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Discrete Mathematics 132 (1994) 291–316

DISCRETE MATHEMATICS

Forbidden subsequences

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Received 26 November 1991; revised 17 November 1992

Abstract

The classification of sets of permutations with forbidden subsequences of length 4 is not yet complete. (In my recent paper *Classification of forbidden subsequences of length 4* submitted to European Journal of Combinatorics, Paris, this classification has been completed.) In this paper we show that $|S_n(4132)|=|S_n(3142)|$ by proving the stronger theorem for the corresponding permutation trees: $T(4132)\cong T(3142)$. We give a new proof of the so-called Schröder result, some results on forbidding entire classes of symmetries of permutation matrices, and some conjectures concerning the basic question: for what permutations τ and σ it is true that $|S_n(\tau)|=|S_n(\sigma)|$ for all $n\in\mathbb{N}$. We also discuss possible attacks on cases similar to the Schröder result by 'visualizing' the structure of the corresponding permutations and generalize the method of permutation trees.

1. Introduction to forbidden subsequences

In this paper the notation $(a_1, a_2, \ldots, a_k)$ represents a permutation of length k, say τ, so that $\tau(i)=a_i$, $i=1,2,\ldots,k$. If $k<10$ we will suppress the commas since this will cause no confusion.

Definition 1.1. Let τ and π be two permutations of lengths k and n, respectively. We say that π is τ-avoiding if there is no subsequence $i_{\tau(1)}, i_{\tau(2)}, \ldots, i_{\tau(k)}$ of $[n]=\{1,2,\ldots,n\}$ such that $\pi(i_1)<\pi(i_2)<\cdots<\pi(i_k)$. If there is such a subsequence, we say that the subsequence $\pi(i_{\tau(1)}), \pi(i_{\tau(2)}), \ldots, \pi(i_{\tau(k.)})$ is of type τ.

Two subsequences τ, σ of length n are of the same type if $\tau(i)<\tau(j)\Leftrightarrow\sigma(i)<\sigma(j)$ for all $1\leqslant i,j\leqslant n$, or equivalently τ and σ have the same pairwise comparisons throughout. We say that a subsequence τ of a permutation π avoids ω provided τ has no subsequence of type ω.

We denote by $S_n(\tau)$ the set of τ-avoiding permutations in S_n, the symmetric group on $[n]$. If Ω is a set of permutations then $S_n(\Omega)=\bigcap_{\tau\in\Omega}S_n(\tau)$. Any permutation $\tau\in\Omega$ is said to be a *forbidden subsequence* (for $S_n(\Omega)$).

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SSDI 0012-365X(93)E0023-W

Example 1.2. The permutation $\omega = (52687431)$ avoids (2413) but does not avoid (3142) because of its subsequence (5283).

The basic problems regarding the sets $S_n(\tau)$ are: find a necessary and sufficient condition on τ and σ so that $|S_n(\tau)| = |S_n(\sigma)|$ for all $n \in \mathbb{N}$; determine $|S_n(\tau)|$ as a function of n; and, discover an explicit bijection between $S_n(\tau)$ and $S_n(\sigma)$ when they have the same cardinality. We will be mainly interested in the first and second of these.

There are two types of numbers that appear frequently in the study of forbidden subsequences; the *Catalan numbers*, c_n, and the *Schröder numbers*, s_n. The Catalan numbers count the number of nondiagonal-crossing lattice paths from the origin (0, 0) to the point (n, n) composed of the vectors (0, 1) and (1, 0). The Schröder numbers count the number of nondiagonal-crossing lattice paths from the origin (0, 0) to the point (n, n) composed of the vectors (0, 1), (1, 0) and (1, 1).

Here follow some useful equations satisfied by the Catalan and the Schröder numbers (see [10]):

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \sum_{j=1}^{n} c_{j-1}c_{n-j},$$

$$s_n = \sum_{i=0}^{n}\binom{2n-i}{i}c_{n-i} = s_{n-1} + \sum_{j=0}^{n-1} s_j s_{n-1-j}\,.$$

Permutations with forbidden subsequences arise naturally in computer science in connection with sorting problems and strings with forbidden subwords. For example, in [2, p. 238] Knuth shows that $S_n(231)$ is the set of permutations that are stack-sortable, so that $|S_n(231)|$ is the number of binary strings of length $2n$, in which 0 stands for a 'move into a stack' and 1 symbolizes a 'move out from the stack'. A bijection between $S_n(123)$ and $S_n(132)$ was suggested by Knuth [2, p. 64], and shown by Rotem [9], and later by Simion and Schmidt in [8], by Richards in [7] and West in [11]. West's bijection has the advantage that the enumerative result $|S_n(123)| = c_n$ follows naturally. In [5] Lovász proves that $|S_n(213)| = c_n$ as well. In [11] West settled a conjecture of Shapiro and Getu that $|S_n(3142, 2413)| = s_{n-1}$, which is known as the Schröder result.

In [6] Macdonald asks the following question: what is $|S_n(2143)|$. The permutations avoiding 2143 are called *vexillary* permutations. They come up in the study of *multi-Schur functions.* A natural question arises: what is the number of permutations of a certain length avoiding a given permutation of length 4. West in his Ph.D. thesis (see [11]) partially classifies these permutations and gives some results on enumerating the sizes of the corresponding S_n-sets. In this paper we continue the classification of forbidden subsequences of length 4 as well as find some sizes of sets $S_n(\Phi)$ where the Φ's are special sets of permutations of length 4.

Definition 1.3. Let Ω be a set of permutations. The permutation tree $T(\Omega)$ is the tree whose nth level has nodes corresponding to all permutations $\tau \in S_n(\Omega)$. The children of a node τ are all permutations in $S_{n+1}(\Omega)$ which are obtained from τ by inserting $n+1$ at the permissible sites. The latter are called the active sites of τ.

Definition 1.4. Let $\tau \in S_n$. The permutation matrix $M(\tau)$ is the $n \times n$ matrix having a 1 in position $(i, \tau(i))$ for $i=1, 2, \ldots, n$, and having 0 elsewhere.

A permutation $\pi \in S_n$ contains a subsequence $\tau \in S_k$ if and only if the corresponding matrix $M(\pi)$ contains the $k \times k$ matrix $M(\tau)$ as a submatrix. In [11] West points out that permutation matrices are useful in establishing three types of bijections between set $S_n(\tau)$ and $S_n(\sigma)$. If $M(\pi)$ contains $M(\tau)$ as a submatrix, then the transpose matrix $M(\pi)^T$ contains $M(\tau)^T$. The same is true when simultaneously reflecting both the matrices $M(\pi)$ and $M(\tau)$ in either a horizontal or a verticle axis of symmetry. The resulting matrices will be denoted by $M(\pi)^-$, $M(\tau)^-$ and $M(\pi)^|$, $M(\tau)^|$, respectively. One notices that the three operations defined above generate a dihedral group acting on the permutation matrices in the obvious way. Thus, we can talk about entire symmetry classes of permutations. Two permutations τ, $\sigma \in S_k$ belong to the same class if and only if they are related by one or more of the following operations: taking inverse, reversal or complement. We denote these by $\tau^{-1}=\sigma$,

$$\tau^| = (\tau(k), \tau(k-1), \ldots, \ldots \tau(1)) = \sigma \quad \text{and}$$

$$\tau^- = (k+1-\tau(1), k+1-\tau(2), \ldots, k+1-\tau(k)) = \sigma,$$

respectively. Since $M(\tau)^T = M(\tau^T)$, $M(\tau)^| = M(\tau^|)$ and $M(\tau)^- = M(\tau^-)$, then $|S_n(\tau)| = |S_n(\tau^{-1})|$, $|S_n(\tau)| = |S_n(\tau^|)|$ and $|S_n(\tau)| = |S_n(\tau^-)|$.

Using the above bijections and the Simion–Schmidt algorithm [8] one can prove that $|S_n(\tau)| = |S_n(\sigma)| = c_n$ for all τ, $\sigma \in S_3$. The permutations in S_4 can also be grouped according as to whether the corresponding $S_n(\tau)$'s sets have the same cardinality for all n. Such sets of permutations will be called *cardinality classes.*

In [11] West found an isomorphism between certain permutation trees:

$$T(1, 2, \ldots, k-2, k-1, k) \cong T(1, 2, \ldots, k-2, k, k-1),$$

and hence he proved the corresponding equalities for the $S_n(\tau)$'s sets. He also showed that

$$|S_n(a_1, a_2, \ldots, a_{k-2}, k-1, k)| = |S_n(a_1, a_2, \ldots, a_{k-2}, k, k-1)|$$

for any $\tau \in S_k$ with $\tau(k-1) = k-1$, $\tau(k) = k$. In Section 3 we will prove that $T(3142) \cong T(4132)$. There is still one open case: whether $|S_n(1234)| = |S_n(1432)|$. Table 1 presents the results for $n=4$ known so far.[1]

The method of the three bijections and the West results mentioned above group together almost all permutations of length 5 and 6 in cardinality classes. We will discuss possible solutions to the remaining unknown cases.

[1] In my recent paper *Classification of forbidden subsequences of length 4* submitted to European Journal of Combinatorics, Paris, this classification has been completed.

2. Some entire symmetry classes for $n=4$

2.1. A new proof of the Schröder result

It seems more natural to forbid entire symmetry classes of permutations rather than just one permutation. For example, there is no known closed formula for $|S_n(\tau)|$, $\tau \in S_4, S_5, \ldots$, while the question of forbidding entire symmetry classes in the case $n=3$ has been completely solved by Simion and Schmidt in [8]. The Schröder result is a special case for $n=4$. The proof presented by West in [10] gives a description of the tree $T(2413, 3142)$ and uses generating functions. In search of a better proof he made the following conjecture (4.2.1, p. 30): Among all the permutations of length n, take those in which 1 appears in position j. For each of these, count 1 less than the number of active sites (with respect to the forbidden patterns 3142 and 2413). Then the total is $s_j \cdot s_{n-j}$. If this conjecture were true, one could easily prove the Schröder result using the recurrence relationship for the s_n's. Unfortunately, it is not. Consider, for example, the case $n=5, j=3$. Let τ_{ch} denote the number of τ's children in $T(2413, 3142)$ and $S_{n,j}$ denote the set of permutations of length n avoiding $(3142, 2413)$ and having 1 in position j. Then

$$\sum_{\tau \in S_{5,3}} (\tau_{\mathrm{ch}}-1) \leqslant |S_{5,3}| \cdot (6-1) \leqslant 4! \cdot 5 = 120 < 132 = 22 \cdot 6 = s_3 \cdot s_2.$$

Our argument remains valid even if we consider all permutations in which k appears in position j for some fixed k and j with $1 \leqslant k, j \leqslant n$.

