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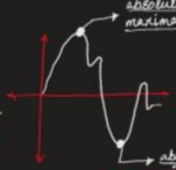
Application Of Derivative

→ Extreme Values of functions :

- Maximum and minimum values are called extreme value or extrema's.
- for absolute maxima, $f(x) \leq f(c)$
  for absolute minima $f(x) \geq f(c)$

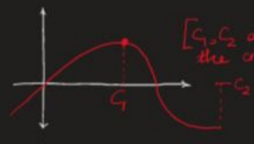


absolute maxima

absolute minima

- finding extrema values
  - First Derivative Test
    - $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$
    - f has a local extrema value if 'c' lies in the domain and $f'(c) = 0$
  - Second Derivative Test
    - $f''(c) > 0$ = Minima
    - $f''(c) < 0$ = Maxima
    - $f''(c) = 0$ = Point of inflection

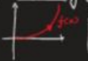
→ Increasing & Decreasing f^n :



[C_1, C_2 are called the critical points]

{the principle of increasing and decreasing f^n is used to find maxima & minima}

1.> increasing function.



$\boxed{f'(x) \geq 0}$

2.> Decreasing function

$\boxed{f'(x) \leq 0}$

⇨ Concavity and Curve Sketching :

Concave up
on an open interval if f' is increasing.
→ if $f'' > 0$, then graph is concave up.

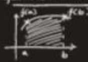
Concave down
on an open interval if f' is decreasing.
→ if $f'' < 0$ on I, then it is concave down.

Mean Value Theorems

- Rolle's Theorum
  1.> f must be continous on [a,b] & differentiable on (a,b)
  2.> $\boxed{f(a) = f(b)}$; $\boxed{f'(c) = 0}$
- Lagrange's Mean Value Theorum
  1.> f must be continous on [a,b] & differentiable on (a,b)
  2.> $\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$

→ Application of Integral -:

- Geometric meaning



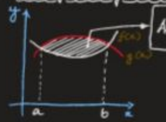
$\boxed{\text{Area of the shaded region} = \int_a^b f(x)\,dx}$

[The area of curve, $y = f(x)$ x-axis and lines $x = a$ & $x = b$]

→ $\text{Area} = \int_a^b y\,dx$ [Bounded along x-axis]

→ $\text{Area} = \int_a^b x\,dy$ [Bounded along y-axis]

- Area of the Region b/w 2 curves



$\boxed{\text{Area} = \int_a^b [g(x) - f(x)]\,dx}$

Top curve — Bottom curve

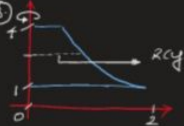
→ Volume of Solid of revolution -

① 

$V = \pi r^2 h$
$= 3.14 \times 3 \times 3 \times 12$
$= 339.12\ ft^3$

② 

$V = \pi r^2 h$
$= \pi (5)^2 \times 6$
$= \pi (25)(6)$
$= 150\pi$

③ 

$x(y) = \frac{2}{y}$

$V = \pi \int_1^4 [f(y)]^2 dy = \pi \int_1^4 \frac{4}{y^2} dy$
$= 4\pi \left[\frac{-1}{y}\right]_1^4$
$= 4\pi \left(\frac{-1}{4} + 1\right)$
$= \frac{3}{4} \times 4\pi$
$= 3\pi$ cubic units.

- The Disk Method

$\boxed{V = \pi \int_a^b [f(x)]^2 dx}$ (horizontal graphs)

[Similarly works for vertical graphs]

- Washer Method :

for horizontal curves

$\boxed{V = \pi \int_a^b [(f(x))^2 - (g(x))^2]\,dx}$

[The formula is similar for vertical curves]

Beta & Gamma Function

- **Beta function:**

  the beta function is defined by:

  $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \;;\; \text{where } m,n > 0$$

→ **Properties of Beta function:**

↳ $\beta(m,n) = \beta(n,m)$

proof: Given, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ — ①

Substitute $y = 1-x$ in eq^n ①

$\Rightarrow dy = -dx$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1} (-dx)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dx$$

$$= \underline{\underline{\beta(n,m)}}$$

↳ $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

proof:

Substitute $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} \times 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin\theta \cos\theta\, d\theta$$

$\pi/2$

$$= 2\int_0 \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

↳ $$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

- Gamma function :

the gamma function is defined as :

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx \;; \text{ where } n > 0$$

→ Properties of Gamma functions :

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx \text{ ——— (1)} \;; \quad \Gamma 1 = \int_0 e^{-x}\,dx = 1$$

↳ Reduction formula :

$\xrightarrow{1^{st}}$ $\Gamma(n+1) = n\Gamma n$

proof : $$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-n}\,dx = \int_0 e^{-x} x^n\,dx$$

$$= x^n(e^{-x})\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\,dx$$

$$= \lim_{x\to 0}\left[\frac{-x^n}{e^x}\right] + n\int_0^\infty e^{-x} x^{n-1}\,dx$$

$$= 0 + n\Gamma n$$

$$= n\Gamma n$$

↳ if n is a negative fractional number, then

$$\boxed{\Gamma n = \frac{\Gamma n+1}{n}}$$

for example :

$$\Gamma\tfrac{1}{2} = \int_0^\infty e^{-x} x^{-1/2}\, dx$$

$$x = y^2 \Rightarrow dx = 2y\, dy$$

$$\Rightarrow \int_0^\infty e^{-y^2} y^{-1} (2y\, dy) = 2\int_0^\infty e^{-y^2} dy$$

$$= 2\left[\frac{\sqrt{\pi}}{2}\right]$$

$$= \underline{\underline{\sqrt{\pi}}}$$

↳ $\Gamma n+1 = n!$, if n is an integer.

proof: we know, $\Gamma n+1 = n\Gamma n$

$$\Rightarrow \Gamma n+1 = n\Gamma n$$
$$= n(n-1)\Gamma n-1$$
$$= n(n-1)(n-2) \_\_\_\_ 3\cdot 2\cdot \Gamma 1$$
$$= n(n-1)(n-2) \_\_\_\_ 3\cdot 2\cdot 1$$
$$= \underline{\underline{n!}}$$

ex : $\int_0^{\pi/2} \sqrt{\tan x}\, dx = \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x\, dx$

$$= 2 \times \frac{1}{2} \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x\, dx$$

