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Adaptive finite elements for a linear parabolic problem

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Abstract

A posteriori error estimates for the heat equation in two space dimensions are presented. A classical discretization is used, Euler backward in time, and continuous, piecewise linear triangular finite elements in space. The error is bounded above and below by an explicit error estimator based on the residual. Numerical results are presented for uniform triangulations and constant time steps. The quality of our error estimator is discussed. An adaptive algorithm is then proposed. Successive Delaunay triangulations are generated, so that the estimated relative error is close to a preset tolerance. Again, numerical results demonstrate the efficiency of our approach.

1. Introduction

A posteriori error estimates are at the base of adaptive finite elements for a great number of problems. This subject has been initiated in [4,5] and extended to linear elliptic problems, see for instance [6,1,21,7,3,16,15] and nonlinear elliptic problems [19,9,22].

Up to now, few a posteriori error estimates have been derived for parabolic problems in more than one space dimension. In [13,14] a posteriori error estimates were derived for linear and nonlinear parabolic problems when using the discontinuous Galerkin method. The L^∞ in time, L^2 in space error was bounded above to an error estimator using sharp a priori estimates for the dual problem. These a posteriori estimates were optimal in the sense that they were bounded above by a priori error estimates. Numerical results were presented.

In [23] the general framework developed in [22] is extended to a wide class of nonlinear parabolic problems. Two different kind of discretizations are studied, the θ-scheme (with extensions to Runge–Kutta schemes) and discontinuous Galerkin methods.

In this paper a posteriori error estimates for a linear heat problem are presented, extending the techniques of elliptic problems [21,7,3]. The Euler implicit time discretization together with continuous, piecewise linear finite elements are used. In the next section the error is bounded above and below in the classical L^2 in time H^1 in space norm by an explicit error estimator. Numerical results for uniform triangulations and constant time steps are reported in Section 3, and compared to the theoretical predictions of Section 2. The quality of our error estimator is discussed. An adaptive algorithm is presented in Section 4, aiming to produce triangulations and time subdivisions for which the relative estimated error is close to a given tolerance. Finally, numerical results on adapted triangulations are presented in Section 5.

We briefly compare the contents of this paper to [13,23]. Note that theoretical results in [13,23] are far more general than ours. However, we believe that our contribution is significant since, for instance, the effectivity index is studied precisely.

In [13], as we already mentioned above, the error is bounded in a different norm. Moreover, their adaptive algorithm aims to control the maximum absolute error, whereas ours controls a relative error.

In [23], the error norm is the same. However, the numerical scheme and the error estimator are slightly different, so as the techniques used in the proofs. Moreover, in [23], the theoretical predictions are not compared to numerical experiments.

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2. A posteriori error estimates

Let Ω be a convex polygon of $\mathbb{R}^2$ with boundary $\partial\Omega$, $T>0$. For any f in $L^2(0,T;L^2(\Omega))$ and u^0 in $L^2(\Omega)$ we are interested in finding $u\colon \Omega\times(0,T)\rightarrow\mathbb{R}$ such that

$$\frac{\partial u}{\partial t}(x,t)-\Delta u(x,t)=f(x,t)\quad (x,t)\in\Omega\times(0,T)\,, \tag{2.1}$$

$$u(x,t)=0\quad (x,t)\in\partial\Omega\times(0,T)\,, \tag{2.2}$$

$$u(x,0)=u^0(x)\quad x\in\Omega\,. \tag{2.3}$$

The usual variational formulation of the above problem (see for instance [12]) consists in seeking $u\in W=\{w\in L^2(0,T;H_0^1(\Omega))$ and $\partial w/\partial t\in L^2(0,T;H^{-1}(\Omega))\}$ such that $u(\cdot,0)=u^0$ and

$$\left\langle\frac{\partial u}{\partial t},v\right\rangle+\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x=\int_\Omega fv\,\mathrm{d}x\quad\forall v\in H_0^1(\Omega)\,,\quad \text{a.e. } t\in(0,T)\,. \tag{2.4}$$

Here, $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Note that, if $u\in W$ then $u\in\mathscr{C}^0([0,T];L^2(\Omega))$, thus the initial condition is to be understood in the L^2 sense.

2.1. The numerical scheme

The Euler implicit scheme is used for the time discretization. Continuous, piecewise linear finite elements are used for the space discretization.

For any $\tau>0$, let $0=t_0<t_1<\cdots<t_N=T$ be a partition of $(0,T)$ into subintervals (t^n-t^{n-1}) such that $\tau=\max\{t^n-t^{n-1},\ n=1,\dots,N\}$. For any $0<h<1$, for any $n=0,1,\dots,N$, let $\mathscr{T}_h^n$ be a mesh of Ω into triangles K with diameter less than h. Let $V_h^n=\{v\in\mathscr{C}^0(\Omega);\ v_{|K}\in\mathbb{P}_1;\ \forall K\in\mathscr{T}_h^n\}\cap H_0^1(\Omega)$ be the usual finite element space of continuous, piecewise linear functions on the triangles of $\mathscr{T}_h^n$, let $r_h^n\colon\mathscr{C}^0(\Omega)\rightarrow V_h^n$ be the corresponding Lagrange interpolant.

Throughout this paper we will assume that, for each $n=1,\dots,N$, the mesh

$$\{(t^{n-1},t^n)\times K;\ K\in\mathscr{T}_h^n\}$$

is regular in the sense of [10]. For any $n=1,\dots,N$, for any $K\in\mathscr{T}_h^n$, let h_K^n (respectively ρ_K^n) be the diameter of the smallest (resp. largest) ball containing (resp. contained in) $(t^{n-1},t^n)\times K$. We assume there is a positive constant β such that

$$\frac{h_K^n}{\rho_K^n}\leq\beta\quad\forall K\in\mathscr{T}_h^n\quad n=1,\dots,N\,. \tag{2.5}$$

This implies that there are two constants c and C independent of h and τ such that

$$c\tau\leq h\leq C\tau\,,$$

and that the triangulations $\mathscr{T}_h^n$, $n=1,\dots,N$ are also regular, uniformly with respect to n.

