Phenomenology of Axion-like Particles in B meson Decays

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Declaration of authorship

I certify that I have written this thesis independently and have not used any sources, resources than those specified. All statements that were taken from other publications, either literally or analogously, are marked. I have not submitted the work in the same or a similar form to any other examination authority.

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ABSTRACT

Axions and Axion-like particles (ALPs) are predicted in many extensions of the Standard Model to address key puzzles in physics, including the nature nature of dark matter and the strong CP problem. ALPs, in particular, emerge as pseudo-Nambu–Goldstone bosons from the spontaneous breaking of a global $U(1)_X$ symmetry. Their weak interactions with ordinary matter and potential stability render them prime candidates for dark matter constituents. Consequently, several experiments have been proposed and conducted to probe their existence, exploring astrophysical observations, particle colliders, and rare meson decay processes.

Rare meson decays present a promising avenue for exploring ALPs' effects due to their sensitivity to new particles and interactions beyond the Standard Model. Recent efforts have focused on light ALPs in $K \longrightarrow \pi\ a$ decays, constraining both flavor-changing and flavor-conserving ALP interactions. However, for heavier masses, decays of heavy mesons such as D and B mesons become crucial, particularly for flavor-changing transitions. The realm of heavy ALPs with flavor-conserving interactions remains largely unexplored. Employing QCD factorization, this study delves into ALP masses ranging from MeV to GeV, aiming to fill gaps in current astrophysical and collider constraints.

This thesis establishes a comprehensive theoretical framework encompassing Effective Field Theories, Axion-like Particles, and non-perturbative inputs in QCD. By meticulously examining heavy-to-light meson transitions, we seek to unravel the intricate interplay between perturbative and non-perturbative dynamics, shedding light on the fundamental nature of ALPs and their potential implications for particle physics.

CONTENTS

1 INTRODUCTION

Axions and Axion-like particles (ALPs) are hypothetical particles proposed in the context of extensions to the Standard Model of particle physics. The concept of axion emerged from attempts to address certain outstanding questions in physics, such as the nature of dark matter and the strong CP problem [1]. The strong CP problem is a puzzle in theoretical physics related to the behavior of quarks under the strong force, one of the four fundamental forces of nature. The problem arises from the unnaturally small size of the θ_{QCD} term, which, if nonzero, would violate CP symmetry [2]. The axions offers a solution to this problem by introducing a global chiral $U(1)$ symmetry which is known as $U(1)_{PQ}$ symmetry and a corresponding scalar field, the axion, which dynamically adjusts the parameter responsible for CP violation in QCD, effectively suppressing CP violation at low energies [3,4].

On the other hand, ALPs are a generalization of axions. It is the pseudo-Nambu–Goldstone boson arises from the spontaneous symmetry breaking of a global $U(1)_X$ extension of the standard model symmetry group. The ALPs enjoy many of the properties of axion as: the shift symmetry and the way they couple to the particles of the standard model. Axions and ALPs are attractive candidates for dark matter because they possess properties that could make them compatible with observations of dark matter. For instance, their interaction with ordinary matter is suppressed by the large ALP decay constant which means that they interact feebly with matter. ALPs also are expected to be stable particles and could have been produced abundantly in the early universe through various mechanisms contribution partially or fully to the dark matter content [5,6]. For these reasons, the axion and ALPs have now become a favoured theoretical candidate for dark matter, motivating several experiments. Searches for axion and ALPs produced in the sun are studied in [7,8]; observations from astrophysical sources in specific of red giants [9] and supernovas [10]; searches closer to our interesting are B meson decays given by BaBar collaboration [11].

Rare meson decays are highly sensitive to the effects of new particles and interactions that may not be accessible in high-energy collider experiments. Since Axion-like particles are predicted to interact weakly with ordinary matter, their influence might manifest most prominently in such rare and subtle decay processes of mesons. Therefore, even small deviations from the expected decay rates could signal the presence of new physics, including the effects of ALPs. Rare meson decays can involve various types of mesons, including pseudo-scalar mesons (e.g., pions, kaons) and vector mesons (e.g., rho mesons). For instance, Examples of rare meson decays include processes such as rare Kaon decays which offer bounds for ALPs with masses $m_a < 300\text{MeV}$ [12]. In the present work, we aim to explore the window of ALP mass range between an MeV and several GeV, thus filling the gap between tight limits derived from astrophysical sources and beam dump experiments (for sub-MeV masses) and collider bounds (for multi-GeV masses) [13]. In order to probe ALP masses in the range of GeV, we should search for heavy meson decays such as B meson.

There are some studies and searches for heavy meson decays into light-meson and ALPs which explores the window of heavy ALP mass ranges. In the reference [13], the authors focused on B meson decays into light-meson mediated by the flavor-changing ALP-quark couplings. It is of our interesting to investigate this kind of process where the flavour-changing is given by the effective weak interactions in combination with ALP-quark flavor-conserving. This approach includes the use of QCD factorization of non-leptonic decays [14, 15]. B meson decays mediated by the combination of flavor-changing given by the ALP-quark and effective weak interactions are suppressed.

Heavy-to-light transition like the one under consideration here involve both perturbative and non-perturbative dynamics. Non-perturbative dynamics become important in certain aspects of B meson decays due to the non-perturbative nature of the strong force at low-energies (long distances), which governs the interactions between quarks and gluons within the meson. In order to compute the decay process, we will need some tools and ingredients: QCD Factorization, form factors, light-cone distribution amplitudes, and partonic amplitudes.

The details of QCD Factorization strongly depends on the number of mesons in the final state but in general heavy- to one light- meson transition can be divided in two pieces: factorizable and non-factorizable contributions. The former contains the contraction of the partonic amplitude (perturbative) times the form factor (non-perturbative) given a Dirac structure. On the other hand, the latter contains a convolution of the partonic amplitude with the light-cone distribution amplitudes of the heavy- and light-mesons.

The rest of the thesis is organized as follows: In Chapter 2 we introduce the standard model of particle physics, with particular focus on the non-perturbative dynamics in QCD. In Chapter 3 we introduce the concepts of Effective Field Theory which is indeed the fundamental concept underlying the search for new physics. After that, in chapter 4 we discuss Axion-like Particles within an effective field theory and how their interactions with the standard model of particles can be described. Chapter 5 is dedicated to the understanding of the QCD factorization of our process and introduces the tools needed to compute the hadronic matrix element. Finally, chapter 6 is where all the fundamental concepts we have introduced and built come together. This chapter is dedicated to the actual calculation of the decay process: $B \longrightarrow K\ a$ at the tree-level.

2 THEORETICAL FRAMEWORK

In this chapter, we will describe the Standard Model of particles physics from the Lagrangian formalism to the spontaneous symmetry breaking. We will also discuss the basics of Quantum Chromodynamics and non-perturbative inputs such as hadronic form factors and decay constants. The aim of this chapter is to build the basis background that we will need through the whole thesis.

2.1 Standard Model of particles

The Standard Model (SM) describes the electromagnetic, weak and strong interaction of the elementary particles in the framework of quantum field theory. It is a gauge theory based on the local symmetry group

$$G = SU(3)_C \times SU(2)_L \times U(1)_Y \; , \tag{2.1}$$

where the subscript C denotes the color; L, the left chirality and Y the weak hypercharge. The symmetry group determines univocally the interactions and the number of gauge bosons that coincides with the number of generator of the group. The symmetry group for Quantum Chromodynamics (QCD) is the special group $SU_(3)$ and the number of generators is $3^2 - 1 = 8$, therefore there are 8 massless gluons that carry the strong interaction. Analogously, for the weak and electromagnetic interaction, three and one gauge bosons respectively. Elementary particles are divided into two: fermions and bosons. The number of gauge bosons is determined by the symmetry group. Instead, the number of scalar bosons and fermions are chosen in a heuristic way. Scalar bosons are chosen to minimally implement the Higgs mechanism. As we will see later in this chapter, only one scalar boson will be needed to generate all the couplings necessary for the particles to gain mass after the SSB. The number of fermions and their properties are determined by the experiments . [16]

2.1.1 Lagrangiana del Modelo Estandar

The Lagrangian density of the SM of particles can be divided in four pieces:

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{\Psi} + \mathcal{L}_{\phi} + \mathcal{L}_{yuk} \; , \tag{2.2}$$

	First generation	Second generation	Third generation
quarks:	u(up)	c (charm)	t (top)
	d(down)	s (strange)	b(bottom)
leptons:	ν_e(electron neutrino)	ν_μ (muon neutrino)	ν_τ (tau neutrino)
	e(electron)	μ (muon)	τ (tau)

Table 2.1: Fermionic content in the Standard Model of Particles.