$2m - 1 = 1/2 \Rightarrow m = 3/4$

$2n - 1 = -1/2 \Rightarrow n = 1/4$

$$\therefore \beta\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{\Gamma 3/4 \; \Gamma 1/4}{\Gamma 1}$$

$$= \Gamma\frac{3}{4}\,\Gamma\frac{1}{4}$$

ex: $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}} \Rightarrow$ put $x^2 = \sin\theta \Rightarrow 2x\,dx = \cos\,d\theta$

$$\Rightarrow dx = \frac{\cos\theta\, d\theta}{2\sqrt{\sin\theta}}$$

$$\Rightarrow \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\cos\theta \times 2\sqrt{\sin\theta}} = \frac{1}{4} \times 2 \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta$$

$$= \frac{1}{4}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4}\frac{\Gamma 1/4 \; \Gamma 1/2}{\Gamma 3/4}$$

$$= \frac{\sqrt{\pi}}{4}\frac{\Gamma 1/4}{\Gamma 3/4}$$

ex: $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = I_1 \times I_2$

put $x^2 = \sin\theta$

$\Rightarrow 2x\,dx = \cos\theta\, d\theta$

$$I_1 = \int_0^{\pi/2} \frac{\sin\theta}{\cancel{\cos\theta}} \times \frac{\cancel{\cos\theta}}{2\sqrt{\sin\theta}}\, d\theta$$

$$= \frac{1}{4} \times 2\int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta$$

$$= \frac{1}{4}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

put $x^2 = \tan\theta \Rightarrow 2x\,dx = \sec^2\theta\, d\theta$

$$\Rightarrow dx = \frac{1}{2\cos^{\cancel{2}}\theta} \times \frac{\cancel{\cos\theta}}{\sin\theta}\, d\theta$$

$$\Rightarrow dx = \frac{d\theta}{2\cos\theta \sin\theta}$$

$$I_2 = \frac{1}{2}\int_0^{\pi/4} \sin^{-1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

put $2\theta = \alpha \Rightarrow d\alpha = 2d\theta$

$$I_2 = \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{d\alpha}{\sqrt{\sin\alpha}} \times \frac{1}{2}$$

$$= \frac{1\times 2}{\sqrt{2}\,2}\int_0^{\pi/2} \sin^{-1/2}\alpha \cos^{1/2}\alpha$$

$$= \frac{1}{4\sqrt{2}}\,\beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$\therefore I = I_1 I_2 = \frac{1}{16\sqrt{2}}\left[\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(5/4)}\right]^2$$

$$= \frac{\pi}{16\sqrt{2}}\frac{(\Gamma(3/4))^2}{\left(\frac{1}{4}\Gamma(1/4)\right)^2}$$

$$= \frac{\pi}{\sqrt{2}}\frac{(\Gamma(3/4))^2}{(\Gamma(1/4))^2}$$

→ Complement/reflection formula

Given: $\int_0^{\infty} \frac{x^{n-1}}{1+x} = \frac{\pi}{\sin n\pi}$, then $\boxed{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$

proof: $\int_0^{\infty} \frac{x^{n-1}}{1+x}\,dx \Rightarrow$ put $x = \tan^2\theta$

$dx = 2\tan\theta \sec^2\theta \, d\theta$

$$\Rightarrow \int_0^{\pi/2} \frac{\tan^{2n-2}\theta}{1+\tan^2\theta} \times 2\tan\theta \sec^2\theta\, d\theta$$

$$\Rightarrow \int_0^{\pi/2} 2\tan^{2n-1}\theta\, d\theta = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{1-2n}\theta\, d\theta$$

$$= \beta(n, 1-n)$$

$$= \Gamma(n)\,\Gamma(1-n)$$

$$\Gamma(n+1-n)$$

$$= \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

LAPLACE Transform

• Laplace Transform of $f(t)$:

$$\boxed{L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt}$$

where, s is a parameter and $L(f(t)) = F(s)$

L → Laplace transformation operator

- $f(t)$ is continous
- $\lim\limits_{t \to \infty} \int_0^\infty e^{-st} f(t)\, dt$ is finite

• **Transforms Of Elementary function**

- $L(1) = \frac{1}{s}$
- $L(t^n)$
  - $\frac{n!}{s^{n+1}}$ $\{n = 1, 2, ---\}$
  - $\frac{\Gamma(n+1)}{s^{n+1}}$
- $L(e^{at}) = \frac{1}{s-a}$
- $L(\sin at) = \frac{a}{s^2 + a^2}$
- $L(\cos at) = \frac{s}{s^2 + a^2}$
- $L(\sinh at) = \frac{a}{s^2 - a^2}$
- $L(\cosh at) = \frac{s}{s^2 - a^2}$

• **Properties of Laplace Transform**

- $L(af + bg - ch) = aL(f) + bL(g) - cL(h)$
- $L(e^{at} f(t)) = \bar{f}(s-a)$
- $L(f(at)) = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$

- Transforms of Periodic function

if $f(t)$ is a periodic function, i.e $f(t+T)=f(t)$ then

$$\boxed{L(f(t)) = \frac{1}{1-e^{sT}}\int_0^T e^{-st} f(t)\,dt}$$

- Other Transforms

$$\boxed{L(f^n(t)) = s^n\bar{f}(s) - s^{n-1}f(0) - \dots - f^{n-1}(0)}$$

$$\boxed{L\left(\int_0^t f(u)\,du\right) = \frac{1}{s}\bar{f}(s)}$$

$$\boxed{L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}(\bar{f}(s))}$$

$$\boxed{L\left(\frac{f(t)}{t}\right) = \int_0^\infty \bar{f}(s)\,ds}$$

→ Inverse Laplace Transform :-

$$L(f(t)) = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\Rightarrow \boxed{L^{-1}[f(s)] = f(s)}$$

Inverse Transform of Elementary function

$$\boxed{L^{-1}\left(\frac{1}{s}\right) = 1}$$

$$\boxed{L^{-1}\left(\frac{1}{s-a}\right) = e^{at}}$$

$$\boxed{L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}}$$

$$\boxed{L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at}$$

$$\boxed{L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at}$$

$$\boxed{L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at}$$

$$\boxed{L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at}$$

$$\boxed{L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{2a}t\sin at}$$

$$\boxed{L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{1}{2a^2}(\sin at - at\cos at)}$$

- Property of inverse laplace :-

I. Shifting property –

↳ if $L^{-1}(f(s)) = f(t)$, then

$$\boxed{L^{-1}(F(s-a)) = e^{at} f(t) = e^{at} F(s)}$$

↳ if $L^{-1}(F(s)) = f(t)$, then

$$\boxed{L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt}$$

↳ if $L^{-1}(F(s)) = f(t)$ and $f(0) = 0$, then

$$\boxed{L^{-1}(s\,F(s)) = \frac{d}{dt}(f(t))}$$

↳ if $L^{-1}\{f(s)\} = f(t)$, then

$$\boxed{t f(t) = L^{-1}\left\{-\frac{d}{ds}(F(s))\right\}}$$

- Convolution Theorum :-

if $L^{-1}(F(s)) = f(t)$ and $L^{-1}(G(s)) = g(t)$, then

$$\boxed{L^{-1}\{F(s)G(s)\} = \int_0^t f(u)\,g(t-u)\,du = f(t) * g(t)}$$

where $f(t) * g(t)$ is called convolution or falting of f and g.