In a personal communication West asked for a more natural proof of the Schröder result. In this section we provide such a new proof. The idea is simple: we reduce the case for n to smaller cases using a specific bijection.

Theorem 2.1. $|S_n(2413, 3142)| = s_{n-1}$.

Proof. Let X_n be the number we are looking for, i.e. $X_n = |S_n(2413, 3142)|$, $X_0 = 1$, $X_1 = 1$. Consider the largest number n in a permutation τ in $S_n(2413, 3142)$. Call a *block* a maximal consecutive subset of $\{1, 2, \ldots, n-1\}$, all of whose elements occur on the same side of n in τ. (For example, the blocks in $\omega = (25687431)$ are $\{1\}$, $\{2\}$, $\{3, 4\}$, $\{5, 6\}$ and $\{7\}$.) For blocks A and B, we say that $A < B$ if the elements of A are less than the elements of B. Note that these blocks partition the list $1, 2, \ldots, n-1$ into consecutive subsequences, and $A < B$ if and only if A corresponds to an earlier subsequence than B.

Now consider two blocks A and B such that $A < B$ and A and B occur in τ before n. Then there must be a block C between A and B ($A < C < B$) that occurs in τ after n. Choose any numbers $a \in A$, $b \in B$ and some $c \in C$. We have $a < c < b$. Since a and b occur before n, and c occurs after n, there are two choices for subsequences consisting of a, b, c and n, namely, (a, b, n, c) and (b, a, n, c). The latter is not possible since τ avoids (3142). Therefore, a must occur in τ before b. Since a and b were arbitrarily chosen from A and B, it follows that the whole block A occurs in τ before the whole block B. Since A and B were two arbitrarily chosen blocks before n so that $A<B$, it follows that if all blocks before n are $A_1<A_2<\cdots<A_k$, then they occur in τ in the same order (i.e. the whole block A_1 occurs before the whole block A_2, etc.).

Using a similar argument with the permutation (2413) instead of (3142), one proves that if all blocks in τ occurring after n are $B_1>B_2>\cdots>B_s$, then they occur in τ in descending order.

Hence we see that τ must satisfy the following conditions (the second is obviously necessary).

(1) τ is of the form $A_1, A_2, \ldots, A_k, n, B_1, B_2, \ldots, B_s$.

(2) The elements in any block form a permutation that avoids (2413) and (3142).

Note that condition (1) is equivalent to saying that n is not an element of any subsequence of τ of type (2413) or (3142).

Conversely, let τ be a permutation of length n which satisfies conditions (1) and (2). We claim that τ is of the desired type. Choose some $c\in[n]$. Delete all numbers in τ larger than c. Call the remaining permutation of $[c]$ τ_c. Observe that τ_c satisfies condition (1) with respect to its largest element c. Indeed, the only problem that can arise is in the image D' (in τ_c) of the block D of τ containing c. The fact that D avoids (2413) and (3142) implies that D' avoids them as well. Since c is the largest element in D', it follows by an argument analogous to the one above that D' satisfies condition (1) with respect to c. It might happen that some part X of D' is concatinated in τ_c to an image Y of a block of τ which was smaller than D. This does not contradict condition (1) for τ_c since $X>Y$ and X is between c and Y. Thus, τ_c satisfies condition (1) with respect to c. By the note above it follows that c cannot lie in a subsequence of type (2413) or (3142) of τ_c. This means that c cannot be the largest element in a subsequence of type (2413) or (3142) of τ. Since this is true for any $c\in[n]$, τ has no subsequences of type (2413) or (3142).

Therefore, $\tau\in S_n(3142, 2413)$.

We established a bijection between $S_n(3142, 2413)$ and the set of permutations τ of length n satisfying conditions (1) and (2) (see Fig. 1). Hence the formula:

$$X_n=2\sum_{i_1+i_2+\cdots+i_m=n-1} X_{i_1}X_{i_2}\cdots X_{i_m}, \quad n\geqslant 2.$$

Here $i_1, i_2, \ldots, i_m$ represent a partition of $1,2,\ldots,n-1$ into blocks of lengths $i_1, i_2, \ldots, i_m$, and the factor of 2 reflects the fact that the A_i's may occur in the list of blocks in either the odd or the even positions for $n\geqslant 2$. The following example illustrates the two cases.

Example 2.2. Let $\tau=(26587431)$ (see Fig. 2). Here the blocks are $A_1=\{2\}$, $A_2=\{5,6\}$, $B_3=\{1\}$, $B_2=\{3,4\}$ and $B_1=\{7\}$. The A_i's occur in even positions in the partition $\{1\}$, $\{2\}$, $\{3,4\}$, $\{5,6\}$, $\{7\}$. On the other hand, in $\sigma=(14378652)$ the A_i's occur in odd positions although the partition of $1,2,\ldots,n-1$ is the same (see Fig. 3).

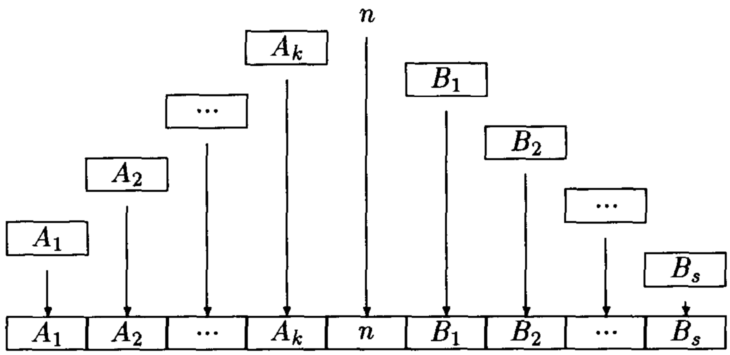

Fig. 1. $\tau \in S_n(3142, 2413)$, A_i's and B_j's avoid (3142, 2413).

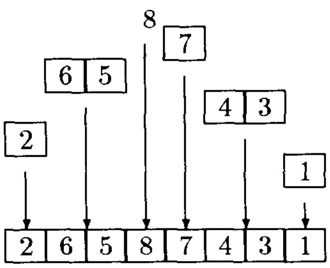

Fig. 2. $\tau = (26587431)$, A_i's in even positions.

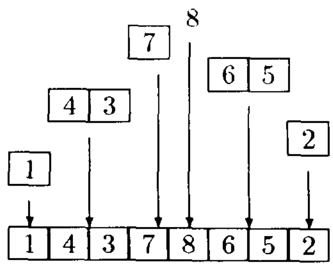

Fig. 3. $\tau = (14378652)$ A_i's in odd positions.

To finish the proof, one notices that summing over the possible i_1's from 1 to $n-1$, the above formula becomes equivalent to:

$$X_n = \sum_{i_1=1}^{n-1} X_{i_1} \cdot 2 \sum_{i_2+\cdots+i_m=n-1-i_1} X_{i_2} \cdots X_{i_m}$$
$$= X_1 X_{n-1} + X_2 X_{n-2} + \cdots + X_{n-2} X_2 + X_{n-1} X_1 + X_{n-1}.$$

Since the Schröder numbers satisfy the same recurrence relationship $s_{n-1} = \sum_{i=1}^{n-1} s_{i-1} s_{n-1-i} + s_{n-2}$, and $s_0 = X_1 = 1, s_1 = X_2 = 2$, by induction it follows that $X_n = s_{n-1}$, i.e. $|S_n(3142, 2413)| = s_{n-1}$. $\square$

Although this proof provides no information about the tree $T(2413, 3142)$, it gives us a picture of exactly what $\tau \in S_n(2413, 3142)$ looks like.

Definition 2.3. We say that two subsequences A and B of a permutation τ are independent on a set of permutations Ω if there are no $a \in A$ and $b \in B$ such that a and b occur in a subsequence of τ of type $\sigma \in \Omega$.

As we saw above, any $\tau \in S_n(2413, 3142)$ can be partitioned into independent consecutive subsequences A_i and B_j on $\{2413, 3142\}$ (see Fig. 1). This describes in a natural way the structure of such permutations and explains why it is easy to find the size of the corresponding set $|S_n(2413, 3142)|$.

2.2. Partial characterization of sets similar to $S_n(2413, 3142)$

In this subsection we apply the idea of partitioning permutations into independent subsequences to two other types of permutations which 'resemble' the elements of $S_n(2413, 3142)$.

Proposition 2.4. *Let* $Y_n = |S_n(3124, 4213)|$. *Then the following is true*:

$$Y_n = 2 \cdot Y_{n-1} + c_1 \cdot Y_{n-1} + c_2 \cdot Y_{n-2} + \cdots + c_{n-2} \cdot Y_2, \quad n \geqslant 3.$$

Proof. Let Θ_1 denote the set $\{(3124), (4213)\}$. Then $|S_n(\Theta_1)| = Y_n$. Any $\tau \in S_n(\Theta_1)$ is of the form:

$$A_1, A_2, \ldots, A_k, 1, B_l, B_{l-1}, \ldots, \mathrm{B}_1,$$

where $A_1 < A_2 < \cdots < A_k$, $B_1, < B_2 < \cdots < B_l$, and all A_i's and B_j's, except the largest block in τ, avoid (312) and (213), respectively. The latter avoids Θ_1 (see Fig. 4).