The first term, $\mathcal{L}_{gauge}$, corresponds to the kinetic term of the gauge bosons.

$$\mathcal{L}_{gauge} = -\frac{1}{4}G^i_{\mu\nu}G^{i\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W^{i\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\ , \tag{2.3}$$

where the tensor fields for $SU(3), SU(2)$ and $U(1)$ respectively are,

$$\begin{aligned} G^i_{\mu\nu} &= \partial_\mu G^i_\nu - \partial_\nu G^i_\mu - g_s f_{ijk} G^j_\mu G^k_\nu\ ;\ \ i,j,k = 1,...,8 \\ W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g_s \epsilon_{ijk} W^j_\mu W^k_\nu\ ;\ \ i,j,k = 1,2,3 \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu\ , \end{aligned} \tag{2.4}$$

For the non-abelian cases $G^i_{\mu\nu}$ and $W^i_{\mu\nu}$, there are self-interactions between the gauge bosons. Meanwhile, the gauge boson corresponding to $U(1)$ has no self-interactions as can be seen in **??**. The tensor fields G^i_μ, W^i_μ *and* B_μ transform under their respective symmetry groups in the following way.

$$\begin{aligned} G'^i_\mu &= G^i_\mu - f_{ijk}\beta^j G^k_\mu - \frac{1}{g_s}\partial_\mu \beta^i\ , \\ W'^i_\mu &= W^i_\mu - \epsilon_{ijk}\beta^j W^k_\mu - \frac{1}{g}\partial_\mu \beta^i\ , \\ B'_\mu &= B_\mu - \frac{1}{g'}\partial_\mu \beta. \end{aligned} \tag{2.5}$$

The second term in Eq. 2.2 is the Lagrangian of fermions that contains the kinetic term with the covariant derivative: $\partial_\mu \longrightarrow D_\mu$ in order to preserve gauge invariance.

$$\mathcal{L}_f = \sum_{m=1}^{F} (\overline{q}_{mL} i\not{D} q_{mL} + \overline{u}_{mR} i\not{D} u_{mR} + \overline{d}_{mR} i\not{D} d_{mR} + \overline{l}_{mL} i\not{D} l_{mL} + \overline{e}_{mR} i\not{D} e_{mR}) \tag{2.6}$$

The expression of the covariant derivative depends on the symmetry group under which the fields transform. It can be seen in Table 2.2 how every fermion field transform under the symmetry group of the SM.

$$D_\mu = \partial_\mu - ig_s\frac{\lambda^a}{2}G^a_\mu - \frac{ig}{2}\vec{\tau}\cdot\vec{W}_\mu - iY_\Psi g' B_\mu\ , \tag{2.7}$$

where the generators of $SU(3)$ in the fundamental representation are the Gell-Mann matrices, $\frac{\lambda}{2}$; the Pauli matrices, $\frac{\vec{\tau}}{2}$ for $SU(2)$, and Y is the hypercharge.

Matter	$SU(3)_c \times SU(2)_L \times U(1)_Y$
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3}, \mathbf{2}, 1/6)$
u_R	$(\mathbf{3}, \mathbf{1}, 2/3)$
d_R	$(\mathbf{3}, \mathbf{1}, \text{-}1/3)$
$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$(\mathbf{1}, \mathbf{2}, \text{-}1/2)$
e_R	$(\mathbf{1}, \mathbf{1}, \text{-}1)$
Φ	$(\mathbf{1}, \mathbf{2}, 1)$

Table 2.2: Quantum numbers of the field the SM group generators of the fermions and Higgs.

The gauge symmetry group of the Standard Model, does not allow for mass terms for the fermions and weak gauge bosons in the Lagrangian. In other words, the fermion lagrangian in Eq. 2.6 describes the massless fermions. This is of course, incompatible with reality, reason why this $SU(2)_L$ group needs to be somehow broken. This can be solved by having a spontaneously broken symmetry through the addition of a self-interacting complex scalar, the Higgs doublet Φ. Thus, we introduce the next term in the SM lagrangian in Eq. 2.2.

$$\mathcal{L}_\phi = (D^\mu \Phi)^\dagger D_\mu \Phi - V(\Phi) \, , \tag{2.8}$$

where the potential is $V(\Phi) = m^2 \Phi^\dagger \Phi + \lambda(\Phi^\dagger \Phi)$. The idea behind the spontaneous symmetry breaking is that the gauge and fermion fields get a mass after the Higgs doublet acquires its *vev*[1]. The covariant derivative of the Higgs contains the electroweak gauge bosons, thus the kinetic term of the Higgs after spontaneous symmetry breaking arise mass terms for the gauge bosons: $W^\pm$ and Z^o. On the other hand, in our Lagrangian construction so far, there is no interaction between the Higgs and the fermions. Therefore, we introduce the last term in Eq. 2.2.

$$\mathcal{L}_{Yuk} = -\sum_{i,j}^{F} Y^u_{ij} \bar{q}^o_{Li} \tilde{\Phi} u^o_{Rj} - \sum_{i,j}^{F} Y^d_{ij} \bar{q}^o_{Li} \Phi d^o_{Rj} - \sum_{i,j}^{F} Y^e_{ij} \bar{l}^o_{Li} \Phi e^o_{Rj} + h.c \, , \tag{2.9}$$

where Y^f, f= u,d are the Yukawa matrices. Similarly to the gauge boson mass, the mass term of the fermions arise after the spontaneous symmetry breaking: $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \longrightarrow SU(3)_c \otimes U(1)_Q$. The remaining symmetry does not have the chirality behaivor and fermion mass terms are allowed.

1 vacuum expectation value.

2.1.2 Cabibbo-Kobayashi-Maskawa Matrix

As we discussed above, the Higgs boson plays is crucial role in the game in order to explain the mass of the particles. It is of our interest to describe the SM Lagrangian after the symmetry breaking. From now on, we will focus more in the quark sector,

$$\mathcal{L}_{SM} \supset -\frac{v}{\sqrt{2}}\left(\bar{u}_{L,i}Y^{u}_{ij}u_{Rj} + \bar{d}_{L,i}Y^{d}_{ij}d_{Rj}\right) - \frac{g}{\sqrt{2}}\bar{u}_{L,i}\gamma^{\mu}W^{-}_{\mu}d_{L,i} + h.c. \tag{2.10}$$

The first term of the above Lagrangian comes from the Yukawa term in Eq. 2.9 and the second one comes from the fermionic kinetic term in Eq. 2.6. In principle, the Yukawa matrices are non-diagonal and a diagonalisation through a bi-unitary transformation is needed: $f^{o}_{L} = V^{f\dagger}_{L} f_{L}, \quad f^{o}_{R} = V^{f\dagger}_{R} f_{R}$ for $f = u, d$. The above Lagrangian in Eq. 2.10 need to be written in the new basis f^{o} which is known as the mass basis.

$$\mathcal{L}_{SM} \supset -\bar{u}^{o}_{L,i}M^{u}_{ij}u^{o}_{Rj} - \bar{d}^{o}_{L,i}M^{d}_{ij}d^{o}_{Rj} - \frac{g}{\sqrt{2}}\bar{u}^{o}_{L,i}\gamma^{\mu}W^{-}_{\mu}(V^{u}_{L}V^{d\dagger}_{L})_{ij}d^{o}_{L,i} + h.c. \tag{2.11}$$

Where two crucial definitions have been introduced: the diagonal mass matrices

$$M^{f} = \frac{v}{\sqrt{2}}V^{f}_{L}Y^{f}V^{f}_{R}, \; f = u, d, \tag{2.12}$$

and the Cabibbo-Kobayashi-Maskawa (CKM) matrix

$$V_{CKM} = V^{u}_{L}(V^{d}_{L})^{\dagger} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{2.13}$$

Since the CKM matrix is a product of two unitary matrices, then it is unitary as well.

$$V_{CKM}V^{\dagger}_{CKM} = \mathbb{1}_{F\times F}. \tag{2.14}$$

Therefore, it has many properties to satisfy.

$$\sum_{m=u,c,t} V_{mi}V^{*}_{mj} = \delta_{ij}, \quad i, j = d, s, b \tag{2.15}$$

$$\sum_{m=d,s,b} V_{in}V^{*}_{jn} = \delta_{ij}, \quad i, j = u, c, t. \tag{2.16}$$

and

$$\lambda^{(q)}_{u} + \lambda^{(q)}_{c} + \lambda^{(q)}_{t} = 0, \text{ where } \lambda^{(q)}_{p} = V_{pb}V^{*}_{pq}. \tag{2.17}$$

The interaction vertices in Eq. 2.11 is known as charged current and they have important consequences. The non-diagonal entries in the CKM matrix describes flavour-changing transition at tree-level given by the charged $W^{\pm}$ boson. The neutral current that describes the interaction of the Z^{o} and γ bosons with the quarks do not have an analogous CKM matrix because the matrix arises by the product of the bi-unitary transformation is one: $V = V_L^u(V_L^u)^\dagger = \mathbb{1}$. Thus, at tree-level does not contribute to the flavour-changing transitions; however, as we will discuss in Chapter 3 the neutral current can contribute to flavour-changing transition at loop level.

2.2 Quantum Chromodynamics

Between the three fundamental forces described by the Standard Model. Quantum Chromodynamics (QCD) is of our main interest. QCD is a gauge field theory that describes the strong interactions between quarks and anti-quarks with gluons. The gauge group under the fermion and gluons transform is the $SU(3)_C$, where C represents the color. The Lagrangian for QCD has been already described in Eq. 2.2 where the quarks transform under the fundamental representation of $SU(3)_C$ (see Table 2.2), meanwhile the gluons transform under the adjoint representation (see Eq. **??**).

2.2.1 Running of the QCD coupling

Similar to the QED [2] renormalization, the strong coupling obeys a Renormalization Group (RG) evolution. The RG evolution is determined by demanding that the physical observables should not depend of, the UV cutoff, the subtraction renormalization point, or the arbitrary scale μ in dimensional regularization.

$$\beta(\alpha_s(\mu)) = \mu \frac{d\alpha_s(\mu)}{d\mu}. \tag{2.18}$$

The treatment of renormalization in QCD is similar to the abelian case. However, due to the self interaction of the gluons and ghost particles, there are additional loop diagrams that contributes to the vacuum polarization of QCD. The RG evolution is given by the Gross-Wilczek-Politzer [19].

