left. + \int_{t_n}^{\theta_n^\nu} \left(\nu_s^a \lambda_s^a + \nu_s^b \lambda_s^b - \phi(Q_s^{t_n, q_n, \nu})^2 \right. \right. \\
&\quad \left. \left. + Q_s^{t_n, q_n, \nu} \mu - \eta \mathfrak{q} \sigma Q_s^{t_n, q_n, \nu} U_s^{t_n, u_n, \nu} \right) ds \right],
\end{aligned}$$

and using equation (34) we obtain

$$\begin{aligned}
V(t_n, q_n, u_n) &\geq \eta + \mathbb{E} \left[V^*(\theta_n^\nu, Q_{\theta_n^\nu}^{t_n, q_n, \nu}, U_{\theta_n^\nu}^{t_n, u_n, \nu}) + \int_{t_n}^{\theta_n^\nu} (\nu_s^a \lambda_s^a + \nu_s^b \lambda_s^b \right. \\
&\quad \left. - \phi(Q_s^{t_n, q_n, \nu})^2 + Q_s^{t_n, q_n, \nu} \mu - \eta \mathfrak{q} \sigma Q_s^{t_n, q_n, \nu} U_s^{t_n, u_n, \nu}) ds \right].
\end{aligned}$$

Since $\eta > 0$ is independent of ν , it follows from the arbitrariness of $\nu \in \mathcal{U}_{t_n}$ that the latter inequality is in contradiction with the second inequality of the dynamic programming principle (23). Then V is a viscosity sub-solution and hence a viscosity solution. \square

3.2.2 Approximate solution to the full informed problem

Next, we study an approximate solution to the HJB in (20) with terminal condition $V(T, q, u) = -\alpha q^2$; in Appendix A.7, we show that this approximation is good when compared to a classical finite-difference scheme. To carry out the approximation, we take $-\delta_\infty = -\infty$, $\underline{q} = -\bar{q} = +\infty$ and $\mathcal{S}^+ = -\mathcal{S}^- = +\infty$. Moreover, we follow the ideas in [9], and consider that $\Delta^- V(t, q, u) := V(t, q - 1, u) - V(t, q, u)$ and $\Delta^+ V(t, q, u) := V(t, q + 1, u) - V(t, q, u)$ are small. Using this, we perform a second order Taylor

expansion around zero for $p \rightarrow \frac{1}{k} \exp(k p)$ of the exponential term. Formally, this means that

$$e^{k[V(t,q+1,u)-V(t,q,u)]} \approx \underbrace{1 + k[V(t,q+1,u) - V(t,q,u)] + \frac{1}{2}(k[V(t,q+1,u) - V(t,q,u)])^2}_{\hat{V}^a(V(q+1) - V(q))},$$

$$e^{k[V(t,q-1,u)-V(t,q,u)]} \approx \underbrace{1 + k[V(t,q-1,u) - V(t,q,u)] + \frac{1}{2}(k[V(t,q-1,u) - V(t,q,u)])^2}_{\hat{V}^b(V(q-1) - V(q))},$$

where we used the notation $V(q)$ instead of $V(t,q,u)$ to make the expression compact. The approximation transforms (20) into the following PDE

$$\begin{aligned} 0 = & \partial_t V - \eta u \partial_u V - \phi q^2 + \frac{1}{2} \partial_{u,u}^2 V + (\mu - \eta \sigma \mathfrak{q} u) q \\ & + \frac{1}{k} e^{-1} (1 + k \Delta^- V + \frac{k^2}{2} (\Delta^- V)^2) (\varphi + \psi e^{-\mathfrak{q} \sigma \gamma u}) \\ & + \frac{1}{k} e^{-1} (1 + k \Delta^+ V + \frac{k^2}{2} (\Delta^+ V)^2) (\varphi + \psi e^{\mathfrak{q} \sigma \gamma u}). \end{aligned}$$

Furthermore, let us consider $\gamma \mathfrak{q} u$ small enough so that we can also perform a first order Taylor expansion around zero for both of the exponential terms of $\pm \sigma \mathfrak{q} \gamma u$. We have that

$$\begin{aligned} 0 = & \partial_t V - \eta u \partial_u V - \phi q^2 + \frac{1}{2} \partial_{u,u}^2 V + (\mu - \eta \sigma \mathfrak{q} u) q \\ & + \frac{1}{k} e^{-1} (1 + k \Delta^- V + \frac{k^2}{2} (\Delta^- V)^2) (\varphi + \psi (1 - \mathfrak{q} \sigma \gamma u)) \\ & + \frac{1}{k} e^{-1} (1 + k \Delta^+ V + \frac{k^2}{2} (\Delta^+ V)^2) (\varphi + \psi (1 + \mathfrak{q} \sigma \gamma u)). \end{aligned} \tag{35}$$

We employ the ansatz that V is quadratic in q , i.e., there exists $A : \mathbb{R}^+ \rightarrow \mathbb{R}$ and $B, C : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$V(t, q, u) = q^2 A(t) + q B(t, u) + C(t, u).$$

We obtain the following characterisation of the approximate solution.

Proposition 10. *Let $A : \mathbb{R}^+ \rightarrow \mathbb{R}$ and $B, C : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a solution to the following system of PDEs*

$$\left\{ \begin{array}{l} 0 = \partial_t A - \phi + 4(\varphi + \psi) e^{-1} k A^2, \\ 0 = \partial_t B + \frac{1}{2} \partial_{u,u}^2 B + (\mu - \eta \sigma \mathfrak{q} u) - \eta u \partial_u B + 4(\varphi + \psi) e^{-1} k A B + 4e^{-1} \psi \mathfrak{q} \sigma \gamma u A \\ \quad + 4e^{-1} k \mathfrak{q} \gamma \sigma u \psi A^2, \\ 0 = \partial_t C + \frac{1}{2} \partial_{u,u}^2 C - \eta u \partial_u C + \frac{1}{k} e^{-1} (2(\varphi + \psi) + 2kA(\psi + \varphi) + 2\psi k B \mathfrak{q} \sigma \gamma u \\ \quad + k^2(\varphi + \psi)(A^2 + B^2) + 2k^2 \psi \mathfrak{q} \sigma \gamma u A B), \end{array} \right. \tag{36}$$

with terminal condition $A(T) = -\alpha$, $B(T, u) = 0$, $C(T, u) = 0$. Define

$$\tilde{V}(t, q, u) = q^2 A(t) + q B(t, u) + C(t, u). \tag{37}$$

It follows that \tilde{V} solves the approximate HJB (35).

Proof. The proof follows by direct substitution. \square

Next, we study the system in (36) in more detail. Standard results from the theory of Riccati differential equations show that if $\sqrt{\phi}$ is not equal to $\sqrt{4(\varphi + \psi)e^{-1}k}\alpha$, then there is a unique non trivial solution to the ODE satisfied by A , given by

$$A(t) = \frac{\sqrt{\phi}}{\sqrt{\kappa}} \frac{1 - e^{2\sqrt{\phi\kappa}(T-t)}\beta}{1 + e^{2\sqrt{\phi\kappa}(T-t)}\beta},$$

where $\beta = \frac{\sqrt{\phi} + \sqrt{\kappa}\alpha}{\sqrt{\phi} - \sqrt{\kappa}\alpha}$ and $\kappa = 4(\varphi + \psi)e^{-1}k$. If $\sqrt{\phi} = \sqrt{4(\varphi + \psi)e^{-1}k}\alpha$, then $A = -\alpha$ solves the ODE.

Then, we propose the ansatz that B is linear in u and C quadratic in u . We have the following result.

Proposition 11. Let $\sqrt{\phi} \neq \sqrt{\kappa}\alpha$ and let

$$b_0, b_1, c_0, c_1, c_2 : [0, T] \rightarrow \mathbb{R}$$

be the unique solution to the system of ODEs

$$\begin{cases} 0 = b'_0 + \mu + 4k(\psi + \varphi)e^{-1}Ab_0, \\ 0 = b'_1 - \eta\sigma q - \eta b_1 + 4(\psi + \varphi)e^{-1}Ak b_1 + 4e^{-1}\psi q\sigma\gamma A + 4e^{-1}k\gamma q\sigma\psi A^2, \\ 0 = c'_0 + c_2 + \frac{e^{-1}}{k}(2(\psi + \varphi) + 2kA(\varphi + \psi) + k^2(\varphi + \psi)(A^2 + b_0^2)), \\ 0 = c'_1 - \eta c_1 + \frac{e^{-1}}{k}(2\psi k\sigma\gamma q b_0 + k^2(\psi + \varphi)(2b_0 b_1) + 2k^2\psi\gamma\sigma q A b_0), \\ 0 = c'_2 - 2\eta c_2 + \frac{e^{-1}}{k}(k^2(\psi + \varphi)b_1^2 + 2k^2\psi\sigma\gamma q A b_1 + 2\psi k\sigma\gamma q b_1), \end{cases} \quad (38)$$

with terminal condition zero. Then

$$B(t, u) := b_0(t) + u b_1(t), \quad (39)$$

$$C(t, u) := c_0(t) + u c_1(t) + u^2 c_2(t), \quad (40)$$

solve the PDEs in (36).

Proof. The proof follows by direct substitution. \square

Existence and uniqueness of the ODE follows by the existence and uniqueness of the Riccati ODE solved by A , and that all the other ODEs are linear. The explicit formula can be obtain through the Duhammel's formula.

4 Partial Information

4.1 Deriving the filtered processes dynamics

In this section we consider the case where the market maker does not have perfect information. More precisely, the information comes only from mid-prices, *i.e.*, $(S_t)_{t \in \mathfrak{T}}$. We define $(\mathcal{F}_t^S)_{t \in \mathfrak{T}}$ the filtration generated by $(S_t)_{t \in \mathfrak{T}}$. We directly remark that $\mathcal{F}_t^S \subset \mathcal{F}_t$ for $t \in \mathfrak{T}$, and that U_t is not \mathcal{F}_t^S -measurable. The goal of this part is to formulate a stochastic control problem with complete information that employs filters to obtain the best estimate of the unobserved fad process.¹² Thus, we face the problem of filtering the fads U_t with knowledge of $(S_u)_{u \in [0,t]}$, and where

$$dU_t = f(U_t)dt + dB_t, \quad (41)$$

$$dS_t = h(U_t)dt + \sigma d\bar{W}_t, \quad (42)$$

with $f(x) = -\eta x$, $h(x) = \mu - \eta \mathbf{q} \sigma x$, and where $\bar{W}_t = \mathbf{p} Z_t + \mathbf{q} B_t$ is a Brownian Motion, with $\langle B, \bar{W} \rangle_t = \mathbf{q} t$. We define the innovation process $(I_t)_{t \in \mathfrak{T}}$ as

$$dI_t = dS_t - \pi_t(h)dt, \quad I_0 \in \mathbb{R}, \quad (43)$$

where $\pi_t(h) := \mathbb{E}[h(U_t) | \mathcal{F}_t^S]$.

The following proposition shows that $\sigma^{-1}(I_t - I_0)$ is a Brownian motion which later on we use to derive the dynamics of the filtered processes.

Proposition 12. *If $\mathbb{E}\left[\int_0^t |h(U_s)| ds\right] < \infty$, then $\tilde{I}_t := \sigma^{-1}(I_t - I_0)$ is a \mathcal{F}_t^S Brownian Motion under \mathbb{P} .*

Proof. The proof is standard so we only describe the steps. First, we prove that I is an $L^2(\Omega, \mathcal{F}^S, \mathbb{P})$ martingale. Second, we show that $\langle I \rangle_t = \sigma^2 t$, and we conclude with the Lévy characterisation of the Brownian Motion; see Proposition 2.30 in [3]. Finally, the condition $\mathbb{E}\left[\int_0^t |h(U_s)| ds\right] < \infty$ holds since $(U_t)_{t \in \mathfrak{T}}$ is a continuous Gaussian process. \square

Proposition 13. *If $\mathbb{P}\left(\int_0^t |\pi_t(h)|^2 ds < \infty\right) = 1$, then for all $L^2(\Omega, \mathcal{F}^S, \mathbb{P})$ -martingales $(\eta_t)_{t \in \mathfrak{T}}$, we have a martingale representation of the form*

$$\eta_t = \eta_0 + \int_0^t \nu_s \sigma^{-1} dI_s,$$

where $(\nu_t)_{t \in \mathfrak{T}}$ is $(\mathcal{F}_t^S)_{t \in \mathfrak{T}}$ -progressively measurable.

Proof. The proof follows from Proposition 2.31 in [3]. The condition $\mathbb{P}\left(\int_0^t |\pi_t(h)|^2 ds < \infty\right) = 1$ holds since we have $\mathbb{E}\left[\int_0^t |\pi_s(h)|^2 ds\right] < \infty$ by Jensen inequality and the fact that $(U_t)_{t \in \mathfrak{T}}$ is a continuous Gaussian process. \square

Next we define

$$\hat{U}_t := \mathbb{E}[U_t | \mathcal{F}_t^S], \quad (44)$$

¹²This is similar to recent works in the mathematical finance literature such as [7, 37].

and $\Gamma_t := \hat{U}_t + \int_0^t \eta \hat{U}_s ds$. We then have the following result.