Conversely, if $\tau \in S_n$ satisfies the conditions above, then $\tau \in S_n(\Theta_1)$. Hence the formula:

$$Y_n = \sum_{i_1 + i_2 + \cdots + i_m = n-1} 2 \cdot c_{i_1} c_{i_2} \cdots c_{i_{m-1}} Y_{i_m}, \quad i_m \geqslant 1, \ n \geqslant 3,$$

where m is the number of blocks in τ and c_i is the ith Catalan number. Summing over the possible i_1's from 0 to $n-2$, the formula above becomes

$$Y_n = 2 \cdot Y_{n-1} + c_1 \cdot Y_{n-1} + c_2 \cdot Y_{n-2} + \cdots + c_{n-2} \cdot Y_2, \quad n \geqslant 3. \qquad \square$$

Let $\Theta_2 = \{(2431), (1342)\}$, $\Theta_3 = \{(2314), (3241)\}$ and $\Theta_4 = \{(1423), (4132)\}$. Since $\Theta_2 = (\Theta_1)^-$, i.e. the elements of Θ_2 are the complements of the elements of Θ_1, and similarly $\Theta_3 = (\Theta_1)^{-1}$, and $\Theta_4 = (\Theta_3)^-$, we have

$$|S_n(\Theta_1)| = |S_n(\Theta_2)| = |S_n(\Theta_3)| = |S_n(\Theta_4)| = Y_n.$$

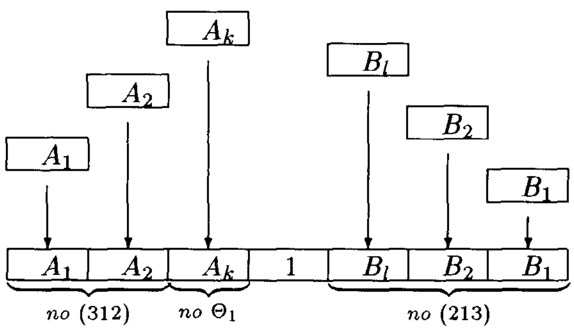

Fig. 4. $\tau \in S_n(\Theta_1)$.

Proposition 2.5. *Let $Z_n = |S_n(1342, 2413)|$. Then the following is true*:

$$Z_n = \sum_{i_1 + \cdots + i_{2k} = n-1} Z_{i_1} Z_{i_2} \cdots Z_{i_k} (Z_{i_{k+1}} + c_{i_{k+1}}) c_{i_{k+2}} \cdots c_{i_{2k}}$$

$$+ 2 \cdot \sum_{i_1 + \cdots + i_{2k+1} = n-1} Z_{i_1} Z_{i_2} \cdots Z_{i_{k+1}} c_{i_{k+1}} c_{i_{k+2}} \cdots c_{i_{2k+1}}.$$

Proof. Let Σ_1 denote the set $\{(1342), (2413)\}$. Any $\tau \in S_n(\Sigma_1)$ is of the form:

$$A_k, A_{k-1}, \dots, A_1, n, B_l, B_{l-1}, \dots, B_1,$$

where $A_1 < A_2 < \cdots < A_k$, $B_1 < B_2 < \cdots < B_l$, the A_i's avoid Σ_1, and the B_j's, except B_1 if it is the smallest block in τ, avoid (231). In the latter case B_1 avoids Σ_1. Conversely, if $\tau \in S_n$ satisfies the conditions above, then $\tau \in S_n(\Sigma_1)$ (see Fig. 5). Hence the desired formula. □

Similarly, let

$\Sigma_2 = \{(4213), (3142)\}$, $\Sigma_3 = \{(2431), (3142)\}$,

$\Sigma_4 = \{(3124), (2413)\}$, $\Sigma_5 = \{(1423), (3142)\}$,

$\Sigma_6 = \{(3241), (2413)\}$, $\Sigma_7 = \{(4132), (2413)\}$,

$\Sigma_8 = \{(2314), (3142)\}$.

Using the three standard bijections we conclude that $|S_n(\Sigma_i)| = Z_n$ for $i = 1, 2, \dots, 8$.

Let Ω be the first symmetry class in Table 1 for $n = 4$. Then notice that Ω is the disjoint union of Θ_i, $i = 1, 2, 3, 4$. Also, the first elements of the Σ_i's, $i = 1, 2, \dots, 8$, are precisely the elements of Ω, while their second elements form the second symmetry class $\{(2413), (3142)\}$ for $n = 4$.

2.3. *Other forbidden entire symmetry classes for $n = 4$*

Theorem 2.6. *Let Ω denote the set of all three cycles in S_4. Then Ω is an entire symmetry class and $|S_n(\Omega)| = 2^n - 2$ for $n > 4$.*

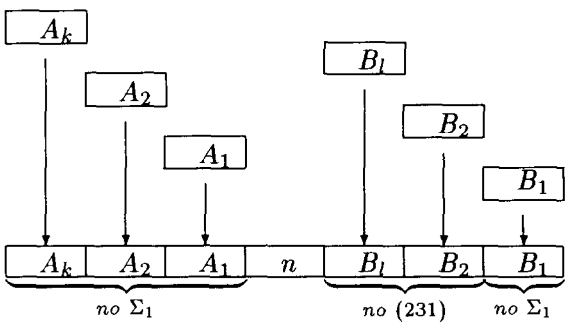

Fig. 5. $\tau \in S_n(\Sigma_1)$.

Proof. As in Theorem 2.1 consider the largest number n in a permutation $\tau \in S_n(\Omega)$. Partition the list $1, 2, \ldots, n-1$ into blocks $A_k > \cdots > A_2 > A_1$ and $B_1 < B_2 < \cdots < B_s$ such that the A's occur before n and the B's occur after n. Since τ avoids (1342) and (2431) the A's are arranged in descending order and the B's are arranged in ascending order. So τ is of the form: $A_k, \ldots, A_2, A_1, n, B_1, B_2, \ldots, B_s$.

Let $1 \in A_1$. Since τ avoids (1423) then n is followed by a decreasing sequence (otherwise τ will have a subsequence $(1, n, b, c)$, $b<c$, which is of the forbidden pattern). Therefore, there is at most one block B_i (the B's are in ascending order), which is arranged in descending order, and hence at most two A_j's. The case that $1 \in B_1$ is symmetric with respect to n and will give exactly the same number of permutations $\tau \in S_n(\Omega)$ (see Fig. 6).

Thus we can assume that $1 \in A_1$.

Case 1: $k=2$ implies that τ is of the form A_2, A_1, n, B_1, where $A_2 > B_1 > A_1$ (see Fig. 7).

If $|B_1| > 1$, choose $b, c \in B_1$, $b > c$. Then b is before c and the subsequence $(a, 1, b, c)$ for an $a \in A_2$ is of the forbidden type (4132). This contradiction shows that $B_1 = \{b\}$. Since the patterns (2314) and (3241) are forbidden, then A_2 must be arranged both in increasing and decreasing order (consider $(a_1, a_2, 1, n)$ and $(a_1, a_2, 4, b)$ for any $a_1, a_2 \in A_2$). Therefore, $|A_2| = 1$, $A_2 = \{n-1\}$ and $B_1 = \{n-2\}$, and τ is of the form $n-1, A_1, n, n-2$.

Since τ avoids (4132) and (4213), the arrangement $n-1, A_1, \ldots$ implies that A_1 avoids (132) and (213). Similarly, since τ avoids (2314) and (3124), the arrangement $\ldots A_1, n, \ldots$ implies that A_1 avoids (231) and (312). Thus, from [8], A_1 is arranged either in decreasing or increasing order and $\tau = (n-1,\ n-3,\ n-4, \ldots, 1,\ n,\ n-2)$ or $(n-1, 1, 2, \ldots, n-3, n, n-2)$. Since $n > 4$, τ contains $(n-1, 2, 1, n-2)$ or $(n-1, 1, 2, n)$ which are of the forbidden types (4213) and (3124). Therefore, $\tau \notin S_n(\Omega)$.

Case 2: $k=1$ implies that τ is of the form A_1, n, B_1, where $A_1 < B_1$ (recall that B_1 is arranged in descending order and may be empty, see Fig. 8). If $|A_1| = i$ then $\tau = (A_1, n, n-1, n-2, \ldots, i+1)$. Since τ avoids (2314) and (3124), then A_1 avoids (231) and (312).

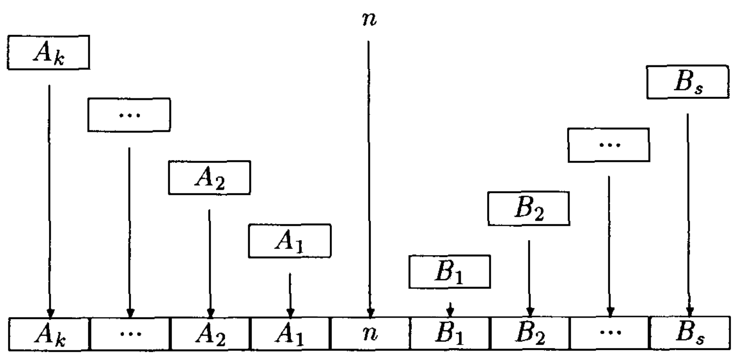

Fig. 6. $\tau \in S_n(\Omega)$.

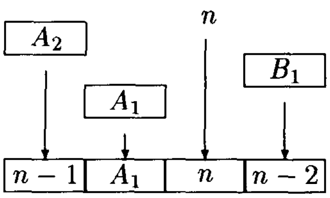

Fig. 7. $\tau \in S_n(\Omega)$, $1 \in A_k$, $k=2$.

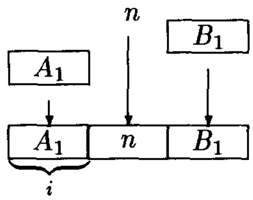

Fig. 8. $\tau \in S_n(\Omega)$, $1 \in A_k$, $k=1$.