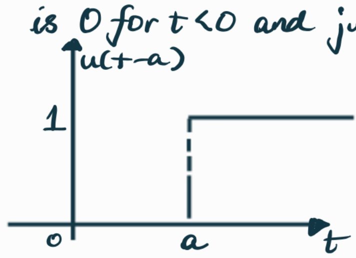
1. Transform of error function

error function is : $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-t^2}\,dt$

∴ The laplace transform of $\operatorname{erf}(x) \rightarrow$ $\boxed{L(\operatorname{erf}(x)) = \frac{1}{s\sqrt{s+1}}}$

- Unit Step function

A function $u(t-a)$ is 0 for $t < 0$ and jumps to 1 when $t \geq a$



$$L(u(t-a)) = \int_0^\infty e^{-st} u(t-a)\,dt$$

$$= \int_{t-a}^{\infty} e^{-st} \cdot 1\,dt = \left| \frac{e^{-st}}{-s} \right|_{t-a}^{\infty}$$

$$\Rightarrow \boxed{L(u(t-a)) = \frac{e^{-as}}{s}}$$

Second Shifting property :

if $L(f(t)) = F(s)$, then

$$\bar{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & , \text{ if } t<a \\ f(t-a) & , \text{ if } t>a \end{cases}$$

$$\boxed{L(f(t-a)u(t-a)) = e^{-as} F(s)}$$

- ## Dirac's delta function

we consider the function,

$$f_k(t-a) = \begin{cases} \frac{1}{\epsilon} & \text{if } a \leqslant t \leqslant a+k \\ 0 & \text{otherwise} \end{cases}$$

impulse function

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a)$$

$$\Rightarrow \delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\therefore \mathcal{L}(\delta(t-a)) = e^{-as}}$$

Multivariable Calculus

- **function of several variable**

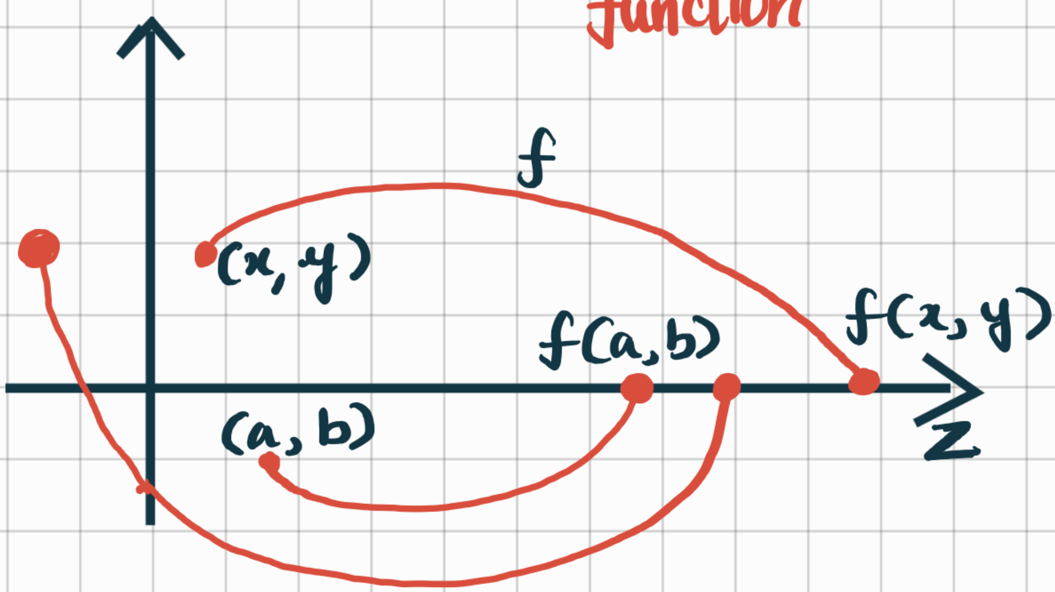
A real valued function f on D is a rule that assigns a unique real number

$$\boxed{w = f(x, x_1, x_2 \cdots x_n)}$$

output function

input function

Domain (set of n-tuples of real no.s)



$$\boxed{z = f(x, y)}$$

↳ **Function of 2 variables:**

A point (x_0, y_0) in a region, R in the xy-plane.

(x_0, y_0) can makeup interior of the region, on the boundary & on the exterior

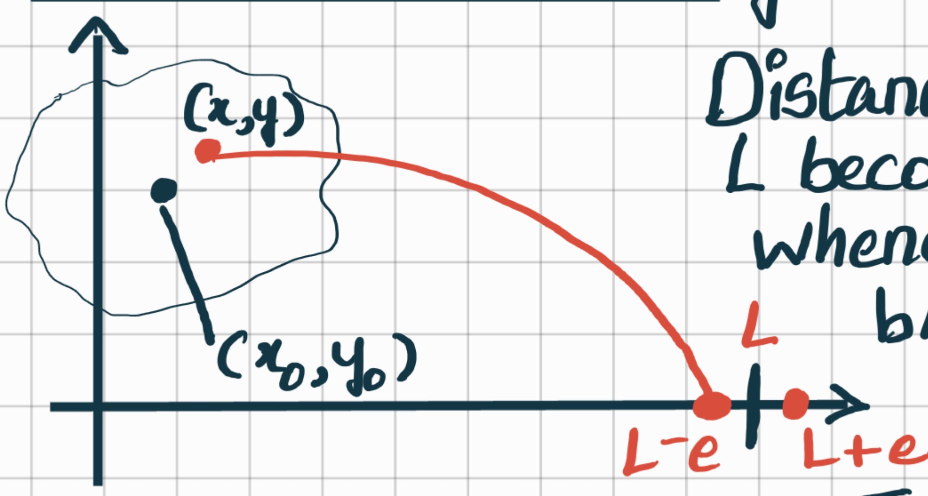
↳ Graphs, Level Curves & Contours:

The set of points in the plane where the function, $f(x,y)$ has a constant value, $f(x,y)=C$ is called a level curve of f.

The graph of f is also called the

$$\boxed{\text{surface } z = f(x,y)}$$

• Limit and Continuity:



Distance b/w $f(x,y)$ and L becomes very small whenever the distance b/w (x,y) & (x_0,y_0)

↳ Properties of Limits
$$\begin{bmatrix} L : \lim\limits_{x,y \to x_0,y_0} f(x,y) \\ M : \lim\limits_{x,y \to x_0,y_0} g(x,y) \end{bmatrix}$$

Sum rule : $L+M$

Difference rule : $L-M$

Quotient rule : $\frac{L}{M}$

Constant Multiple rule : $k \cdot L$

Multiple rule : $L \cdot M$

Power rule : L^n

Root rule : $\sqrt[n]{L} = L^{1/n}$

- Continuity :

→ A function is continuous if it satisfies the following conditions :

↳ $f(x)$ exists at (x_0, y_0)

↳ $\lim_{(x,y) \to (x_0, y_0)} f(x)$ exists and is finite

→ Polynomial and rational functions are continous at every point.

→ Algebraic operations on continous functions gives a continous function ex : $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} \\ 0 & (x,y)=(0,0) \end{cases}$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \underset{''\,''}{Lt} \frac{2x(mx)}{x^2+m^2x^2}$$

$$= \underset{''\,''}{Lt} \frac{2m\cancel{x^2}}{\cancel{x^2}(1+m^2)}$$

$$= \frac{2m}{1+m^2}$$

• Partial Derivatives

Let $z=f(x,y)$ then the partial derivatives is defined as :

$$\boxed{\frac{\delta f}{\delta x}(x,y) = \frac{f(x+h,y)-f(x,y)}{h}}$$

ex: $f(x,y) = x^2y - y^3 + \ln(x)$

$f_{xx} = 2y - \frac{1}{x^2}$

$f_{yy} = -6y$

$\Rightarrow f_{xy} = f_x f_y = 2xy + \frac{1}{x}(x^2 - 3y^2)$

Laplace Equation:

Partial derivatives occur in partial D.E that express certain physical laws.