Proposition 14. *The process $(\Gamma_t)_{t \in \mathfrak{T}}$ is an $(\mathcal{F}_t^S)_{t \in \mathfrak{T}}$ -martingale under \mathbb{P} .*

Proof. By construction Γ_t is \mathcal{F}_t^S -measurable for all $t > 0$. Moreover, since $U \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, then, by the Jensen inequality, $\hat{U} \in L^1(\Omega, \mathcal{F}^S, \mathbb{P})$. Then, take $0 \leq s \leq t$, it follows that

$$\mathbb{E}[\Gamma_t | \mathcal{F}_s^S] = \mathbb{E}[\mathbb{E}[U_t | \mathcal{F}_t^S] | \mathcal{F}_s^S] + \int_0^s \eta \hat{U}_r dr + \int_s^t \eta \mathbb{E}[\hat{U}_r | \mathcal{F}_s^S] dr. \quad (45)$$

The first term equals $\mathbb{E}[U_t | \mathcal{F}_s^S]$, using $\mathcal{F}_s^S \subset \mathcal{F}_t^S$ and the tower property. The third equals $\int_s^t \eta \mathbb{E}[U_r | \mathcal{F}_s^S] dr$ by the same argument. Moreover,

$$\mathbb{E}[U_t | \mathcal{F}_s^S] = \mathbb{E}\left[-\eta \int_s^t U_r dr | \mathcal{F}_s^S\right] + \mathbb{E}[B_t - B_s | \mathcal{F}_s^S] + \hat{U}_s = \mathbb{E}\left[-\eta \int_s^t U_r dr | \mathcal{F}_s^S\right] + \hat{U}_s,$$

by the tower property. Finally, it follows that $\mathbb{E}[\Gamma_t | \mathcal{F}_s^S] = \Gamma_s$ and $(\Gamma_t)_{t \in \mathfrak{T}}$ is an $(\mathcal{F}_t^S)_{t \in \mathfrak{T}}$ -martingale under \mathbb{P} . \square

The following theorem presents the dynamic of the filtering process $(\hat{U}_t)_{t \in \mathfrak{T}}$ which we use to formulate a stochastic control problem with complete information.

Theorem 3. *The process \hat{U}_t satisfies*

$$d\hat{U}_t = -\eta \hat{U}_t dt + \sigma^{-1}(-\eta \mathbf{q} \hat{P}_t + \mathbf{q}) dI_t, \quad (46)$$

where $\hat{P}_t = \mathbb{E}[(U_t - \hat{U}_t)^2]$ is the conditional variance and is the unique solution of the Riccati equation

$$\frac{d}{dt} \hat{P}_t = -\eta^2 \mathbf{q}^2 \hat{P}_t^2 - \hat{P}_t (2\eta - 2\eta \mathbf{q}^2) + \mathbf{p}^2, \quad \hat{P}_0 = 0. \quad (47)$$

Given the dynamics of $(\hat{U}_t)_{t \in \mathfrak{T}}$, the dynamics of $(S_t)_{t \in \mathfrak{T}}$ can be rewritten using the Brownian motion $(\tilde{I}_t)_{t \in \mathfrak{T}}$ as

$$dS_t = \sigma d\tilde{I}_t + (\mu - \eta \sigma \mathbf{q} \hat{U}_t) dt.$$

Following the motivation in the previous section, our goal is to solve a control problem of the form

$$\sup_{\delta^{a,b}} \mathbb{E} \left[X_T + Q_T S_T - \alpha Q_T^2 - \phi \int_0^T Q_u^2 du \right]. \quad (48)$$

In line with the full information model, here, we model $(N_t^a)_{t \in \mathfrak{T}}$ and $(N_t^b)_{t \in \mathfrak{T}}$ to be counting processes with intensities

$$\lambda_t^a = \left(\varphi e^{-k\delta_t^a} + \psi e^{-k\delta_t^a - \gamma(\mathbf{q} \sigma \hat{U}_t) \vee S^-} \right) \mathbf{1}_{\{Q_{t-} > q\}}, \quad (49)$$

$$\lambda_t^b = \left(\varphi e^{-k\delta_t^b} + \psi e^{-k\delta_t^b + \gamma(\mathbf{q} \sigma \hat{U}_t) \wedge S^+} \right) \mathbf{1}_{\{Q_{t-} < \bar{q}\}}, \quad (50)$$

where $\varphi, \psi, k, \mathcal{S}^+, \mathcal{S}^-$ are as before.¹³ The inventory Q_t and the cash X_t are defined as in (4) and (7). Next, we state the derived full information control problem we associate with the partial information case. The agent wishes to solve the control problem

$$V(t, q, \hat{u}) = \sup_{\delta^{a,b} \in \mathcal{A}_{t,T}} \mathbb{E}_{t,q,u} \left[-\alpha Q_T^2 + \int_t^T (\delta_s^a \lambda_s^a + \delta_s^b \lambda_s^b - \phi Q_s^2 + (\mu - \eta q \sigma \hat{U}_s) Q_s) ds \right], \quad (51)$$

where $\mathcal{A}_{t,T}$ are controls bounded from below by $-\delta_\infty$ and predictable with respect to the filtration $(\hat{\mathcal{F}}_t)_{t \in \mathfrak{T}}$ generated by $(\tilde{I}_t)_{t \in \mathfrak{T}}$, $(\hat{U}_t)_{t \in \mathfrak{T}}$ and $(N_t^{a,b})_{t \in \mathfrak{T}}$ and where

$$\begin{aligned} d\hat{U}_t &= -\eta \hat{U}_t dt - (\hat{P}_t q \eta - q) d\tilde{I}_t, & \hat{U}_0 &= 0, \\ dQ_t &= dN_t^b - dN_t^a, & Q_0 &\in \mathcal{Q}. \end{aligned}$$

The processes $(N_t^{a,b})_{t \in \mathfrak{T}}$ count the number of orders filled by the market maker and have controlled stochastic intensities $(\lambda_t^{a,b})_{t \in \mathfrak{T}}$ defined in (49) and (50).

Before characterising the solution to the partial information problem, it is important to provide a rigorous characterisation of the probability space we employ.

4.2 Formal derivation of the probability space

Consider Ω_d the set of increasing piecewise constant càdlàg functions from $[0, T]$ into \mathbb{N} with jumps equal to one and Ω_c the set of continuous functions from $[0, T]$ into \mathbb{R} . We define $\Omega = \Omega_c \times \Omega_d^2$ as the sample space. We let $(\tilde{I}, N_t^a, N_t^b)_{t \in \mathfrak{T}}$ be the canonical process on Ω . The associated filtration is $\hat{\mathcal{F}} = (\mathcal{F}_t^c \otimes \mathcal{F}_t^d \otimes \mathcal{F}_t^d)_{t \in \mathfrak{T}}$ where $(\mathcal{F}_t^d)_{t \in \mathfrak{T}}$ (resp. $(\mathcal{F}_t^c)_{t \in \mathfrak{T}}$) is the right continuous completed filtration associated with N^a (or N^b) (resp. \tilde{I}). We denote by \mathbb{P}_0 the probability measure on (Ω, \mathcal{F}) such that $(M_s^a = N_s^a - \int_0^t \mathbf{1}_{\{N_{s-}^b - N_{s-}^a > q\}} ds, M_s^b = N_s^b - \int_0^t \mathbf{1}_{\{N_{s-}^b - N_{s-}^a < \bar{q}\}} ds)_{s \in \mathfrak{T}}$ are martingales and $(\tilde{I}_t)_{t \in \mathfrak{T}}$ is a Brownian motion.¹⁴ We also assume independence between the processes $(N_t^a)_{t \in \mathfrak{T}}$, $(N_t^b)_{t \in \mathfrak{T}}$ and the Brownian motion. Next, we define $\mathbb{X}_t = (\hat{U}_t, Q_t)$ to be the state of the system, where the dynamics of $(\mathbb{X}_t)_{t \in \mathfrak{T}}$ are given by

$$\begin{cases} dQ_t = dN_t^b - dN_t^a, & Q_0 \in \mathcal{Q}, \\ d\hat{U}_t = -\eta \hat{U}_t dt + (q - \hat{P}_t q \eta) d\tilde{I}_t, & \hat{U}_0 = 0, \end{cases} \quad (52)$$

and we define an equivalent probability measure \mathbb{P}^δ through the Doléans-Dade exponential of

$$\mathcal{Z}_t^\delta = \int_0^t (\lambda_s^a - 1) dM_s^a + \int_0^t (\lambda_s^b - 1) dM_s^b.$$

¹³One way to motivate this choice of stochastic intensity is that informed traders filter the fad, and then, market arrivals follow the same structure as before, but now depending on the filter of the fad. Another way to understand this structure is that the market maker uses this model as an approximation of reality; that is, the market maker faces informed traders that know exactly what the fad is, but without the knowledge of $(U_t)_{t \in \mathfrak{T}}$, she uses the filter $(\hat{U}_t)_{t \in \mathfrak{T}}$ as a proxy. In Section 6 we explore an alternative formulation.

¹⁴The construction of $(N_t^a)_{t \in \mathfrak{T}}$ and $(N_t^b)_{t \in \mathfrak{T}}$ follows the same idea as in the full information setting.

Under \mathbb{P}^δ , $(\tilde{I}_t)_{t \in \mathfrak{T}}$ is a Brownian Motion, and $(N_t^a)_{t \in \mathfrak{T}}$ and $(N_t^b)_{t \in \mathfrak{T}}$ are Poisson processes with stochastic intensities given by

$$\lambda^a(\hat{U}_t, Q_{t-}, \delta_t^a) := \lambda_t^a = (\varphi + \psi e^{-\gamma(\sigma \mathbf{q} \hat{U}_t \vee \mathcal{S}^-)}) e^{-k \delta_t^a} \mathbf{1}_{\{Q_{t-} > \underline{q}\}}, \quad (53)$$

$$\lambda^b(\hat{U}_t, Q_{t-}, \delta_t^b) := \lambda_t^b = (\varphi + \psi e^{\gamma(\sigma \mathbf{q} \hat{U}_t \wedge \mathcal{S}^+)}) e^{-k \delta_t^b} \mathbf{1}_{\{Q_{t-} < \bar{q}\}}. \quad (54)$$

We refer to the expectation with respect to \mathbb{P}^δ as \mathbb{E} instead of \mathbb{E}^δ .

4.3 Characterisation of the solution

The corresponding HJB equation associated with the above control problem is

$$\begin{aligned} 0 = & \partial_t V(t, q, \hat{u}) - \phi q^2 + +q(\mu - \sigma \eta \mathbf{q} \hat{u}) \\ & - \partial_u V(t, q, \hat{u}) \eta \hat{u} + \frac{1}{2} \partial_{u,u}^2 V(t, q, \hat{u}) (\hat{P}_t \mathbf{q} \eta - \mathbf{q})^2 \\ & + \sup_{\delta^a} \{(\varphi e^{-k \delta^a} + \psi e^{-k \delta^a - \gamma(\mathbf{q} \sigma \hat{u}) \vee \mathcal{S}^-})(\delta^a + V(t, q-1, \hat{u}) - V(t, q, \hat{u}))\} \mathbf{1}_{q > \underline{q}} \\ & + \sup_{\delta^b} \{(\varphi e^{-k \delta^b} + \psi e^{-k \delta^b + \gamma(\mathbf{q} \sigma \hat{u}) \wedge \mathcal{S}^+})(\delta^b + V(t, q+1, \hat{u}) - V(t, q, \hat{u}))\} \mathbf{1}_{q < \bar{q}}, \end{aligned} \quad (55)$$

where the optimal controls in feedback form are given by

$$\begin{aligned} \delta^{a,*} &= \left(\frac{1}{k} - V(t, q-1, u) + V(t, q, u) \right) \vee -\delta_\infty^a, \\ \delta^{b,*} &= \left(\frac{1}{k} - V(t, q+1, u) + V(t, q, u) \right) \vee -\delta_\infty^b. \end{aligned}$$

This transforms the HJB equation into the following PDE

$$\begin{aligned} 0 = & \partial_t V(t, q, u) - \phi q^2 + q(\mu - \sigma \eta \mathbf{q} u) \\ & - \partial_u V(t, q, u) \eta u + \frac{1}{2} \partial_{u,u}^2 V(t, q, u) (\hat{P}_t \mathbf{q} \eta - \mathbf{q})^2 \\ & + \frac{1}{k} \exp(-1 + k[V(t, q-1, u) - V(t, q, u)]) (\varphi + \psi e^{-\gamma(\mathbf{q} \sigma u) \vee \mathcal{S}^-}) \mathbf{1}_{q > \underline{q}} \mathbf{1}_{\{\delta^{a,*} > -\delta_\infty^a\}} \\ & + (\varphi e^{k \delta_\infty^a} + \psi e^{k \delta_\infty^a - \gamma(\mathbf{q} \sigma \hat{u}) \vee \mathcal{S}^-})(-\delta_\infty^a + V(t, q-1, \hat{u}) - V(t, q, \hat{u})) \mathbf{1}_{q > \underline{q}} \mathbf{1}_{\{\delta^{a,*} = -\delta_\infty^a\}} \\ & + (\varphi e^{k \delta_\infty^b} + \psi e^{k \delta_\infty^b + \gamma(\mathbf{q} \sigma \hat{u}) \wedge \mathcal{S}^+})(-\delta_\infty^b + V(t, q+1, \hat{u}) - V(t, q, \hat{u})) \mathbf{1}_{q < \bar{q}} \mathbf{1}_{\{\delta^{b,*} = -\delta_\infty^b\}} \\ & + \frac{1}{k} \exp(-1 + k[V(t, q+1, u) - V(t, q, u)]) (\varphi + \psi e^{\gamma(\mathbf{q} \sigma u) \wedge \mathcal{S}^+}) \mathbf{1}_{q < \bar{q}} \mathbf{1}_{\{\delta^{b,*} > -\delta_\infty^b\}}, \end{aligned} \quad (56)$$

with terminal condition $V(T, q, u) = -\alpha q^2$.