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_o)}{1 + \frac{\alpha_s(\mu_o)}{4\pi}\beta_o \log(\mu^2/\mu_o^2)}, \tag{2.19}$$

where $\beta_o = 11 - \frac{2}{3}N_f$ and N_f is the number of quark flavors. In order to obtain the given evolution of $\alpha_s(\mu)$ at a general scale μ, one requires the value of the coupling constant at a fixed scale. The most common value of reference comes from Z decays [20].

$$\alpha_s(m_Z) = 0.1179 \pm 0.0010 \tag{2.20}$$

2 We have not discussed the renormalization of QED but it could be found in many standard books [17,18]

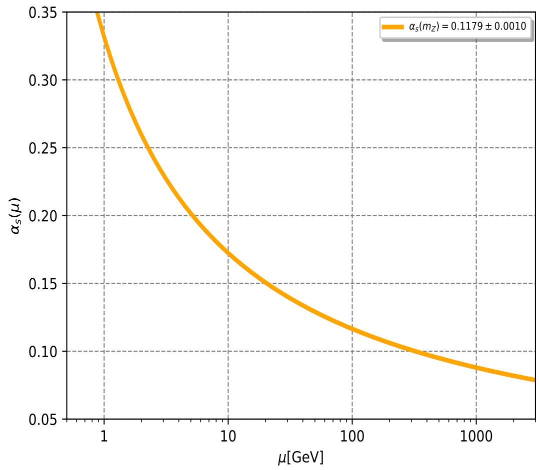


Figure 2.1: Running coupling of the strong force given by the evolution in Eq. 2.19.

Within the SM of particles, we have the upper constrain that the number of quark flavors is six. Thus, β_o is positive, giving a peculiar RG evolution as we can see in Fig. 2.1. As we can see in Fig. 2.1 the coupling constant grows (decreases) as we go to lower (higher) energies, it means that the strong interaction is stronger (weaker) at longer (shorter) distances. For instance, the value of $\alpha_s \sim 0.5$ GeV for values of the renormalization scale, $\mu \sim 1$ GeV. This puts at risk the convergence of the perturbative expansion. In general, we can define an scale called $\mu = \Lambda_{QCD}$ where the strong coupling diverges and this is when the denominator of the evolution of the strong coupling vanishes.

$$\frac{\alpha_s(\mu_o)}{4\pi}\beta_o \log\left(\Lambda^2_{QCD}/\mu_o^2\right) = -1. \tag{2.21}$$

In other words, at this scale $\mu \sim \Lambda_{QCD}$ the strong coupling is too large that we start entering to the so-called non-perturbative regime of QCD.

2.3 Non-perturbative dynamics

As we discussed above, QCD can not be treated in a fully perturbative way, which means that we need to introduce non-perturbative inputs to describe QCD at low energies. In the hadronic decay processes of our interests that are given by effective weak interaction, the amplitude will generally require the parametrization of hadronic matrix elements, of a generic current insertion, with the vacuum or hadrons as external states. We aim to shortly introduce the definition of the decay constants, hadrons form factors in particular of heavy-to light-pseudoscalar-meson.

2.3.1 Meson decay constants

The decay constant of a meson, denoted by f_M quantifies the amplitude for the transition of a meson state into the vacuum state through a given current insertion. For a pseudoscalar-meson M with quark-constituents $(M = (q\bar{q}_1, q_2))$, the decay constant is defined by

$$\langle 0| \bar{q}_1(0)\gamma_\mu\gamma_5 q_2(0) |M(p)\rangle = ip_\mu f_M, \tag{2.22}$$

where p_μ is the four-momentum of the meson. Since the decay constant is a physical observable, then it should be independent of the renormalization scale.

2.3.2 Hadron Form Factors

Another quantity that often appears in the parametrization of hadronic matrix elements is the hadronic form factors. Generally, a form factor describes the non-perturbative dynamics contained in the matrix element for a given current insertion in between two hadron states

$$\langle M(k)| \bar{q}_1\Gamma^\mu q_2 |B(p)\rangle \equiv F^\mu_{B\to M} = \sum_i V_i^\mu f_i(q^2) \tag{2.23}$$

where Γ^μ is a generic Dirac structure and V_i^μ contains the most general Lorentz decomposition for a given Γ^μ. Besides, form factors are rather universal quantities and factorize from the more specific short-distance dynamics.

2.3.2.1 $B \longrightarrow M$ form factors

We are interested in the form factors of a B meson decay into pseudoscalar light-meson

$$\langle M(k)| \bar{q}b \left|\bar{B}(p)\right\rangle = \frac{m_B^2 - m_M^2}{m_b - m_q} f_o, \tag{2.24}$$

$$\langle M(k)| \bar{q}\gamma_\mu b \left|\bar{B}(p)\right\rangle = f_+ \left[p^\mu + k^\mu - \frac{m_B^2 - m_M^2}{q^2} q^\mu\right] + f_o(q^2)\frac{m_B^2 - m_M^2}{q^2} q^\mu. \tag{2.25}$$

$$\langle M(k)| \bar{q}\sigma_{\mu\nu}q_\nu b \left|\bar{B}(p)\right\rangle = \frac{if_T(q^2)}{m_B + m_M} \left[q^2(p^\mu + k^\mu) - (m_B^2 - m_M^2)q^\mu\right] \tag{2.26}$$

where m_B and m_M are the masses of the B meson and the pseudoscalar meson respectively and the momentum transfer: $q = p - k$. It seems to be that the form factor exhibits a singularity when the transfer momentum is zero; however, this is removed by

$$f_+(0) = f_o(0). \tag{2.27}$$

3

EFFECTIVE FIELD THEORIES

Effective Field Theories (EFTs) are widely used in particle physics and beyond. In this chapter, we review the concepts, techniques, and applications of the EFT framework. We will offer a few illustrative examples, such as the Fermi theory, the Effective Weak Interactions, the Heavy-Quark Effective Theory, and the Soft-Collinear Effective Theory. This chapter follows the structure in [21].

3.1 Idea of Effective Theories

The fundamental concept underlying Effective Field Theories (EFTs) revolves around the notion that phenomena tend to simplify when observed from a certain distance. To illustrate this idea we can remember the approach to study the potential in electrodynamics. For instance, an arrangement of electric charges confined within a region of space with a characteristic size approximately equal to L. When analyzing their effects from a distance "r" significantly greater than L, the intricate details of the charge distribution become less significant. Instead, we can simplify the system by considering only a handful of terms in the multipole expansion: the total charge, the dipole moment, the quadrupole moment, and so forth. The accuracy of our simplification is governed by the ratio between these two parameters (L/r) raised to the power of n, where n represents the number of multipoles we choose to include.

We can implement this magnificent idea to field theory. Let us consider a theory with a large fundamental scale Λ. It could be the mass of a heavy particle or the energy where a symmetry breaking happens, etc... Let us suppose that we are interested on phenomenology at energies much less than the large scale: $E << \Lambda$. We will explain a guideline with three steps to build a Low-energy Theory, which is an EFT, from a High-energy theory which is a called an UV theory.

- Choose a cutoff:
  First, we need to choose an energy scale $\mu < \Lambda$ which is our the threshold of ignorance. We can split up our field into high- and low-frequency modes.

$$\Phi = \Phi_L + \Phi_H \tag{3.1}$$

where our whole field is

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(e^{-ik\cdot x} a_{\boldsymbol{k}} + e^{ik\cdot x} b^{\dagger}_{\boldsymbol{k}} \right). \tag{3.2}$$

We are interested in the physics given by the low-frequency field: Φ_L and its correlation function are given by

$$\langle 0| \, T \left\{ \Phi_L(x_1) ... \Phi_L(x_n) \right\} |0\rangle = \frac{1}{Z[0]} \left(-i \frac{\delta}{\delta J_L(x_1)} \right) ... \left(-i \frac{\delta}{\delta J_L(x_n)} \right) Z[J_L] \mid_{J_L=0}, \tag{3.3}$$

where the generating functional of the theory is

$$Z[J_L] = \int \mathcal{D}\Phi_L \mathcal{D}\Phi_H \exp\Big(i \; S(\Phi_L, \Phi_H) + i \int d^D x J_L(x) \Phi_L(x) \Big). \tag{3.4}$$

- Integrating out:

  The next step is to integrate out the high-frequency modes.

$$Z[J_L] = \int \mathcal{D}\Phi_L \exp\Big(i \; S_\mu(\Phi_L) + i \int d^D x J_L(x) \Phi_L(x) \Big) \tag{3.5}$$

  where

$$\exp(i \; S_\mu(\Phi_L)) = \int \mathcal{D}\Phi_H \exp(i S(\Phi_L, \Phi_H)) \tag{3.6}$$

  We see that our generating functional now depends only on the low-frequency fields after integrating out the heavy fields. It is important to stress that S_μ is non-local on scales $\delta x^\mu \propto \frac{1}{\mu}$ because there are no longer high-frequency modes.

- Operator-product-expansion.