We now present the numerical scheme. Assuming $u^0\in\mathscr{C}^0(\Omega)$, we can compute $u_h^0=r_h^0u^0$. Then, for $n=1,\dots,N$, we are seeking, $u_h^n\in V_h^n$, an approximation of $u(\cdot,t^n)$, such that

$$\frac{1}{t^n-t^{n-1}}\int_\Omega(u_h^n-u_h^{n-1})v\,\mathrm{d}x+\int_\Omega\nabla u_h^n\cdot\nabla v\,\mathrm{d}x=\int_\Omega f^nv\,\mathrm{d}x\quad\forall v\in V_h^n\,. \tag{2.6}$$

When writing the last integral in (2.6) we have assumed that $f\in\mathscr{C}^0([0,T];L^2(\Omega))$ and $f^n=f(\cdot,t^n)$. Introducing the continuous, piecewise linear approximation in time defined by

$$u_{h\tau}(x,t)=\frac{t-t^{n-1}}{t^n-t^{n-1}}u_h^n(x)+\frac{t^n-t}{t^n-t^{n-1}}u_h^{n-1}(x)\quad t^{n-1}\leq t\leq t^n\,,\quad x\in\Omega\,, \tag{2.7}$$

we can rewrite (2.6) as

$$\int_\Omega \frac{\partial u_{h\tau}}{\partial t} v \, dx + \int_\Omega \nabla u_h^n \cdot \nabla v \, dx = \int_\Omega f^n v \, dx \quad \forall v \in V_h^n , \quad t^{n-1} \leq t \leq t^n . \tag{2.8}$$

In the sequel, we shall use the standard notations for the norms (respectively semi-norms), that is $\|\cdot\|_{0,D}$, $\|\cdot\|_{1,D}$, $\|\cdot\|_{2,D}$ (respectively $\|\cdot|_{1,D}$, $|\cdot|_{2,D}$) for the spaces $L^2(D)$, $H^1(D)$, $H^2(D)$.

2.2. The error estimator

For any $n = 1, \ldots, N$, for any triangle K of the current triangulation $\mathcal{T}_h^n$, let E_K^n be the set of its three edges. For each interior edge ℓ of $\mathcal{T}_h^n$, let us choose an arbitrary normal direction $\boldsymbol{n}$, let $[\nabla u_{h\tau} \cdot \boldsymbol{n}]_\ell$ denote the jump of $\nabla u_{h\tau} \cdot \boldsymbol{n}$ across the edge ℓ, and let

$$J_\ell^n = [\nabla u_{h\tau} \cdot \boldsymbol{n}]_\ell \quad \text{if } \ell \text{ is an interior edge of } \mathcal{T}_h^n , \tag{2.9}$$

$$J_\ell^n = 0 \qquad \text{else} . \tag{2.10}$$

For any $n = 1, \ldots, N$, for any triangle $K \in \mathcal{T}_h^n$, let

$$\Pi_K^n v = \frac{1}{(t^n - t^{n-1})|K|} \int_{t^{n-1}}^{t^n} \int_K v \, dx \, dt$$

be the $L^2((t^{n-1}, t^n) \times K)$-projection onto the constants. For any edge ℓ of $\mathcal{T}_h^n$, let

$$\Pi_\ell^n v = \frac{1}{(t^n - t^{n-1})|\ell|} \int_{t^{n-1}}^{t^n} \int_\ell v \, ds \, dt$$

be the $L^2((t^{n-1}, t^n) \times \ell)$-projection onto the constants. We now introduce the error estimator defined by

$$\begin{aligned}
(\eta_K^n)^2 &= \int_{t^{n-1}}^{t^n} \left\{ |K| \left\| \Pi_K^n\left(f - \frac{\partial u_{h\tau}}{\partial t} \right) \right\|_{0,K}^2 + \frac{1}{2} \sum_{\ell \in E_K^n} |\ell| \, \|\Pi_\ell^n(J_\ell^n)\|_{0,\ell}^2 \right\} dt , \\
(\varepsilon_K^n)^2 &= \int_{t^{n-1}}^{t^n} \|\nabla(u_h^n - u_h^{n-1})\|_{0,K}^2 \, dt , \qquad (2.11) \\
(\gamma_K^n)^2 &= \int_{t^{n-1}}^{t^n} \|f - f^n\|_{0,K}^2 \, dt .
\end{aligned}$$

Note that, from the definition of Π_K^n and Π_ℓ^n, we have

$$(\eta_K^n)^2 = (t^n - t^{n-1}) \left\{ |K|^2 \left(\Pi_K^n\left(f - \frac{\partial u_{h\tau}}{\partial t} \right) \right)^2 + \frac{1}{2} \sum_{\ell \in E_K^n} |\ell|^2 (\Pi_\ell^n(J_\ell^n))^2 \right\} .$$

For each time interval (t^{n-1}, t^n), $n = 1, \ldots, N$, the error estimator is defined by the three following contributions:

$$\begin{aligned}
(\eta^n)^2 &= \sum_{K \in \mathcal{T}_h^n} (\eta_K^n)^2 , \\
(\varepsilon^n)^2 &= \sum_{K \in \mathcal{T}_h^n} (\varepsilon_K^n)^2 \qquad (2.12) \\
(\gamma^n)^2 &= \sum_{K \in \mathcal{T}_h^n} (\gamma_K^n)^2 .
\end{aligned}$$

Finally the global space–time error estimator is defined by

$$\eta^2 = \sum_{n=1}^N (\eta^n)^2 , \qquad \varepsilon^2 = \sum_{n=1}^N (\varepsilon^n)^2 . \qquad \gamma^2 = \sum_{n=1}^N (\gamma^n)^2 . \tag{2.13}$$

Note that the first term of (2.11) is similar to the error estimator for the Laplace problem. It can be seen as a measure, in the $L^2(0,T;H^{-1}(\Omega))$ norm of the equation residual. The second and third terms of (2.11) arise from the use of the Euler scheme for time discretization.

Our goal is now to obtain upper and lower bounds to the error e defined by

$$e = u - u_{h\tau}\,.$$

2.3. An upper bound

Proceeding as for the Laplace problem [7] we can prove the following theorem.

THEOREM 2.1. Assume that $f \in H^1((0,T)\times\Omega)$, that the meshes

$$\{(t^{n-1},t^n)\times K;\ K\in\mathcal{T}_h^n\}$$

$n=1,\dots,N$, are regular in the sense of (2.5), and that the triangulations are nested, that is $V_h^{n-1}\subset V_h^n$, for $n=1,\dots,N$. Then, there is a constant C depending on the size of Ω and on the constant β such that, for h small enough, we have

$$\|e(\cdot,t^n)\|_{0,\Omega}^2 + \int_{t^{n-1}}^{t^n}\|\nabla e\|_{0,\Omega}^2\,dt \le \|e(\cdot,t^{n-1})\|_{0,\Omega}^2 + C\big((\eta^n)^2+(\varepsilon^n)^2+(\gamma^n)^2+h^4|f|_{1,(t^{n-1},t^n)\times\Omega}^2\big)\,, \tag{2.14}$$

for $n=1,\dots,N$. Moreover, if there exists two positive constants C^* and $s\in(0,1]$ not depending on h such that

$$\int_0^T\|\nabla e\|_{0,\Omega}^2\,dt \ge C^*h^{2s}\,, \tag{2.15}$$

then, for h small enough, we have

$$\int_0^T\|\nabla e\|_{0,\Omega}^2\,dt \le C(\eta^2+\varepsilon^2+\gamma^2)\,. \tag{2.16}$$

PROOF. Let I^n be defined by

$$I^n = \frac{1}{2}\big(\|e(\cdot,t^n)\|_{0,\Omega}^2 - \|e(\cdot,t^{n-1})\|_{0,\Omega}^2\big) + \int_{t^{n-1}}^{t^n}\|\nabla e\|_{0,\Omega}^2\,dt\,.$$

