The last piece to calculate has the following expression

$$\begin{aligned}
\mathrm{tr}(\Phi_K \mathbb{M}_3 \Phi_B \mathbb{N}_3) &= \frac{\delta_{ik}\delta_{lj}}{N_c^2}\frac{f_K f_B}{16} m_B \cdot K \frac{4m_s}{m_a^2 + 2u m_B E}\left[-2E(\kappa_s - \kappa_S) m_B \Phi_+^B \Phi_K \bar{u}(2E)\right] \\
&= \frac{\delta_{ik}\delta_{lj}}{N_c^2}\frac{f_K f_B}{16} m_B \cdot K \frac{4m_s \bar{u}}{m_a^2 + 2u m_B E}(-4E^2)(\kappa_s - \kappa_S) m_B \Phi_+^B \Phi_K \\
&= \frac{-\lambda_t C_8}{2}\left(\frac{g_s^2}{4\pi^2}\right)\left(\frac{\delta_{ik}\delta_{lj}(t^a)^2}{N_c^2}\right)\frac{f_K f_B}{16} m_B^3 \cdot (4m_s)(4E) \\
&\qquad \frac{(\kappa_s - \kappa_S)}{f}\int d\omega \frac{\Phi_+^B}{\omega}\int du \frac{\Phi_K}{m_a^2 + 2u m_B E} \\
&= -\lambda_t C_8 \left(\frac{g_s^2}{4\pi^2}\right)\left(\frac{N_c^2 - 1}{2N_c^2}\right)\frac{f_K f_B}{4} m_B^2 \cdot \left(m_B^2 - m_a^2\right)\frac{(\kappa_s - \kappa_S)}{f}\frac{m_s}{\omega_o}\frac{\mathcal{I}}{m_B^2}
\end{aligned} \tag{6.82}$$

where the integral is defined as

$$\mathcal{I} = \int du \frac{6u(1-u)}{u + \bar{u}\cdot r} = \frac{3}{m_B^2}\left[\frac{r^2 - 2r\log(r) - 1}{(r-1)^3}\right] = \frac{3\mathbb{L}(r)}{m_B^2}, \qquad r = \frac{m_a^2}{m_B^2} \quad . \tag{6.83}$$

This integral is the same integral as for the case of ALP-gluon case (see Eq. 6.69).

$$\mathrm{tr}(\Phi_K \mathbb{M}_3 \Phi_B \mathbb{N}_3) = -\lambda_t C_8 \left(\frac{g_s^2}{4\pi^2}\right)\left(\frac{N_c^2 - 1}{2N_c^2}\right)\frac{f_K f_B}{4}\left(m_B^2 - m_a^2\right)\frac{(\kappa_s - \kappa_S)}{f}\frac{3m_s}{\omega_o}\left[\frac{r^2 - 2r\log(r) - 1}{(r-1)^3}\right] \tag{6.84}$$

Then, summing up the three pieces.

$$\begin{aligned}
\mathbb{B} = -\lambda_t C_8 \left(\frac{\alpha_s}{\pi}\right)\left(\frac{N_c^2 - 1}{2N_c^2}\right)\frac{f_K f_B}{4}\Bigg\{ & m_B^2 \frac{\mu_K}{\lambda_o}\frac{(\kappa_S - \kappa_B)}{f} + 3m_B^2 \frac{(2\kappa_S - \kappa_B - \kappa_b)}{f} \\
& + 3m_a^2 \frac{(\kappa_b - \kappa_B)}{f} + 3(m_B^2 - m_a^2)\frac{m_s}{\lambda_o}\frac{(\kappa_s - \kappa_S)}{f}\mathbb{L}\left(\frac{m_a^2}{m_B^2}\right)\Bigg\} .
\end{aligned} \tag{6.85}$$

The amplitude $\mathbb{B}$ at the end is of order one since the order λ terms in the NUMERATOR are compensated by the order λ term in the DENOMINATOR. To illustrate this, we see in Eq. 6.85 that in the first term the λ scale is canceled by μ_K and λ_o. The second and third term already simplified this scaling and in the fourth term the λ_o compensates the scaling of m_s.