  Last step is to expand the non-local action in function of local operators. This process is called operator-product expansion (OPE).

$$S_\mu(\Phi_L) = \int d^D x \mathcal{L}^{eff}_\mu(x), \tag{3.7}$$

  where

$$\mathcal{L}^{eff}_\mu(x) = \sum_i g_i \mathcal{Q}_i(\Phi_L(x)). \tag{3.8}$$

  This object is called the "effective Lagrangian". It is an infinite sum over local operators $\mathcal{Q}_i$ multiplied by coupling constants g_i , which are called Wilson coefficients. In general, all operators allowed by the symmetries of the theory are generated in the construction of the effective Lagrangian and appear in this sum.

- Power counting:

In principle, the operator-product expansion can give an infinite operators in the serie and the question if all the operators contributes might arise. Therefore an EFT should come with a set of power counting rules which allow us to distinguish which operators are important in the low energy and at which order the expansion should be truncated. Thus, we introduce the "naive dimensional analysis". Let us define the mass dimension of a Wilson coefficient as $[g_i] = -\gamma_i$. Then, the coefficient can be written as

$$g_i = C_i \, \Lambda^{-\gamma_i}, \tag{3.9}$$

with dimensionless coefficients C_i. We naturally assume that these coefficients are order one since the only fundamental scale in the theory is Λ. Then, at low energy: $E << \mu < \Lambda$, the contribution of an effective operator $\mathcal{Q}_i$ to an observable is expected to scales as the following

$$C_i \left(\frac{E}{M}\right)^{\gamma_i} = \begin{cases} \mathcal{O}(1), \ \gamma_i = 0, \\ << 1, \ \gamma_i > 0, \\ >> 1, \ \gamma_i < 0, \end{cases} \tag{3.10}$$

It yields that the operators whose coupling with dimension $\gamma_i < 0$ are the ones important in the effective theory.

3.1.1 Fermi Theory

As an illustration of these steps, we discuss the Fermi theory. The fermi theory is a low-energy effective theory of the Standard Model below the W boson mass scale. At this scale, the Z boson and the Higgs are also integrated out. The relevant degrees of freedom that are considered in the effective Lagrangian are the gluon, photon, leptons and quarks.

Let us study the muon decay process: $\mu^-(p) \longrightarrow e^-(k_1)\bar{\nu}_e(k_2)\nu_\mu(k_3)$. Within the Standard Model of particles, this decay is mediated by the charged current interaction.

$$\mathcal{L}_{SM} = \frac{g}{\sqrt{2}}(\bar{\nu}_\mu \bar{\sigma}_\rho \mu + \bar{\nu}_e \bar{\sigma}_\rho e) W^+_\rho + \text{h.c.}, \tag{3.11}$$

where g is the coupling of the weak interaction. The amplitude of this decay reads,

$$\mathcal{M} = \frac{g^2}{2} \nu_\mu \bar{\sigma}_\rho \mu \frac{1}{q^2 - m_W^2} \bar{e} \bar{\sigma}_\rho \nu_e \tag{3.12}$$

The first step is to find the large fundamental scale, in this case is the mass of the W boson: $m_W \approx 80 \ GeV$. In the physical process, the momentum transfer is small in comparison to the large scale: $0 < q^2 < m_\mu^2$. It yields $q^2/m_W^2 < 10^{-6}$. Then, we can expand the propagator in powers of q^2/m_W^2. This is an alternative way of integrating out the heavy modes.

$$\mathcal{M} \approx -\frac{g_L^2}{2m_W^2}\left[\nu_\mu \bar{\sigma}_\rho \mu\right]\left[\bar{e}\bar{\sigma}_\rho \nu_e\right]\left[1 + \mathcal{O}(q^2/m_W^2)\right]. \tag{3.13}$$

This amplitude up to leading order does not have the pole corresponding to the W boson propagation. We can reproduce this amplitude by means of an effective lagrangian with a local four-fermion operator.

$$\mathcal{L}_{EFT} \supset \frac{C}{\Lambda^2}(\bar{n}_\mu \bar{\sigma}_\rho \mu)(\bar{e}\bar{\sigma}_\rho \nu_e) + h.c. \tag{3.14}$$

Where $C = \frac{-g_L^2}{2}$ is the Wilson coefficient and the new scale is $\Lambda = m_W$.

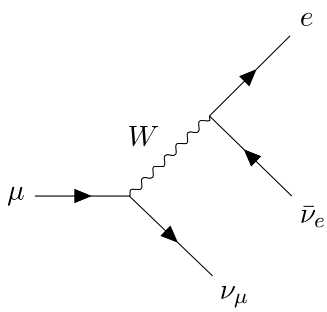


Figure 3.1: Diagrammatic representation of the muon decay: $\mu \longrightarrow e + u + d$.

3.2 Effective Weak Interactions

We have used the Fermi theory to describe the muon decay but this effective theory can be generalized to describe many more processes, for example transitions mediated by the weak interaction. We have integrated out the heavy weak boson "informally"; however, it can be integrated out by the formalism of path integrals and strictly follow the described steps to build an effective theory. The Lagrangian in the Standard Model that describes the weak decays is given by the charged current Lagrangian.

$$\mathcal{L}_W = -\frac{-1}{2}(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)(\partial^\mu W^{-\mu} - \partial^\nu W^{-\mu}) + m_W^2 W_\mu^+ W_\mu^- + \frac{g_2}{2\sqrt{2}}(j_\mu^+ W^{+\mu} + j_\mu^- W^{-\mu}), \tag{3.15}$$

in unitary gauge where the current is described by

$$j_\mu^+ = \sum_{i,j} V_{ij}\bar{u}_i \gamma_\mu (1-\gamma_5) d_j + \sum_i \bar{\nu}_i \gamma_\mu (1-\gamma_5) e_i \tag{3.16}$$

$$j_\mu^- = (j_\mu^+)^\dagger \tag{3.17}$$

We can rewrite this Lagrangian in a convenient way

$$\int d^4x\mathcal{L}_W(x) = \int d^4xd^4yW^+_\mu(x)K^{\mu\nu}W^-_\nu + \frac{g_2}{2\sqrt{2}}\int d^4x\left[j^+_\mu(x)W^{+\mu} + j^-_\mu(x)W^{-\mu}(x)\right] \tag{3.18}$$

where

$$K^{\mu\nu} = \delta^{(4)}\delta(x-y)\left[g^{\mu\nu}(\Box_y + m_W^2) - \partial^\mu_y\partial^\nu_y\right] \tag{3.19}$$

It can be proven that the inverse of this operator is the vector boson propagator and using the Gaussian integral, the generating function take the following expression

$$\begin{aligned} Z_{weak} &\propto \int \mathcal{D}W^+\mathcal{D}W^- \exp\left(i\int d^4x\mathcal{L}_W(x)\right), \\ &\propto \exp\left(-i\frac{g_2^2}{8}\int d^4xd^4yj^{-\mu}(x)\Delta_{\mu\nu}(x,y)j^{+\nu}(y)\right) \end{aligned} \tag{3.20}$$

At this point we can expand the boson propagator for momentum less than the "W" mass: $|k^\mu| << m_W$.

$$\begin{aligned} \Delta_{\mu\nu}(x,y) &= \int \frac{d^4k}{(2\pi)^4}\exp(-ik\cdot(x-y))\left(\frac{g_{\mu\nu}}{m_W^2} + \frac{g_{\mu\nu}k^2 - k_\mu k_\nu}{m_W^4} + ...\right) \\ &= \left[\frac{g_{\mu\nu}}{m_W^2} + \frac{1}{m_W^4}(-g_{\mu\nu}\Box + \partial_\mu\partial_\nu) + ...\right]\delta^{(4)}(x-y) \end{aligned} \tag{3.21}$$

We can see that this series expansion in the propagator offers infinite operators which is exactly the Operator Product Expansion. Therefore, the effective Lagrangian we find is

$$\mathcal{L}_{eff} = \mathcal{L}^{light}_{SM}(x) - \frac{g_2^2}{8m_W^2}\left[j^-_\mu j^{+\mu} + \frac{1}{m_W^2}j^-_\mu(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)j^+_\nu(x) + ...\right], \tag{3.22}$$

where light Lagrangian contains the lighter fields with mass less than the W mass. We also have introduced the Fermi constant.