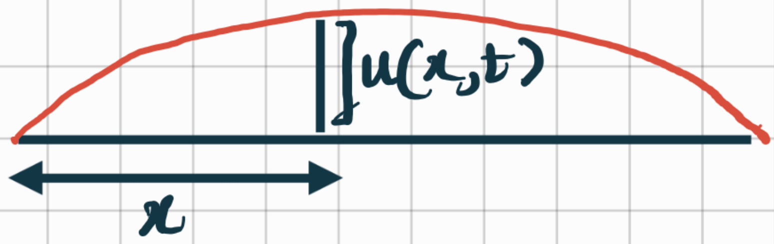
$$\boxed{\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \text{----} = 0}$$

ex: $u(x,y) = e^x \sin y$

$\Rightarrow \frac{\delta u}{\delta x} = e^x \sin y \Rightarrow \frac{\delta^2 u}{\delta x^2} = e^x \sin y$

$\frac{\delta^2 u}{\delta y^2} = -e^x \sin y$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}}$$ → Wave equation



- Clairaut's Theorum

if $f(x,y)$ and partial derivatives f_x, f_y, f_{xx} and f_{yy} are defined throughout (a,b) & is continous across the same, then

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$$

Chain Rule :

if $w = f(x,y)$ is diffruntiable and $x = x(t)$ & $y = y(t)$ are also diffrentiable wrt t.
Hence, the composition of $w = f(x(t), y(t))$ is a diffrentiable func. of t, then

$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}}$$

ex: $w = xy$, $x = \cos t$, $y = \sin t$
$t = \frac{\pi}{2}$

$$\Rightarrow \frac{dw}{dt} = y(-\sin t) + x\cos t$$
$$= \cos^2 t - \sin^2 t$$
$$= \cos 2t = \cos 2\left\{\frac{\pi}{2}\right\} = \underline{\underline{-1}}$$

ex $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$, $t = 0$

$$\frac{dw}{dt} = y(-\sin t) + x(\cos t) + 1(1)$$
$$= \cos^2 t - \sin^2 t + 1$$
$$= \cos 2t + 1 = 1 + 1 = \underline{\underline{2}}$$

Jacobians

in 1D problems, we have a simple change of variables, like

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(x(u))\,du \frac{\delta x}{\delta u} \longrightarrow \text{1-D Jacobian}$$

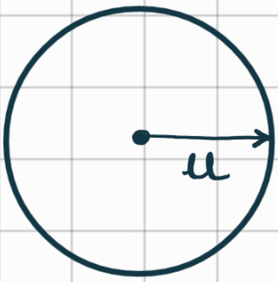
for a continous 1 to 1 transformation from $(x, y) \longrightarrow (u, v)$, then $x = x(u, v)$ and $y = y(u, v)$,

$$\iint_R f(x, y)\,dx\,dy = \iint_{R'} f(x(u,v), y(u,v))\,du\,dv \frac{\delta(x,y)}{\delta(u,v)}$$

2-D Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

The terms x_u, y_u, x_v, y_v are called Jacobians.



Area of the circle, $A = \iint_A dx\, dy$

$x = r\cos\theta$

$y = r\sin\theta$

$$\begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \theta} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

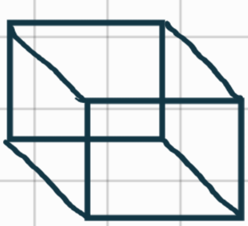
$$= \underline{\underline{r}}$$

$$\int_{r=0}^{r=x}\int_{\theta=0}^{\theta=2\pi} r\, dr\, d\theta = \int_0^x r\, dr \int_0^{2\pi} d\theta = \left[\frac{r^2}{2}\right]_0^x \left[\theta\right]_0^{2\pi}$$

$$= \frac{x^2}{2} \times 2\pi$$

$$= \underline{\underline{\pi x^2}}$$

$\longrightarrow$ $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$



Volume of small cubes

$$\iiint_V f(x,y,z)\, dx\, dy\, dz = \iiint_V F(u,v,w) \underbrace{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|}_{\text{3-D Jacobian}} du\, dv\, dw$$

where,
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \underline{\underline{r^2\sin\theta}}$$

• **Properties of Jacobians**

→ if $u = u(x,y)$ & $v = r(x,y)$, then

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1}$$

→ **Chain Rule**: if $u = u(r,s)$, $v = (r,s)$ and $r = r(x,y)$, $s = s(x,y)$, then

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}}$$

ex : Find $\dfrac{\partial(u,v)}{\partial(x,y)}$, when $u = 3x + 5y$
$v = 4x - 3y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 4 & -3 \end{vmatrix}$$

$$= 3(-3) - 4(5)$$
$$= -9 - 20$$
$$= \underline{\underline{-29}}$$

Application of Multivariable Calculus

Module - 4

Taylor's expansion for 2 variables

We know that Taylor's theorum for 1 variable is :

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \text{------}$$

Now we can find Taylor's expansion for 2 variables is :

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{2!}\left[h f_{xx}(a,b) + k f_{yy}(a,b)\right] + \text{-----}$$

example : Expand the function $\sin xy$ in terms of $x-1$ and $y - \frac{\pi}{2}$ upto second term.

function	value at $(1, \frac{\pi}{2})$
$f(x,y) = \sin xy$	1
$f_x = y\cos xy$	0
$f_y = x\cos xy$	0
$f_{xx} = -y^2 \sin xy$	$-\pi^2/4$
$f_{xy} = -xy\sin xy + \cos xy$	$-\pi/2$
$f_{yy} = -x^2 \sin xy$	-1

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x + k f_y\right] + \frac{1}{2!}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]$$

$$= 1 + \left[(x-1)(0) + \left(y-\frac{\pi}{2}\right)(0)\right] + \frac{1}{2}\left[(x-1)^2\left\{\frac{-\pi^2}{4}\right\} + \left(y-\frac{\pi}{2}\right)^2(-1) + 2(x-1)\left(y-\frac{\pi}{2}\right)\left\{\frac{-\pi}{2}\right\}\right]$$

$$= 1 + \frac{1}{2}\left[\frac{-\pi^2}{4}(x-1)^2 - \pi(x-1)\left(y-\frac{\pi}{2}\right) - \left(y-\frac{\pi}{2}\right)^2\right]$$

example: $x^2y + 3y - 2$ in powers of $x-1$ and $y+2$

function	value at $(1,-2)$
$f = x^2y + 3y - 2$	-10
$f_x = 2xy$	-4
$f_y = x^2 + 3$	4
$f_{xx} = 2y$	-4
$f_{yy} = 0$	0
$f_{xy} = 2x$	2

$$f(x,y) = f(a,b) + [hf_x + kf_y] + \frac{1}{2!}\left[h^2f_{xx} + k^2f_{yy} + 2hk\,f_{xy}\right]$$

$$= -10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2}\left[(x-1)^2(-4) + (y+2)^2(0) + 2(x-1)(y+2)(2)\right]$$

$$= -2 + 4(y-x) + \frac{1}{2}\left[-4(x-1)^2 + 4(x-1)(y+2)\right]$$

- Maxima and Minima of 2 variables

$f(a,b) > f(a+h, b+k) \longrightarrow$ Maximum value

$f(a,b) < f(a+h, b+k) \longrightarrow$ Minimum value

where h & k are small +ve/−ve values.