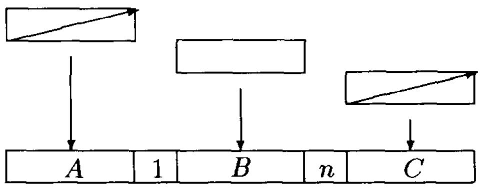

Fig. 9. $\tau \in S_n(\Phi)$, $B \neq \emptyset$.

Conversely, if A_1 avoids (231) and (312) then τ avoids Ω (any $\sigma \in \Omega$ contains a subsequence of type (312) or (231)). We are left to calculate the number of ways in which A_1 can be arranged so that $A_1 \in S_n(312, 231)$. From Rotem [9], $|S_i(312, 231)| = 2^{i-1}$. Hence the number of permutations avoiding Ω is:

$$\sum_{i=1}^{n-1} |S_i(312, 231)| = \sum_{i=1}^{n-1} 2^{i-1} = 2^{n-1} - 1.$$

To finish the proof, we multiply the last result by 2 because of the symmetry mentioned above. Therefore,

$$|S_n(\Omega)| = 2 \cdot (2^{n-1} - 1) = 2^n - 2, \quad \text{for } n > 4. \qquad \square$$

For completeness we remark that, by inspection, $|S_4(\Omega)| = 16$ and $|S_n(\Omega)| = n!$ for $n = 1, 2, 3$.

Using an argument similar to the one in Theorem 2.6, we can prove the following result.

Theorem 2.7. *Let Φ be the entire symmetry class* $\{(1243), (4312), (2134), (3421)\}$. *Then* $|S_n(\Phi)| = 14n$ *for* $n \geqslant 6$.

Proof. Let $\tau \in S_n(\Phi)$. For reasons of symmetry we may assume that 1 is before n in τ. Call the sets of number before 1, between 1 and n, and after n, A, B and C, respectively.

Case 1: $B \neq \emptyset$. Let $b \in B$. Since τ avoids (1243) and (2134), then $A > B > C$. Avoiding (4312) implies that A is arranged in ascending order (otherwise, there is a subsequence from A, A, 1, B of type (4312)). Similarly, C is arranged in ascending order (see Fig. 9).

(i) If $A \neq \emptyset$ and $C \neq \emptyset$, then A and C consist of one element each. For if A has at least two elements $a_1 < a_2$, and c is some element of C, then (a_1, a_2, b, c) is a subsequence of τ of type (3421), a contradiction. Similarly, one proves that $|C| = 1$. Hence τ is of the form $(n-1, 1, B, n, 2)$. Since τ avoids Φ, then B avoids (213), (132), (231) and (312). From [8], B is arranged in monotone order. Hence

$$\tau = (n-1, 1, 3, 4, \ldots, n-2, n, 2) \quad \text{or} \quad \tau = (n-1, 1, n-2, n-3, \ldots, 3, n, 2).$$

(ii) Let $A = \emptyset$ and $C \neq \emptyset$. If $|C| > 1$, choose $c_1 < c_2$ in C. If $b_1 < b_2$ are elements in B, then b_1 is before b_2 (otherwise, (b_2, b_1, c_1, c_2) is of type (4312)). Thus, B is arranged in ascending order and

$$\tau = (1, i+1, i+2, \ldots, n, 2, 3, \ldots, i) \quad \text{for } i = 3, 4, \ldots n-2.$$

If $|C| = 1$, then $C = \{2\}$, τ is of the form $(1, B, n, 2)$, and B avoids (132), (213) and (231). From [8, p. 397], B is of the form $D, 3, E$ where $D > E > \{3\}$, D is decreasing and E is increasing. Since B also avoids (4312), then either $|D| < 2$ or $E = \emptyset$. Thus,

$$\tau = (1, n-1, 3, 4, \ldots, n-2, n, 2), \qquad \tau = (1, 3, 4, \ldots, n-1, n, 2)$$
$$\text{or} \quad \tau = (1, n-1, n-2, \ldots, 3, n, 2).$$

Similarly, if $A \neq \emptyset$ and $C = \emptyset$, then

$$\tau = (1+1, i+2, \ldots, n-1, 1, 2, \ldots, i, n) \quad \text{for } i = 2, 3, \ldots, n-3,$$
$$\tau = (n-1, 1, 3, 4, \ldots, n-2, 2, n), \qquad \tau = (n-1, 1, 2, \ldots, n-2, n)$$
$$\text{or} \quad \tau = (n-1, 1, n-2, n-3, \ldots, 2, n).$$

(iii) Let $A = \emptyset$ and $C = \emptyset$. Then τ is of the form $(1, B, n)$, where B avoids (132) and (213). From [8, p. 393], B is of the form $(F, 2, G)$, where $F > G > \{2\}$ and G is increasing. Since B avoids (4312), we have that either F is increasing of $G = \emptyset$. In the first case,

$$\tau = (1, i+1, i+2, \ldots, n-1, 2, 3, \ldots i, n) \quad \text{for} \quad i = 3, 4, \ldots, n-1.$$

In the second case, τ is of the form $(1, F, 2, n)$, where F avoids (132), (213) and (231). Similarly, as in the case (ii) for the set B,

$$\tau = (1, n-1, 3, 4, \ldots, n-2, 2, n), \qquad \tau = (1, 3, 4, \ldots, n-1, 2, n)$$
$$\text{or} \quad \tau = (1, n-1, n-2, \ldots, 3, 2, n).$$

This exhausts all possible cases when $B \neq \emptyset$.

Case 2: $B = \emptyset$. Then τ is of the form $(A, 1, n, C)$. Let τ' denote the permutation obtained form τ by removing n.

(i) If there is a number between 1 and $n-1$ in τ', then as in Case 1, τ' will be of the form $(H, 1, I, n-1, J)$ with $H > I > J$, where H and J are arranged in ascending order, or $(D, n-1, E, 1, F)$ with $D < E < F$, where D and F are arranged in descending order. So, τ is of the form $(H, 1, n, I, n-1, J)$ or $(D, n-1, E, 1, n, F)$ with I and E nonempty. In the first case the sequence $(H \ldots n \ldots n-1 \ldots)$ implies that H is arranged in descending

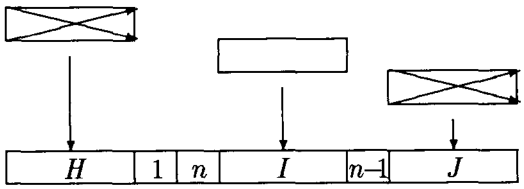

Fig. 10. $\tau \in S_n(\Phi)$, $B \neq \emptyset$, $n-1$ after 1, $I = \emptyset$.

order (consider (1243)), and the sequence $(\ldots n \ldots n-1 \ldots J)$ implies that J is arranged in descending order (consider (4312), see Fig. 10)).

Therefore, H and J have at most one element each. Since $I \neq \emptyset$, then at least one of H and J is empty (otherwise, a subsequence from H, n, I, J will be of type (3421)). Thus, τ is of the form $(n-2, 1, n, I, n-1)$, $(1, n, \mathrm{I}, n-1, 2)$ or $(1, n, I, n-1)$. In the first form $(n-2 \ldots n, I \ldots)$ implies that I is increased (consider (3421)), while in the second form I avoids all nonmonotone permutations of length 3 and hence I is arranged in monotone order (see [8]). Therefore,

$$\tau = (n-2, 1, n, 2, 3, \ldots, n-3, n-1), \qquad \tau = (1, n, 3, 4, \ldots, n-2, n-1, 2)$$

$$\text{or} \quad \tau = (1, n, n-2, n-3, \ldots, 3, n-1, 2).$$

The third form $(1, n, I, n-1)$ implies that I avoids (312), (132) and (213). As before, (see Case 1, (ii), $|C| = 1$),

$$\tau = (1, n, 2, 3, \ldots, n-2, n-1), \qquad \tau = (1, n, 3, 4, \ldots, n-2, 2, n-1),$$

$$\text{or} \quad \tau = (1, n, n-2, n-1, \ldots, 2, n-1).$$

Similarly, when $n-1$ is before 1,

$$\tau = (n-1, n-3, n-4, \ldots, 1, n, n-2), \qquad \tau = (2, n-1, n-2, \ldots 3, 1, n),$$

$$\tau = (2, n-1, 3, 4, \ldots, n-2, 1, n), \qquad \tau = (n-1, n-2, \ldots, 1, n),$$

$$\tau = (n-1, 2, n-2, n-3, \ldots, 3, 1, n), \quad \text{or} \quad \tau = (n-1, 2, 3, \ldots, n-2, 1, n).$$

(ii) If there is no number between 1 and $n-1$ in τ', then τ is of the form $(A, 1, n, n-1, C)$ or $(A, n-1, 1, n, C)$. In the first case $(\ldots n, n-1 \ldots)$ implies that A and C are arranged in descending order (consider (1243) and (4312)). Since τ avoids (3421), then for any $a \in A$ and $c_1 > c_2 \in C$ we have that $a < c_1$ (otherwise, (a, n, c_1, c_2) is of the forbidden type). So, $A < C - \{c\}$ where c is the smallest element in C. Similarly, using (4312), one proves that $A - \{a\} < C$ where a is the largest element in A (see Figs. 11(a) and (b)). Thus,

$$\tau = (i, i-1, \ldots, 2, 1, n, n-1, \ldots, i+1) \quad \text{for } i = 1, 2, \ldots, n-2 \quad \text{or}$$

$$\tau = (i+1, i-1, i-2, \ldots, 2, 1, n, n-1, \ldots, i+2, i) \quad \text{for } i = 2, 3, \ldots, n-3.$$

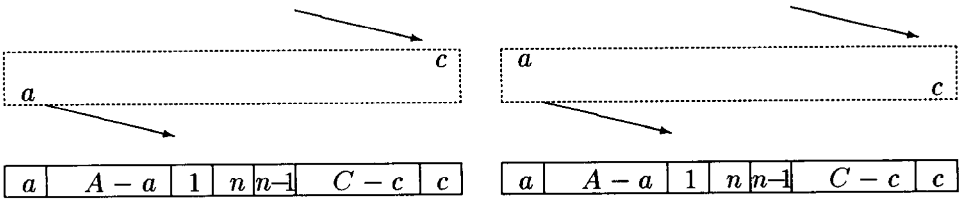

Fig. 11(a) and (b). $\tau \in S_n(\Phi)$, $B \neq \emptyset$, $n-1$ next to n, $a<c$ or $a>c$.