By definition of e:

$$I^n = \int_{t^{n-1}}^{t^n}\left\{\left\langle\frac{\partial u}{\partial t},e\right\rangle - \left\langle\frac{\partial u_{h\tau}}{\partial t},e\right\rangle + \int_\Omega(\nabla u\cdot\nabla e - \nabla u_{h\tau}\cdot\nabla e)\,dx\right\}dt\,.$$

Using (2.4) we have

$$I^n = \int_{t^{n-1}}^{t^n}\int_\Omega\left(\left(f-\frac{\partial u_{h\tau}}{\partial t}\right)e - \nabla u_{h\tau}\cdot\nabla e\right)dx\,dt\,. \tag{2.17}$$

Using (2.7) and (2.8) we also have

$$\int_\Omega\left(\left(f-\frac{\partial u_{h\tau}}{\partial t}\right)e - \nabla u_{h\tau}\cdot\nabla e\right)dx = \int_\Omega\left(\left(f-\frac{\partial u_{h\tau}}{\partial t}\right)(e-v) - \nabla u_{h\tau}\cdot\nabla(e-v)\right.$$
$$\left. + \frac{t^n-t}{t^n-t^{n-1}}\nabla(u_h^n-u_h^{n-1})\cdot\nabla v + (f-f^n)v\right)dx \quad \forall\, v\in V_h^n \quad \text{a.e. } t\in(t^{n-1},t^n)\,.$$

We proceed as in the elliptic case [7] and choose $v=\pi_h^n e(\cdot,t)$ a.e. $t\in(t^{n-1},t^n)$ in the above equation, where π_h^n is Clément's interpolant on triangulation $\mathcal{T}_h^n$ [11,8]. Integrating over (t^{n-1},t^n) and putting into (2.17) we thus obtain

$$I^n = \int_{t^{n-1}}^{t^n} \int_\Omega \left(\left(f - \frac{\partial u_{h\tau}}{\partial t} \right)(e - \pi_h^n e) - \nabla u_{h\tau} \cdot \nabla(e - \pi_h^n e) + \frac{t^n - t}{t^n - t^{n-1}} \nabla(u_h^n - u_h^{n-1}) \cdot \nabla(\pi_h^n e) + (f - f^n)\pi_h^n e \right) \mathrm{d}x \, \mathrm{d}t \, .$$

As usual, we split the integral over Ω into integrals over triangles. Since the triangulation $\mathcal{T}_h^n$ is nested into $\mathcal{T}_h^{n-1}$, we can integrate by parts the $\nabla u_{h\tau} \cdot \nabla(e - \pi_h^n e)$ term and the $\Delta u_{h\tau}$ term is zero on each $K \in \mathcal{T}_h^n$. We obtain

$$I^n = \sum_{K \in \mathcal{T}_h^n} \int_{t^{n-1}}^{t^n} \left\{ \int_K \Pi_K^n \left(f - \frac{\partial u_{h\tau}}{\partial t} \right)(e - \pi_h^n e) \, \mathrm{d}x + \frac{1}{2} \sum_{\ell \in E_K^n} \int_\ell \Pi_\ell^n (J_\ell^n)(e - \pi_h^n e) \, \mathrm{d}s + \int_K \frac{t^n - t}{t^n - t^{n-1}} \nabla(u_h^n - u_h^{n-1}) \cdot \nabla(\pi_h^n e) \, \mathrm{d}x + \int_K (f - f^n)(\pi_h^n e) \, \mathrm{d}x \right\} \mathrm{d}t + \delta^n(e - \pi_h^n e) \, ,$$

where we have set

$$\delta^n(w) = \sum_{K \in \mathcal{T}_h^n} \int_{t^{n-1}}^{t^n} \left\{ \int_K \delta_K^n(w) \, \mathrm{d}x + \frac{1}{2} \sum_{\ell \in E_K^n} \int_\ell \delta_\ell^n(w) \, \mathrm{d}s \right\} \mathrm{d}t \, ,$$

and

$$\delta_K^n(w) = (I - \Pi_K^n)\left(f - \frac{\partial u_{h\tau}}{\partial t} \right) w \, , \quad \delta_\ell^n(w) = (I - \Pi_\ell^n)(J_\ell^n) w \, .$$

The standard approximations results for Clément's interpolant [11,8,7], insure the existence of a constant C depending only on β such that

$$\|e - \pi_h^n e\|_{0,K} \leqslant C|K|^{1/2} \|e\|_{1,\Delta K} \, ,$$
$$\|e - \pi_h^n e\|_{0,\ell} \leqslant C \, |\ell|^{1/2} \|e\|_{1,\Delta K} \, ,$$

where ΔK denotes the set of triangles having a common edge or vertex with K. Using the Cauchy–Schwarz inequality and the above results, there is a constant C depending on only β such that

$$I^n \leqslant C \sum_{K \in \mathcal{T}_h^n} \int_{t^{n-1}}^{t^n} \left\{ |K| \left\| \Pi_K^n \left(f - \frac{\partial u_{h\tau}}{\partial t} \right) \right\|_{0,K} + \frac{1}{2} \sum_{\ell \in E_K^n} |\ell|^{1/2} \|\Pi_\ell^n (J_\ell^n)\|_{0,\ell} + \|\nabla(u_h^n - u_h^{n-1})\|_{0,K} + \|f - f^n\|_{0,K} \right\} \|e\|_{1,\Delta K} \, \mathrm{d}t + |\delta^n(e - \pi_h^n e)| \, . \tag{2.18}$$

We now bound the last term of (2.18). The standard approximations results of Π_K^n [3] insure the existence of a constant C independent of β and h_K^n such that

$$\left\| (I - \Pi_K^n)\left(f - \frac{\partial u_{h\tau}}{\partial t} \right) \right\|_{0,(t^{n-1},t^n)\times K} \leqslant C h_K^n \left| f - \frac{\partial u_{h\tau}}{\partial t} \right|_{1,(t^{n-1},t^n)\times K} .$$

Using the Cauchy–Schwarz inequality and the above approximation results, there is a constant C depending only on β such that

$$\int_{t^{n-1}}^{t^n} \int_K \delta_K^n(e - \pi_h^n e) \, \mathrm{d}x \, \mathrm{d}t \leqslant C(h_K^n)^2 \left| f - \frac{\partial u_{h\tau}}{\partial t} \right|_{1,(t^{n-1},t^n)\times K} \left(\int_{t^{n-1}}^{t^n} \|e\|_{1,\Delta K}^2 \, \mathrm{d}t \right)^{1/2} .$$

Using (2.7) and (2.11), we have

$$\left| \frac{\partial u_{h\tau}}{\partial t} \right|_{1,(t^{n-1},t^n)\times K} = \left| \frac{u_h^n - u_h^{n-1}}{t^n - t^{n-1}} \right|_{1,(t^{n-1},t^n)\times K} = \frac{\varepsilon_K^n}{t^n - t^{n-1}} \, ,$$

so that

$$\int_{t^{n-1}}^{t^n} \int_K \delta_K^n(e - \pi_h^n e) \, \mathrm{d}x \, \mathrm{d}t \leqslant C(h_K^n)^2 \left(|f|_{1,(t^{n-1},t^n)\times K} + \frac{\varepsilon_K^n}{t^n - t^{n-1}} \right) \left(\int_{t^{n-1}}^{t^n} \|e\|_{1,\Delta K}^2 \, \mathrm{d}t \right)^{1/2} . \tag{2.19}$$