$f(a,b)$ is said to be the **extremum** value of $f(x,y)$ if it is either maxima or minima.

Saddle point/minimax : is a point where function is neither maximum nor minimum (or) At such point, f is maximum from one direction & is minimum from the other.

Sufficient condition for sadde point : if $p = f_x(a,b)$ & $q = f_y(a,b)$ and $p, q = 0$ then it is called **stationary point**

let $r = f_{xx}(a,b)$, $s = f_{xy}(a,b)$, $t = f_{yy}(a,b)$ and $\Delta = rt - s^2$

if $\Delta > 0$:
- $r < 0 \longrightarrow$ **maximum value**
- $r > 0 \longrightarrow$ **minimum value**

$\Delta < 0 \longrightarrow$ $f(x,y)$ has neither maximum nor minimum value

$\Delta = 0 \longrightarrow$ case fails to find the nature of the function.

example : Examine the extrema of $f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$

$f_x = 2x + y - \frac{1}{x^2}$, $f_y = x + 2y - \frac{1}{y^2}$

$$\Rightarrow (p,q) = 0 \Rightarrow 2x + y - \frac{1}{x^2} = 0 \text{ --- } (1)$$

$$x + 2y - \frac{1}{y^2} = 0 \text{ --- } (2)$$

$$(1) - (2) \longrightarrow x - y + \frac{1}{y^2} - \frac{1}{x^2} = 0$$

$$\Rightarrow x - y + \frac{x^2 - y^2}{x^2y^2} = 0$$

$$\Rightarrow (x-y)x^2y^2 + x^2 - y^2 = 0$$

$$\Rightarrow (x-y)[x^2y^2 + x + y] = 0$$

$$\Rightarrow x - y = 0 \Rightarrow \underline{\underline{x = y}} \text{ --- } (3)$$

$$\therefore (1) \Rightarrow 3x - \frac{1}{x^2} = 0 \Rightarrow 3x^3 = 1 \Rightarrow x = \left(\frac{1}{3}\right)^{1/3} = y$$

$$r = f_{xx} = 2 + \frac{2}{x^3} = 8 \qquad t = 2 + \frac{2}{y^3} = 8$$

$s = 1$

$$\therefore \Delta = rt - s^2 = 8(8) - 1^2 = 64 - 1 = 63 > 0$$

$r > 0$

$\Rightarrow f(x,y)$ attains minimum value at $\left(\left\{\frac{1}{3}\right\}^{1/3}, \left\{\frac{1}{3}\right\}^{1/3}\right)$

ex : $f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$f_y = 2yx^3 - 2yx^4 - 3y^2x^3$

$p = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$\Rightarrow 3 - 4x - 3y = 0$

$\Rightarrow 4x + 3y = 3$ —— (1)

$q = 2yx^3 - 2yx^4 - 3y^2x^3 = 0$

$\Rightarrow 2 - 2x - 3y = 0$

$\Rightarrow 2x + 3y = 2$ —— (2)

$$\begin{array}{r} 4x + 3y = 3 \\ 2x + 3y = 2 \\ - \quad - \quad - \\ \hline 2x = 1 \end{array}$$

$2x = 1 \Rightarrow x = \frac{1}{2}$ and $y = \frac{1}{3}$

$r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$ $\quad t = 2x^3 - 2x^4 - 6yx^3$

$s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$

$$\Delta = rt - s^2 = \left[\frac{1}{3} - \frac{2}{3} - \frac{1}{27}\right]\left[\cancel{\frac{1}{4}} - \frac{1}{8} - \cancel{\frac{1}{4}}\right] - \left\{\frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right\}^2$$

$$\Rightarrow \Delta = \frac{-\cancel{10}^{5}}{27} \times \left(\frac{-1}{\cancel{8}_{4}}\right) - \frac{1}{144} = \frac{5}{108} - \frac{1}{144} \longrightarrow \text{positive}$$

ex : $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f_x = 3x^2 + 3y^2 - 30x + 72 = 0$

$\Rightarrow x^2 + y^2 - 10x + 24 = 0$ —— (1)

$f_y = 6xy - 30y = 0$

$\Rightarrow 6x\cancel{y} = 30\cancel{y}$

$\Rightarrow x = 5$, $25 + y^2 - 50 + 24 = 0$

$y^2 = 1$

$y = 1$

$r = f_{xx} = 6x - 30$ $\quad s = 6y$

$t = f_{yy} = 6x - 30$

$\Delta = rt - s^2 = (6x-30)^2 - 36y^2$
$= (6(5)-30)^2 - 36(1)^2$
$= -36 \longrightarrow$ The function $f(x,y)$ has no extremum value.

<u>ex</u>: $f = \sin x + \sin y + \sin(x+y)$, where $x \geq 0$ and $y \leq \pi$

$f_x = \cos x + \cos(x+y)$
$f_y = \cos y + \cos(x+y)$

$p = q = 0 \Rightarrow \cos x + \cos(x+y) = \cos y + \cos(x+y)$
$\Rightarrow \cos x = \cos y$
$\Rightarrow x = y \longrightarrow f_x = \cos x + \cos(x+x) = 0$
$\Rightarrow \cos x + \cos 2x = 0$
$\Rightarrow \cos x = -\cos 2x$
$x = y = \pi \pm 2x$
$\Rightarrow x = y = \frac{\pi}{3}, -\pi$

$r = -\sin x - \sin(x+y) \quad s = -\sin(x+y)$
$t = -\sin y - \sin(x+y)$

$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$: $r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$, $s = -\sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2}$

$t = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

$\Delta_1 = rt - s^2 = -\sqrt{3}(-\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4}$ (maximum)

$(-\pi, -\pi)$: $r = s = t = 0$
hence, $\Delta_2 = 0$ (cannot determine)

- **Lagrangian Multiplier method :**

Suppose, we need to find the maxima and minima of $f(x,y,z)$ where x,y,z are subject to a constraint equation, $\phi(x,y,z)=0$.

we define a function, $\boxed{g(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)}$ —— (1.)

where, λ is the Lagrange Multiplier which is independant by x,y,z.

The condition for max. & min value :-

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 0 \quad \text{———} \quad (2.)$$

Find λ, x, y, z by solving eqn (1.) & (2.). The point (x,y,z) may be maximum, minimum or neither depending upon the physical condition.