4.3.1 Viscosity solution: incomplete information problem

Again, the value function V is a viscosity solution of the HJB equation (55). We remark that the steps we followed in the proof of the viscosity solution of Section 3 still hold in this setting. We omit the details for brevity.

4.3.2 Approximate solution to the partial informed problem

Similar to Section 3.2.2, we take $-\delta_\infty^{a,b} = -\infty$, $\bar{q} = -\bar{q} = +\infty$ and $\mathcal{S}^+ = -\mathcal{S}^- = +\infty$. We assume that $V(t, q-1, u) - V(t, q, u)$ and $V(t, q+1, u) - V(t, q, u)$ are small and we perform a second order Taylor expansion around zero for $p \rightarrow \frac{1}{k} \exp(k p)$ as before. Furthermore, we perform a first order Taylor expansion of the exponential functions of $\mp q \gamma \sigma u$. To simplify the notation further, we define $x_1 := \mu - \sigma q \eta u$, and $x_2 := -\hat{P}_t q \eta + q$. As a consequence of the second order approximation, and employing an ansatz for V that is quadratic in q , we show that the approximate solution satisfies a system of PDEs that can be easily solved numerically. The result is summarised in the following proposition.

Proposition 15. *Let $A : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $B, C : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a solution to the system*

$$\begin{cases} \partial_t A = -\phi + 4e^{-1}(\varphi + \psi)kA^2, \\ \partial_t B = \partial_u B \eta u - \frac{1}{2}\partial_{u,u}^2 B x_2^2 + x_1 + \psi e^{-1}(4A q \gamma \sigma u + 4kq \sigma \gamma u A^2) \\ \quad + 4e^{-1}(\varphi + \psi)kAB, \\ \partial_t C = \partial_u C \eta u - \frac{1}{2}\partial_{u,u}^2 C x_2^2 + \psi e^{-1}(-2B q \sigma \gamma u + 2ABkq \sigma \gamma u) \\ \quad + \frac{e^{-1}(\varphi+\psi)}{k}(2 - 2kA + k^2 A^2 + k^2 B^2), \end{cases} \quad (57)$$

with terminal conditions

$$A(T) = \alpha, B(T, u) = 0, C(T, u) = 0. \quad (58)$$

Then, $\check{V}(t, q, u) = -q^2 A(t) - q B(t, u) - C(t, u)$ is a solution to the approximation of the PDE in (56).

Proof. The proof follows by direct substitution of the derivatives of \check{V} in the quadratic approximation we obtained for the PDE satisfied by V . \square

Next, we use an ansatz in which the function B is linear in u and C is quadratic in u . We have the following result.

Proposition 16. *Let $A : [0, T] \rightarrow \mathbb{R}$ solve the Riccati ODE*

$$A' = -\phi + 4e^{-1}(\varphi + \psi)kA^2, \quad A(T) = \alpha. \quad (59)$$

Let $(b_0, b_1, c_0, c_1, c_2) : \mathbb{R}^+ \rightarrow \mathbb{R}$ solve the system of ODEs

$$\begin{cases} b'_0 = \mu + 4e^{-1}(\varphi + \psi)kb_0A, \\ b'_1 = \eta b_1 - \eta \sigma q + \psi e^{-1}(-4A \gamma \sigma q + 4kq \gamma \sigma A^2) + 4e^{-1}(\varphi + \psi)kb_1A, \\ c'_0 = -x_2^2 c_2 + \frac{e^{-1}}{k}(\varphi + \psi)(2 - 2kA + k^2 A^2 + k^2 b_0^2), \\ c'_1 = \eta c_1 - 2e^{-1}\psi b_0 q \gamma \sigma + \Psi e^{-1}2k \gamma \sigma q b_0 A + 2\frac{e^{-1}}{k}(\varphi + \psi)k^2 b_0 b_1, \\ c'_2 = 2\eta c_2 - 2\psi e^{-1}b_1 \gamma \sigma q + 2\psi e^{-1}A k \gamma \sigma q b_1 + \frac{e^{-1}}{k}(\varphi + \psi)b_1^2 k^2, \end{cases} \quad (60)$$

Define $\hat{B}, \hat{C} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$\begin{aligned}\hat{B}(t, u) &= u b_1(t) + b_0(t), \\ \hat{C}(t, u) &= u^2 c_2(t) + u c_1(t) + c_0(t).\end{aligned}$$

It follows that (A, \hat{B}, \hat{C}) solves the system of PDEs in (57).

4.4 Connection with the separation principle

In this section we computed the filter of the process $(U_t)_{t \in \mathfrak{T}}$ with respect to the filtration generated by $(S_t)_{t \in \mathfrak{T}}$, and then we constructed new stochastic processes $(N_t^{a,b})_{t \in \mathfrak{T}}$ where their stochastic intensities depend on $(\hat{U}_t)_{t \in \mathfrak{T}}$ instead of $(U_t)_{t \in \mathfrak{T}}$. Then, as shown above, the optimal strategies are akin to those from the full information setup but using $(\hat{U}_t)_{t \in \mathfrak{T}}$ instead of $(U_t)_{t \in \mathfrak{T}}$. This modelling approach is similar to employing the separation principle. Quoting [8], “the result says that it is optimal to estimate the state and then apply the rule to the estimate, instead of the state itself”. Of course, this is not the optimal approach when one works in a non-linear-quadratic framework; however, as we show below, there is non-negligible economic value when employing this modelling approach.

5 Sensitivity analysis and simulations

In this section we investigate the optimal strategies derived in the previous sections. Our main goals are: (i) to understand the value of information for the market maker, (ii) to investigate the dependence of the optimal quotes on key model parameters, and (iii) to study the performance of the market maker as a function of the percentage of informed traders in the market. To this end, we develop a sensitivity analysis of the optimal solutions with respect to key parameters and we simulate the model. Similar to [10], we consider the following baseline set of parameters: $\alpha = 0.001, \sigma = 1, \phi = 0.1, \varphi = 15, k = 1, \gamma = 1, T = 1, \mu = 0, \eta = 10$. The value of q modulates the extent to which the fad influences the price process. We set $q = 0.6$ and $p = \sqrt{1 - q^2}$. Note that $k = \gamma$ and, therefore, informed traders are equally sensitive to the displacement of the quotes and to the fad.

Note that the baseline expected number of market arrivals on the bid and ask side, i.e., (5)-(6) without displacements, is $\varphi T + \psi \int_0^T \mathbb{E}[e^{\pm \gamma \sigma q U_t}] dt$. In what follows, we set

$$\psi = \frac{30 - \varphi T}{\int_0^T \mathbb{E}[e^{\pm \gamma \sigma q U_t}] dt} \tag{61}$$

so that the baseline proportion of market arrivals from informed and uninformed traders is constant (50-50%) and the expected arrivals are 30.

In this setting, the fad only affects the market maker through the arrival rates of orders (5)-(6) because the mark-to-market of the inventory in (8) at time T is evaluated at the mid-price and not at the fundamental. In Section A.5 we study the case where the market marker values the inventory at fundamental.

5.1 Optimal strategies under full information

In Figure 1 we show the optimal ask and bid displacements of the market maker (in solid and dashed lines respectively) in the full information setting as a function of the fad (in the x -axis) varying the inventory level in the colour bar (the lighter the colour, the higher the inventory). The left panel shows the optimal displacements when $q = 0.3$, the middle panel shows the optimal displacements when $q = 0.6$, and the right panel shows the optimal displacements when $q = 0.9$.

As expected, and confirming classical results, the higher is the market maker's inventory the lower is the ask displacement ($\delta^{a*,\text{FI}}$) and the higher the bid displacement ($\delta^{b*,\text{FI}}$).

The effect of the fad on the displacements is asymmetric. The ask displacement ($\delta^{a*,\text{FI}}$) is decreasing in the fad U_t , the opposite is observed for the bid displacement ($\delta^{b*,\text{FI}}$). The rationale of this result is that as U_t increases, λ^a decreases and λ^b increases due to the informed traders' market activity. Given the performance criterion (8), the market maker values her open inventory at mid-price, and therefore she decreases the price of liquidity in the ask and increases the price of liquidity in the bid to manage the inventory in an optimal way. Moreover, in agreement with this argument, the absolute value of the slope of the displacements with respect to the fad component decreases as q becomes smaller. Indeed, the closer q is to zero, the less impact the fad has on the mid-price process and on the arrival rates of informed traders (informed and uninformed traders behave in a similar way), and therefore the dependence of the optimal displacements on the fad weakens. In the limit, when $q = 0$, the slope is zero.

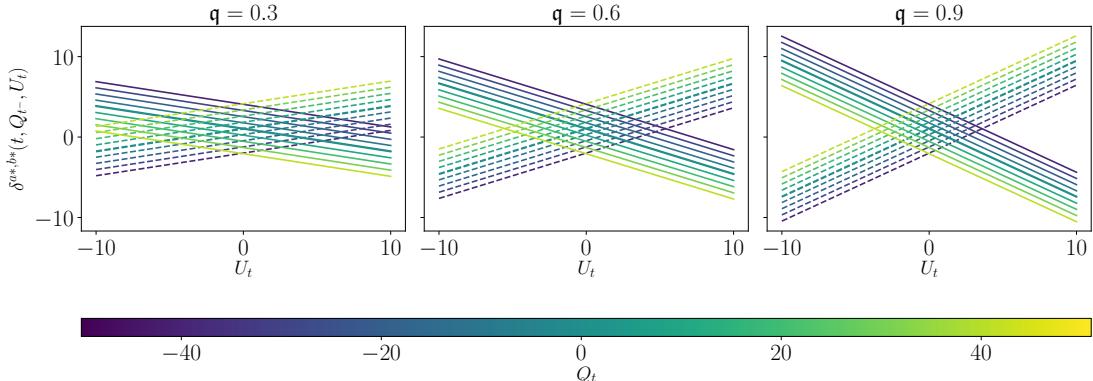


Figure 1: Ask and bid displacements as a function of the fad (x -axis) and the inventory of the market maker (colour bar) under full information. Solid lines: optimal ask displacement; dashed lines: optimal bid displacement. Left panel is for $q = 0.3$, middle panel is for $q = 0.6$, and right panel is for $q = 0.9$.

For a given path of the fad and of the asset price, Figure 2 shows the dynamics of the inventory $(Q_t)_{t \in \mathfrak{T}}$ (in red) and the fad $(U_t)_{t \in \mathfrak{T}}$ (in blue) for two values of γ ($\gamma = 0.1$ and $\gamma = 10$) which modulates the sensitivity of informed traders to the fad. Note that the baseline sensitivity is $\gamma = k = 1$. We observe that a high value of γ leads to a high sensitivity in the number of trades (by informed traders) with respect to U_t . On the contrary, when γ is small, the executed trades are less related to changes of the value of U_t . There is also a higher correlation between the fad $(U_t)_{t \in \mathfrak{T}}$ and the inventory $(Q_t)_{t \in \mathfrak{T}}$ as γ increases.¹⁵

¹⁵The mean correlation between the realizations of the paths of U and Q (1000 timesteps) over 1000 simulations is 0.49 when

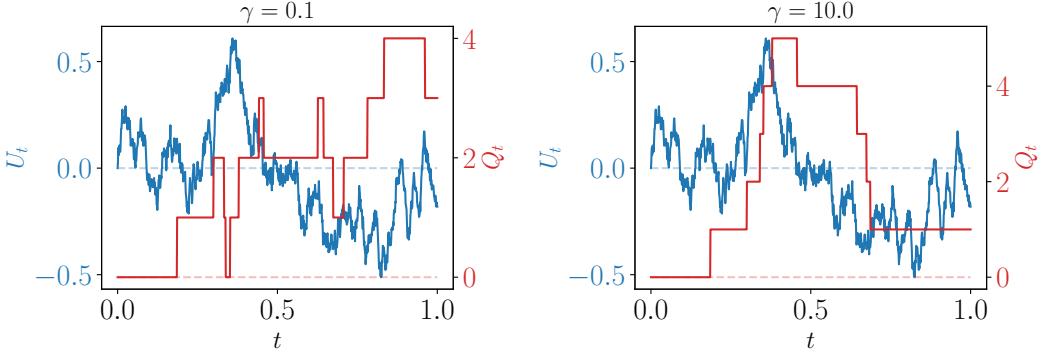


Figure 2: Sample paths followed by $(Q_t)_{t \in \mathfrak{T}}$ and $(U_t)_{t \in \mathfrak{T}}$ for $\gamma = 0.1$ (left panel) and $\gamma = 10$ (right panel).

5.2 Simulations of optimal strategies under partial information

Figure 3 shows a simulation path of $(U_t)_{t \in \mathfrak{T}}$ and its filter $(\hat{U}_t)_{t \in \mathfrak{T}}$ for three values of q . We observe that the closer q is to one, the closer the filter is to the fad process. Indeed, the more the price is driven by the fad, the more the filter mimics the true trajectory because the signal is more precise. On the other hand, when q approaches zero, knowledge of the price is not valuable to filter the fad, and so the filter becomes uninformative and collapses around the mean of the fad, which is zero.