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8m_W^2}. \tag{3.23}$$

For instance, for the decay process of our interest which is the B meson into Kaon has a flavor changing: $b \longrightarrow s\ \bar{u}\ u$. This interaction can be described by the following Lagrangian.

$$\mathcal{L}_{eff} = -\frac{G_F}{\sqrt{2}}V_{us}V^*_{ub}C_1\ \bar{s}^i\gamma_\mu(1-\gamma_5)u^i\ \bar{u}^j\gamma_\mu(1-\gamma_5)\ b^j \tag{3.24}$$

We can see that the dimension of this operator is six because of the four spinors. This term is the first contribution of the OPE, but is it not the only one at tree-level. Derivatives are not allowed since this a four-fermion process and derivatives will give higher dimension to the operator. The left-handed projectors between the spinors come from the chirality nature of the weak interaction. The color indices has to be contracted at the end to have color singlet, there is another option where we permute the color indices.

$$\mathcal{L}_{eff} = -\frac{G_F}{\sqrt{2}} V_{us} V_{ub}^* \{ C_1 \; \bar{s}^i \gamma_\mu (1-\gamma_5) u^i \; \bar{u}^j \gamma_\mu (1-\gamma_5) \; b^j + \\ C_2 \; \bar{s}^i \gamma_\mu (1-\gamma_5) u^j \; \bar{u}^j \gamma_\mu (1-\gamma_5) \; b^i \} \tag{3.25}$$

For a general process where the weak interaction gives a flavour changing decay in the quark sector, there are more operators apart of these two, which are customarily written as:

$$\begin{aligned} \mathcal{Q}_1^{(p)} &= (\bar{s}_i p_i)_{V-A} \, (\bar{p}_j b_j)_{V-A} \,, \quad p = u, c \\ \mathcal{Q}_2^{(p)} &= (\bar{s}_i p_j)_{V-A} \, (\bar{p}_j b_i)_{V-A} \,, \quad p = u, c \end{aligned} \tag{3.26}$$

The "QCD penguin operators".

$$\begin{aligned} \mathcal{Q}_3 &= (\bar{s}_i b_i)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_j q_j)_{V-A} \\ \mathcal{Q}_4 &= (\bar{s}_i b_j)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_j q_i)_{V-A} \\ \mathcal{Q}_5 &= (\bar{s}_i b_i)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_j q_j)_{V+A} \\ \mathcal{Q}_6 &= (\bar{s} i b_j)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_j q_i)_{V+A} \\ \mathcal{Q}_{7\gamma} &= \frac{e m_b}{8\pi^2} \bar{s} \; \sigma^{\mu\nu} (1+\gamma_5) b F_{\mu\nu} \\ \mathcal{Q}_{8g} &= \frac{g_s m_b}{8\pi^2} \bar{s}_i \; \sigma^{\mu\nu} (1+\gamma_5) t^a_{ij} b_j G^a_{\mu\nu} \end{aligned} \tag{3.27}$$

Where the convention of the signs are following the reference [22]. Besides, we have used the short-hand convention that

$$(\bar{q}_1 q_2)_{V\pm A} \equiv \bar{q}_1 \gamma_\mu (1 \pm \gamma_5) q_2 \quad , \quad \sigma_{\mu\nu} = \frac{i}{2} \left[\gamma_\mu, \gamma_\nu \right] . \tag{3.28}$$

The operators $\mathcal{Q}_1$ and $\mathcal{Q}_1$ are generated from the graphs of type (a) in Fig. 3.2. The gluon propagator in the second graph of type (a) permute the color indices between the spinors. The operators $\mathcal{Q}_{3,\ldots,6}$ are generated from the graphs type (b) which are the so-called QCD penguin operators. In addition, the operator $\mathcal{C}_{7,8}$ are generated by the graph of type (d)

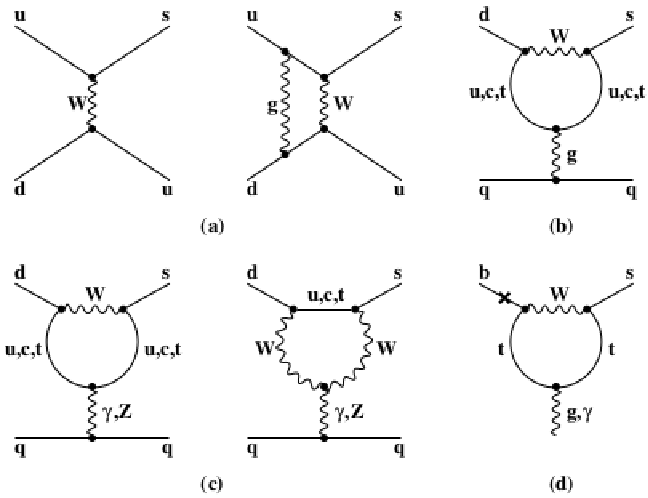


Figure 3.2: Diagrammatic graphs in the Standard Model which generate, after integrating out the W mass, the several operators in the Effective Weak Theory. This image was taken from [22]

Certain characteristics of the Standard Model of particles are implicitly addressed in the above discussions. These include the restriction that only left-handed fields participate in flavor-changing weak interactions, the identical coupling of light (nearly massless) quarks in strong interactions, and the identical coupling of both up-type (u, c) and down-type (d, s, b) quark fields to the weak force. The unitarity of the CKM matrix implies $\lambda_u + \lambda_c + \lambda_t = 0$, where $\lambda^p = V_{pb}V^*_{ps}$. We will use this relation to eliminate CKM factors involving couplings of the top quark. The final result for the effective weak Lagrangian reads

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}}\lambda^p \left[\sum_{p=u,c} \left(C_1 Q_1^{(p)} + C_2 Q_2^{(p)} \right) + \sum_{i=3,\ldots,8} C_i Q_i \right] \tag{3.29}$$

3.3 Heavy-Quark Effective Theory

It is of our interest to describe the B meson decays and heavy-quark mesons offer relevant opportunities for employing Effective Field Theory (EFT) techniques due to the significant hierarchy $m_b >> \Lambda_{QCD}$, facilitating a natural separation of scales. Physics at the scale of m_b primarily involves non-perturbative dynamics in QCD, whereas heavy-quark systems inherently encompass aspects of hadronic physics due to the confinement behavior of quark at scales larger than Λ_{QCD}. Effectively separating the short-distance and long-distance effects associated with these scales is indispensable for any precise characterization in heavy-quark physics.

The heavy bound state, made up of a light quark system and one heavy quark, is moving with velocity v_μ, then the 4-momentum of the bound state, in a reference frame in which the heavy meson is at rest, is

$$p_Q^\mu = M_Q v^\mu = m_Q v^\mu + k^\mu, \tag{3.30}$$

where "v" is the 4-velocity of the bound state ($v^2 = 1, v^o > 0$) and the $k \approx \Lambda_{QCD}$ is the residual momentum. The mass M_Q of the bound state is essentially the same as the mass of the heavy quark m_Q. Clearly, $k^\mu = (M_Q - m_Q)v^\mu$ which is order λ. It is also understood from the expression of the momentum that the heavy quark is nearly on-shell. This off-shell momentum results from the interaction with the light-quark. A near on-shell Dirac spinor can be descomposed into two large and two small components

$$Q(x) = e^{-im_Q v\cdot x}\left[h_v(x) + H_v(x)\right], \tag{3.31}$$

where

$$h_v(x) = e^{im_Q v\cdot x}\frac{1+\not{v}}{2}Q(x), \tag{3.32}$$

$$H_v(x) = e^{im_Q v\cdot x}\frac{1-\not{v}}{2}Q(x). \tag{3.33}$$

are the large (upper) components and small (lower) components respectively and satisfy the following relations.

$$\not{v}h_v(x) = h_v(x) \quad , \quad \not{v}H_v(x) = -H_v(x) \tag{3.34}$$

Inserting the large and small components in the Dirac Lagrangian,

$$\begin{aligned}\mathcal{L}_Q &= \bar{Q}\left(i\not{D} - m_Q\right)Q \\ &= \bar{h}_v i\not{D} h_v + \bar{H}(i\not{D} - 2m_Q)H_v + \bar{h}_v i\not{D} H_v + \bar{H}_v i\not{D} h_v \\ &= \bar{h}_v iv\cdot D\; h_v + \bar{H}_v(-iv\cdot D - 2m_Q)H_v + \bar{h}_v i\vec{\not{D}} H_v + \bar{H}_v i\vec{\not{D}} h_v.\end{aligned} \tag{3.35}$$

Where $i\vec{\not{D}}^\mu = iD^\mu - v^\mu iv\cdot D$ is the spatial covariant derivative. We can compute the equation of motions from this Lagrangian, using the Euler-Lagrange equation for $\bar{H}_v$ we find

$$\begin{aligned}\frac{\delta\mathcal{L}_Q}{\delta\bar{H}_v} = 0 \longrightarrow -\left(iv\cdot D + 2m_Q\right)H_v + i\vec{\not{D}}h_v = 0 \\ \left(iv\cdot D + 2m_Q\right)H_v = i\vec{\not{D}}h_v.\end{aligned} \tag{3.36}$$

Since there are no derivatives in the Lagrangian, then the first term of the Euler-Lagrange equation is the one non-vanishing. Solving this equation, we can write the large components in function of the small components in order to integrate it out.

$$H_v = \frac{1}{2m_Q + iv\cdot D}i\vec{\not{D}}h_v = \frac{1}{2m_Q}\sum_{n=0}^{\infty}\left(-\frac{iv\cdot D}{2m_Q}\right)^n i\vec{\not{D}}h_v \tag{3.37}$$

Where we have used the fact that the mass of the heavy quark is larger in comparison to $v \cdot D$ and we have expanded in series of $\frac{v \cdot D}{2m_Q}$. Besides, we can replace this equality into the Lagrangian and integrate out the large components.

$$\mathcal{L}_{HQET} = \bar{h}_v iv \cdot D_s h_v + \mathcal{O}\left(\frac{1}{m_Q}\right), \tag{3.38}$$

$$\mathcal{L}_{HQET} = \bar{h}_v (iv^\mu \partial_\mu + gT_a v^\mu A^a_\mu) h_v + \mathcal{O}\left(\frac{1}{m_Q}\right), \tag{3.39}$$

Where $iD^\mu_s = i\partial^\mu + g_s A^\mu_s$ contains only the soft gluon field because hard gluons have been already integrated out. This leading term in the HQET Lagrangian exhibits a spin-flavor symmetry and it manifestly invariant under rotations in the flavor space. Regarding the spin symmetry, since there are v^μ structure instead of Dirac matrices, interactions of the heavy with gluons will not affect the spin. One can define a spin operator in this framework: $S^i = \frac{1}{2}\gamma_5 \not{v} \not{e}^i$ which satisfies the SU(2) algebra and commute with the four-velocity. An infinitesimal transformation in the field: $h_v \to (1 + i\boldsymbol{\alpha} \cdot \boldsymbol{S}) h_v$ will leave the Lagrangian invariant since the spin operator commutes with $v \cdot D$. It also preserves the on-shell condition of the small component: $\not{v}(1 + i\boldsymbol{\epsilon} \cdot S) h_v = h_v$. Therefore, in the infinite mass limit, the properties of hadronic systems containing a single heavy quark are insensitive to the spin and flavor of the heavy quark [23].