Similarly, in the case $\tau=(I, n-1, 1, n, J)$ we have:

$$\tau=(i+1, i+2, \dots n-2, n-1, 1, n, 2, 3, \dots i) \quad \text{for } i=1, 2, \dots, n-2 \quad \text{or}$$

$$\tau=(i, i+2, i+3, \dots, n-1, 1, n, 2, 3, \dots, i-1, i+1) \quad \text{for } i=2, 3, \dots, n-3.$$

One readily checks that all listed forms of τ are admissible, i.e. they avoid Φ, and that they are distinct for $n \geqslant 6$. Counting them twice for symmetry reasons, we get that

$$|S_n(\Phi)|=14n \quad \text{for } n \geqslant 6. \qquad \square$$

If we select only the distinct permutations for $n=5$ from the general results in the proof, we get that $|S_5(\Phi)|=54$.

Remark 2.8. By a theorem of Erdős and Szekeres in [1, p. 160] it follows that any permutation of length at least 10 will have an increasing or decreasing subsequence of 4 terms, i.e. $|S_n(1234, 4321)|=0$ for $n \geqslant 10$. Note that this is the third entire symmetry class for $n=4$ (see Table 1).

So far we have managed to provide nice formulas for the first four entire symmetry class for $n=4$. One of the remaining three classes, $\Phi'=\{(2143), (3412)\}$ satisfies the following interesting property.

Theorem 2.9. *A permutation τ belongs to $\Phi'=\{(2143), (3412)\}$ if and only if τ can be partitioned into two, not necessarily nonempty, subsequences τ_1 and τ_2 such that τ_1 is increasing and τ_2 is decreasing.*

Proof. We say that a permutation τ satisfies property P if τ can be partitioned into two disjoint subsequences τ_1 and τ_2 such that τ_1 is increasing and τ_2 is decreasing. We denote this by $\tau=\tau_1 \oplus \tau_2$. Let $\tau \in S_n$ be a permutation satisfying P. Suppose that $\tau \notin S_n(\Phi')$. Then τ has a subsequence $\tau^0=(a, b, c, d)$ of type (2143) or (3412) which satisfies property P (τ^0 inherits the partition of τ). But neither (2143) nor (3412) satisfies P, which implies that τ^0 does not satisfy P, a contradiction. Thus, $\tau \in S_n(\Phi')$.

Conversely, let $\tau \in S_n(\Phi')$. We will show by induction on n that τ satisfies P. For $n=1, 2$ the statement is trivial. Let $n \geqslant 3$. For reasons of symmetry we can assume that 1 is before n in τ, i.e. τ is of the form $(A, 1, B, n, C)$. If $C=\emptyset$, let $\tau'=\tau-\{n\} \in S_{n-1}(\Phi')$. By

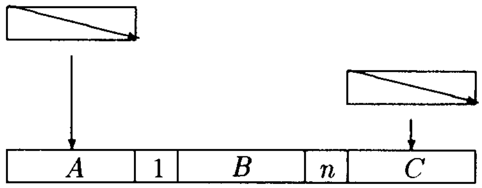

Fig. 12. $\tau \in S_n(\Phi')$, 1 before n, $A, C \neq \emptyset$.

induction, $\tau' = \tau_1' \oplus \tau_2'$. Then $\tau = (\tau_1' + \{n\}) \oplus \tau_2'$ ($\tau_1' + \{n\}$ means τ' is followed by n). Thus, we can assume that $C \neq \emptyset$. For reasons of symmetry we can also assume that $A \neq \emptyset$. Then, as in Theorem 2.7, $A > C$ and A and C are arranged in decreasing order (see Fig. 12).

Then $n-1 \in A \cup B$, i.e. $n-1$ is before n. If $n-1 \in A$, then $n-1$ is the first element of τ. Consider $\tau' = \tau - \{n-1\}$. We can think of τ' as an element of $S_{n-1}(\Phi')$. By induction, $\tau' = \tau_1' \oplus \tau_2'$, where n is either the first element of τ_2' or the last element of τ_1'. In the second case $\tau = \tau_1' \oplus (\{n-1\} + \tau_2')$. If n is the first element of τ_2', then everything before n in τ' is arranged in increasing order. Since C is arranged in decreasing order, we can assume that $\tau' = \tau_1' \oplus \tau_2'$ where τ_1' is increasing and consists of all the elements in τ' before n, while τ_2' is decreasing and consists of n and all elements in τ' after n. Then $\tau = (\tau_1' + \{n\}) \oplus (\{n-1\} + (\tau_2' - \{n\}))$.

Suppose now that $n-1 \in B$. Then $\tau' = \tau - \{n\} = \tau_1' \oplus \tau_2'$ by induction. Here $n-1$ is either the final element of τ_1' or the first element of τ_2'. In the former case, $\tau = (\tau_1' + \{n\}) \oplus \tau_2'$ ($n-1$ is before n). In the latter case $A \cup \{1\}$ (which is before $n-1$) is arranged in increasing order, i.e. A is empty, a contradiction.

This exhausts all possible cases and hence the theorem follows. □

3. Continuing the classification of the case $n=4$

In [11] West proved that $T(1234) \cong T(1243) \cong T(2143)$ and that the isomorphisms are unique. He further conjectured that $T(4132) \cong T(3142)$ (see Table 1). In this section we verify this conjecture.

Theorem 3.1. $T(4132) \cong T(3142)$.

Proof. Let us label the two trees as follows. Suppose that a node τ of one of the trees has k active sites. To each active site s associate the number i_s which is 1 more than the number of previous active sites q such that any number in τ between q and s is greater than any number in τ after s (symbolically, $[q,s] > [s,\infty)$). The label of the node τ is defined to be $(i_1, i_2, \ldots, i_k)$ (see Fig. 13).

Example 3.2. $\tau = (1342) \in T(3142)$ has 5 active sites. Consider its fourth active site, $s=4$. If q is the first active site, then the number 1 is between the qth and sth active sites

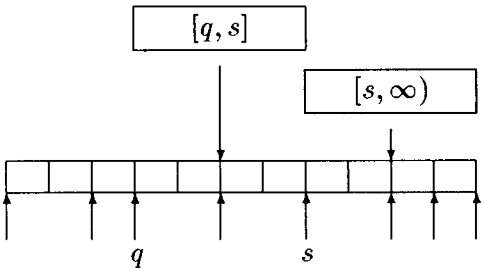

Fig. 13. Labelling of $\tau \in T(4132)$ or $T(3142)$.

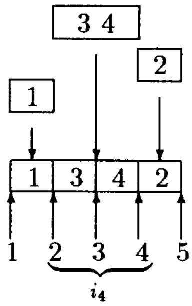

Fig. 14. $(1342) \in T(3142)$.

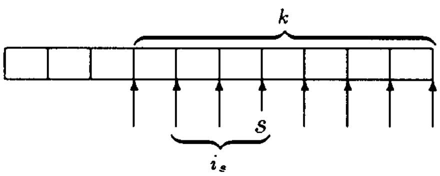

Fig. 15. Active sites of $\tau \in T(4132)$.

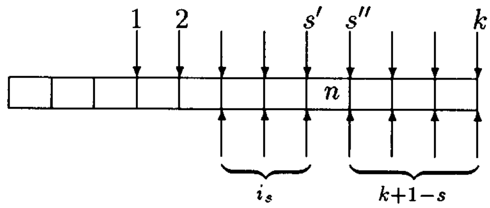

Fig. 16. Active sites of $\tau(s) \in T(4132)$.

while the number 2 is after the sth active site, i.e. condition $[q,s] > [s,\infty)$ is not satisfied. Therefore, the qth active site is not counted in i_4. Since $\{3,4\} > \{2\}$ the second and third active sites are counted in i_4. Thus, $i_4 = 3$. The permutation τ is labelled $(1,1,1,3,5)$ (see Fig. 14).

Lemma 3.3 (Parent–children relationship in $T(4132)$). *The sth child* $(s = 1, 2, \ldots, k)$ *of a node* $\tau \in T(4132)$ *with label* $(i_1, i_2, \ldots, i_k)$ *has label*

$$(\overbrace{1, 1, \ldots, 1,}^{i_s} \quad \hat{i}_s, \hat{i}_{s+1}, \ldots, \hat{i}_k),$$

where for $j = s, s+1, \ldots, k$

$$\hat{i}_j = \begin{cases} i_j & \text{if } i_j \leqslant j - s, \\ i_j + 1 & \text{if } j - s < i_j \leqslant i_s + j - s, \\ i_s + 1 + j - s & \text{if } i_s + j - s < i_j. \end{cases}$$

Proof. To see this, first notice that the active sites of $\tau \in T(4132)$ are the last k active sites (see Fig. 15).

Active site 1 is the leftmost active site such that there is no pattern (132) to the right of it. If site q is counted in i_s, then any site between q and s will be counted too. Thus, i_s represents a set of consecutive sites to the left of and including site s.

Consider the sth child of τ. Call it $\tau(s)$ (see Fig. 16 — all arrows pointing downwards stand for active sites of τ and all arrows pointing upwards stand for active sites of $\tau(s)$). Observe that $n+1$ can be inserted in all sites after n since they were active for τ and n and $n+1$ cannot both occur in a pattern (4132) if n is before $n+1$. Also, $n+1$ can be inserted in the i_s sites to the left of n since they were active for τ and the definition of i_s excluded the possibility of $n+1$ and n to occur in a pattern (4132) if $n+1$ is before n. Thus, we found all active sites of $\tau(s)$.