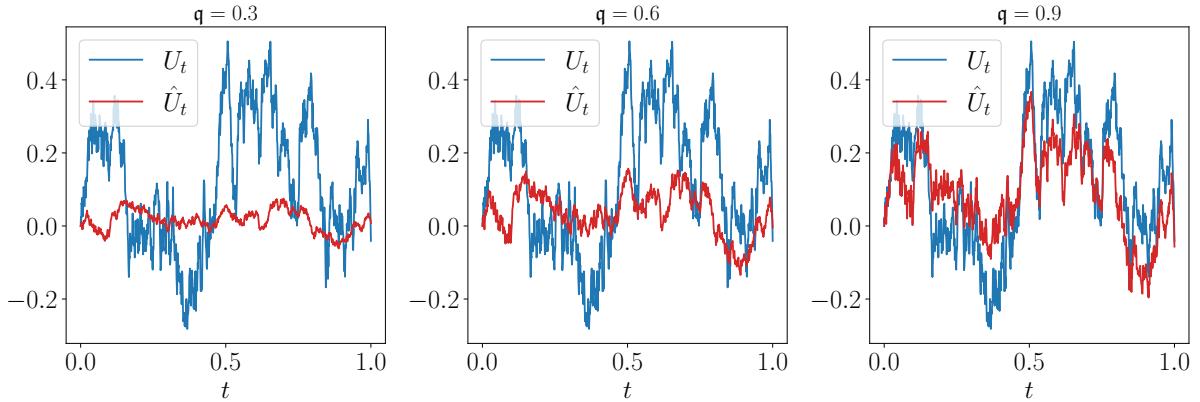


Figure 3: Comparison between the process $(U_t)_{t \in \mathfrak{T}}$, in blue, and the filter $(\hat{U}_t)_{t \in \mathfrak{T}}$, in red, for (from left to right) $q = 0.3$, $q = 0.6$, and $q = 0.9$.

The displacements obtained under partial information coincide with those obtained under full information as a function of $(\hat{U}_t)_{t \in \mathfrak{T}}$ instead of $(U_t)_{t \in \mathfrak{T}}$. Indeed, the PDE associated with the value function under partial information differs from the full information value function only in the volatilities of the processes $(U_t)_{t \in \mathfrak{T}}$ and $(\hat{U}_t)_{t \in \mathfrak{T}}$. However, the volatility processes do not appear in the linear and quadratic terms in q (of the approximations), thus, their effects vanish when we compute the optimal displacements. Therefore, the pictures of the partial information displacements coincide with those in Figure 1, as a function of \hat{U} instead of U .

$\gamma = 0.1$ and 0.57 when $\gamma = 10$; the correlation between the changes of the processes also increases.

5.3 Performance evaluation

We compare three strategies using the performance criterion in (8) as the yardstick: (i) the optimal strategy obtained for the full information setting (FI, $\delta^{*,\text{FI}}(t, Q_{t-}, U_t)$), (ii) the Cartea–Jaimungal–Penalva strategy [15, Chapter 10] (CJP, $\delta^{*\text{CJP}}(t, Q_{t-})$), and (iii) the optimal strategy from the incomplete information setting (PI, $\delta^{*,\text{PI}}(t, Q_{t-}, \hat{U}_t)$). The ground truth in the simulations is that of the full information case.

We consider the above set of model parameters with $Q_0 = 0$, $S_0 = 100$, $X_0 = 0$, and we carry out 100,000 simulations. We discretise $[0, T]$ in 1,000 timesteps. In Table 1 we report the mean performance and standard deviation (in brackets) of the three strategies as the parameters of the model change. On each line, as a parameter (q, γ, η) changes, ψ (the parameter that captures the presence of informed traders in the market) is calibrated according to (61) in a way that the expected number of market arrivals (the arrival rate without considering the effect of the displacements) in the full information setting is 30 and the informed-uninformed composition is 50-50%. The bottom rows of the table stress the market composition.

As a parameter changes, we have three different types of effects: (i) the evaluation of the performance criterion changes, (ii) the expected number of market arrivals changes, (iii) the informed-uninformed traders market composition changes. By imposing (61), we neutralize the latter two effects and we concentrate on the first one. In Appendix A.6 we do not keep the composition of the market and the number of trades fixed; the main results of the analysis are confirmed.

The three strategies range according to the information available to the market maker. The first strategy is for when the market maker knows the ground truth, *i.e.*, she knows the composition of the order flow (5)-(6) and of the asset price dynamics (1)-(3) and she observes the order flow arrival as well as the realizations of the noise components (B_t and Z_t), which allows her to disentangle the fundamental information from the fad component (fully informed market maker). Note that the market maker is not able to identify whether the trade comes from an informed or an uninformed trader. The second strategy is for when the market maker observes the order flow, the asset price, and does not know of the existence of a fad component that affects both the asset price and the arrival rates (uninformed market maker). The market maker believes in a misspecified model ($q = 0$, $\gamma = 0$) but the simulation path for the asset price and the arrival rate reflect the existence of the fad. The third strategy is for when the market maker knows the model, observes the order flow and the asset price, and filters the fad from price because she does not observe it. We remark that she knows that there is a fad component in the market and its relevance on the price process, *i.e.*, the value of the parameters q and γ (partially informed market maker).

The market maker's performance in the three settings depends on her knowledge of the model (e.g., about the presence of a fad) and of the information at her disposal. To make the point more explicit, the optimal strategy $\delta^{*,\text{CJP}}(t, Q_{t-})$ is obtained assuming that the market maker ignores the existence of a fad in the market and, therefore, she ignores $(U_t)_{t \in \mathfrak{T}}$, however, she receives market orders according to the rate $(\varphi + \psi e^{\mp q \gamma U_t}) e^{-k \delta_t^{\text{CJP}}(t, Q_{t-})}$ and, therefore, her performance is affected by the fad. As already

observed, the fad component impacts the optimal displacements: in the full information setting, the optimal ask displacement ($\delta^{*,\text{FI}}(t, Q_{t-}, U_t)$) is decreasing in U_t , while the optimal bid displacement is increasing in U_t balancing the activity of informed traders on the two sides of the market.

For all parameter sets, the optimal strategy under full information outperforms the other two, and the optimal strategy of the partial information setting outperforms the CJP-strategy.

parameters	Performance criterion $X_T + Q_T S_T - \alpha Q_T^2 - \phi \int_0^T Q_u^2 du$		
	$\delta^{*,\text{FI}}(t, Q_{t-}, U_t)$	$\delta^{*\text{CJP}}(t, Q_{t-})$	$\delta^{*,\text{PI}}(t, Q_{t-}, \hat{U}_t)$
$\varphi = 15, \eta = 10, \gamma = 1, q = 0.6$	21.33 (4.94)	21.18 (4.94)	21.20 (4.95)
$q = 0.0$	21.34 (5.10)	21.34 (5.10)	21.34 (5.10)
$q = 0.2$	21.34 (5.08)	21.32 (5.08)	21.32 (5.09)
$q = 0.4$	21.34 (5.03)	21.27 (5.03)	21.28 (5.03)
$q = 0.6$	21.33 (4.94)	21.18 (4.94)	21.20 (4.95)
$q = 0.8$	21.31 (4.82)	21.06 (4.82)	21.14 (4.82)
$q = 1.0$	21.30 (4.64)	20.91 (4.65)	21.30 (4.64)
$\gamma = 0$	21.46 (4.97)	21.34 (4.97)	21.36 (4.97)
$\gamma = 1$	21.33 (4.94)	21.18 (4.94)	21.20 (4.95)
$\gamma = 2$	21.17 (4.92)	21.01 (4.93)	21.03 (4.93)
$\gamma = 3$	21.00 (4.91)	20.82 (4.93)	20.85 (4.93)
$\eta = 2.5$	21.23 (4.97)	20.99 (5.00)	21.05 (5.01)
$\eta = 5.0$	21.29 (4.95)	21.09 (4.97)	21.12 (4.97)
$\eta = 7.5$	21.32 (4.95)	21.15 (4.95)	21.17 (4.96)
$\eta = 10.0$	21.33 (4.94)	21.18 (4.94)	21.20 (4.95)
$\eta = 12.5$	21.33 (4.94)	21.21 (4.94)	21.23 (4.94)
$\varphi, \psi : 0\% \text{ informed}$	21.46 (4.97)	21.34 (4.97)	21.36 (4.97)
$\varphi, \psi : 25\% \text{ informed}$	21.39 (4.96)	21.27 (4.96)	21.28 (4.96)
$\varphi, \psi : 50\% \text{ informed}$	21.33 (4.94)	21.18 (4.94)	21.20 (4.95)
$\varphi, \psi : 75\% \text{ informed}$	21.26 (4.93)	21.10 (4.94)	21.12 (4.94)
$\varphi, \psi : 100\% \text{ informed}$	21.17 (4.92)	21.01 (4.93)	21.02 (4.93)

Table 1: Average performance (with standard deviation) for a market maker who follows either (i) the full information optimal strategy, (ii) the optimal strategy of Cartea-Jaimungal-Penalva (CJP), or (iii) the imperfect information optimal strategy. The performance for the baseline set of parameters is provided in the first row, then, the performance where a parameter changes (one at a time) is reported in the following rows. On each line (varying q , γ , or η) ψ is rescaled as in (61). The last rows deal with variations of φ, ψ keeping the expected number of market arrivals at 30.

Effect of q : The sensitivity of the performance with respect to q is shown in Figure 4. In the limit, we observe two interesting results. As $q \rightarrow 0$, the gap between the strategies vanishes conforming with Proposition 21 in the Appendix. This is because the impact of the fad on the mid-price and on the arrival rates vanishes and therefore the three strategies render the same performance. As $q \rightarrow 1$, the optimal strategy in the full information setting and in the partial information setting tend to coincide in agreement with Proposition 22 in the Appendix. This is because in the limit, the filter matches the fad. Instead, the performance of the CJP strategy deteriorates as q increases because it becomes more

misspecified.

The performance under full information weakly declines in q . The rationale for this effect is that as q increases the arrival of informed traders becomes more sensitive to the fad component. That is, all else being equal, the (expected) fifteen arrivals from informed traders occur when the value of the fad is further away from zero and the fundamental value differs more from the mid-price. Thus, market arrivals from the informed traders tend to be more “sharp” or more “toxic” as q increases and, therefore, hurt the market maker more. Note that the market maker can only use the bid and ask displacement to manage the inventory from trading with both, informed and uninformed traders. The classical adverse selection effect applies here: the market maker copes with both informed and uninformed traders setting the bid and the ask price, and when trading becomes more “toxic”, the market maker sets a larger bid-ask spread and experiences a poorer performance. As a result, it can be shown numerically that the bid-ask spread increases with q .

Under partial information, the performance decreases and then increases; the minimum in the figure is around $q = 0.8$. When q is low, the filter is imprecise and becomes stable around its long run mean (which is zero), as a consequence the performance of the partial information strategy looks similar to the CJP-strategy, see Figure 3. As q increases, the quality of the filter improves and the market maker reacts to the fad in a more precise manner. This reduces the performance gap with respect to the performance obtained under full information.

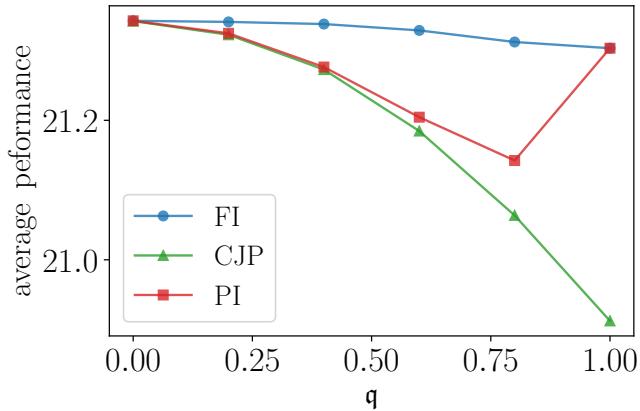


Figure 4: Performance of the strategies as a function of q .

Effect of γ : As γ increases, the performance of the three strategies decreases quasi linearly; see Figure 5. The rationale of this result is that, all else being equal, the higher the value of γ the higher the expected number of arrivals coming from a counting process with intensity $\exp(\pm\gamma q \sigma U_t)$. Since ψ is rescaled according to (61), the informed-uninformed composition in terms of market arrivals remains constant as γ changes. However, as informed traders become more sensitive to the fad process (γ increases), the market maker is penalised because the informed traders send orders that are sharper (or more toxic) and the market maker only uses the bid and the ask displacements to trade with both informed and uninformed traders.

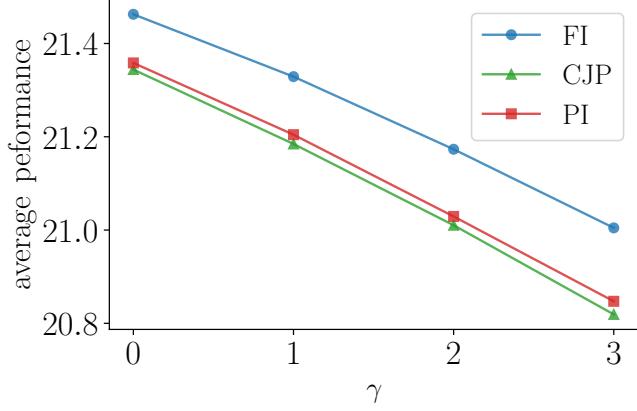


Figure 5: Performance of the strategies as a function of γ .