The Lagrangian contains terms up to $\mathcal{O}(1)$ but we are able to include power corrections by taking higher-order terms in the expansion.

$$\mathcal{L}_{HQET} = \bar{h}_v iv \cdot D_s h_v + \frac{1}{2m_Q}\left[\bar{h}_v (i\vec{\not{D}})^2 h_v + C_{mag}(\mu) \frac{g_s}{2} \bar{h}_v \sigma_{\mu\nu} G^{\mu\nu}_s h_v\right] + \mathcal{O}\left(\frac{1}{m_Q^2}\right). \tag{3.40}$$

The first correction is the kinetic operator and the second one is the chromomagnetic interaction. Note that the HQET Lagrangian up to $\mathcal{O}(1/m_Q)$ do not longer respect the spin-flavor symmetry because of the chromomagnetic operator which contains a more complicated Dirac structure that do not commute with the spinor.

3.3.1 Scaling of the fields

It is convenient to insert the power counting $\lambda = \frac{\Lambda_{QCD}}{m_Q}$ to the fields and operators. In order to do that we assign the residual moment to be order of $\Lambda_{QCD} = \lambda m_Q \propto \lambda$. Since the hard fields have been integrated out, there are only soft fields which carry a momentum of order λ. We can find the scaling of the small components by considering the two-point correlation function in position space.

$$\langle 0 | T\{h_v(x) \bar{h}_v(0)\} | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \frac{i}{v \cdot k + i\epsilon} \propto \lambda^4 \cdot \frac{1}{\lambda} \propto \lambda^3, \tag{3.41}$$

from which we conclude that the small component follows the scaling: $h_v \propto \lambda^{3/2}$. For soft gluons, we follow the same procedure.

$$\langle 0|\, T\{A_s^\mu(x)A_s^\nu(0)\}\, |0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{-ig_{\mu\nu}}{k^2+i\epsilon} \propto \lambda^4 \cdot \frac{1}{\lambda^2} \propto \lambda^2. \tag{3.42}$$

from which we conclude that the soft gluon follows that $A_s^\mu \propto \lambda$. Then, the covariant derivative defined in HQET has a homogeneous scaling $D^\mu \propto \partial_\mu + A_s^\mu \propto \lambda$.

3.3.2 Decoupling transformation

The Effective field theory that we have build so far is a theory of soft fields where the hard interactions (coupling of heavy quarks and hard gluons) have been integrated out. As a consequence of working with soft fields only, the gauge transformation applied to the soft fields is not longer a symmetry; however, we can redefine a *Residual Gauge Transformation*: $U_s(x)$ for the soft fields that must satisfy the following conditions.

$$h_v(x) \longrightarrow U_s(x)h_v(x), \tag{3.43}$$

$$A_s^\mu(x) \longrightarrow U_s(x)A_s^\mu(x)U^\dagger(x) + \frac{i}{g_s}U_s(x)\left[\partial^\mu, U_s^\dagger(x)\right]. \tag{3.44}$$

The interactions between the soft gluon and soft spinor can be removed by a field redefinition. To see this, we introduce a time-like Wilson line

$$\begin{aligned} h_v(x) &= S_v(x)h_v^{(0)}(x), \\ h_v(x) &= P\exp\bigg(ig_s\int_{-\infty}^{0} dt\ v\cdot A_s(x+tv)\bigg), \end{aligned} \tag{3.45}$$

where the symbol P is the ordering operator applied to the gauge fields which are ordered from left to right in the order of decreasing t values. The Wilson lines are introduced because it has a beautiful property when a derivative is applied.

$$\begin{aligned} iv\cdot\partial S_v &= i(v\cdot\partial_\mu)\cdot P\exp\bigg(ig_s\int_{-\infty}^{0} dt\ v\cdot A_s(x+tv)\bigg), \\ &= (i^2 g_s\ v\cdot A_s)\ P\exp\bigg(ig_s\int_{-\infty}^{0} dt\ v\cdot A_s(x+tv)\bigg), \\ &= -g_s\ v\cdot A_s\ S_v \end{aligned} \tag{3.46}$$

It can be interpreted as the Wilson line is a constant under this derivative. We can plug the redefinition of the soft field into the HQET Lagrangian and use this property.

$$\begin{aligned}
\mathcal{L}_{HQET} &= \bar{h}_v iv \cdot D_s h_v \\
&= \bar{h}_v^o\ S_v^\dagger\ (iv \cdot \partial + g_s v \cdot A_s)\ S_v\ h_v^o \\
&= \bar{h}_v^o\ S_v^\dagger\ [-g_s\ v \cdot A_s S_v h_v^o + iS_v v \cdot \partial\ h_v^o + g_s v \cdot A_s S_v h_v^o] \\
&= \bar{h}_v^o (S_v^\dagger S_v) iv \cdot \partial h_v^o \\
&= \bar{h}_v^o (iv \cdot \partial) h_v^o.
\end{aligned} \tag{3.47}$$

We find a Lagrangian for HQET with no interaction.

3.4 Soft-Collinear Effective Theory

In Quantum Chromodynamics (QCD), a persistent puzzle revolves around how to systematically quantify the effects of long distances in processes where a local Operator Product Expansion (OPE) is not allowed. The OPE serves as a reliable framework for expanding matrix elements in terms of powers and logarithms of large scales, such as a heavy mass. However, complications arise when dealing with processes involving energetic light particles. In such scenarios, certain components of the momentum vector are large but $p^2 \approx 0$ remains small. These "jet-like" processes make it challenging to disentangle short-distance phenomena from long-distance effects. [21]
Let us start with the decay in the rest frame of a heavy meson into two light meson: $H \longrightarrow M_1 + M_2$. Then, we can define two light-like vectors in the axis of the jets.

$$p_1^\mu = (E, 0, 0, \sqrt{E^2 - m_1^2})\ \ ,\ \ p_2^\mu = (M - E, 0, 0, -\sqrt{E^2 - m_1^2}) \tag{3.48}$$

Using the relativistic equation: $p^2 = m^2$ we can find a relation of the energy in the final state of one of the light-mesons in function of the masses.

$$E = \frac{M^2 + m_1^2 - m_2^2}{2M} >> \Lambda, \tag{3.49}$$

Since $M >> m_1,\ m_2$, we can see that the light meson is high-energetic and therefore we introduce a parameter that is the ratio between the light-mass and the high-energy,

$$\lambda \sim \frac{m_y}{Q} << 1, \tag{3.50}$$

where we define $Q = 2E$ and two light-like vectors along the jet axis and a perpendicular component in order to descompose any momentum

$$p^\mu = \frac{\bar{n}^\mu}{2}(n \cdot p) + \frac{n^\mu}{2}(\bar{n} \cdot p) + p_\perp^\mu \equiv \frac{\bar{n}^\mu}{2} p^+ + \frac{n^\mu}{2} p^- + p_\perp^\mu. \tag{3.51}$$

where $n^\mu = (1,0,0,1)$, $\bar{n}^\mu = (1,0,0,-1)$ and $p_\perp^\mu = (0,p_x,p_y,0)$. We can induce some properties of this decomposition.

$$n\cdot\bar{n} = \bar{n}\cdot n = 2 \;\;,\;\; n\cdot p^\perp = \bar{n}\cdot p^\perp = \bar{n}^2 = n^2 = 0 \tag{3.52}$$

Let us compute the light-like components of one of the meson and see which components are small or large.

$$\begin{aligned} n\cdot p_1 &= E - \sqrt{E^2 - m_1^2} \approx E - E\left(1 - \frac{m_1^2}{2E} + ...\right) = \frac{m_1^2}{Q} \sim \lambda^2\cdot Q. \\ \bar{n}\cdot p_1 &= E + \sqrt{E^2 - m_1^2} \approx E + E\left(1 - \frac{m_1^2}{2E} + ...\right) = Q \sim 1\cdot Q. \\ p_1^\perp &= 0 \end{aligned} \tag{3.53}$$

The perpendicular component of the meson is zero since the light-like vector axis coincide with the jet axis; however, the partons inside the meson might have a perpendicular component with the restriction that the sum of the perpendicular components of all the partons vanish. Similarly for the second meson: $n\cdot p_2 \sim 1\cdot E$, $\bar{n}\cdot p_2 \sim \lambda^2\cdot E$ and $p_2^\perp = 0$. The partons inside the light-mesons have the following scales in the light-components.