6.4 Phenomenology

Once we have calculated the three contributions at tree-level from the Fig. 6.1, we should superpose them as a whole amplitude. The decay rate expression includes the integration of the phase space of a two body decay and is given by

$$\Gamma(B \longrightarrow K\ a) = \frac{|\mathbb{A}+\mathbb{B}+\mathbb{C}|^2}{16\pi m_B}\lambda^{1/2}\left(\frac{m_K^2}{m_B^2}, \frac{m_a^2}{m_B^2}\right), \tag{6.86}$$

where:

$$\lambda(u,v) = 1 - 2(u+v) + (u-v)^2. \tag{6.87}$$

where summarizing each contributions, we have

$$\begin{aligned}
\mathbb{A} &= \frac{G_F}{\sqrt{2}} \frac{\lambda_u(N_c C_1 + C_2) + (-\lambda_t)(N_c C_4 + C_3)}{N_c}(f_K f_B)\frac{(m_B^2 - m_a^2)}{2}\frac{(\kappa_b - \kappa_S)}{f}, \\
\mathbb{B} &= (-\lambda_t)\frac{G_F}{\sqrt{2}}C_8\left(\frac{\alpha_s}{\pi}\right)\left(\frac{N_c^2-1}{2N_c^2}\right)\frac{f_K f_B}{4}\left\{m_B^2\frac{\mu_K}{\omega_o}\frac{(\kappa_S - \kappa_B)}{f} + 3m_B^2\frac{(2\kappa_S - \kappa_B - \kappa_b)}{f}\right. \\
&\left. +3m_a^2\frac{(\kappa_b - \kappa_B)}{f} + 3(m_B^2 - m_a^2)\frac{m_s}{\omega_o}\frac{(\kappa_s - \kappa_S)}{f}\mathbb{L}(r)\right\}, \\
\mathbb{C} &= (-\lambda_t)\frac{G_F}{\sqrt{2}}C_8\left(\frac{g_s^2}{4\pi^2}\right)\left(\frac{N_c^2-1}{2N_c^2}\right)\left(\frac{f_B f_K}{4}\right)\left(\frac{C_g}{f}\frac{\alpha_s}{\pi}\right)(m_B)\frac{3}{2\omega_o}(m_B^2 - m_a^2)\mathbb{L}(r).
\end{aligned} \tag{6.88}$$

where during the calculation of every graph, we have omitted for simplicity the common factor $\frac{G_f}{\sqrt{2}}$ which we have brought it back now. The amplitude $\mathbb{A}$ represents the contribution from the four-fermion operators. Whereas, the amplitudes $\mathbb{B}$ and $\mathbb{C}$ represent the contribution from the dipole operator which in principle arise from integration out of the W boson in loops (see Fig. 3.2).

We can see in Fig. 6.3 that the biggest contribution is given by the four-fermion operator (see Fig. 6.1 (a)). On the other hand, the contribution of the amplitudes given by the dipole operator are around two order of magnitude and this can be explained because the effective dipole operator is suppressed by the α_s.