Module - 5

MULTIPLE INTEGRALS

The definite integral $\int_a^b f(x)\,dx$ defined as the limit of sum :-

$$\iint_R f(x,y)\,dA = \lim_{\substack{n\to\infty \\ \delta A\to 0}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

for the purpose of evaluation, The integral is written as :

$$\boxed{I = \iint f(x,y)\,dx\,dy}$$

ex: $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

$\Rightarrow \int_0^5 dx \int_0^{x^2} x(x^2+y^2)\,dy$

$\Rightarrow \int_0^5 dx \left[x^3 y + \frac{xy^3}{3}\right]_0^{x^2}$

$\Rightarrow \int_0^5 \left[x^5 + \frac{x^7}{3}\right] dx \Rightarrow \left[\frac{x^6}{6} + \frac{x^8}{24}\right]_0^5 \Rightarrow 18880 \cdot 223$

→ Change of Order of integration -

ex: By changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin px \,dx\,dy$ show that $\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}$

$\Rightarrow \int_0^\infty \int_0^\infty e^{-xy} \sin px \,dx\,dy = \int_0^\infty \int_0^\infty \left[e^{-xy} \sin px \,dx\right] dy$

$$\Rightarrow \int_0^\infty \left| \frac{-e^{-xy}}{p^2+y^2}(p\cos px + y\sin px) \right|_0^\infty dy$$

$$\Rightarrow \int_0^\infty \frac{p}{p^2+y^2}\,dy \Rightarrow \left| \tan^{-1}\left(\frac{y}{p}\right) \right|_0^\infty = \frac{\pi}{2} \quad \text{— ①}$$

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy = \int_0^\infty \sin px \left[\int_0^\infty e^{-xy}\,dy \right] dx$$

$$= \int_0^\infty \sin px \left| \frac{e^{-xy}}{-x} \right|_0^\infty dx$$

$$= \int_0^\infty \frac{\sin px}{x}\,dx \quad \text{— ②}$$

$\therefore$ LHS = RHS $\longrightarrow$ Henced proved

ex: Change the order of integration of $I = \int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy$ and hence evaluate the same.

$$I_1 = \int_0^1 dy \int_{x^2}^{2-x} xy \, dx \quad ; \quad I_2 = \int_0^1 dx \int_{x^2}^{2-x} xy \, dy$$

$$\Rightarrow I = I_1 + I_2$$

$$= \int_0^1 dy \int_{x^2}^{2-x} xy\,dx + \int_0^1 dx \int_{y^2}^{2-y} xy\,dy$$

$$= \int_0^1 \left| \frac{x^2}{2} \cdot y \right|_{x^2}^{2-x} dy + \int_0^1 \left| \frac{xy^2}{2} \right|_{y^2}^{2-x} dx$$

$$= \int_0^1 \frac{y^2}{2}\,dy + \int_1^2 y(2-y)^2\,dy$$

$$= \frac{3}{8}$$

→ Double Integrals in Polar Coordinates -

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$, we first integrate w.r.t r between limits $r=r_1$ & $r=r_2$ and resulting expression is then integrated w.r.t θ.

ex: Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the cardiod $r=a(1-\cos\theta)$ above the initial line.

$$\Rightarrow \int_0^{\pi} \sin\theta \left[\int_0^{r=a(1-\cos\theta)} r\,dr\right] d\theta$$

$$\Rightarrow \int_0^{\pi} \sin\theta \left|\frac{r^2}{2}\right|_0^{a(1-\cos\theta)} d\theta$$

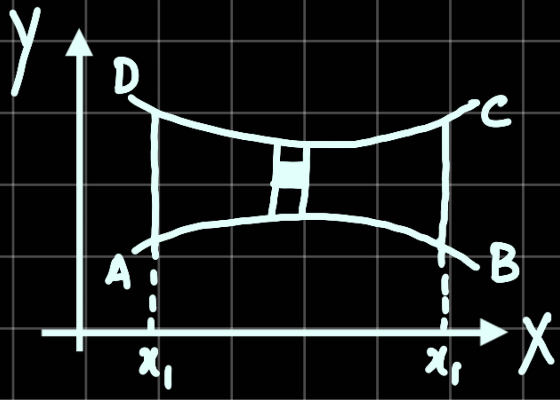
$$\Rightarrow \int_0^{\pi} \sin\theta\cdot(1-\cos\theta)^2\cdot\frac{a^2}{2}\,d\theta$$

$$\Rightarrow \frac{a^2}{2}\left|\frac{(1-\cos\theta)^3}{3}\right|_0^{\pi}$$

$$\Rightarrow \frac{a^2}{2}\times\frac{8}{3} \Rightarrow \frac{4a^2}{3}$$

→ Area enclosed by Plane Curves

- Cartesian coordinates -

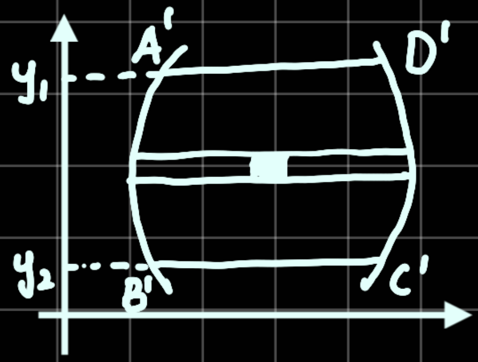


$y=f_1(x)$ and $y=f_2(x)$

$x=x_1$ and $x=x_2$

$$\rightarrow A = \int_{y_1}^{y_2}\int_{f_1(y)}^{f_2(y)} dx\,dy$$

- **Polar coordinates**



$$A = \int\int r \, dr \, d\theta$$

→ TRIPLE INTEGRALS ←

The limit of sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the triple integral of $f(x,y,z)$ over region V :-

$$\iiint f(x,y,z) \, dV$$

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x,y,z) \, dx \, dy \, dz$$

ex : Evaluate $\displaystyle\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z) \, dx \, dy \, dz$

$$\Rightarrow \quad I = \int_{-1}^{1} \int_{0}^{z} \left| xy + \frac{y^2}{2} + zy \right|_{x-z}^{x+z} dx \, dz$$

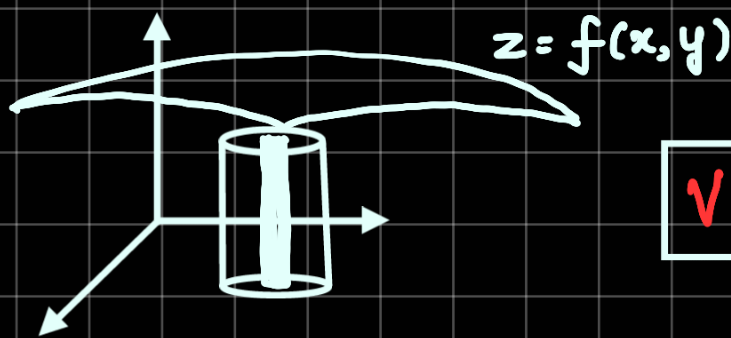
$$= \int_{-1}^{1} \left| \frac{x^2 z}{2} + z^2 x + \frac{x^2 z}{2} \right|_{0}^{z} dz$$

$$= \left| \frac{z^3}{2} + z^3 + \frac{z^3}{2} \right|_{-1}^{1}$$

$$= \underline{\underline{0}}$$

→ Volume of Solids

- **Using double integral** -



$$\boxed{V = \iint f(x,y)\,dx\,dy}$$

ex : Find the volume of the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$

$$\Rightarrow V = 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} z \,dx\,dy$$

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$= 2\int_{-2}^{2} (4-y)\sqrt{4-y^2}\,dy$$