The argument is similar to the one explaining the effect of a change of q . As γ increases, the market maker becomes more sensitive to “toxic” order flow. This is confirmed by Figure 6 where we show the difference between the optimal bid (and ask) displacements for $\gamma = 10$ and $\gamma = 1$. In agreement with what is shown in Figure 1 and its interpretation, the market maker’s sensitivity with respect to U_t becomes stronger as γ increases and, as shown above, this leads to a poorer performance. There is an asymmetric effect of the fad on the displacements: when the inventory is zero and U_t is positive, as γ increases, the optimal bid (resp. ask) displacement becomes larger (resp. smaller) because the arrival rate of informed traders on that side of the market increases (resp. decreases). The reverse is observed for a negative U_t . It can be shown numerically that the bid-ask spread increases as γ goes up.

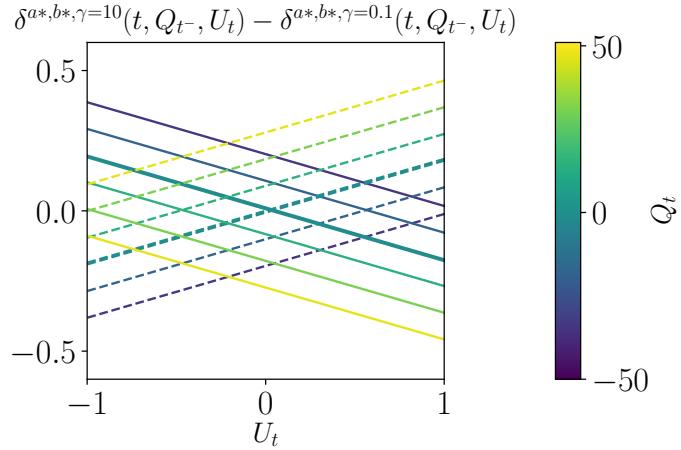


Figure 6: Difference in bid and ask displacements for $\gamma = 10$ and $\gamma = 1$ as a function of the fad (x -axis) and the inventory of the market maker (colour bar). Solid lines: optimal ask displacement. Dashed lines: optimal bid displacement.

Effect of η : The parameter η governs the speed of mean reversion of the fad. When η increases, the fad reverts more quickly to zero, while when η decreases towards zero, the fad becomes pure noise. As η increases, the performances of the three strategies increase. The rationale is that as η increases, the variance of the fad decreases and it becomes less persistent and, therefore, the market maker is

exposed to less risk managing the inventory. This argument holds true for all the three settings. We also observe that the performance gap between the full information and the other two strategies decreases as η increases. The reason for this result is that when η is high, the fad collapses around zero and, therefore, the model with no information becomes less misspecified. Confirming this interpretation, it can be shown that the bid-ask spread decreases as η increases.

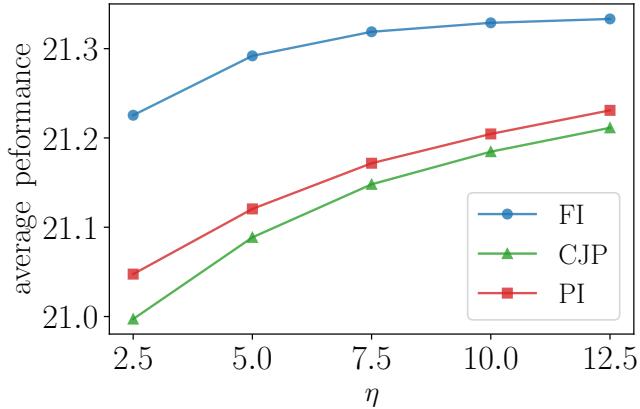


Figure 7: Performance of the strategies as a function of η .

Effect of φ : We consider the change in the percentage of informed traders keeping constant the expected number of market arrivals. To this end, given a value for $\varphi \in [0, 30]$, we compute ψ according to (61). The percentage of informed traders is $(1 - \varphi/30) \times 100\%$.

In Figure 8 we show the quoted bid-ask spread of the market maker and her performance as a function of the percentage of informed traders in the market. The bid-ask spread increases in the fraction of informed traders. This effect is due to classical adverse selection effect going back to [30]: the market maker reacts to the presence of more informed traders by widening the spread.¹⁶ We also observe that the bid-ask spread increases when the running penalty on inventory increases. This is because a higher running penalty makes the market maker more averse to inventory risk.

As far as the performance is concerned, we observe that it decreases with the fraction of informed traders. The root cause of this result is that as the fraction of informed traders increases, the fraction of "toxic" trades, e.g., those arriving from informed traders and referring to the fundamental value, goes up. This jeopardizes the performance of the market maker.

¹⁶Quoting [30]: "Generally, ask prices increase and bid prices decrease if the insiders' information becomes better, or the insiders become more numerous relative to liquidity traders".

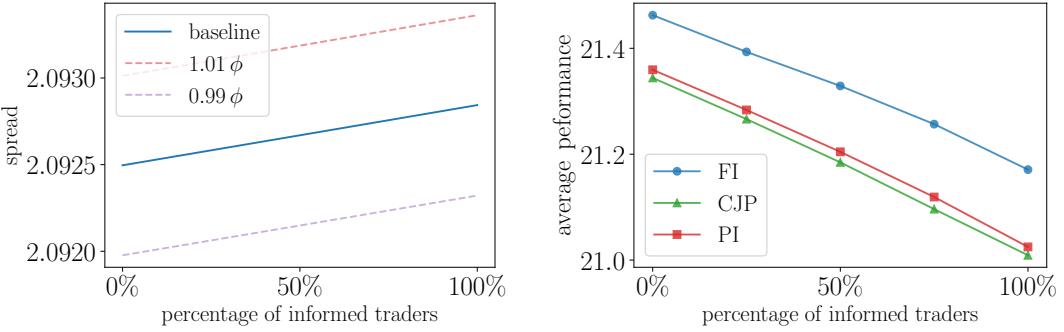


Figure 8: Left panel: bid-ask spread of the informed market maker as a function of the percentage of informed traders such that the expected number of market arrivals at T in the full information setting is 30. The parameter ϕ controls the running penalty on inventory. The baseline value is $\phi = 0.1$. Right panel: performance of the strategies as a function of the percentage of informed traders.

5.4 Misspecification of parameters

Finally, we investigate the effect of a misspecification by the market maker of the values of the key model parameters. We limit our analysis to the full information setting. In Table 2 we provide the difference in % between the performance associated with the optimal strategy in the full information setting, and the performance for the same strategy when the parameters are misspecified by $\pm 50\%$. For example, if the market maker computes the optimal strategy considering $q = 0.3$ rather than the true value ($q = 0.6$) then the performance loss is 0.158% (see first value in the third column). For the baseline set of model parameters identified at the beginning of this section, we observe that a misspecification of q induces significant changes in profitability, while the strategy is more robust with respect to a misspecification of η , γ , or φ . We can conclude that the key feature for the market maker is how the fad impact the asset price rather than the informed-uninformed composition of the market. We remark that these results hold true for the parameters reported at the start of the section. They may change as the reference parameters change.

	Robustness (in %)	
baseline	over-estimation by 50%	under-estimation by 50%
$q = 0.6$	0.160%**	0.158%**
$\gamma = 1$	0.000%	0.002%
$\eta = 10$	0.004%	0.014%
$\varphi = 15$	0.004%	0.003%

Table 2: Percentage loss in the performance (8) between the baseline optimal strategy in the full informed setting, and a strategy that misspecifies the parameter by $\pm 50\%$. The number of simulations is 1,000,000, “* *” means significant at 1%.

6 Future Research Direction

6.1 Beyond first approximation of the intensities

As a first order approximation, we modelled $(N_t^{a,b})_{t \in \mathfrak{T}}$ as counting processes with intensities

$$\hat{\lambda}_t^a = (\varphi e^{-k\delta_t^a} + \psi \exp(-k\delta_t^a - \gamma(\sigma \mathbf{q} \hat{U}_t) \vee \mathcal{S}^-)) \mathbf{1}_{\{Q_{t-} > \underline{q}\}}, \quad (62)$$

$$\hat{\lambda}_t^b = (\varphi e^{-k\delta_t^b} + \psi \exp(-k\delta_t^b + \gamma(\sigma \mathbf{q} \hat{U}_t) \wedge \mathcal{S}^+)) \mathbf{1}_{\{Q_{t-} < \bar{q}\}}. \quad (63)$$

Here, we construct an exact intensity process $(\psi \hat{\lambda}_t^{a,b})_{t \in \mathfrak{T}}$ for the counting process $(N_t^{a,b})_{t \in \mathfrak{T}}$ in the special case $\mathcal{S}^+ = -\mathcal{S}^- = +\infty$.¹⁷

Proposition 17. *Let $\hat{\lambda}_t^a = e^{-k\delta_t^a} (\varphi + \psi \mathbb{E}[e^{-\gamma \sigma \mathbf{q} U_t} | \mathcal{F}_t^S]) \mathbf{1}_{\{Q_t > \underline{q}\}}$. Define $(Z_t)_{t \in \mathfrak{T}}$ as $Z_t := e^{-\gamma \sigma \mathbf{q} U_t}$. Then, it follows that*

$$dZ_t = Z_t \left(\eta \gamma \sigma \mathbf{q} U_t + \frac{(\gamma \sigma \mathbf{q})^2}{2} dt - \gamma \sigma \mathbf{q} dB_t \right) =: h(U_t, Z_t) dt + \sigma(Z_t) dB_t. \quad (64)$$

Define $\hat{Z}_t := \mathbb{E}[Z_t | \mathcal{F}_t^S]$, we have that

$$d\hat{Z}_t = \left(\frac{(\gamma \sigma \mathbf{q})^2}{2} \hat{Z}_t + \eta \gamma \sigma \mathbf{q} (Q_t + \hat{Z}_t \hat{U}_t) \right) dt - (\mathbf{q}^2 \sigma \gamma \hat{Z}_t + \eta \mathbf{q} Q_t) d\tilde{I}_t, \quad (65)$$

where $Q_t = \mathbb{E}[(U_t - \hat{U}_t)(Z_t - \hat{Z}_t) | \mathcal{F}_t^S]$ and $(\tilde{I}_t)_{t \in \mathfrak{T}}$ is the Brownian Motion defined by the filtering process.

Proof. Consider the process $(\Gamma_t)_{t \in \mathfrak{T}}$ as $\Gamma_t := \hat{Z}_t - 1 - \int_0^t h(\widehat{U_s}, \widehat{Z}_s) ds$. Then, $(\Gamma_t)_{t \in \mathfrak{T}}$ is an L^2 -bounded (\mathcal{F}_t^S) -martingale starting from zero.

Indeed, consider $0 \leq s \leq t \leq T$, then, using the tower property, we have

$$\mathbb{E}[\Gamma_t | \mathcal{F}_s^S] = \mathbb{E}[Z_t | \mathcal{F}_s^S] - \int_0^s h(\widehat{U}_u, \widehat{Z}_u) du - \int_s^t \mathbb{E}[h(U_u, Z_u) | \mathcal{F}_s^S] du - 1, \quad (66)$$

which implies

$$\mathbb{E}[Z_t | \mathcal{F}_s^S] = \hat{Z}_s + \mathbb{E} \left[\int_s^t h(U_u, Z_u) du + \sigma(Z_u) dB_u \middle| \mathcal{F}_s^S \right]. \quad (67)$$

Using the tower property, we find

$$\mathbb{E} \left[\int_s^t \sigma(Z_u) dB_u \middle| \mathcal{F}_s^S \right] = \mathbb{E} \left[\mathbb{E} \left[\int_s^t \sigma(Z_u) dB_u \middle| \mathcal{F}_s \right] \middle| \mathcal{F}_s^S \right], \quad (68)$$

¹⁷This could be understood as follows: the market maker faces informed traders that know exactly what the fad is, but the market maker does not observe U_t . Given knowledge of the price, the market maker estimates the intensity of the market arrivals with $\mathbb{E}[\psi e^{\mp \mathbf{q} \gamma U_t} | \mathcal{F}_t^S]$.

and since $t \rightarrow \int_0^t \sigma(Z_s) dB_s$ is a $(\mathcal{F}_t)_t$ -martingale, we have that

$$\mathbb{E} \left[\int_s^t \sigma(Z_u) dB_u \middle| \mathcal{F}_s^S \right] = 0. \quad (69)$$

Finally, we find

$$\mathbb{E}[\Gamma_t | \mathcal{F}_s^S] = \hat{Z}_s - 1 - \int_0^s h(\widehat{U}_u, \widehat{Z}_u) du = \Gamma_s, \quad (70)$$

which means that $(\Gamma_t)_{t \in \mathfrak{T}}$ is an \mathcal{F}^S -martingale, with initial value zero. Moreover, by construction, we also have that $(\Gamma_t)_{t \in \mathfrak{T}}$ is bounded in $L^2(\Omega, \mathcal{F}^S, \mathbb{P})$.