Let $j=1,2,\ldots,i_s$. Since n is after the jth active site, then there does not exist an active site q before s with $[q,j]>[j,\infty)$. Then the jth active site of $\tau(s)$ is associated with 1. This justifies the first i_s 1's in $\tau(s)$'s label.

Let $j>i_s$. Every such active site in $\tau(s)$ corresponds to exactly one active site in τ after n, namely, its preimage. Consider the jth active site in $\tau, j=s,s+1,\ldots,k$.

Case 1: If $i_j\leqslant j-s$, then inserting n in sth active site of τ will not affect the active sites q before jth active site with $[q,j]>[j,\infty)$ (see Fig. 17).

Hence the new number associated with the old jth active site in $\tau(s)$ will be $\hat{i}_j=i_j$.

Case 2: If $j-s<i_j\leqslant i_s+j-s$, then inserting n in sth active site of τ will add 1 more active site q before the jth active site with $[q,j]>[j,\infty)$. This comes from the 'split' of the old sth active site into the two new sites surrounding the number n (see Fig. 18). Hence the new number associated in $\tau(s)$ with the old jth active site will be $\hat{i}_j=i_j+1$.

Case 3: If $i_s+j-s<i_j$, then as in Case 2, inserting n will add to $\hat{i}_j$ one more active site q from the 'split' of the old sth active site. It will also remove some of the first q's for the old jth active site since they will not be active in $\tau(s)$ (they will be before the first i_s active sites of $\tau(s)$, see Fig. 19). Hence the new number associated with the old jth active site in $\tau(s)$ will be $\hat{i}_j=i_s+1+j-s$.

This finishes the proof of Lemma 3.3. □

Proof of Theorem 3.1 (*continuation*). For convenience we introduce the following notation. If a node τ in $T(4132)$ has label

$$(1,\ldots,1,l_{m+1},1,\ldots,1,l_{m+2},1,\ldots,1,l_{n-1},1,\ldots,1,l_n),$$

with $2\leqslant l_{m+1}<l_{m+2}<\cdots<l_n=n$ and every l_j is in l_jth position, we abbreviate the label as $\langle l_{m+1},l_{m+2},\cdots,l_n\rangle$.

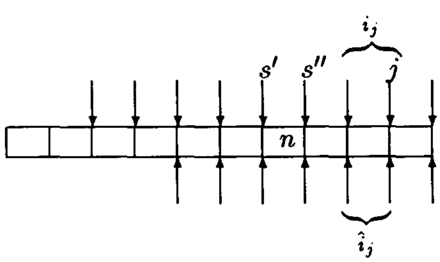

Fig. 17. $\hat{i}_j=i_j, j>i_s, i_j\leqslant j-s$.

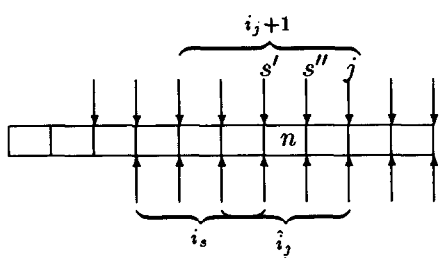

Fig. 18. $\hat{i}_j=i_j+1, j-s<i_j\leqslant i_s+j-s$.

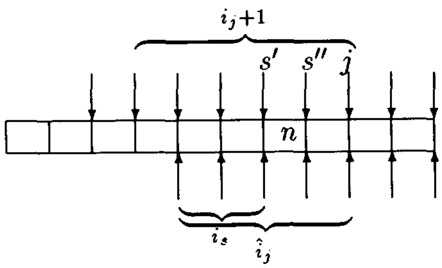

Fig. 19. $\hat{i}_j = i_s + 1 + j - s$, $i_j > i_s + j - s$.

Corollary 3.4. *The sth child a node τ in $T(4132)$ with label $\langle l_{m+1}, l_{m+2}, \ldots, l_n\rangle$ is labelled as follows*:

(1) $\langle 2, l_{i+1}+2-s, l_{i+2}+2-s, \ldots, l_n+2-s\rangle$ *for* $l_i < s < l_{i+1}$, *where* $i = m, m+1, \ldots, n-1$ *and* $l_m = 0$;

(2) $\langle l_q+1, l_{q+1}+1, \ldots, l_n+1\rangle$ *for* $q = m+1, m+2, \ldots, n$ $(s = l_{m+1}, l_{m+2}, \ldots, l_n)$.

Proof. Consider $\tau(s)$. Let $l_i < s < l_{i+1}$ for $i = m+1, \ldots, n-1$. Then $i_s = 1, s > 1$. By Lemma 3.3, $\tau(s)$ will be labelled $(1, 2, \hat{i}_{s+1}, \ldots, \hat{i}_n)$ where for $j > s$:

$$\hat{i}_j = \begin{cases} 1 & \text{if } i_j = 1, \\ l_j + 2 - s & \text{if } i_j = l_j > 1 \text{ (then } j = l_j). \end{cases}$$

(For $i_j = 1$ we apply the first case in Lemma 3.3, for $i_j > 1$ we apply the third case in Lemma 3.3, and for the 2 in the label we apply the second case in Lemma 3.3.)

Similarly, in the special case when $i = m$, $l_m < s < l_{m+1}$ (then $i_s = 1$) we have the same label-pattern as above.

It is easy to check that in the new label any number t different from 1 is in tth position. Also, for $j = n$, $l_n + 2 - s = n + 2 - s > 1$ is the last number in the label. Thus, the label satisfies the conditions in Corollary 3.4 and can be abbreviated as

$$\langle 2, l_{i+1}+2-s, l_{i+2}+2-s, \ldots, l_n+2-s\rangle.$$

Similarly, one can prove that if $s = l_{m+1}, l_{m+2}, \ldots, l_n$ then the label of $\tau(s)$ can be abbreviated as

$$\langle l_q+1, l_{q+1}+1, \ldots, l_n+1\rangle \quad \text{for } q = m+1, m+2, \ldots, n,$$

($i_s = s > 1$ and the second case of Lemma 3.3). This exhausts all possible cases for s and hence the corollary is proved. □

Example 3.5. See Fig. 20.

Lemma 3.6 (Parent–children relationships in T(3142)). *The sth child* ($s = 1, 2, \ldots$, k) *of a node τ with label $(i_1, i_2, \ldots, i_k)$ has label*

$$(\underbrace{1, 1, \ldots, 1}_{s}, \hat{i}_s, \hat{i}_{s+1}, \ldots, \hat{i}_k),$$

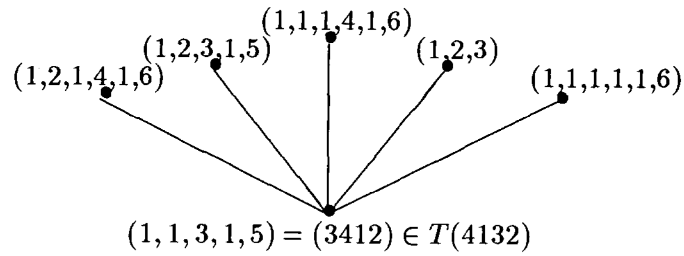

Fig. 20.

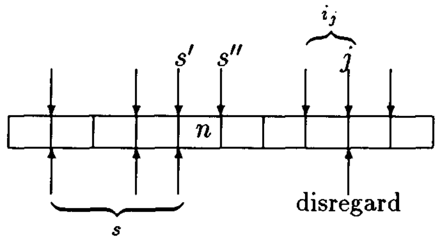

Fig. 21. $\tau(s)\in T(3142)$, $i_j<j-s$.

where for $j=s,s+1,\ldots,k$

$$\hat{i}_j \begin{cases} \textit{disregard } \hat{i}_j & \textit{if } i_j\leqslant j-s,\\ =i_j+1-|\{i_l : s<l<j, i_l\leqslant l-s\}| & \textit{if } i_j>j-s. \end{cases}$$

Proof. We begin with the fact that the first s active sites of τ are also active in $\tau(s)$ because n and $n+1$ cannot occur in a pattern (3142) if $n+1$ is before n. All of them will be associated in $\tau(s)$ with 1's because of the number n occurring after them (see the proof of Lemma 3.3). This justifies the first s 1's in $\tau(s)$'s label.

Any old active site j after n will also be active in $\tau(s)$ if and only if $i_j>j-s$. The last means that the sth active site of τ is counted in i_j, i.e. it's not possible for n and $n+1$ to occur in a pattern (3142) if n is before $n+1$ ($[s,j]>[j,\infty)$). Therefore we will disregard any $\hat{i}_j$ if $j\geqslant s$ and $i_j\leqslant j-s$ (see Fig. 21).

On the other hand, in the case $j>s$ and $i_j>j-s$, inserting n in sth active site of τ will add 1 more active site q before the old jth active site such that $[q,j]>[j,\infty)$. This comes from the split of the old sth active site into two new sites surrounding the number n. Notice that any active site q counted in i_j forces any other active site t between qth and jth old active sites to be counted in i_j ($[q,j]>[j,\infty)$, q before t before $j \Rightarrow [t,j]>[j,\infty)$). Thus, the i_j represents consecutive active sites before and including the jth active site (see Fig. 22).

Since all the old active sites before sth active site are also active in $\tau(s)$, then the interval represented by i_j will start with one of these first active sites. Therefore, the number $\hat{i}_j$ associated in $\tau(s)$ with the old jth active site will count the one additional

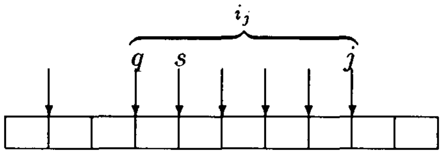

Fig. 22. $\tau \in T(3142)$, i_j-interval.

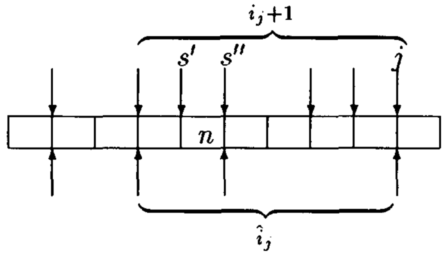

Fig. 23. $\tau(s) \in T(3142)$, $i_j > j-s$.