Similarly, using the Cauchy–Schwarz inequality, the standard approximation results of Π^n_ℓ and π^n_h, there is a constant C depending only on β such that

$$\int_{t^{n-1}}^{t^n}\int_\ell \delta^n_\ell(e-\pi^n_h e)\,\mathrm{d}s\,\mathrm{d}t \leq C(h^n_K)^{3/2}|J^n_\ell|_{1,(t^{n-1},t^n)\times\ell}\left(\int_{t^{n-1}}^{t^n}\|e\|^2_{1,\Delta\ell}\,\mathrm{d}t\right)^{1/2}.$$

Using the definitions (2.7), (2.9) and (2.11), we obtain

$$|J^n_\ell|^2_{1,(t^{n-1},t^n)\times\ell} = \int_{t^{n-1}}^{t^n}\int_\ell\left(\frac{[\nabla(u^n_h-u^{n-1}_h)\cdot\boldsymbol{n}]}{t^n-t^{n-1}}\right)^2\mathrm{d}s\,\mathrm{d}t$$
$$\leq \frac{|\ell|}{|K|(t^n-t^{n-1})^2}(\varepsilon^n_K)^2+\frac{|\ell|}{|K'|(t^n-t^{n-1})^2}(\varepsilon^n_{K'})^2,$$

where K and K' are the two triangles having ℓ as common edge. Thus, using the regularity assumption (2.5), we have

$$\int_{t^{n-1}}^{t^n}\int_\ell \delta^n_\ell(e-\pi^n_h e)\,\mathrm{d}s\,\mathrm{d}t \leq C(\varepsilon^n_K+\varepsilon^n_{K'})\left(\int_{t^{n-1}}^{t^n}\|e\|^2_{1,\Delta K}\,\mathrm{d}t\right)^{1/2}. \tag{2.20}$$

The inequalities (2.19), (2.20) and the definition of δ^n yield

$$|\delta^n(e-\pi^n_h e)| \leq C\sum_{K\in\mathcal{T}^n_h}\left((h^n_K)^2|f|_{1,(t^{n-1},t^n)\times K}+(1+h^n_K)\varepsilon^n_K\right)\left(\int_{t^{n-1}}^{t^n}\|e\|^2_{1,\Delta K}\,\mathrm{d}t\right)^{1/2}. \tag{2.21}$$

Estimate (2.21) in (2.18), and the Cauchy–Schwarz inequality yields

$$I^n \leq C\sum_{K\in\mathcal{T}^n_h}\left((\eta^n_K)^2+(\varepsilon^n_K)^2+(\gamma^n_K)^2+(h^n_K)^4|f|^2_{1,(t^{n-1},t^n)\times K}\right)^{1/2}\left(\int_{t^{n-1}}^{t^n}\|e\|^2_{1,\Delta K}\,\mathrm{d}t\right)^{1/2},$$

provided h is small enough. It suffices to apply the discrete Cauchy–Schwarz inequality, then Young's inequality to obtain (2.14).

We now turn to estimate (2.16). Summing (2.14) for $n=1,\ldots,N$, we obtain

$$\|e(\cdot,T)\|^2_{0,\Omega}+\int_0^T\|\nabla e\|^2_{0,\Omega}\,\mathrm{d}t \leq \|e(\cdot,0)\|^2_{0,\Omega}+C(\eta^2+\varepsilon^2+\gamma^2+h^4|f|^2_{1,(0,T)\times\Omega}). \tag{2.22}$$

Using the standard interpolation results (remember that $e(\cdot,0)=u^0-r^0_h u^0$) and assumption (2.15), there is a constant C independent of h such that

$$\|e(\cdot,0)\|^2_{0,\Omega}+h^4|f|^2_{1,(0,T)\times\Omega} \leq Ch^{2(2-s)}\int_0^T\|\nabla e\|^2_{0,\Omega}\,\mathrm{d}t. \tag{2.23}$$

Estimate (2.23) in (2.22), and Young's inequality yield the result for h small enough. □

REMARK 2.1. From a computational point of view, all the integrals in (2.6) are evaluated numerically using the trapeze formula. Using standard interpolation results we know that, if $f\in\mathscr{C}^0([0,T];H^2(\Omega))$ then there is a constant C such that for all $0<h<1$, for all $v\in V^n_h$, for all triangle K of the triangulation $\mathcal{T}^n_h$, we have

$$\int_{t^{n-1}}^{t^n}\int_K(f^n v-r^n_h(f^n v))\,\mathrm{d}x \leq Ch^2\int_{t^{n-1}}^{t^n}\|f^n\|_{2,K}\|v\|_{1,K}\,\mathrm{d}t,$$
$$\int_{t^{n-1}}^{t^n}\int_K(u^n_h v-r^n_h(u^n_h v))\,\mathrm{d}x \leq Ch^2\int_{t^{n-1}}^{t^n}\|u^n_h\|_{1,K}\|v\|_{1,K}\,\mathrm{d}t.$$

Moreover, if the triangulation $\mathcal{T}^n_h$ is nested into $\mathcal{T}^{n-1}_h$ we also have

$$\int_{t^{n-1}}^{t^n}\int_K(u^{n-1}_h v-r^n_h(u^{n-1}_h v))\,dx \leq Ch^2\int_{t^{n-1}}^{t^n}\|u^{n-1}_h\|_{1,K}\|v\|_{1,K}\,\mathrm{d}t.$$

Thus, the use of numerical integration has the effect of adding terms of order h^4 in estimate (2.14). Then, estimate (2.16) still holds provided (2.15) does and h is small enough.

2.4. A lower bound

We now turn to a lower bound of the error. For this purpose we extend the ideas of [21,3].

THEOREM 2.2. Assume that $f \in H^1((0,T)\times\Omega)$, *that the meshes*

$$\{(t^{n-1},t^n)\times K;\ K\in\mathcal{T}_h^n\}$$

$n=1,\dots,N$, *are regular in the sense of (2.5), and that the triangulations are nested, that is* $V_h^{n-1}\subset V_h^n$, *for* $n=1,\dots,N$. *Then, there is a constant C depending on the size of* Ω *and on the constant* β *such that*

$$(\eta^n)^2 \le C\left(\int_{t^{n-1}}^{t^n} \|\nabla e\|_{0,\Omega}^2\,dt + (\varepsilon^n)^2 + h^4|f|_{1,(t^{n-1},t^n)\times\Omega}^2\right) \tag{2.24}$$

Moreover, if (2.15) holds then, for h small enough, we have

$$\eta^2 \le C\left(\int_0^T \|\nabla e\|_{0,\Omega}^2\,dt + \varepsilon^2\right). \tag{2.25}$$

To prove Theorem 2.2, we need the following lemma.