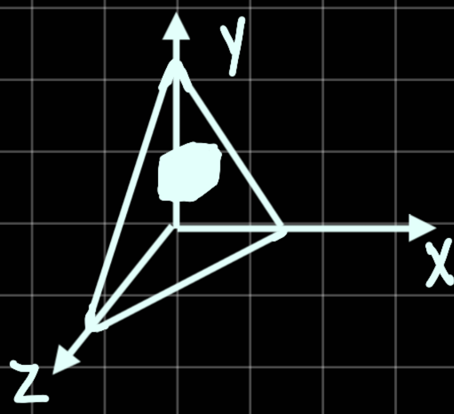
$$= 2\int_{-2}^{2} 4\sqrt{4-y^2}\,dy - 2\int_{-2}^{2} y\sqrt{4-y^2}\,dy$$

$$= 8\int_{-2}^{2}\sqrt{4-y^2}\,dy$$

$$= 8\left|\frac{y}{2}\sqrt{4-y^2} + 2\sin^{-1}\frac{y}{2}\right|_{-2}^{2}$$

$$= \underline{\underline{16\pi}}$$

- Using triple integral -



$$\boxed{V = \iiint dx\, dy\, dz}$$

ex: Calculate the volume of the solid bounded by the planes $x=0$, $y=0$, $x+y+z=a$ and $z=0$

$$\Rightarrow \quad V = \int_0^a \int_0^a \int_0^{a-x-y} dz\, dy\, dx$$

$$= \int_0^a \int_0^a (a-x-y)\, dy\, dx$$

$$= \int_0^a \left| ay - xy - \frac{y^2}{2} \right|_0^a dx$$

$$= \int_0^a \frac{(a-x)^2}{2}\, dx$$

$$= \left| \frac{-(a-x)^3}{3} \right|_0^a \times \frac{1}{2}$$

$$= \frac{a^3}{6}$$

→ Change of Variables -

- **In double integral** -

Let the variables x, y be changed to the new variables u, v by the transformation:

$$x = \phi(u,v) \quad , \quad y = \psi(u,v)$$

$$\iint_R f(x,y)\, dx\, dy = \iint_{R'} f\left[\phi(u,v), \psi(u,v)\right] |J|\, du\, dv$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)}$ → Jacobian

Similarly for triple integrals;

- **Particular Cases** -

i) Change cartesian coordinates to polar coordinates :

for example, we have $x = r\cos\theta$ and $y = r\sin\theta$
and, $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = r$

$$\boxed{\therefore \iint_R f(x,y)\, dx\, dy = \iint_{R'} f(r\cos\theta, r\sin\theta) \cdot r\, dr\, d\theta}$$

ii) To change rectangular coordinates to cylindrical coordinates

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z, \quad \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho$$

$$\boxed{\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint \rho\cdot f(\rho\cos\phi, \rho\sin\phi, z)\,d\rho\,d\phi\,dz}$$

iii) To change rectangular coordinate to spherical coordinates

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

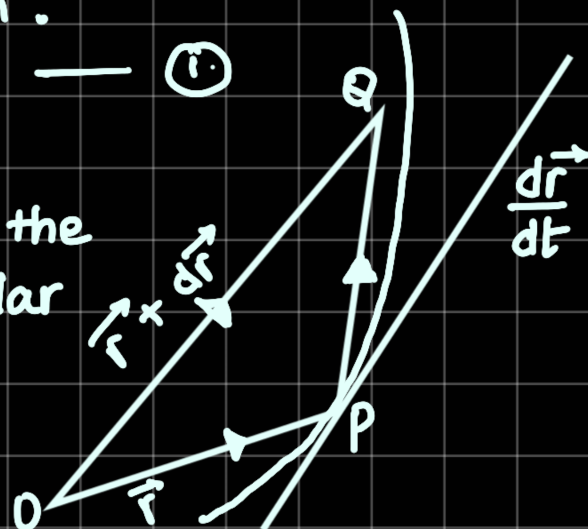
$$\boxed{\iiint f(x,y,z)\,dx\,dy\,dz = \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\cdot r^2\cos\theta\,dr\,d\theta\,d\phi}$$

Vector Diffrentiation

Module - 6

Let $\vec{r} = \vec{f}(t)$ be a single valued continous vector field point function of a scalar variable. Let $\overrightarrow{OP}$ represent the vector $\vec{r}$ corresponding to a certain value t to the scalar variable t, Then :

$$\vec{r} = \vec{f}(t) \quad \text{—— (1)}$$



Let $\overrightarrow{OQ}$ be the vector $\vec{r} + \delta\vec{r}$ corresponding to the value $t + \delta t$ of the scalar variable t, Then

$$\vec{r} + \delta\vec{r} = f(t + \delta t) \quad \text{—— (2)}$$

$$\boxed{\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t} = \frac{d}{dt}(\vec{f}(t))}$$

Some results of diffrentiation -

1. $\frac{d}{dt}(\vec{r} - \vec{s}) = \frac{dr}{dt} - \frac{ds}{dt}$
2. $\frac{d}{dt}(s\vec{r}) = \frac{ds}{dt}\vec{r} + s\frac{d\vec{r}}{dt}$
3. $\frac{d}{dt}(\vec{a} \times \vec{b}) = \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}$
4. $\frac{d}{dt}(\vec{a} \times (\vec{b} \times \vec{c})) = \frac{d\vec{a}}{dt} \times (\vec{b} \times \vec{c}) + \vec{a} \times \left(\frac{d\vec{b}}{dt} \times \vec{c}\right) + \vec{a} \times \left(\vec{b} \times \frac{d\vec{c}}{dt}\right)$
5. $\vec{V} = \frac{d\vec{r}}{dt}$
6. $\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt}$

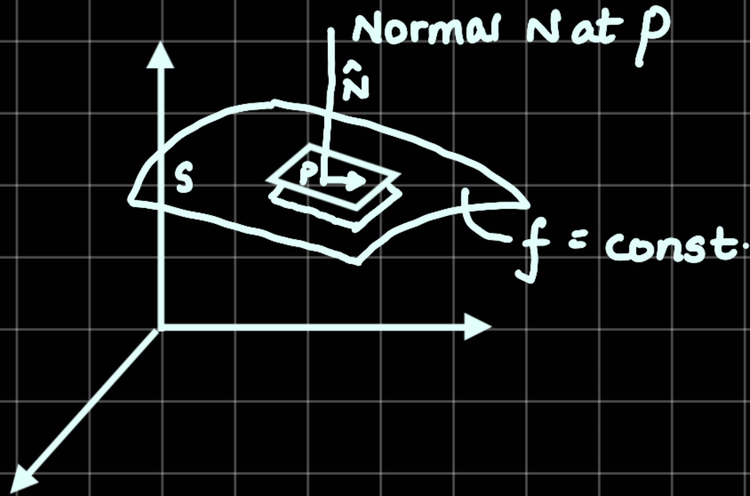
There are two types of functions:

1. Scalar point function
2. Vector point function.

→ **Gradient / Slope of scalar point functions** –

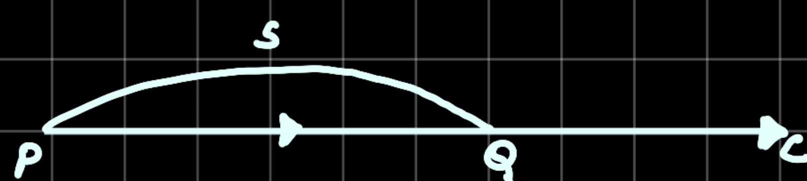
If $f(x,y,z)$ be a scalar function & continously diffrentiable then the vector:

$$\boxed{\text{grad} f = \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}}$$



$\vec{\nabla} f$ of scalar field of $f(x,y,z)$ at P is the vector normal to the surface $f = \text{const.}$ & the magnitude is equal to directional derivative $\frac{df}{dN}$ in that dir^n.