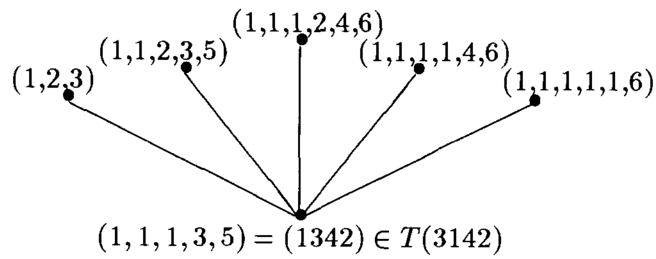

Fig. 24.

active site (see above) coming form the 'split' of the sth active site, and all the sites counted in i_j without the disregarded old active sites between the old sth and jth active sites. This yields the formula: $\hat{i}_j = i_j + 1 - \{$the number of i_l's with $s < l < j$ and $i_l \leqslant l - s$ for $j > s$ and $i_j > j - s\}$ (see Fig. 23). □

Example 3.7. See Fig. 24.

As before, if a node τ in $T(3142)$ has label

$$(\overbrace{1, 1, \ldots, 1}^{m}, \quad l_{m+1}, l_{m+2}, \ldots, l_n),$$

with $2 \leqslant l_{m+1} < l_{m+2} < \cdots < l_n = n$, we abbreviate it as $\langle l_{m+1}, l_{m+2}, \ldots, l_n \rangle$.

Corollary 3.8. *The sth child a node τ in $T(3142)$ with label $\langle l_{m+1}, l_{m+2}, \ldots, l_n \rangle$ is labelled as follows:*

(1) $\langle 2, l_{i+1} + 1 - i + s, l_{i+2} + 1 - i + s, \ldots, l_n + 1 - i + s \rangle$ *for* $i + 1 - l_{i+1} < s \leqslant i - l_i$, *where* $i = m, m+1, \ldots, n-1$ *and* $l_m = 0$;

(2) $\langle l_s + 1, l_{s+1} + 1, \ldots, l_n + 1 \rangle$ *for* $s = m+1, m+2, \ldots, n$.

Proof. Consider $\tau(s)$. Since $l_{i+1}-l_i \geqslant 1$ for $i=m,m+1,\ldots,n-1$, then $n-l_n \leqslant n-1-l_{n-1} \leqslant \cdots \leqslant m+1-l_{m+1} < m$. Let $i+1-l_{i+1} < s \leqslant i-l_i$, where $i=m+1,m+2,\ldots,n-1$. By Lemma 3.6, $\tau(s)$ will be labelled

$$(\underbrace{1,1,\ldots,1,}_{s}\quad \hat{i}_s, \hat{i}_{s+1}, \ldots, \hat{i}_n),$$

where $i_s=2$. All 1's after the sth position in τ's label will be disregarded in $\tau(s)$'s label ($i_j=1 \leqslant j-s$). Also $l_{s+1},\ldots,l_i$ will be disregarded in $\tau(s)$'s label:

$$s \leqslant i-l_i < j-l_j \;\Rightarrow\; i_j=l_j<j-s \quad \text{for } j=s+1,\ldots,i.$$

Moreover for any $j>i$:

$$j-l_j \leqslant i+1-l_i+1 < s \;\Rightarrow\; i_j=l_j>j-s.$$

Therefore the old jth active site ($j=i+1,\ldots,n$) will be an active site in $\tau(s)$. Thus for any $j>i$ the disregarded old active sites between the old sth and jth active sites are exactly the $s+1$th, $\ldots$, ith old active sites (they are associated with 1's or $l_{s+1},\ldots,l_i$).

By Lemma 3.6, $\hat{i}_j=l_j+1-(i-s)$. Hence $\tau(s)$'s label is:

$$(\underbrace{1,1,\ldots,1,}_{s}\quad 2,\ l_{i+1}+1-i+s,\ldots,l_n+1-i+s).$$

Using the properties of τ's label (see Corollary 3.6), one can easily prove that $\tau(s)$'s label satisfies the same properties and hence it can be abbreviated as

$$\langle 2, l_{i+1}+1-i+s, l_{i+2}+1-i+s, \ldots, l_n+1-i+s \rangle .$$

for $i+1-l_{i+1} < s \leqslant i-l_i$, $i=m+1,\ldots,n-1$. The special case $i=m$, $m+1-l_{m+1} < s \leqslant m-l_m = m$ and the cases $s=m+1,\ldots,n$ are approached in a similar way and they yield the corresponding formulas in Corollary 3.6. This exhausts all possible cases for s and hence proves the corollary. □

Proof of Theorem 3.1 (*conclusion*). To finish the proof of Theorem 3.1, notice that the first case in Corollary 3.8 is equivalent to:

$$\langle 2, l_{i+1}+2-s, l_{i+2}+2-s, \ldots, l_n+2-s \rangle .$$

for $l_i < s < l_{i+1}$, where $i=m,m+1,\ldots,n-1$ and $l_m=0$ (in the original formula substitute $(i+1)-s$ for s). Also notice that the labels in the second cases of Corollary 3.4 and Corollary 3.8 are essentially the same (in Corollary 3.8 substitute q for s). Thus the abbreviated labels of $T(4132)$ and $T(3142)$ satisfy the same 'parent–children' relationships in the sense that parents with the same abbreviated labels have children with the same abbreviated labels. Now it is obvious how to construct the isomorphism between the two trees: using induction on the levels of the trees we associate two nodes $\tau \in T(4132)$ and $\sigma \in T(3142)$ if and only if their parents are associated and the nodes have the same abbreviated labels. (Note that the roots of the two trees

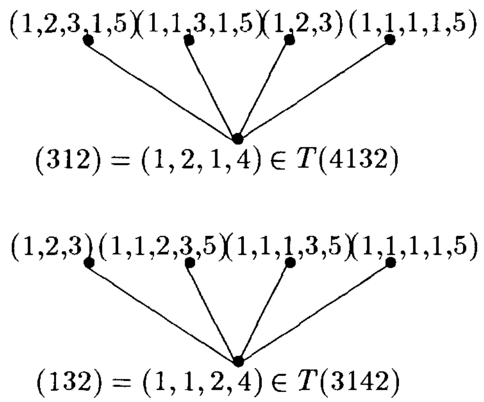

Fig. 25.

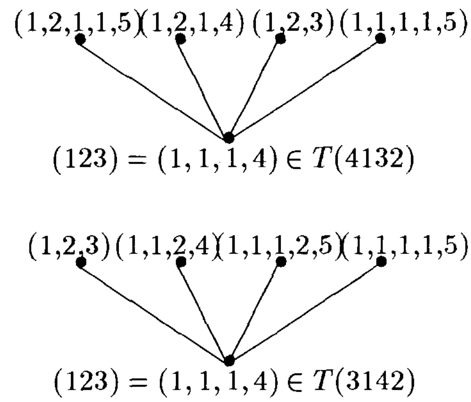

Fig. 26.

have the same label (1,2) and therefore the same abbreviated label $\langle 2 \rangle$.) Hence, $T(4132) \cong T(3142)$. □

Examples 3.9 and 3.10. See Figs. 25 and 26.

Example 3.11. See Fig. 27.

As one can see from Examples 3.9–3.11, the isomorphism between $T(4132)$ and $T(3142)$ works in a very nice way. If $\sigma \in T(3142)$ and $\tau \in T(3142)$ are associated, then their labels have the same numbers j arranged as follows: τ's label is a strictly increasing sequence of integers with the exception of the initial 1's, while in σ's label for all $j \geqslant 1$ the j's are in the corresponding jth positions and all other positions are filled in with 1's.

Note that the isomorphism between $T(4132)$ and $T(3142)$ is unique. This follows from the fact that the children of a node in any of the two trees have distinct abbreviated labels. An immediate consequence of Theorem 3.1 is the following.

Corollary 3.12. $|S_n(4132)| = |S_n(3142)|$.

The results so far for the case $n=4$ are summarized in Table 1 (see [11]). There each entire symmetry class is enclosed in a box. Any two symmetry classes Π_1 and Π_2 for which it is proved that $|S_n(\tau_1)| = |S_n(\tau_2)|$ for every $\tau_1 \in \Pi_1$, $\tau_2 \in \Pi_2$, i.e. they belong to the same cardinality class, have no separating line for the values of the $S_n(\tau)$'s. The *'s refer to West's Theorem 3.1.5 in [11], namely that $T(1234) \cong T(1243) \cong T(2143)$, and the ** refers to Theorem 3.1.