LEMMA 2.3. Assume that the meshes

$$\{(t^{n-1},t^n)\times K;\ K\in\mathcal{T}_h^n\}$$

$n=1,\dots,N$, *are regular in the sense of (2.5). Then, there is a constant C depending on* β, *and a function* $w\in H_0^1((0,T)\times\Omega)$ *such that, for all* $n=1,\dots,N$, *for all* $K\in\mathcal{T}_h^n$, *we have*

$$\text{(i)}\quad w(\cdot,t^n)=0\,, \tag{2.26}$$

$$\text{(ii)}\quad |K|\int_{t^{n-1}}^{t^n}\left\|\Pi_K^n\left(f-\frac{\partial u_{h\tau}}{\partial t}\right)\right\|_{0,K}^2 dt = \int_{t^{n-1}}^{t^n}\int_K \Pi_K^n\left(f-\frac{\partial u_{h\tau}}{\partial t}\right) w\,dx\,dt\,, \tag{2.27}$$

$$\text{(iii)}\quad |\ell|\int_{t^{n-1}}^{t^n}\|\Pi_\ell^n(J_\ell^n)\|_{0,\ell}^2\,dt = \int_{t^{n-1}}^{t^n}\int_\ell \Pi_\ell^n(J_\ell^n) w\,ds\,dt \quad \forall\,\ell\in E_K^n\,, \tag{2.28}$$

$$\text{(iv)}\quad |w|_{1,(t^{n-1},t^n)\times K} \le C\eta_K^n\,, \tag{2.29}$$

PROOF OF THEOREM 2.2. Using (2.12), (2.27) and (2.28) we have

$$(\eta^n)^2 = \sum_{K\in\mathcal{T}_h^n}(\eta_K^n)^2 = \sum_{K\in\mathcal{T}_h^n}\left(\int_{t^{n-1}}^{t^n}\int_K\left(f-\frac{\partial u_{h\tau}}{\partial t}\right)w\,dx\,dt + \frac{1}{2}\sum_{\ell\in E_K^n}\int_{t^{n-1}}^{t^n}\int_\ell J_\ell^n w\,ds\,dt\right) - \delta^n(w)\,, \tag{2.30}$$

where δ^n is defined as in the proof of Theorem 2.1. Integrating by parts the second term of the right-hand side of (2.30) we obtain

$$(\eta^n)^2 = \sum_{K\in\mathcal{T}_h^n}\int_{t^{n-1}}^{t^n}\int_K\left(\left(f-\frac{\partial u_{h\tau}}{\partial t}\right)w - \nabla u_{h\tau}\cdot\nabla w\right)dx\,dt - \delta^n(w)\,.$$

Making use of (2.4), we thus have

$$(\eta^n)^2 = \int_{t^{n-1}}^{t^n}\left(\left\langle\frac{\partial e}{\partial t}, w\right\rangle + \int_\Omega \nabla e\cdot\nabla w\,dx\right)dt - \delta^n(w)\,.$$

Integrating by parts the first term of the above equation and using (2.26) we have

$$(\eta^n)^2 = \int_{t^{n-1}}^{t^n} \int_\Omega \left(-e\frac{\partial w}{\partial t} + \nabla e \cdot \nabla w\right) \mathrm{d}x\, \mathrm{d}t - \delta^n(w)\,.$$

Applying Cauchy–Schwarz, Poincaré, Young inequalities, and using (2.29), there is a constant C depending only on β and on the size of Ω such that

$$(\eta^n)^2 \leq C \int_{t^{n-1}}^{t^n} \|\nabla e\|^2_{0,\Omega}\, \mathrm{d}t + |\delta^n(w)|\,. \tag{2.31}$$

Let us now estimate $|\delta^n(w)|$. By definition of Π^n_K

$$\int_{t^{n-1}}^{t^n} \int_K \delta^n_K(w)\, \mathrm{d}x\, \mathrm{d}t = \int_{t^{n-1}}^{t^n} \int_K \delta^n_K(w - \Pi^n_K w)\, \mathrm{d}x\, \mathrm{d}t\,.$$

Using the standard interpolation results of Π^n_K, there is a constant C such that

$$\int_{t^{n-1}}^{t^n} \int_K \delta^n_K(w)\, \mathrm{d}x\, \mathrm{d}t \leq C(h^n_K)^2 \left| f - \frac{\partial u_{h\tau}}{\partial t} \right|_{1,(t^{n-1},t^n)\times K} |w|_{1,(t^{n-1},t^n)\times K}\,.$$

Similarly, using the standard interpolation results of Π^n_ℓ and a trace inequality, there is a constant C depending only on β such that

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \int_\ell \delta^n_\ell(w)\, \mathrm{d}s\, \mathrm{d}t &= \int_{t^{n-1}}^{t^n} \int_\ell \delta^n_\ell(w - \Pi^n_\ell w)\, \mathrm{d}s\, \mathrm{d}t \\ &\leq C(h^n_K)^2 |J^n_\ell|_{1,(t^{n-1},t^n)\times \ell} |w|_{1,(t^{n-1},t^n)\times \ell} \\ &\leq C(h^n_K)^{3/2} |J^n_\ell|_{1,(t^{n-1},t^n)\times \ell} |w|_{1,(t^{n-1},t^n)\times K}\,.\end{aligned}$$

Proceeding as in the proof of Theorem 2.1 we obtain

$$\delta^n(w) \leq C(h^4 |f|^2_{1,(t^{n-1},t^n)\times\Omega} + (\varepsilon^n)^2)^{1/2} \left(\sum_{K\in\mathcal{T}^n_h} |w|^2_{1,(t^{n-1},t^n)\times K} \right)^{1/2}. \tag{2.32}$$

Finally, (2.29) in (2.32), and (2.32) in (2.31) yields (2.24).

To obtain (2.25), we proceed as in the proof of Theorem 2.1, sum (2.24) for $n = 1, \ldots, N$, and make use of (2.15). □

PROOF OF LEMMA 2.3. For each triangle K, let φ^K_1, φ^K_2, φ^K_3 be the usual hat functions. Let $\psi_K = \varphi^K_1 \varphi^K_2 \varphi^K_3$ be the bubble function attached to K, ψ_{ℓ_i}, be the bubble functions to edge ℓ_i, $i = 1, 2, 3$, and defined by

$$\psi_{\ell_1} = \varphi_1\varphi_2\,, \qquad \psi_{\ell_2} = \varphi_2\varphi_3\,, \qquad \psi_{\ell_3} = \varphi_3\varphi_1\,,$$

and let ψ_n be the bubble function corresponding to the time interval (t^{n-1}, t^n). Following [21,3] we postulate that w is of the form

$$w(x,t) = C_0 \psi_K(x)\psi_n(t) + \sum_{i=1}^3 C_{\ell_i} \psi_{\ell_i}(x)\psi_n(t)\,,$$

for $(x,t) \in \bar{K} \times [t^{n-1}, t^n]$. Clearly, (2.28) yields

$$C_{\ell_i} = \frac{(t^n - t^{n-1})|\ell_i|^2 J_{\ell_i}}{\displaystyle\int_{t^{n-1}}^{t^n} \int_{\ell_i} \psi_{\ell_i}\psi_n\, \mathrm{d}x\, \mathrm{d}t}\,.$$