Since $(\Gamma_t)_{t \in \mathfrak{T}}$ satisfies condition of Proposition 2.31 in [3], then, there exists an $(\mathcal{F}_t^S)_{t \in \mathfrak{T}}$ -progressively measurable process $(\nu_t)_{t \in \mathfrak{T}}$ such that

$$\Gamma_t = \int_0^t \nu_s d\tilde{I}_s. \quad (71)$$

Similar calculations to those in Section 4.1 imply that $\nu_t = \mathbf{q} \sigma(Z_t) - \eta \mathbf{q} Q_t$, where $Q_t := \mathbb{E}[(U_t - \hat{U}_t)(Z_t - \hat{Z}_t) | \mathcal{F}_t^S]$. Then, we get

$$d\hat{Z}_t = (\widehat{\sigma(Z_t)} - \eta \mathbf{q} Q_t) d\tilde{I}_t + h(\widehat{U}_t, \widehat{Z}_t) dt, \quad (72)$$

where

$$h(\widehat{U}_t, \widehat{Z}_t) = \eta \gamma \sigma \mathbf{q} \widehat{Z}_t \widehat{U}_t + \frac{(\gamma \sigma \mathbf{q})^2}{2} \widehat{Z}_t = \frac{(\gamma \sigma \mathbf{q})^2}{2} \widehat{Z}_t + \eta \gamma \sigma \mathbf{q} (Q_t + \widehat{Z}_t \hat{U}_t),$$

and where

$$\widehat{\sigma(Z_t)} = -\gamma \sigma \mathbf{q} \widehat{Z}_t.$$

Finally, we have

$$d\hat{Z}_t = \left(\frac{(\gamma \sigma \mathbf{q})^2}{2} \widehat{Z}_t + \eta \gamma \sigma \mathbf{q} (Q_t + \widehat{Z}_t \hat{U}_t) \right) dt - (\mathbf{q}^2 \gamma \sigma \widehat{Z}_t + \eta \mathbf{q} Q_t) d\tilde{I}_t. \quad (73)$$

□

The computation for λ^b is similar. Since one cannot find closed-form expressions for the process $(Q_t)_{t \in \mathfrak{T}}$, it would be interesting to study non-linear filtering approximation methods, such as those developed in [24].

6.2 Beyond the Kalman-Bucy filter

Building alternative filters is a promising avenue of future research. Information about the fads U_t can be extracted from prices (as done in this paper), or through arrivals of liquidity taking orders. Recall, that in the full information model the arrivals were modulated by the fads in the asset price. Thus, following [19], and similar to [12], one can discretise U_t and model it as a continuous time Markov chain (CTMC) that we introduce next.

Let us consider that the fad process $(U_t)_{t \in \mathfrak{T}}$ is a continuous-time Markov Chain with generator L taking values in $G = \{\theta^1, \dots, \theta^J\}$. Instead of modelling the stochastic intensity of filled orders, we keep

track of both arrivals and fills as two separate (but related) counting processes. Here, we filter the fads at the level of market arrivals and we formulate the effect of the control at the level of the fills. The disentanglement between arrivals and fills allows us to employ the results in Theorem 3.1 in [19] and in [20].

We let $M^{a,b}$ be the market arrivals which happen with stochastic intensity

$$\lambda_t^a = \sum_{j=1}^J \lambda_t^{a,j} \mathbb{1}_{\{U_t=\theta^j\}}, \quad (74)$$

$$\lambda_t^b = \sum_{j=1}^J \lambda_t^{b,j} \mathbb{1}_{\{U_t=\theta^j\}}, \quad (75)$$

where

$$\lambda_t^{a,j} = \varphi + \psi e^{-\gamma(\mathfrak{q}\sigma\theta^j) \vee S^-} \quad (76)$$

$$\lambda_t^{b,j} = \varphi + \psi e^{\gamma(\mathfrak{q}\sigma\theta^j) \wedge S^+}. \quad (77)$$

Upon arrivals, the fills probabilities are

$$\exp(-k\delta_t^a), \quad \text{and} \quad \exp(-k\delta_t^b), \quad (78)$$

and we keep track of all filled trades in the controlled counting processes $(N_t^{a,b})_{t \in \mathfrak{T}}$.

Now, we denote the filtration generated by the Poisson processes $(M_t^{a,b})_{t \in \mathfrak{T}}$ by $(\mathcal{F}_t^M)_{t \in \mathfrak{T}}$. As before, the full filtration of all processes involved is $(\mathcal{F}_t)_{t \in \mathfrak{T}}$.

We are interested in the process $(\pi_t^j)_{t \in \mathfrak{T}}$ defined by

$$\pi_t^j := \mathbb{E}[\mathbb{1}_{\{U_t=\theta^j\}} | \mathcal{F}_t^M]. \quad (79)$$

Conditional on knowing the dynamic of $(\pi_t^j)_{t \in \mathfrak{T}}$, we would be able to transform our partial-information optimal control problem into a full-information one. We have the following result.

Theorem 4. *For all $j \in [\![1, J]\!]$, the process $(\pi_t^j)_{t \in \mathfrak{T}}$ admits the following representation*

$$\pi_t^j = \Delta_t^j / \sum_{j=1}^J \Delta_t^j, \quad (80)$$

where for all $j \in [\![1, J]\!]$, the process $(\Delta_t)_{t \in \mathfrak{T}}$ satisfies

$$d\Delta_t^j = \Delta_{t-}^j (\lambda_{t-}^{a,j} - 1) (dM_t^a - dt) + \Delta_{t-}^j (\lambda_{t-}^{b,j} - 1) (dM_t^b - dt) + \sum_{i=1}^J \Delta_{t-}^i L_{i,j} dt \quad (81)$$

Proof. This is direct consequence of Theorem 3.1 in [18]. \square

Armed with the above filter, we formulate the following control problem. We write the price dynam-

ics of $(S_t)_{t \in \mathfrak{T}}$ as

$$dS_t = dP_t^S + \sigma \mathfrak{p} dZ_t = \sum_{j=1}^J \sigma \mathfrak{q} \theta^j d1_{\{U_t=\theta^j\}} + \mu dt + \sigma \mathfrak{p} dZ_t, \quad (82)$$

and assuming that the process $(Z_t)_{t \in \mathfrak{T}}$ is independent of $(\mathcal{F}_t^M)_{t \in \mathfrak{T}}$, we have

$$\hat{S}_t = \mu t + \sum_{j=1}^J \sigma \mathfrak{q} \theta^j \pi_t^j. \quad (83)$$

Proposition 18. Let $\hat{\lambda}_t^{a,b}$ be given by

$$\hat{\lambda}_t^{a,b} = \sum_{j=1}^J \lambda_t^{a,b,j} \pi_t^j. \quad (84)$$

The process $(M_t^{a,b})_{t \in \mathfrak{T}}$ is an $(\mathcal{F}_t^M)_{t \in \mathfrak{T}}$ -Poisson processes with stochastic intensity $\hat{\lambda}_t^{a,b}$.

Proof. Let us consider the compensated processes

$$D_t^{a,b} := M_t^{a,b} - \int_0^t \lambda_s^{a,b} ds. \quad (85)$$

We remark that those are $(\mathcal{F}_t)_{t \in \mathfrak{T}}$ -martingales. Next, we consider the processes $(\hat{D}_t^{a,b})_{t \in \mathfrak{T}}$ defined by

$$\hat{D}_t^{a,b} = D_t^{a,b} + \int_0^t (\lambda_s^{a,b} - \hat{\lambda}_s^{a,b}) ds = M_t^{a,b} - \int_0^t \hat{\lambda}_s^{a,b} ds, \quad (86)$$

and we show that it is a $(\mathcal{F}_t^M)_{t \in \mathfrak{T}}$ -martingale. Since $(M_t^{a,b})_{t \in \mathfrak{T}}$ and $(\hat{\lambda}_t^{a,b})_{t \in \mathfrak{T}}$ are $(\mathcal{F}_t^M)_{t \in \mathfrak{T}}$ -adapted, then $(\hat{D}_t^{a,b})_{t \in \mathfrak{T}}$ is $(\mathcal{F}_t^M)_{t \in \mathfrak{T}}$ -adapted. By the definition of the processes we have that $(\hat{D}_t^{a,b})_{t \in \mathfrak{T}} \in L^1(\Omega, (\mathcal{F}_t^M)_{t \in \mathfrak{T}}, \mathbb{P})$. Let $0 \leq s \leq t \leq T$, we have

$$\mathbb{E}[M_t^{a,b} - \int_0^t \hat{\lambda}_u^{a,b} du | \mathcal{F}_s^N] = \hat{D}_s^{a,b} + \mathbb{E}[D_t^{a,b} - D_s^{a,b} | \mathcal{F}_s^N] - \mathbb{E}\left[\int_s^t (\lambda_u^{a,b} - \hat{\lambda}_u^{a,b}) du | \mathcal{F}_s^M\right].$$

However, by the tower property and the fact that $(D_t^{a,b})_{t \in \mathfrak{T}}$ is a martingale, we have

$$\mathbb{E}[D_t^{a,b} - D_s^{a,b} | \mathcal{F}_s^M] = \mathbb{E}[\mathbb{E}[D_t^{a,b} - D_s^{a,b} | \mathcal{F}_s] | \mathcal{F}_s^M] = 0.$$

Again, by the tower property,

$$\mathbb{E}\left[\int_s^t (\lambda_u^{a,b} - \hat{\lambda}_u^{a,b}) du | \mathcal{F}_s^M\right] = \mathbb{E}\left[\mathbb{E}\left[\int_s^t (\lambda_u^{a,b} - \hat{\lambda}_u^{a,b}) du | \mathcal{F}_u^M\right] | \mathcal{F}_s^M\right] = 0.$$

Finally, we end up with $\mathbb{E}[\hat{D}_t^{a,b} | \mathcal{F}_s^M] = \hat{D}_s^{a,b}$, and then $(\hat{D}_t^{a,b})_{t \in \mathfrak{T}}$ is a $(\mathcal{F}_t^M)_{t \in \mathfrak{T}}$ -martingale.

Finally, by the Watanabe's characterization theorem, $(M_t^{a,b})_{t \in \mathfrak{T}}$ are $(\mathcal{F}_t^M)_{t \in \mathfrak{T}}$ Poisson processes, with stochastic intensities $(\hat{\lambda}_t^{a,b})_{t \in \mathfrak{T}}$. \square

Once we have this, we can rewrite the dynamics of the processes as $(\mathcal{F}_t^M)_{t \in \mathfrak{T}}$ -predictable processes.

$$d\hat{S}_t = d\hat{P}_t^S = \mu dt + \sum_{j=1}^J \sigma \mathbf{q} \theta^j d\pi_t^j. \quad (87)$$

The processes $(M_t^{a,b})_{t \in \mathfrak{T}}$ are $(\mathcal{F}_t^M)_{t \in \mathfrak{T}}$ -adapted Poisson processes with stochastic intensities $(\hat{\lambda}_t^{a,b})_{t \in \mathfrak{T}}$ given by

$$\hat{\lambda}_t^{a,b} = \sum_{j=1}^J \lambda_t^{a,b,j} \pi_t^j, \quad (88)$$

and the process $(\hat{U}_t)_{t \in \mathfrak{T}}$ is given by

$$\hat{U}_t = \sum_{j=1}^J \theta^j \pi_t^j. \quad (89)$$

Since all the processes are now $(\mathcal{F}_t^M)_{t \in \mathfrak{T}}$ -predictable, we are able to formulate the full information optimal control problem similar to that in the previous setting. The optimal control problem is

$$V(t, x, q, s, (\Delta^i)_{i \in [\![1, J]\!]}) = \sup_{\delta^{a,b}} \mathbb{E} \left[\hat{X}_T + Q_T \hat{S}_T - \alpha Q_T^2 - \phi \int_t^T Q_u^2 du \right], \quad (90)$$

with

$$\begin{cases} dS_t = \mu dt + \sum_{j=1}^J \sigma \mathbf{q} \theta^j d\pi_t^j, \\ d\hat{U}_t = \sum_{j=1}^J \theta^j d\pi_t^j, \\ dQ_t = d\tilde{N}_t^b - d\tilde{N}_t^a, \\ d\hat{X}_t = (\hat{S}_t + \delta_t^a) d\tilde{N}_t^a - (\hat{S}_t - \delta_t^b) d\tilde{N}_t^b, \\ \pi_t^i = \Delta_t^i / \sum_{j=1}^J \Delta_t^j, \\ d\Delta_t^j = \Delta_{t-}^j (\lambda_{t-}^{a,j} - 1) (dM_t^a - dt) + \Delta_{t-}^j (\lambda_{t-}^{b,j} - 1) (dM_t^b - dt) + \sum_{i=1}^J \Delta_{t-}^i L_{i,j} dt, \end{cases} \quad (91)$$

where $(\tilde{N}_t^{a,b})_{t \in \mathfrak{T}}$ are Poisson process with intensity¹⁸ $\left(\hat{\lambda}_t^{a,b} e^{-k \delta_t^{a,b}} \mathbb{1}_{\{Q_t \in \mathcal{Q}\}} \right)_{t \in \mathfrak{T}}$ and where the controls belong to the set of $(\mathcal{F}_t^{M,N})_{t \in \mathfrak{T}}$ -predictable and bounded processes, where $(\mathcal{F}_t^{M,N})_{t \in \mathfrak{T}}$ is the filtration generated by the Poisson processes $(M_t^{a,b})_{t \in \mathfrak{T}}$ and $(\tilde{N}_t^{a,b})_{t \in \mathfrak{T}}$. We leave the computation of the optimal strategies for future research.