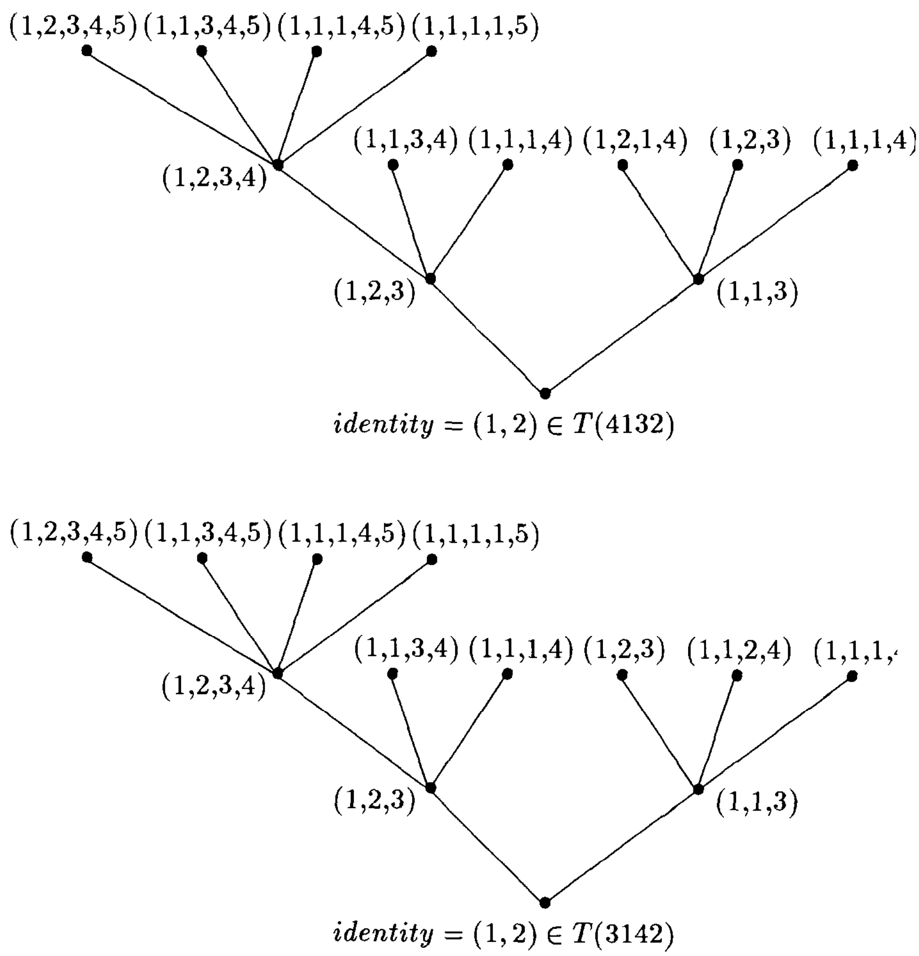

Fig. 27. Isomorphism between $T(4132)$ and $T(3142)$.

4. Speculation on the general case $|S_n(\tau)| = |S_n(\sigma)|$

West's conjecture 3.3.4 in [10, p. 45] states that

$$|S_n(a_1, a_2, \ldots, a_k, k+1, k+2, \ldots, n)| = |S_n(a_1, a_2, \ldots, a_k, n, n-1, \ldots, k+1)|.$$

In particular, for $n=4$, $k=1$, $a_1=1$ the conjecture covers the only missing case in the classification of forbidden permutations of length 4: $|S_n(1234)| = |S_n(1432)|$ (see Table 1). For $n=5$ and $n=6$ the only missing cases are $|S_n(12345)| = |S_n(12543)| = |S_n(15432)|$ and $|S_n(abc456)| = |S_n(abc654)|$, respectively, and they are also covered by the above conjecture. An interesting fact which distinguishes the classification of the case $n=4$ from the classification of the cases $n=1,2,3,4,5,6$ is the extra case in Theorem 3.1 which is not covered by the three

Table 1
Known results for $n=4$

<table>
<tr><th colspan="3">τ</th><th>$S_5(\tau)$</th><th>$S_6(\tau)$</th><th>$S_7(\tau)$</th><th>$S_8(\tau)$</th><th>$S_9(\tau)$</th><th>$S_{10}(\tau)$</th></tr>
<tr><td colspan="3">1342
1423
2341
2431
3124
3241
4132**
4213</td><td rowspan="2">103</td><td rowspan="2">512</td><td rowspan="2">2740</td><td rowspan="2">15485</td><td rowspan="2">91245</td><td rowspan="2">555662</td></tr>
<tr><td colspan="3">2413
3142**</td></tr>
<tr><td>1234*
4321</td><td>1243*
2134
3421
4312</td><td>2143*
3412</td><td>103</td><td>513</td><td>2761</td><td>15767</td><td>94359</td><td>586590</td></tr>
<tr><td colspan="3">1432
2314
3214
4123</td><td>103</td><td>513</td><td>2761</td><td>15767</td><td>94359</td><td>586590</td></tr>
<tr><td colspan="3">1324
4231</td><td>103</td><td>513</td><td>2762</td><td>15793</td><td>94776</td><td>591950</td></tr>
</table>

bijections inverse, reversal and complementation, or by the switch of the last two and largest elements of a permutation (Theorem 3.1.6 in [11]).

Although for many permutations τ and σ for which $|S_n(\tau)|=|S_n(\sigma)|$ it is also true that $T(\tau)\cong T(\sigma)$, there are some cases even within the symmetry classes when this does not hold true. For example, $T(1342)\ncong T(4132)$ even though $(1342), (4213)\in\Omega$ (see Theorem 2.6). One might think that a possible attack on West's Conjecture 3.3.4 would be to prove the isomorphism of the corresponding trees. It turns out that this fails to work even for $n=4$ because $T(1234)\ncong T(1432)$. Nor can any isomorphism between permutation trees from the corresponding cardinality classes be found. This suggests that we probably have restricted the notion of trees too much and this will not help us in finding general necessary and sufficient conditions for $\tau, \sigma\in S_k$ to belong to the same cardinality class.

The way we defined a tree $T(\tau)$ seems random: we chose n to insert in the nodes of the $n-1$st level of $T(\tau)$ in order to determine their children. We may as well have chosen to insert any fixed number j starting from $j-1$ level by first lifting by 1 all elements of a node which are greater than $j-1$. We will denote the tree so defined by $T_j(\tau)$. We may also want to insert $n-j$ to the nodes of level $n-1$ starting from level j by first lifting by 1 all elements of these nodes which are greater than $n-j-1$. The tree defined this way will be denoted by $T_{-j-1}(\tau)$. To be precise, we define the first $j-1$st and jth levels of $T_j(\tau)$ and T_{-j-1}, respectively, as we defined them in $T(\tau)$. This does not put a restriction on the definition of the trees. Indeed, when studying a specific case for forbidden permutations, $\tau, \sigma \in S_k$, we can ignore the first k levels of $T_j(\tau)$, $T_{-j-1}(\tau)$ and $T_j(\sigma)$, $T_{-j-1}(\sigma)$ since they will trivially have the same number of nodes. This will be also ignored when establishing isomorphisms between trees.

Example 4.1. For any permutation τ and any $j \in N$ it is true that:

$$T_{-1}(\tau) = T(\tau), \quad T_j(\tau) \cong T_{-j}(\tau^{-}) \quad \text{and} \quad T_j(\tau) \cong T_j(\tau^{\text{l}}).$$

All trees $T_j(\tau)$ and $T_{-j}(\tau)$ will have the same nodes on the nth level, namely all $\omega \in S_n(\tau)$. The difference will occur in the branches of the trees.

The above discussion suggests that there may be a more general approach to the problem of defining the cardinality classes. The necessary and sufficient conditions for $|S_n(\tau)| = |S_n(\sigma)|$ may be invariant in some sense with respect to all trees for τ and σ.

Consider the number of permutations of length m avoiding τ and having exactly l children in T_j. We will denote this by $\Psi_j\{\tau; m; l\}$. It turns out that in many cases these numbers are invariant for all trees T_j and all permutations τ and σ such that $\tau = (a_1, a_2, \ldots, a_k, k+1, k+2, \ldots, n)$, $\sigma = (a_1, a_2, \ldots, a_k, n, n-1, \ldots, k+1)$ (see West's Conjecture 3.3.4 in [10]).

Example 4.2. Let A denote the union of the symmetry classes of (1234), (1243), (2143) and (1432) (see Table 1). Then for all $\tau \in A$, all $j \in [-6, 6]$ and fixed $l \in \mathbb{N}$, fixed $m \in [5, 10]$ the $\Psi_j\{\tau; m, l\}$'s are equal. A similar example works for all $\tau \in B$ where B is the union of the symmetry classes of (12345), (12354), (12543) and (15432).

These data support the following conjecture.

Conjecture 4.3. For any m, $l \in \mathbb{N}$ and any tree T_j

$$\Psi_j\{(a_1, a_2, \ldots, a_k, k+1, k+2, \ldots, n); m; l\}$$
$$= \Psi_j\{(a_1, a_2, \ldots, a_k, n, n-1, \ldots, k+1); m; l\}.$$

Furthermore, the equality holds true if the above permutations are replaced by any permutations belonging to their symmetry classes (we assume that $k < n-1$).

A natural question arises. What about the remaining permutations which lie outside the symmetry classes mentioned in the conjecture? It turns out that they do not satisfy such a strong relationship, although the argument restricted to the trees T_1 and T_n works in many cases.

Example 4.4. $\Psi_2\{(1342);4;3\}=8\neq 6=\Psi_{-1}\{(1342);4;3\}$.

On the other hand, for all τ, σ in the symmetry class of (1342) or in $\{(3142), (2413)\}$ (recall that this is an entire cardinality class) it is true that

$$\Psi_1\{\tau;m;l\}=\Psi_1\{\sigma;m;l\},$$

where $m\in[5,10]$ and $l\in\mathbb{N}$. Since $T_1(\tau)\cong T_{-1}(\tau^-)$ (see Example 3.7) these numbers are also equal to the corresponding numbers for the trees $T_{-1}(\tau)$.

The examples can be summarized in the following conjecture.

Conjecture 4.5. For two permutations τ and σ $|S_n(\tau)|=|S_n(\sigma)|$ for all $n\in\mathbb{N}$ if and only if $\Psi_{-1}\{\tau;m;l\}=\Psi_{-1}\{\sigma;m;l\}$ for all $m,l\in\mathbb{N}$.

Note that the equalities for the Ψ's in the two conjectures are much weaker than saying that the corresponding trees are isomorphic. Still, since

$$|S_n(\tau)|=\sum_{l=1}^{n+1}\Psi_j\{\tau,n,l\}.$$

Conjecture 4.3, if true, would easily imply that

$$|S_n(a_1,a_2,\ldots,a_k,k+1,k+2,\ldots,n)|=|S_n(a_1,a_2,\ldots,a_k,n,n-1,\ldots,k+1)|.$$

Furthermore, Conjecture 4.5, if true, would give an insight about the structure of the sets $|S_n(\tau)|$ and of the corresponding trees $T_j(k)$.

Acknowledgement

This research was done at the University of Minnesota, Duluth, under the supervision of Joseph Gallian. I would like to thank David Moulton for his understanding and help throughout my work on the project, and Julian West for providing references [10] and [11] and suggesting some of the problems in this paper.

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