Directional directive →



$$f(s) = f(x,y,z) = f|x(s), y(s), z(s)|$$

$$\Rightarrow \frac{df}{ds} = \frac{df}{dx}\cdot\frac{dx}{ds} + \frac{df}{dy}\cdot\frac{dy}{ds} + \frac{df}{dz}\cdot\frac{dz}{ds}$$

Here, $\frac{df}{ds}$ is the directional derivative of f at P in the dir^n $\hat{b}$.

$$\frac{df}{ds} = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right)$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\hat{b}$$

$$= \nabla f\cdot\hat{b} \longrightarrow \boxed{\frac{df}{ds} = \nabla f\left[\frac{\vec{a}}{|\vec{a}|}\right]}$$

- **Properties of Gradient** -

1. $(\vec{a}\cdot\nabla)f = \vec{a}\cdot(\nabla f)$

2. $\phi(x,y,z) = c \longrightarrow \nabla\phi = 0$

3. $\nabla(f \pm g) = \nabla f \pm \nabla g$

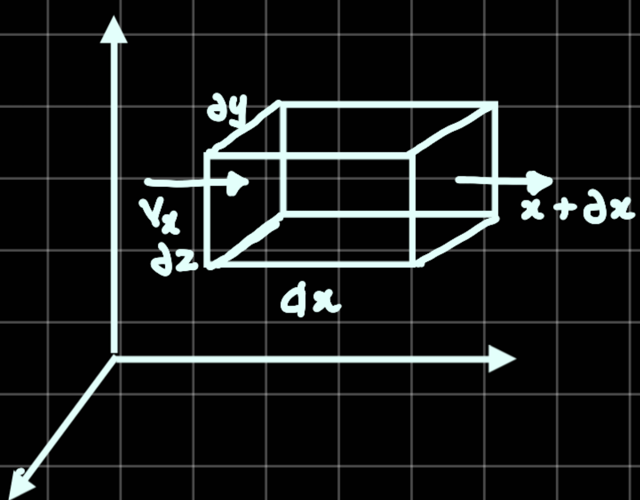
4. $\nabla(fg) = f\nabla g + g\nabla f$

5. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

→ Divergence of vector point function -

If $\vec{f}(x,y,z)$ is a continously differentiable vector point function then divergence of $\vec{f}$:

$$\boxed{\text{div}\,\vec{f} = \nabla\vec{f} = \frac{\partial \vec{f}}{\partial x}\hat{i} + \frac{\partial \vec{f}}{\partial y}\hat{j} + \frac{\partial \vec{f}}{\partial z}\hat{k}}$$



div$\vec{f}$ basically represent the rate of loss of function per unit time.

→ Curl of a vector -

If $\vec{f}$ is any continously diffrentiable vector point function then curl of $\vec{f}$ is :

$$\boxed{\text{Curl}\,\vec{f} = \vec{\nabla}\times\vec{f} = \frac{\partial \vec{f}}{\partial x}\times\hat{i} + \frac{\partial \vec{f}}{\partial y}\times\hat{j} + \frac{\partial \vec{f}}{\partial z}\times\hat{k}}$$

$$\text{curl}\,\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}$$

→ Vector Identities -

1. $\text{grad } uv = u \text{ grad } v + v \text{ grad } u$

2. $\text{grad}(\vec{a} \cdot \vec{b}) = \vec{a} \times \text{curl } \vec{b} + \vec{b} \times \text{curl } \vec{a} + (\vec{a} \cdot \nabla)\vec{b} + (\vec{b} \cdot \nabla)\vec{a}$

3. $\text{div}(u\vec{a}) = u \text{ div } \vec{a} + \vec{a} \cdot \text{grad } u$

4. $\text{div}(\vec{a} \times \vec{b}) = \vec{b} \times \text{curl } \vec{b} - \vec{a} \times \text{curl } \vec{b}$

5. $\text{curl}(u\vec{a}) = u \, \text{curl } \vec{a} + \text{grad } u \times \vec{a}$

6. $\text{curl}(\vec{a} \times \vec{b}) = \vec{a} \text{ div } \vec{b} - \vec{b} \text{ div } \vec{a} + (\vec{b} \cdot \nabla)\vec{a} - (\vec{a} \cdot \nabla)\vec{b}$

7. $\text{div grad } f = \nabla^2 f$

8. $\text{curl grad } f = 0$

9. $\text{div curl } \vec{f} = 0$

10. $\text{grad div } \vec{f} = \text{curl curl } \vec{f} + \frac{\partial^2 \vec{f}}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Vector Integration

Module - 7

- **Line integral**

Let $\vec{F}(\vec{r})$ be a continous vector point function, then $\int_C \vec{F}(\vec{r})\, d\vec{r}$ is known as the line integral of $\vec{F}(\vec{r})$ along C.

$$\int_C \vec{F}(\vec{r})\, d\vec{r} = \int_C (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$\Rightarrow \boxed{\int_C \vec{F}(\vec{r})\, d\vec{r} = \int_C [F_1\,dx + F_2\,dy + F_3\,dz]}$$

- **Surface integral**

Let $\vec{F}(\vec{r})$ be a continous vector point function. Let $\vec{F}(u,v)$ be a smooth surface such that $\vec{F}(u,v)$ possesses continous derivatives. Then the surface integral of $\vec{F}(\vec{r})$ over S :

$$\boxed{\int_S \vec{F}(\vec{r})\, dA = \int_S \vec{F}(\vec{r}) \cdot \hat{n}\, dS = \int_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)}$$

- **Volume integral** –

Let $\vec{f}(\vec{r})$ be a continous vector function & let a volume V be enclosed by a surface S.

$$\boxed{\iiint_V \vec{F}(\vec{r})\, d\vec{r} = \iiint_V [F_1\, dx\, dy\, dz + F_2\, dx\, dy\, dz + F_3\, dx\, dy\, dz]}$$

→ Green's Theorem

If C be a regular closed curve in the xy-plane bounding a region S and $P(x,y)$ and $Q(x,y)$ be continously diffrentiable functions and on C then

$$\boxed{\iint_C (P\,dx + Q\,dy) = \iint_S \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\,dy}$$

$$\boxed{\oint_C \vec{F}\,d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot k\,dS} \rightarrow \text{Vector form}$$

→ Stoke's Theorem

If $\vec{F}(\vec{r})$ is any continously diffrentiable vector functions and S is a surface enclosed by a curve C, then

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,dS}$$

$$\int_C F_1\,dx + F_2\,dy + F_3\,dz = \iint_S \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy\,dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz\,dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy.$$

Cartesian form of Stoke's Theorem

→ Gauss Divergence Theorem

If $\vec{F}$ is a continously diffrentiable vector point func. in a region V and S is the closed surface enclosing the region V, then

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V \text{div}\,\vec{F}\, dV}$$

$$\int_V \text{div}\,\vec{F}\, dV = \int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

Cartesian form of

Gauss Divergence Theorem