$$\begin{aligned} &\text{inside jet 1:}\quad (n\cdot p_i, \bar{n}\cdot p_i, p_i^\perp) \sim (\lambda^2, 1, \lambda)E:\ \text{collinear particles} \\ &\text{inside jet 2:}\quad (n\cdot p_i, \bar{n}\cdot p_i, p_i^\perp) \sim (1, \lambda^2, \lambda)E:\ \text{anti-collinear particles} \\ &\qquad p_i^2 = (n\cdot p_i)(\bar{n}\cdot p_i) + p_{\perp,i}^2 \sim \lambda^2 E^2\ \text{light mesons.} \end{aligned} \tag{3.54}$$

Just from the fact that the mass of the meson is much smaller than the mass of the heavy meson, we find that the partons inside the light meson are collinear-like. The partons inside the heavy meson has the following decomposition into light-like components,

$$p_i^\mu \sim E \longrightarrow (n\cdot p_i, \bar{n}\cdot p_i, p_i^\perp) \sim (1,1,1)\cdot E \tag{3.55}$$

In principle, the heavy meson has no perpendicular components but the partons inside might have it. In a pedagogical way, we introduce the other scaling

$$\begin{aligned} p^\mu &\sim (\lambda^2, 1, \lambda) \longrightarrow \text{collinear:} \\ p^\mu &\sim (1, \lambda^2, \lambda) \longrightarrow \text{anti-collinear:} \\ p^\mu &\sim (1,\ 1,\ 1) \longrightarrow \text{hard:} \\ p^\mu &\sim (\lambda, \lambda, \lambda) \longrightarrow \text{soft:} \\ p^\mu &\sim (\lambda, \lambda, \lambda) \longrightarrow \text{ultra-soft:} \\ p^\mu &\sim (\lambda,\ 1, \lambda) \longrightarrow \text{hard-collinear:} \\ p^\mu &\sim (\ 1, \lambda, \lambda) \longrightarrow \text{hard-anticollinear:} \end{aligned} \tag{3.56}$$

3.4.1 Effective Lagrangian

The goal of this effective theory is to integrate out the hard-modes and build an effective Lagrangian made of collinear, anti-collinear, soft quarks and gluons. From momentum conservation, there are some restrictions in the interaction between the fields in this region scales.

In analogy with Heavy-Quark Effective Theory, we can decompose a collinear spinor into "large" and "small" components using projector operators.

$$P_n = \frac{\not{n}\bar{\not{n}}}{4} \quad , \quad P_{\bar{n}} = \frac{\bar{\not{n}}\not{n}}{4} \tag{3.57}$$

which satisfy the properties of a projector,

$$P_n^2 = P_n, \quad P_{\bar{n}}^2 = P_{\bar{n}}, \quad P_n P_{\bar{n}} = 0, \quad P_n + P_{\bar{n}} = 1 \tag{3.58}$$

We define the large and small components

$$\xi_n = P_n \Psi \quad , \quad \eta_n = P_{\bar{n}} \Psi, \tag{3.59}$$

Where $\not{n}\xi_n = 0$ and $\bar{n}\eta_n = 0$. We can do the power counting of these fields through the two-point correlation function in the massless case for fermions and gluons.

$$\begin{aligned}
\langle 0| T\left\{\Psi(x)\bar{\Psi}(0)\right\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i\not{p}}{p^2+i\epsilon} \sim \lambda^4 \cdot 1 \cdot \frac{(\lambda^2, 1, \lambda)}{\lambda^2} \\
\langle 0| T\left\{A_c^{\mu,a}(x) A_c^{\nu,b}(0)\right\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i\delta_{ab}}{p^2+i\epsilon} \left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right].
\end{aligned} \tag{3.60}$$

Fixing the Feynman gauge for the vector boson would be wrong because it will kill the term that is proportional to $p_\mu p_\nu$. Then, decomposing the whole fields in QCD into SCET collinear fields, we find

$$\begin{aligned}
\langle 0| T\left\{\xi(x)\bar{\xi}(0)\right\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i\bar{n}\cdot p}{p^2+i\epsilon} \frac{\not{n}}{2} \sim \lambda^4 \cdot 1 \cdot \lambda^{-2} \sim \lambda^2 \\
\langle 0| T\left\{\eta(x)\bar{\eta}(0)\right\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{in\cdot p}{p^2+i\epsilon} \frac{\not{n}}{2} \sim \lambda^4 \cdot 1 \cdot \frac{\lambda^2}{\lambda^2} \sim \lambda^4 \\
\langle 0| T\left\{n\cdot A_c^\mu(x)\; n\cdot A_c^\nu(0)\right\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{p^2+i\epsilon} (1-\xi)\frac{(n\cdot p)^2}{p^2} \sim \lambda^4 \cdot \lambda^{-2} \cdot \frac{\lambda^4}{\lambda^2} \\
\langle 0| T\left\{\bar{n}\cdot A_c^\mu(x)\; \bar{n}\cdot A_c^\nu(0)\right\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{p^2+i\epsilon} (1-\xi)\frac{(\bar{n}\cdot p)^2}{p^2} \sim \lambda^4 \cdot \lambda^{-2} \cdot \frac{\lambda^0}{\lambda^2} \\
\langle 0| T\left\{A_\perp^\mu(x)\; A_\perp^\nu(0)\right\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{p^2+i\epsilon} (1-\xi)\frac{p_\perp^\mu p_\perp^\nu}{p^2} \sim \lambda^4 \cdot \lambda^{-2} \cdot \frac{\lambda^2}{\lambda^2}
\end{aligned} \tag{3.61}$$

Then, we find that the dimensional analysis of the SCET fields are

$$\xi \sim \lambda, \quad \eta \sim \lambda^2, \quad (n\cdot A_c, \bar{n}\cdot A_c, A^\mu_{c,\perp}) \sim (1, \lambda^2, \lambda). \tag{3.62}$$

We also find that the collinear covariant derivative: $iD_\mu = i\partial_\mu + g_s A_c^\mu \cdot t \sim (\lambda^2, 1, \lambda)$. Besides, similarly we can find the power counting for the ultra-soft quark and gluons: $q_{us} \sim \lambda^3$ and $A_{us} \sim \lambda^2$. Then, the Dirac spinors and gluon fields decompose into SCET fields are

$$\begin{aligned} \Psi &\longrightarrow \xi + \eta + q_{us} \sim \lambda + \lambda^2 + \lambda^3 \\ A_\mu &\longrightarrow A_c^\mu + A_{us}^\mu \sim (\lambda^2, 1, \lambda) + \lambda^2 \end{aligned} \tag{3.63}$$

The leading power counting in the Dirac Lagrangian is four. The $\mathcal{O}(5)$ and higher-terms in the Lagrangian involves only an ultra-soft field; however, we will focus in the leading-order SCET.

$$\mathcal{L}_{SCET} = \mathcal{L}_c + \mathcal{L}_{us} + \mathcal{L}_{c+us}, \tag{3.64}$$

where

$$\begin{aligned} \mathcal{L}_c &= \bar{\xi}\, \frac{\bar{\not{n}}}{2} in \cdot D_c\, \xi + \bar{\eta}\frac{\not{n}}{2} i\bar{n} \cdot D_c\, \eta + \bar{\xi}(i\not{D}_c^\perp - m)\eta + \bar{\eta}(i\not{D}_c^\perp - m)\xi \sim \lambda^4, \\ \mathcal{L}_{c+us} &= \bar{\xi}\, \frac{\bar{\not{n}}}{2} g_s(n \cdot A_{us})\, \xi \sim \lambda^4, \end{aligned} \tag{3.65}$$

Now the question of what else we can integrate out at this point emerges from the analogy to HQET where the large components were integrated out using the equation of motion.

$$\begin{aligned} \frac{\delta \mathcal{L}_c}{\delta \bar{\eta}} = 0 &\longrightarrow \frac{\not{n}}{2} i\bar{n} \cdot D_c \eta + (i\not{D}_c^\perp - m)\xi = 0 \\ \eta &= -\frac{1}{i\bar{n} \cdot D_c + i\delta}\, \frac{\bar{\not{n}}}{2}(i\not{D}_c^\perp - m)\, \xi \end{aligned} \tag{3.66}$$

When in the last line we have added a regulator $i\delta$ in order to solve the equation of motion for the small component. We can replace the expression of the small component in function of the large component in the SCET Lagrangian.

$$\mathcal{L}_c = \bar{\xi}\frac{\bar{\not{n}}}{2} in \cdot D_c \xi - \bar{\xi}(i\not{D}_c^\perp - m)\frac{\bar{\not{n}}}{2}\frac{1}{i\bar{n} \cdot D_c + i\delta}(i\not{D}_c^\perp - m)\xi. \tag{3.67}$$

4 AXION-LIKE PARTICLES

In this chapter, we will focus on Axion-like Particles (ALPs) as an extension of the Standard Model. During the chapter, we will describe the motivation of ALPs and why they are attractive nowadays. In addition, we will introduce an effective Lagrangian of standard model plus the ALPs (we borrow most of the material from [24]). At the end, we will discuss the current theoretical prediction of ALPs in meson decays.