7 Conclusions

We computed approximate closed-form solutions for a market making problem à la Avellaneda-Stoikov when the market is populated by informed and uninformed traders.¹⁹ To the best of our knowledge this is the first time that informed and uninformed trading is incorporated in the optimal market making

¹⁸Similarly to the previous cases, we assume that the inventory of the market maker is in \mathcal{Q} .

¹⁹Here, information is knowledge about the fad in the market.

problem. Our work is also the first to build a bridge for the classical results of [30] in the continuous-time optimal market making problem of [2].

We carried out an extensive sensitivity analysis to find the value of information by comparing strategies with increasing levels of information. We have shown that the toxicity of the order flow deteriorates the market maker's performance and widens the bid-ask spread. Our results can be easily generalised to the case of market making with signals.

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A Appendix

A.1 Gaussian processes properties

Let us present few results related to the Kalman-Bucy filter.

Proposition 19. *If we define the process $(\mathbb{X}_t)_{t \in \mathfrak{T}}$ as*

$$d\mathbb{X}_t = (\mu - \mathbb{A} \mathbb{X}_t) dt + \Sigma_t dW_t, \quad \mathbb{X}_0 \in \mathbb{R}^d, \quad (92)$$

where $(W_t)_{t \in \mathfrak{T}}$ is a p -dimensional Brownian motion, $(\mathbb{X}_t)_{t \in \mathfrak{T}} \in \mathbb{R}^d$, $\Sigma \in C^0(\mathbb{R}; \mathbb{R}^{d \times p})$, $\mathbb{A} \in \mathbb{R}^{d \times d}$, and $\mu \in \mathbb{R}^d$, and such that

$$\int_0^t \|e^{-\mathbb{A}(t-s)} \Sigma_s\|^2 ds < \infty,$$

then

$$\mathbb{X}_t = \int_0^t e^{-\mathbb{A}(t-s)} \Sigma_s dW_s + e^{-\mathbb{A}t} \mathbb{X}_0 + \int_0^t e^{-\mathbb{A}(t-s)} \mu ds. \quad (93)$$

Then the process $(\mathbb{X}_t)_{t \in \mathfrak{T}}$ is a Gaussian process. For instance, if we consider two one-dimensional processes $(X_t)_{t \in \mathfrak{T}}$ and $(Y_t)_{t \in \mathfrak{T}}$ such that their joint dynamic is given by (92), then they are jointly Gaussian. Consequently, and using Gaussian property, it holds that the conditional expectation of the process $(X_t)_{t \in \mathfrak{T}}$ given the natural filtration generated by $(Y_t)_{t \in \mathfrak{T}}$ is also Gaussian. Finally, we have the following corollary.

Corollary 20. *The process $(\tilde{X}_t)_{t \in \mathfrak{T}}$ defined by*

$$\tilde{X}_t := X_t - \mathbb{E}[X_t | \sigma(Y_u)_{u \in [0,t]}], \quad (94)$$

is also a Gaussian process, independent of $\sigma(Y)$.

A.2 Proof of Theorem 3

Proof. The above Kalman-Bucy filtering equations are derived as follows.

Since $\mathbb{E}\left[\int_0^t |\pi_s(h)|^2 ds\right] < \infty$ for all $t > 0$, we can use Proposition 13, and given that $(\Gamma_t)_{t \in \mathfrak{T}} \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, we have that

$$\Gamma_t = \hat{U}_t + \eta \int_0^t \hat{U}_s ds = \int_0^t \nu_s \sigma^{-1} dI_s,$$

where our goal is to find ν . Let $(\xi_t)_{t \in \mathfrak{T}}$ be an $(\mathcal{F}_t^S)_{t \in \mathfrak{T}}$ progressively measurable process in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, and define $\eta_t := \int_0^t \sigma^{-1} \xi_s dI_s$. Since $(\tilde{I}_t)_{t \in \mathfrak{T}}$ is a Brownian Motion and $(\xi_t)_{t \in \mathfrak{T}}$ is square-integrable, we have that $(\eta_t)_{t \in \mathfrak{T}}$ is a martingale, and we get

$$\mathbb{E}[\Gamma_t \eta_t] = \mathbb{E}\left[\int_0^t \nu_s \xi_s ds\right] = \mathbb{E}[\hat{U}_t \eta_t + \eta \int_0^t \hat{U}_s \eta_s ds]. \quad (95)$$

Using the martingale property of $(\eta_t)_{t \in \mathfrak{T}}$, we get that

$$\mathbb{E}[\hat{U}_s \eta_t] = \mathbb{E}[\hat{U}_s \eta_s]. \quad (96)$$

Hence, we get $\mathbb{E}[\Gamma_t \eta_t] = \mathbb{E}[\hat{U}_t \eta_t + \eta \int_0^t \hat{U}_s \eta_s ds]$. Moreover, we have

$$d\eta_t = \sigma^{-1} \xi_t (dY_t - \pi_t(h) dt) = -\sigma^{-1} \xi_t (\sigma q \eta \tilde{U}_t dt - \sigma d\bar{W}_t),$$

where $\tilde{U}_t := U_t - \hat{U}_t$.

Then, using integration by parts, we find that

$$\mathbb{E}[U_t \eta_t] = \mathbb{E}\left[\int_0^t -\eta \eta_s U_s ds - \xi_s U_s q \eta \tilde{U}_s ds + \xi_s q ds\right]. \quad (97)$$

Now, using that $U_t = \tilde{U}_t + \hat{U}_t$ and that \tilde{U}_t is independent²⁰ of \mathcal{F}_t^S we find that

$$\mathbb{E}\left[\int_0^t \xi_s U_s \tilde{U}_s q \eta ds\right] = \mathbb{E}\left[\int_0^t \xi_s \hat{P}_s q \eta ds\right]. \quad (98)$$

Thus, on the one hand

$$\mathbb{E}[\hat{U}_t \eta_t] = -\eta \mathbb{E}\left[\int_0^t \eta_s U_s ds\right] - \mathbb{E}\left[\int_0^t \xi_s \hat{P}_s \eta q ds\right] + \mathbb{E}\left[\int_0^t \xi_s q ds\right], \quad (99)$$

and on the other hand

$$\mathbb{E}[\hat{U}_t \eta_t] = \mathbb{E}\left[\int_0^t \xi_s v_s ds\right] - \eta \mathbb{E}\left[\int_0^t \hat{U}_s \eta_s ds\right]. \quad (100)$$

Since $\mathbb{E}[\hat{U}_s \eta_s] = \mathbb{E}[U_s \eta_s]$, and given that $(\xi_t)_{t \in \mathfrak{T}}$ is arbitrary, we get that

$$v_t = -\hat{P}_t \eta q + q. \quad (101)$$

Now we can rewrite Γ_t as

$$\Gamma_t = \int_0^t -\hat{P}_s \sigma^{-1} q \eta dI_s + \sigma^{-1} q I_t = \hat{U}_t + \eta \int_0^t \hat{U}_s ds, \quad (102)$$

and compute the dynamics of \hat{U}_t as

$$d\hat{U}_t = -\eta \hat{U}_t dt - \sigma^{-1} (\hat{P}_t q \eta - q) dI_t. \quad (103)$$

Next, we compute the dynamics of \hat{P}_t . Since

$$dU_t = -\eta U_t dt + dB_t,$$

²⁰See Appendix A for the details.

we have that

$$\begin{aligned} d\tilde{U}_t &= -\eta \tilde{U}_t dt + dB_t - (\hat{P}_t \eta q - q) q \eta \tilde{U}_t dt + (\hat{P}_t \eta q - q) d\bar{W}_t \\ &= -\eta \tilde{U}_t (1 + q(\hat{P}_t \eta q - q)) dt + (1 + (\hat{P}_t \eta q - q) q) dB_t + (\hat{P}_t \eta q - q) p dZ_t. \end{aligned}$$

Now, we compute $\frac{d}{dt} \mathbb{E}[\tilde{U}_t^2] = \frac{d\hat{P}_t}{dt}$ and we use that $\mathbb{E}[\tilde{U}_0^2] = 0$. We find that

$$\begin{aligned} \hat{P}_t &= \mathbb{E}\left[\int_0^t -2\eta \hat{P}_s (1 + q(\hat{P}_s \eta q - q)) ds\right] \\ &\quad + \mathbb{E}\left[\int_0^t (1 + (\hat{P}_s \eta q - q) q)^2 ds\right] + \mathbb{E}\left[\int_0^t ((\hat{P}_s \eta q - q) p)^2 ds\right] \\ &= -2 \int_0^t \eta \hat{P}_s (1 + q(\hat{P}_s \eta q - q)) ds \\ &\quad + \int_0^t (1 + (\hat{P}_s \eta q - q) q)^2 ds + \int_0^t ((\hat{P}_s \eta q - q) p)^2 ds. \end{aligned}$$

Hence, we get that \hat{P}_t solves the following Riccati equation

$$\frac{d\hat{P}_t}{dt} = -2\eta \hat{P}_t (1 + q(\hat{P}_t \eta q - q)) + (1 + (\hat{P}_t \eta q - q) q)^2 + ((\hat{P}_t \eta q - q) p)^2. \quad (104)$$

Existence and uniqueness of the Riccati equation follows by reverting the time and Lemma 2.1 in [42]. \square

A.3 CJP-strategy

Under the CJP-strategy we consider a model that follows from Chapter 10 of [15], where we consider an uninformed market maker that does not model the market arrivals and the mid-price with fads.²¹ According to her beliefs, the mid-price $(S_t)_{t \in \mathfrak{T}}$ follows

$$dS_t = \mu dt + \sigma dB_t, \quad (105)$$

and his inventory $(Q_t)_{t \in \mathfrak{T}}$ and wealth $(X_t)_{t \in \mathfrak{T}}$ are defined through Poisson processes $(N_t^{a,b})_{t \in \mathfrak{T}}$ with intensity

$$\lambda_t^a = \kappa e^{-k\delta_t^a}, \quad \lambda_t^b = \kappa e^{-k\delta_t^b}, \quad (106)$$

and where $\kappa \in \mathbb{R}^+$ is such that the expected number of market arrivals at T in this setting is equal to the expected number of arrivals at T in the full information setting.²²

Considering this model, the market maker computes her optimal quotes $(\delta_t^{*,a}, \delta_t^{*,b})_{t \in \mathfrak{T}}$ as solution of the control problem (14).

²¹That is, she considers $q = 0$.

²²We take $\kappa = \varphi + \frac{\psi}{T} \int_0^T \mathbb{E}[e^{\pm \gamma \sigma q U_t}] dt$.

A.4 Limit cases for $q \in \{0, 1\}$

In this subsection, we present two limit results linking the PnL of our strategies when $q \in \{0, 1\}$. To simplify the calculations, we only consider the cases $\delta_\infty = \infty$, $\mathcal{Q} = \mathbb{Z}$ and $\mathcal{S}^+ = -\mathcal{S}^- = +\infty$.

Let us start with $q = 0$. In this case, we have the following property:

Proposition 21. *When $q = 0$, the PnL of the full information optimal strategy is equal to the PnL of the partial information optimal strategy and the CJP-strategy.*

Proof. Consider $q = 0$ in the HJB equation of the full informed problem, we obtain

$$0 = \partial_t V - \eta u \partial_u V - \phi q^2 + \frac{1}{2} \partial_{u,u}^2 V + q \mu + \frac{e^{-1}}{k} e^{k \Delta^- V} (\varphi + \psi), \quad (107)$$

where the processes satisfy

$$\left\{ \begin{array}{l} dU_t = -\eta U_t dt + dB_t, \\ dS_t = \mu dt + \sigma dZ_t, \\ dQ_t = dN_t^b - dN_t^a, \end{array} \right. \quad (108)$$

and where $(N_t^{a,b})_{t \in \mathfrak{T}}$ are Poisson processes with stochastic intensity

$$\lambda_t^{a,b} = (\varphi + \psi) e^{-k \delta_t^{a,b}}.$$

On the other hand, the HJB equation satisfies by in the partial informed setting is given by

$$0 = \partial_t V - \phi q^2 + q \mu - \eta u \partial_u V + \frac{e^{-1}}{k} (\varphi + \psi) (e^{k \Delta^- V} + e^{k \Delta^+ V}), \quad (109)$$

where the processes satisfy

$$\left\{ \begin{array}{l} \hat{U}_t = 0, \\ dS_t = \mu dt + \sigma dZ_t, \\ dQ_t = dN_t^b - dN_t^a, \end{array} \right. \quad (110)$$

and where $(N_t^{a,b})_{t \in \mathfrak{T}}$ are Poisson processes with stochastic intensity

$$\lambda_t^{a,b} = (\varphi + \psi) e^{-k \delta_t^{a,b}}.$$

The equality $\hat{U}_t = 0$ comes from the independence of the two Brownian Motions $(B_t)_{t \in \mathfrak{T}}$ and $(Z_t)_{t \in \mathfrak{T}}$.