Then, (2.27) leads to

$$C_0 = \frac{(t^n - t^{n-1})|K|^2 \Pi_K^n \left(f - \frac{\partial u_{h\tau}}{\partial t}\right) - \sum_{i=1}^{3} C_{\ell_i} \int_{t^{n-1}}^{t^n} \int_K \psi_{\ell_i} \psi_n \,\mathrm{d}x\,\mathrm{d}t}{\int_{t^{n-1}}^{t^n} \int_K \psi_K \psi_n \,\mathrm{d}x\,\mathrm{d}t}$$

Using the properties of the bubble functions, there is a constant C depending only on β such that

$$C_0 + C_{\ell_i} = C\left(|K| \Pi_K^n \left(f - \frac{\partial u_{h\tau}}{\partial t}\right) + |\ell_i| \Pi_{n,\ell_i}(J_{\ell_i}) \right).$$

It remains to check (2.29). Using the definition of w, the properties of the bubble functions and the above estimate, there is a constant C depending only on β such that

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \int_K \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t &\leq C \frac{|K|^3 \left(\Pi_K^n\left(f - \frac{\partial u_{h\tau}}{\partial t}\right)\right)^2 + |K|\,|\ell_i|^2 (\Pi_{n\ell_i} J_{\ell_i})^2}{t^n - t^{n-1}} \\ &\leq C \frac{|K|}{(t^n - t^{n-1})^2} (\eta_K^n)^2,\end{aligned}$$

and

$$\int_{t^{n-1}}^{t^n} \int_K |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C(\eta_K^n)^2,$$

which yields the result. □

3. Numerical results: uniform triangulations and constant time steps

In the previous section we have shown that, under suitable assumptions, the true error $\|e\|$ defined by

$$\|e\| = \left(\int_0^T \|\nabla e(\cdot, t)\|_{0,\Omega}^2 \,\mathrm{d}t \right)^{1/2}, \tag{3.1}$$

was bounded above and below by η, ε, γ defined in (2.13). Looking forward to designing an adaptive algorithm we now define the estimated error by a linear combination of η, ε, γ, that is

$$(\omega_\eta \eta^2 + \omega_\varepsilon \varepsilon^2 + \omega_\gamma \gamma^2)^{1/2}.$$

Here, ω_η, ω_ε and ω_γ are three parameters that will be chosen such that the estimated error is a good approximation of the true error.

For this purpose, we set the calculation domain $\Omega = (0, 1) \times (0, 1)$, the final time $T = 1$, and define the solution of (2.1)–(2.3) by

$$\begin{aligned} &\text{case (1a)} \quad u(x_1, x_2, t) = \sin(\pi t/2), \\ &\text{case (1b)} \quad u(x_1, x_2, t) = \sin(10\pi t/2), \\ &\text{case (2a)} \quad u(x_1, x_2, t) = \sin(\pi x_1/2)\sin(\pi x_2/2), \\ &\text{case (2b)} \quad u(x_1, x_2, t) = \sin(10\pi x_1/2)\sin(10\pi x_2/2). \end{aligned}$$

In cases (1a) and (1b) the error due to space discretization should be small compared to the error due to time discretization. This should be the opposite in cases (2a) and (2b). We have reported in Tables 1 and 2 the error estimators η, ε, γ, and the true error $\|e\|$ when using uniform triangulations and constant time steps, for both cases (1a) and (2a). All the integrals involved in the computation of the true error are evaluated numerically using Simpson's formula. All the integrals involved in the computation of the estimated error are evaluated numerically using the trapeze formula. In case (1a), it can be seen that only γ is relevant, η and ε being small, and we roughly have $\|e\| \approx \gamma/10$. In case (2a), only η is relevant, ε and γ being small, and we roughly have

Table 1
Convergence results when using uniform triangulations and constant time steps, case (1a)

Triangulation	h	τ	η	ε	γ	$\|e\|$
10×10	0.1	0.1	0.013	0.0017	0.12	0.015
80×80	0.0125	0.1	0.0017	0.0017	0.12	0.015
10×10	0.1	0.0125	0.0016	0.000027	0.015	0.0019
80×80	0.0125	0.0125	0.00022	0.000028	0.015	0.0019

Table 2
Convergence results when using uniform triangulations and constant time steps, case (2a)

Triangulation	h	τ	η	ε	γ	$\|e\|$
10×10	0.1	0.1	6.61	0.029	0	1.39
80×80	0.0125	0.1	0.85	0.00046	0	0.174
10×10	0.1	0.0125	6.61	0.0066	0	1.39
80×80	0.0125	0.0125	0.85	0.00010	0	0.174

Table 3
Convergence results when using uniform triangulations and constant time steps, case (3a)

Triangulation	h	τ	$\eta/5$	ε	$\gamma/10$	$\|e\|$	e.i.
10×10	0.1	0.1	0.12	0.79	0.58	0.48	2.1
20×20	0.05	0.1	0.060	0.79	0.58	0.48	2.1
40×40	0.025	0.1	0.030	0.79	0.58	0.48	2.1
80×80	0.0125	0.1	0.015	0.79	0.58	0.48	2.1
10×10	0.1	0.05	0.080	0.41	0.31	0.28	1.9
20×20	0.05	0.05	0.042	0.41	0.31	0.28	1.9
40×40	0.025	0.05	0.020	0.41	0.31	0.28	1.9
80×80	0.0125	0.05	0.010	0.41	0.32	0.28	1.9
10×10	0.1	0.025	0.067	0.21	0.16	0.16	1.7
20×20	0.05	0.025	0.034	0.21	0.16	0.15	1.7
40×40	0.025	0.025	0.017	0.21	0.16	0.15	1.8
80×80	0.0125	0.025	0.0085	0.21	0.16	0.15	1.8
10×10	0.1	0.0125	0.063	0.11	0.081	0.10	1.5
20×20	0.05	0.0125	0.032	0.11	0.081	0.083	1.6
40×40	0.025	0.0125	0.016	0.11	0.081	0.078	1.7
80×80	0.0125	0.0125	0.0080	0.11	0.081	0.077	1.7

$\|e\|\approx \eta/5$. For the sake of concision we do not present the results for cases (1b) and (2b) since the conclusions are essentially the same. Indeed, we also obtained $\|e\|\approx\gamma/10$ for case (1b) and $\|e\|\approx\eta/5$ for case (2b).