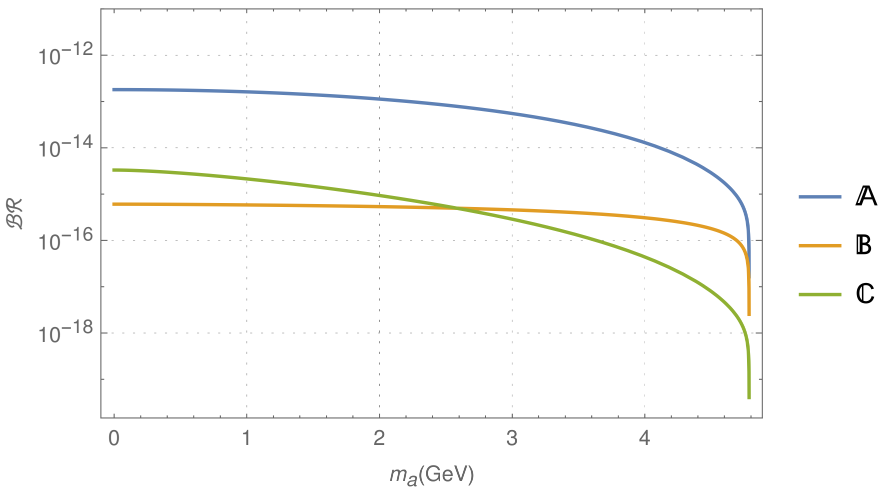

Figure 6.3: Contribution to the Branching Ratio of each graph at tree-level (see Fig. 6.1) in function on the ALP-mass. The assumption of the ALP-couplings being naturally order $\mathcal{O}(1)$ has been taken.

ALP couplings	m_a = 0 GeV	m_a = 1 GeV	m_a = 4 GeV
κ_S	$(1.07^{+0.15}_{-0.18}) \cdot 10^4$	$(1.07^{+0.22}_{-0.13}) \cdot 10^4$	$(3.70^{+0.53}_{-0.62}) \cdot 10^4$
κ_s	$(2.92^{+0.42}_{-0.49}) \cdot 10^6$	$(3.65^{+0.52}_{-0.61}) \cdot 10^6$	$(2.54^{+0.36}_{-0.43}) \cdot 10^7$
κ_B	$(2.08^{+0.30}_{-0.35}) \cdot 10^5$	$(2.10^{+0.30}_{-0.35}) \cdot 10^5$	$(2.76^{+0.39}_{-0.46}) \cdot 10^5$
κ_b	$(1.11^{+0.16}_{-0.19}) \cdot 10^4$	$(1.12^{+0.17}_{-0.20}) \cdot 10^4$	$(4.15^{+0.59}_{-0.70}) \cdot 10^4$
C_g	$(8.33^{+1.19}_{-1.40}) \cdot 10^4$	$(10.39^{+1.49}_{-1.74}) \cdot 10^4$	$(7.24^{+1.04}_{-1.21}) \cdot 10^5$

Table 6.1: Fitting values of the ALP-couplings at 1σ to the $\mathcal{BR}(B^+ \to K^+\nu\bar{\nu}) = (2.3 \pm 0.5(stat)^{+0.5}_{-0.4}(syst)) \times 10^{-5}$ [34].

Besides, one will expect that the amplitude $\mathbb{C}$ is notably suppressed by the orders in α_s since the ALP-gluon coupling carries itself one order of magnitude in α_s. However, due to the scale of this amplitude is inversely proportional to ω_o, at the end of the day it has bigger contributions in comparison to $\mathbb{B}$ for light ALPs. Clearly, we can see in Fig. 6.3 that the Branching Ratio ($\mathcal{BR}$) has an inverse dependence on the ALP-mass. In other words, heavy ALPs close to the threshold, given by the kinematic, offer small values of $\mathcal{BR}$ which will lead to large values to the ALP-coupling. On the other hand, light ALPs give larger $\mathcal{BR}$ which will lead smaller[1] ALP-couplings.

In order to calculate the decay rate of our process, we have to compute the square of the amplitude which bring itself terms proportional to ALP-couplings mixing. Obviously these mixing terms will bring complexity to the phenomenology and due to the several ALP-couplings in the game, we proceed to work in the ideal scenario where the ALPs couple to one quark only. It is relevant to stress that the assumption of this ideal scenario does not imply that the ALP will not couple to more quarks at other energy scales. On the other hand, due to the RG evolution of the ALP-couplings, non-zero ALP couplings will be introduced [24].