We take V to be independent of u , and we obtain

$$0 = \partial_t V - \phi q^2 + q \mu + \frac{e^{-1}}{k} (\varphi + \psi) (e^{k \Delta^- V} + e^{k \Delta^+ V}), \quad (111)$$

that can be solved following the ideas developed in Chapter 10 of [15]. This is exactly the CJP-strategy.²³ Since the common value function is independent of u and since the optimal quotes are given by $\delta_t^{a,b} = \frac{1}{k} - \Delta^\pm V(t, q)$, then, neither the value of u nor the value of s impacts the quotes. Finally, and following our previous notations, we have that an equality of the PnL following the strategies $\delta^{*,\text{FI}}(t, Q_{t-}, U_t)$, $\delta^{*,\text{CJP}}(t, Q_{t-})$, $\delta^{*,\text{PI}}(t, Q_{t-}, \hat{U}_t)$. \square

Now, we consider $q = 1$. In this case, we have the following property:

Proposition 22. *When $q = 1$, the PnL of the optimal strategy in the full information setting and in the partial information setting are the same.*

Proof. Consider $q = 1$, then the HJB equation in the full informed setting becomes

$$\begin{aligned} 0 = & \partial_t V - \eta u \partial_u V - \phi q^2 + \frac{1}{2} \partial_{u,u}^2 V + (\mu - \eta \sigma u) q \\ & + \frac{e^{-1}}{k} e^{k \Delta^- V} (\varphi + \psi e^{-\gamma \sigma u}) + \frac{e^{-1}}{k} e^{k \Delta^- V} (\varphi + \psi e^{\gamma \sigma u}), \end{aligned} \quad (112)$$

where the processes satisfy

$$\left\{ \begin{array}{l} dU_t = -\eta U_t dt + dB_t, \\ dS_t = (\mu - \eta \sigma U_t) dt + \sigma dB_t, \\ dQ_t = dN_t^b - dN_t^a, \end{array} \right. \quad (113)$$

and where $(N_t^{a,b})_{t \in \mathcal{T}}$ are Poisson processes with intensity

$$\lambda_t^{a,b} = (\varphi + \psi e^{\mp \gamma \sigma U_t}) e^{-k \delta_t^{a,b}}.$$

The HJB equation satisfied in the partial informed setting is given by

$$\begin{aligned} 0 = & \partial_t V - \eta u \partial_u V - \phi q^2 + \frac{1}{2} \partial_{u,u}^2 V + (\mu - \eta \sigma u) q \\ & + \frac{e^{-1}}{k} e^{k \Delta^- V} (\varphi + \psi e^{-\gamma \sigma u}) + \frac{e^{-1}}{k} e^{k \Delta^- V} (\varphi + \psi e^{\gamma \sigma u}), \end{aligned} \quad (114)$$

since we have $\hat{P}_t = 0$ in this setting. And the processes satisfy

$$\left\{ \begin{array}{l} \hat{U}_t = U_t, \\ dS_t = (\mu - \eta \sigma U_t) dt + \sigma dB_t, \\ dQ_t = dN_t^b - dN_t^a, \end{array} \right. \quad (115)$$

²³Indeed, in this case $\kappa = \varphi + \psi$.

where $(N_t^{a,b})_{t \in \mathcal{T}}$ are Poisson processes with intensity

$$\lambda_t^{a,b} = (\varphi + \psi e^{\mp \gamma \sigma U_t}) e^{-k \delta_t^{a,b}}.$$

We then have $V^{\text{FI}} = V^{\text{PI}}$, and since the dynamics of their processes are the same, we have equality of the PnL. \square

A.5 Evaluation at the fundamental price

In our analysis, we assumed that the residual inventory at time T is evaluated at the mid-price. In what follows, we assume that the market maker evaluates the final inventory at the fundamental price ($S_T - \sigma q U_T$) and therefore the performance criterion to be considered in the maximization problem becomes:

$$X_T + Q_T (S_T - \sigma q U_T) - \alpha Q_T^2 - \phi \int_0^T Q_u^2 du \quad (116)$$

instead of (8).

Table 3 reports the mean (and standard deviation) of the performance of the optimal strategies when the mark-to-market of the terminal inventory is computed at the fundamental value (100,000 simulations). Here, the optimal strategies are those computed when the performance criterion is that in (116) following the same steps as above.

The main insights we collected from Table 1 are confirmed in Table 3. Notice that evaluating the final inventory at the fundamental leads to a very small increase in the performance of the market. The full information strategy outperforms the partial information strategy, which in turn outperforms the CJP strategy. The effect associated with a stress of key model parameters is also similar.

parameters	Performance criterion $X_T + Q_T (S_T - \sigma q U_T) - \alpha Q_T^2 - \phi \int_0^T Q_u^2 du$		
	$\delta^{*,\text{FI}}(t, Q_{t-}, U_t)$	$\delta^{*\text{CJP}}(t, Q_{t-})$	$\delta^{*,\text{PI}}(t, Q_{t-}, \hat{U}_t)$
$\varphi = 15, \eta = 10, \gamma = 1, q = 0.6$	21.33 (4.93)	21.16 (4.93)	21.19 (4.93)
$q = 0$	21.34 (5.10)	21.34 (5.10)	21.34 (5.10)
$q = 0.2$	21.34 (5.08)	21.32 (5.08)	21.32 (5.08)
$q = 0.4$	21.34 (5.03)	21.26 (5.03)	21.27 (5.03)
$q = 0.6$	21.33 (4.93)	21.16 (4.93)	21.19 (4.93)
$q = 0.8$	21.33 (4.79)	21.03 (4.79)	21.12 (4.79)
$q = 1.0$	21.33 (4.60)	20.86 (4.61)	21.33 (4.60)
$\gamma = 0$	21.49 (4.96)	21.34 (4.96)	21.36 (4.95)
$\gamma = 1$	21.33 (4.93)	21.16 (4.93)	21.19 (4.93)
$\gamma = 2$	21.16 (4.91)	20.97 (4.91)	20.99 (4.92)
$\gamma = 3$	20.98 (4.90)	20.76 (4.90)	20.79 (4.91)
$\eta = 2.5$	21.29 (4.94)	20.78 (4.97)	20.87 (4.99)
$\eta = 5.0$	21.32 (4.93)	21.02 (4.94)	21.06 (4.95)
$\eta = 7.5$	21.33 (4.93)	21.11 (4.93)	21.14 (4.94)
$\eta = 10.0$	21.33 (4.93)	21.16 (4.93)	21.19 (4.93)
$\eta = 12.5$	21.34 (4.93)	21.20 (4.93)	21.22 (4.93)
$\varphi, \psi : 0\% \text{ informed}$	21.49 (4.96)	21.34 (4.96)	21.36 (4.95)
$\varphi, \psi : 25\% \text{ informed}$	21.41 (4.94)	21.26 (4.94)	21.28 (4.94)
$\varphi, \psi : 50\% \text{ informed}$	21.33 (4.93)	21.16 (4.93)	21.19 (4.93)
$\varphi, \psi : 75\% \text{ informed}$	21.25 (4.91)	21.07 (4.92)	21.09 (4.93)
$\varphi, \psi : 100\% \text{ informed}$	21.16 (4.91)	20.97 (4.91)	20.99 (4.92)

Table 3: Average profit and loss (with standard deviation) when the final inventory is evaluated at the fundamental value of the asset for a market maker who follows either (i) the full information optimal strategy $\delta^{*,\text{FI}}(t, Q_{t-}, U_t)$ observing the process U_t , (ii) the optimal strategy of Cartea-Jaimungal-Penalva (CJP), or (iii) the imperfect information optimal strategy $\delta^{*,\text{PI}}(t, Q_{t-}, \hat{U}_t)$ using the filtered process \hat{U}_t . The baseline set of parameters is provided in the first row then the parameter changes are reported. On each line (varying q , γ , or η) ψ is rescaled as in (61). The last rows deal with variations of φ, ψ keeping the expected number of market arrivals at 30.

A.6 Further robustness checks

As a robustness check, we repeat the analysis developed in Section 5 without rescaling ψ ; thus, the expected number of market arrivals is not fixed at 30 as the parameters change. As the number of trade arrivals changes, the figures are different from those in Section 5 but the order (according to performance) among the three strategies is confirmed as well as the main results.

Effect of q : We observe an increase of the performance for the FI strategy, while the CJP-strategy performances decreases. The behaviour of the three strategies observed in Section 5 when $q \rightarrow 0$ and $q \rightarrow 1$ is confirmed. The performance increase is due to the higher number of market arrivals. A higher number of market arrivals leads to a higher expected profitability because the market maker benefits from the round trip trades made by liquidity takers.²⁴

²⁴The expected number of market arrivals (before sieving according to displacements) goes from 29.86 (when $q = 0$) to 30.22

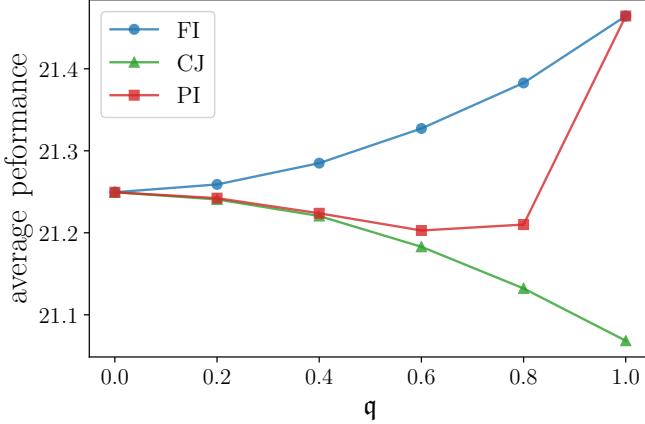


Figure 9: Performance of the strategies as a function of q .

Effect of γ : We observe a decrease of the performances up until $\gamma = 1$, followed by an increase in the performances as gamma increases ($\gamma > 1$). The decrease for $\gamma \leq 1$ is due to the higher toxicity of the trades sent by the informed traders (in line with the previous results) while the performance increase for $\gamma \geq 1$ is due to the higher number of market arrivals that lead to an increase in profitability that overshadows the increase of toxicity in the order flow of informed traders.²⁵

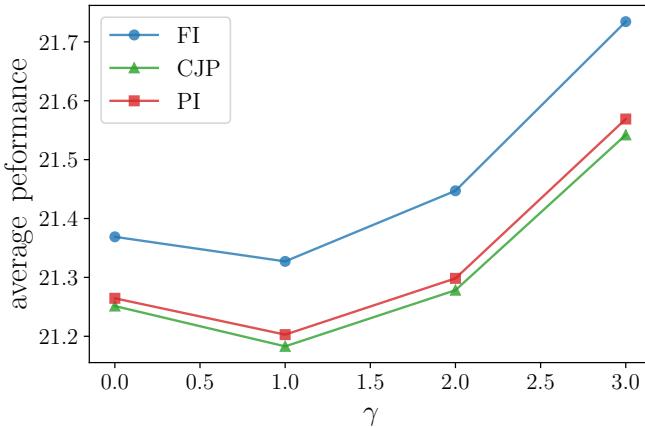


Figure 10: Performance of the strategies as a function of γ .

Effect of η : For the full information case we observe a decrease of the performance as η increases. This is due to the lower arrivals of orders. For the other two strategies, we refer to the arguments made for the closing of the gap in Figure 7.

(when $q = 1$).

²⁵The expected number of market arrivals (before sieving according to displacements) goes from 29.86 (when $\gamma = 0$) to 31.06 (when $\gamma = 3$).

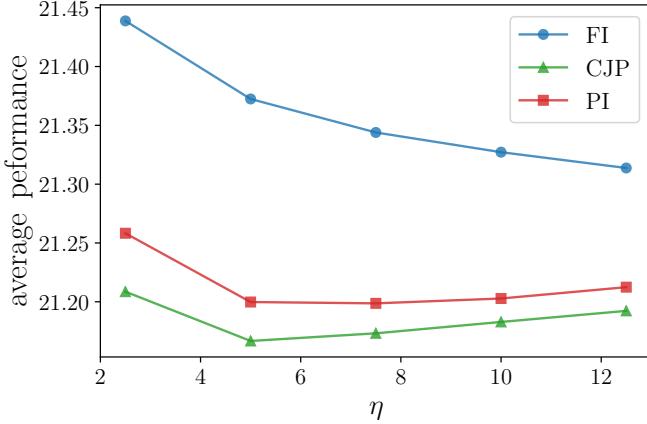


Figure 11: Performance of the strategies as a function of η .

A.7 How good is the closed-form second order approximation?

Figure 12 shows the bid and ask optimal displacements obtained using a finite-difference scheme for the solution of the PDE (20) and the second order approximation closed-form solution in (38). We observe that the differences are small justifying the second-order approximation of the solution.

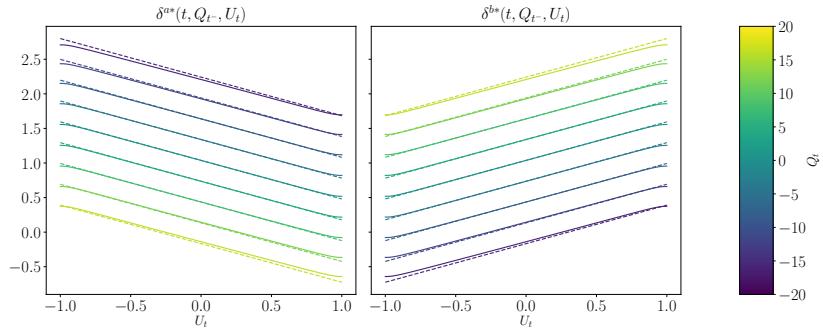


Figure 12: Ask and bid displacements as a function of the fad (x -axis) and the inventory of the market maker (colour bar) under full information. Solid lines: numerical simulation of the PDE; dashed lines: numerical simulation of the linear-quadratic approximation. Left panel is for the optimal ask displacement, and right panel is for the optimal bid displacement.