From these experiments we choose $\omega_\eta = 1/5^2$, $\omega_\varepsilon = 1$, and $\omega_\gamma = 1/10^2$, in other words we believe that

$$\left(\left(\frac{\eta}{5}\right)^2 + \varepsilon^2 + \left(\frac{\gamma}{10}\right)^2\right)^{1/2} \tag{3.2}$$

is a good approximation of the true error $\|e\|$. Then, we define the effectivity index e.i. by

$$\text{e.i.} = \frac{\left(\left(\frac{\eta}{5}\right)^2 + \varepsilon^2 + \left(\frac{\gamma}{10}\right)^2\right)^{1/2}}{\|e\|}.$$

When the solution u of (2.1)–(2.3) is a function of the time and space variables, the error e is expected to be a combination of η, ε, and γ. However, in some cases the error due to time discretization can be greater than the error due to space discretization, and vice versa. In order to illustrate these phenomena, we now define the solution of (2.1)–(2.3) by

Table 4
Convergence results when using uniform triangulations and constant time steps, case (3b)

Triangulation	h	τ	$\eta/5$	ε	$\gamma/10$	$\|e\|$	e.i.
10×10	0.1	0.1	5.4	8.5	17.0	5.5	3.7
20×20	0.05	0.1	2.9	7.9	17.0	3.2	6.1
40×40	0.025	0.1	1.5	7.8	17.0	2.3	8.6
80×80	0.025	0.1	0.74	7.7	17.0	1.9	10.0
10×10	0.1	0.05	5.4	4.6	9.4	5.8	2.0
20×20	0.05	0.05	2.9	4.3	9.4	3.0	3.6
40×40	0.025	0.05	1.5	4.2	9.4	1.6	6.7
80×80	0.025	0.05	0.75	4.2	9.4	0.90	11.0
10×10	0.1	0.025	5.4	2.4	4.8	6.0	1.3
20×20	0.05	0.025	2.9	2.2	4.8	3.0	2.0
40×40	0.025	0.025	1.5	2.2	4.8	1.5	3.6
80×80	0.025	0.025	0.75	2.2	4.8	0.78	6.9
10×10	0.1	0.0125	5.4	1.2	2.4	6.1	1.0
20×20	0.05	0.0125	2.9	1.1	2.4	3.1	1.3
40×40	0.025	0.0125	1.5	1.1	2.4	1.5	2.0
80×80	0.025	0.0125	0.75	1.1	2.4	0.77	3.6

$$\text{case (3a)}\quad u(x_1,x_2,t)=\sin(10\pi t/2)\sin(\pi x_1/2)\sin(\pi x_2/2)\,,$$

$$\text{case (3b)}\quad u(x_1,x_2,t)=\sin(10\pi t/2)\sin(10\pi x_1/2)\sin(10\pi x_2/2)\,.$$

As will be explained in the sequel, for the case (3a) the error due to time discretization is greater than the error due to time discretization. This is the opposite for problem (3b).

The numerical results are shown in Tables 3 and 4. We have reported $\eta/5$, ε, $\gamma/10$, $\|e\|$, and e.i. For the test case (3a), the error is closely correlated to ε and γ. Moreover, since the error is divided by two each time the time step is, we conclude that the error is mainly due to time discretization. On the other side, for the test case (3b), the error is correlated to η, and the error is mainly due to space discretization.

These observations do not contradict the upper and lower bounds obtained in Section 2. However, for problems with strong variations in time we observe that $\|e\|\approx\gamma/10$ whereas for problems with strong variations in space we observe that $\|e\|\approx\eta/5$. Moreover, from the theoretical predictions of Section 2, we also know that the effectivity index should depend on the ratio between the space step h and the time step τ. For instance, when $h=\tau$ in Table 4, the effectivity index stays between 3.6 and 3.7. Thus, we believe that the estimate error (3.2) is a fairly good estimator of the true error $\|e\|$.

4. An adaptive algorithm

Let η, ε, γ be the error estimator defined in (2.13), let $\|e\|$ be the true error defined in (3.1). In the previous section have chosen three parameters ω_η, ω_ε, and ω_γ such that $(\omega_\eta\eta^2+\omega_\varepsilon\varepsilon^2+\omega_\gamma\gamma^2)^{1/2}$ was a good approximation of $\|e\|$. Our goal is now to define an adaptive algorithm using this error estimator.

Let $u_{h\tau}$ be the approximated solution defined in (2.7), our estimated relative error e.r.e. is defined by

$$\text{e.r.e.}=\frac{(\omega_\eta\eta^2+\omega_\varepsilon\varepsilon^2+\omega_\gamma\gamma^2)^{1/2}}{\left\{\int_0^T \|\nabla u_{h\tau}(\cdot\,,t)\|_{0,\Omega}^2\,\mathrm{d}t\right)^{1/2}}\,. \tag{4.1}$$

Let TOL be a preset tolerance, $0<\alpha<1$ a given parameter. Our aim is to generate a sequence of sub-intervals (t^{n-1},t^n) and Delaunay triangulations $\mathcal{T}_h^n$, $n=1,\dots,N$ such that the estimated relative error is close to a preset tolerance, that is

$$(1-\alpha)\mathrm{TOL}\leqslant\text{e.r.e.}\leqslant(1+\alpha)\mathrm{TOL}\,. \tag{4.2}$$

When (4.2) is achieved, we hope to control t.r.e, the true relative error defined by

$$\text{t.r.e.} = \frac{\int_0^T \|\nabla e\cdot,t)\|_{0,\Omega}^2\,\mathrm{d}t)^{1/2}}{\left(\int_0^T \|\nabla u_{h\tau}(\cdot,t)\|_{0,\Omega}^2\,\mathrm{d}t\right)^{1/2}}\,. \tag{4.3}$$

For instance, when choosing TOL = 0.1, our adaptive algorithm will generate a numerical solution for which the estimated relative error will be close to 10%. Our hope is that the true relative error will also be close to 10%.

Note that, since the successive Delaunay triangulations are not necessarily nested, the theoretical results of Section 2 do not apply. However, we will see in the next section that this algorithm still gives good results.

We now present a sufficient condition to achieve (4.2). Let us define l.b. and r.b. (standing for left bound and right bound) by

$$\begin{aligned} \text{l.b.} &= (1-\alpha)^2\mathrm{TOL}^2 \int_{t^{n-1}}^{t^n} \|\nabla u_{h\tau}(\cdot,t)\|_{0,\Omega}^2\,\mathrm{d}t\,,\\ \text{r.b.} &= (1+\alpha)^2\mathrm{TOL}^2 \int_{t^{n-1}}^{t^n} \|\nabla u_{h\tau}(\cdot,t)\|_{0,\Omega}^2\,\mathrm{d}t\,. \end{aligned} \tag{4.4}$$