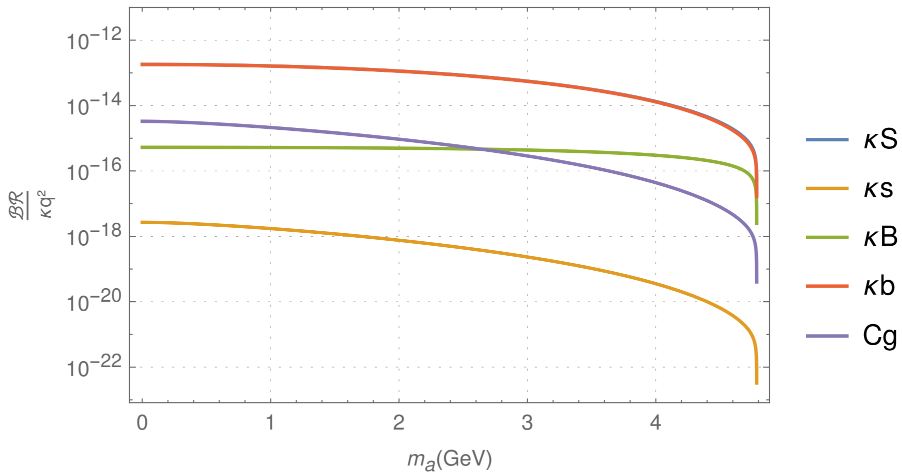

Figure 6.4: Dependency of the ratio between Branching ratio and ALP-coupling on the ALP-mass in the scenario where the ALPs couple to one quark at a time.

1 Still large but smaller in comparison to heavy ALPs

We can see in Fig. 6.4 that the ratio between the theoretical prediction of $\mathcal{BR}$ and the ALP-coupling in the one-coupling scenario. As expected the $\mathcal{BR}$ has an inverse dependency on the ALP-mass. In addition, the smallest ratio is given by the coupling between the ALP and the right-handed strange quark which will lead to a large κ_s. Besides, we can see that the contribution from κ_b and κ_S has unexpectedly the same dependency on the ALP-mass. In addition, the biggest ratio in Fig. 6.4 is given by these two ALP-couplings: κ_b and κ_S which will lead to the smallest ALP-couplings.

<table>
<tr><td colspan="6">Wilson Coefficients [15]</td></tr>
<tr><td>C_1</td><td>C_2</td><td>C_3</td><td>C_4</td><td colspan="2">C_8</td></tr>
<tr><td>1.08</td><td>-0.19</td><td>0.01</td><td>-0.04</td><td colspan="2">0.01</td></tr>
<tr><td colspan="6">Masses [GeV] [20]</td></tr>
<tr><td colspan="2">m_B</td><td>m_s</td><td colspan="2">m_K</td><td>m_u</td></tr>
<tr><td colspan="2">5.279</td><td>0.093</td><td colspan="2">0.493</td><td>0.002</td></tr>
<tr><td colspan="6">LCDA's parameters [GeV]</td></tr>
<tr><td colspan="2">f_K [35]</td><td>ω_o [33]</td><td>μ_K [20]</td><td colspan="2">f_B [36]</td></tr>
<tr><td colspan="2">0.160</td><td>0.46</td><td>2.55</td><td colspan="2">0.194</td></tr>
<tr><td colspan="6">Other parameters</td></tr>
<tr><td>λ_u [20]</td><td>λ_t [20]</td><td>f</td><td>α_s (at 5GeV) [20]</td><td>τ_{B^-} [GeV^{-1}] [20]</td><td>G_f[GeV^{-2}] [20]</td></tr>
<tr><td>(3.4 - i 7.4) $\cdot 10^{-4}$</td><td>-0.04</td><td>1 TeV</td><td>0.21 GeV</td><td>2.49 10^{12}</td><td>1.16 10^{-5}</td></tr>
</table>

Table 6.2: Input parameters used to calculate the Branching Ratio of the decay.