If, for $n = 1, 2, \ldots, N$, we have

$$\text{l.b.} \leq \omega_\eta(\eta^n)^2 + \omega_\varepsilon(\varepsilon^n)^2 + \omega_\gamma(\gamma^n)^2 \leq \text{r.b.}\,, \tag{4.5}$$

then, summing from $n = 1$ to $n = N$, we obtain

$$(1-\alpha)^2\mathrm{TOL}^2 \int_0^T \|\nabla u_{h\tau}(\cdot,t)\|_{0,\Omega}^2\,\mathrm{d}t \leq \omega_\eta\eta^2 + \omega_\varepsilon\varepsilon^2 + \omega_\gamma\gamma^2 \leq (1+\alpha)^2\mathrm{TOL}^2 \int_0^T \|\nabla u_{h\tau}(\cdot,t)\|_{0,\Omega}^2\,\mathrm{d}t\,,$$

and (4.2) is achieved. Our adaptive algorithm thus consists, for $n = 1, \ldots, N$, in finding a time t^n and a Delaunay triangulation $\mathcal{T}_h^n$ such that (4.5) is satisfied. The adaptive algorithm is similar to the one exposed in [13] and is presented in Table 5. Note that η^n and ε^n are used to control the space step, whereas ε^n and γ^n are used to control the time step. This choice is justified by the numerical results of the previous section. The LAPACK numerical library [2] is used to solve the linear systems and the cost of generating Delaunay triangulations is small, $O(N_v^{5/4})$ according to [20], N_v being the number of vertices. Thus, we can afford a new triangulation to be generated at each time step. The generation of the Delaunay triangulation is performed by the

Table 5
The adaptive algorithm

Set $\mathcal{T}_h^0$, u_h^0, $n = 1$, t, Δt.	Initializations
Do while $t \leq T$:	
Compute u_h^n, $(\eta^n)^2$, $(\varepsilon^n)^2$, $(\gamma^n)^2$, l.b., r.b.	
If $\frac{1}{2}\omega_\varepsilon(\varepsilon^n)^2 + \omega_\gamma(\gamma^n)^2 <$ l.b.	The current time step is too small
$\Delta t := 2\,\Delta t$	Same time iteration with bigger time step
Else if $\frac{1}{2}\omega_\varepsilon(\varepsilon^n)^2 + \omega_\gamma(\gamma^n)^2 \leq$ r.b.	The time step is correct, check the space step
If $\omega_\eta(\eta^n)^2 + \frac{1}{2}\omega_\varepsilon(\varepsilon^n)^2 <$ l.b.	The triangulation is too fine
continue	Same time with coarser triangulation
Else if $\omega_\eta(\eta^n)^2 + \frac{1}{2}\omega_\varepsilon(\varepsilon^n)^2 \leq$ r.b.	The triangulation is correct
$t := t + \Delta t$	Increment the current time
$n := n + 1$	
Else	The triangulation is too coarse
continue	Same time with finer triangulation
Else	The time step is too large
$\Delta t := \Delta t/2$	Same time iteration with smaller time step
End If	
Gen_Del_Tri	Generate a new Delaunay triangulation $\mathcal{T}_h^n$
End Do	

Table 6
Results when using the adaptive algorithm

TOL	av(N_v)	N	e.r.e.	t.r.e.	e.i.
1.0	67	10	0.86	0.40	2.1
0.5	113	20	0.46	0.26	1.8
0.25	288	40	0.24	0.15	1.6
0.125	956	80	0.12	0.078	1.6
0.0625	3497	160	0.061	0.040	1.5

Gen_Del_Tri routine. Such a triangulation is obtained by equidistributing locally the error indicator (see [18,17] for details).

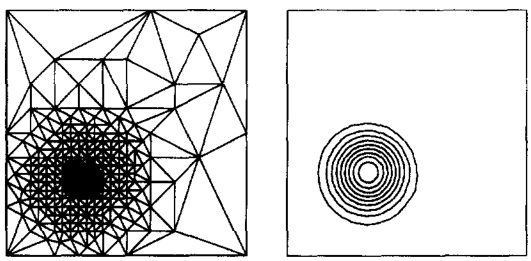
5. Numerical results: the adaptive algorithm

In order to test our adaptive algorithm, we consider the case when the solution of (2.1)–(2.3) is defined by

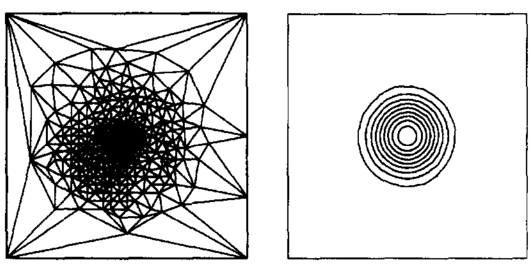
$$u(x_1, x_2, t) = \beta(t) * \exp(-50 * r^2(x_1, x_2, t))) , \tag{5.1}$$

with

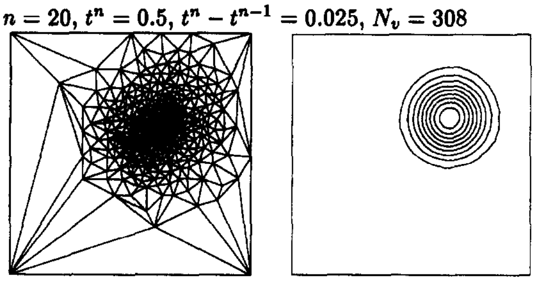
$$r^2(x_1, x_2, t) = (x_1 - 0.4 * t - 0.3)^2 + (x_2 - 0.4 * t - 0.3)^2 ,$$



$n = 4$, $t^n = 0.1$, $t^n - t^{n-1} = 0.025$, $N_v = 266$



$n = 20$, $t^n = 0.5$, $t^n - t^{n-1} = 0.025$, $N_v = 308$



$n = 36$, $t^n = 0.9$, $t^n - t^{n-1} = 0.025$, $N_v = 312$

Fig. 1. Adapted triangulations and isovalues of $u_{h\tau}$ (from levels 0.1 to 0.9 step 0.1) at time 0.1, 0.5, 0.9 (TOL = 0.25, $\alpha = 0.5$).

and

$$\beta(t) = 1 - \exp(-50 * (0.98 * t + 0.01)^2 \qquad \text{if } t < 0.5\,,$$

$$\beta(t) = 1 - \exp(-50 * (1 - (0.98 * t + 0.01))^2) \quad \text{else}\,.$$

Thus, u is a Gaussian function, which center moves from point (0.3, 0.3) at $t = 0$ to point (0.7, 0.7) at $t = 1$.

In Table 6 we have reported the average number of vertices, av(N_v), the number of time steps N, the true relative error t.r.e., the estimated relative error e.r.e., and the effectivity index e.i., when running the adaptive algorithm for different values of TOL. The initial triangulation was the 10×10 uniform triangulation, the initial time step was 0.1, α was set to 0.5 in (4.2).

The effectivity index is between 1.5 and 2.1 for these calculations. This surprisingly good result is probably due to the fact that the aim of the algorithm is to equidistribute both errors due to time and space discretization. Roughly speaking, half of the error is due to time discretization, the other half to space discretization. It should be noted that each time TOL is divided by two, the error is roughly divided by two, the average number of time steps is multiplied by two, the average number of vertices by four.

The triangulations and numerical solutions at times 0.1, 0.5, 0.9, are shown in Fig. 1, when TOL = 0.25.

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