4.1 Motivation

Axions and Axion-like particles (ALPs) have become highly attractive in particle physics and cosmology because of their potential to address several fundamental questions and gaps in our understanding of the universe. Here's a detailed explanation of why Axions and ALPs are currently of great interest, mostly due to their relation to the strong CP problem and dark matter:

4.1.1 Strong CP Problem and PQ Solution

Among the several shortcomings of the Standard Model of particles, QCD suffers from the "Strong CP Problem". There is a term that satisfies the gauge and Lorentz symmetry in QCD but it is not usually included in the description of the standard model. This θ_{QCD} term has the following expression:

$$\mathcal{L}_{QCD} = \mathcal{L}^o_{QCD} + \frac{\theta_{QCD}}{32\pi^2} G^i_{\mu\nu}\tilde{G}^{i\mu\nu}, \tag{4.1}$$

where the dual tensor field is defined: $\tilde{G}^i_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha\beta}G^{i\alpha\beta}$¸ . Also, $\mathcal{L}^o_{QCD}$ is the Lagrangian of QCD described in the standard model (see Eq. 2.2). In principle, this term is not CP invariant and gives rise to an electric dipole moment for the neutron [25].

$$d_n \approx 2.410^{-16}\theta_{QCD}\; e \text{ cm}. \tag{4.2}$$

The upper bound for dipole moment given by [26] is: $|d_n| < 2.9 \cdot 10^{-26} e$ cm. This yields that,

$$\theta_{QCD} \leq 10^{-10}. \tag{4.3}$$

Understanding the smallness of θ_{QCD} is the so-called strong CP problem. The Peccei-Quinn mechanism offers a solution to explain CP conserving in the presence of pseudoscalar particles [1]. The axion [3, 4] is a pseudoscalar field that emerges from the spontaneous symmetry breaking of $U(1)_{PQ}$ which interacts with $G\tilde{G}$. This coupling dynamically regulates θ_{QCD} to be zero through the influence of QCD non-perturbative effects, particularly instantons. The basic concept suggests the existence of the axion a, which enjoys a shift symmetry: $a \longrightarrow a + c$. Setting $\theta_{QCD} = C\ \frac{a}{f_a}$, where f_a is the axion decay constant, and C represents the "color anomaly,". Consequently, any violation from shift symmetry caused solely by quantum effects can be absorbed by a shift in a [2].

4.2 Effective Lagrangian of ALPs

The ALPs is a singlet of the Standard Model gauge group that satisfies a shift symmetry $a \longrightarrow a + c$. The couplings of the ALP with the SM fields at UV scale are given by the following Lagrangian up to dimension five [24].

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{D\leq 5} =&\mathcal{L}_{SM} + \frac{1}{2}\partial_\mu a \partial^\mu a - \frac{m_{a,0}^2}{2}a^2 + \frac{\partial_\mu a}{f}\sum_F \bar{\Psi}_F \boldsymbol{c}_F \gamma^\mu \Psi_F \\ &+ c_{GG}\frac{\alpha_s}{4\pi}\frac{a}{f}G^a_{\mu\nu}\tilde{G}^{\mu\nu,a} + c_{WW}\frac{\alpha_2}{4\pi}\frac{a}{f}W^a_{\mu\nu}\tilde{W}^{\mu\nu,a} + c_{BB}\frac{\alpha_1}{4\pi}\frac{a}{f}B_{\mu\nu}\tilde{B}^{\mu\nu},\end{aligned} \tag{4.4}$$

where $\alpha_i = \frac{g_i^2}{4\pi}$ and $\tilde{G}^{a,\mu\nu} = \frac{1}{2}\epsilon^{\rho\sigma\mu\nu}G^a_{\rho\sigma}$. The sum runs over the SM chiral fermions: $\{Q_L,\ u_R,\ d_R\}$. This Lagrangian is the most general shift-invariant Lagrangian one can write, with the only exception of the mass term, which gives an explicit symmetry breaking. The 5-dimension operators have a suppression given by the UV scale: $\Lambda = 4\pi f$. where f is the ALP decay constant. Therefore, operators with higher-dimension such as (4+n)-operators will be suppressed by the large scale $\frac{1}{\Lambda^n}$ and they could be neglected.

In principle, the shift symmetry protects the ALP to be massless but it can get mass by explicit soft symmetry breaking and/or by non-perturbative QCD dynamics.

$$m_a^2 = m_{a,o}^2\left[1 + \mathcal{O}\left(\frac{f_\pi^2}{f^2}\right)\right] + c_{GG}^2\frac{f_\pi^2 m_\pi^2}{f^2}\frac{2m_u m_d}{(m_u+m_d)^2}. \tag{4.5}$$

Redudant Operator

We can see that there is no interaction between the ALP and the Higgs, however there is one operator up to dimension five that, in principle, should be consider.

$$\mathcal{L}_{\text{eff}}^{D<5} \supset c_\Phi \mathcal{O}_\Phi = c_\Phi \frac{\partial_\mu a}{f}\left(\Phi^\dagger i D^\mu \Phi + \text{h.c.}\right). \tag{4.6}$$

The reason why this operator is not consider in the Lagrangian above is because is a redundant operator. Redefining the fields, we can see that the ALP-higgs operator is already considered in the ALP-fermion operator. Let us do the redefinition: $\Phi \longrightarrow e^{ic_\Phi a/f}\Phi$ for the

Higgs and $\Psi \longrightarrow e^{i\beta_\Psi c_\Phi a/f}\Psi$ for the fermions. These redefinition's are constrained by the SM couplings. In order to not break the invariance in Yukawa interaction we have

$$\beta_\mu - \beta_Q = -1, \qquad \beta_d - \beta_Q = 1, \qquad \beta_e - \beta_L = 1, \tag{4.7}$$

and in order to have gauge anomaly cancellation in the SM, we have

$$3\beta_Q + \beta_L = 0 \tag{4.8}$$

The ALP-fermion coupling changes like

$$\boldsymbol{c}_\Psi \longrightarrow \boldsymbol{c}_\Psi + \beta_\Psi \cdot c_\Phi \cdot \mathbb{I}, \tag{4.9}$$

It follows from the discussion that the redundant operator can be re-expressed as

$$\mathcal{O}_\Phi = \mathcal{O}_\Phi + \sum_F \beta_F \mathcal{O}_F, \qquad \text{with} \quad \mathcal{O}_F = \frac{\partial_\mu a}{f}\bar{\Psi}_F \gamma^\mu \Psi_F, \tag{4.10}$$

where $\mathcal{O}_\phi$ vanishes because of the equation of motion, so we have that the ALP-Higgs operator is a linear combination of the ALP-fermion operator. However, as we will see the ALP-Higgs coupling can offers a different type of UV divergence and it has to be consider as well.

4.2.1 Effective ALP Lagrangian below the electroweak scale

Since it is of our interests to describe the heavy- to light- meson transition by the effective weak interactions, it is necessary to describe the effective ALP Lagrangian in the scale below the electroweak. The top quark, the Higgs boson and the $W^\pm$, Z^o gauge bosons are integrated out at scale $\mu < \mu_{EW}$.

$$\mathcal{L}_{eff}^{D\leq 5}(\mu \lesssim \mu_{EW}) = \frac{1}{2}\partial_\mu a \partial^\mu a - \frac{m_{a,0}^2}{2}a^2 + \mathcal{L}_{ferm} + c_{GG}\frac{\alpha_s}{4\pi}\frac{a}{f}G_{\mu\nu}^a \tilde{G}^{\mu\nu,a} + c_{\gamma\gamma}\frac{\alpha_1}{4\pi}\frac{a}{f}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{4.11}$$

$$\mathcal{L}_{eff}^{D\leq 5}(\mu \lesssim \mu_{EW}) = \frac{1}{2}\partial_\mu a \partial^\mu a - \frac{m_{a,0}^2}{2}a^2 + \mathcal{L}_{ferm} + \tag{4.12}$$

$$c_{GG}\frac{\alpha_s}{4\pi}\frac{a}{f}G_{\mu\nu}^a \tilde{G}^{\mu\nu,a} + c_{\gamma\gamma}\frac{\alpha_1}{4\pi}\frac{a}{f}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{4.13}$$

where the ALP-fermion Lagrangian is given by

$$\mathcal{L}_{ferm}(\mu) = \frac{\partial^\mu a}{f}\left[\bar{u}_L \boldsymbol{\kappa_U}(\mu)\gamma_\mu u_L + \bar{u}_R \boldsymbol{\kappa_u}(\mu)\gamma_\mu u_R + \bar{d}_L \boldsymbol{\kappa_D}(\mu)\gamma_\mu d_L + \bar{d}_R \boldsymbol{\kappa_d}(\mu)\gamma_\mu d_R + ...\right], \tag{4.14}$$

where the points represent to the ALP-lepton couplings. The quarks field are written in the fermion mass basis, therefore the matrices κ_Q and $\boldsymbol{c}_Q$ are related by the bi-linear transformation matrix.

$$\kappa_U = U_u^\dagger \boldsymbol{c}_Q U_u \quad , \quad \kappa_D = U_d^\dagger \boldsymbol{c}_Q U_d \ . \tag{4.15}$$

and,

$$c_{\gamma\gamma} = C_{WW} + c_{BB} \tag{4.16}$$

Since the flavour-changing transition in our process will be mediated by the effective weak interaction, it will be needed the flavor-conserving ALP coupling. Therefore, we define.

$$c_{f_if_i}(\mu) \equiv \kappa_f(\mu)_{ii} - \kappa_F(\mu)_{ii}. \tag{4.17}$$

4.2.2 Alternative Lagrangian for ALPs

The interaction between ALPs and fermions are proportional to the ALPs momentum due to the derivative coupling; however, an important freedom in constructing the effective Lagrangian is that we can always perform a field redefinition. Therefore, we can have an alternative and equivalent Lagrangian for the ALPs [24].

$$\begin{aligned}\mathcal{L}_{eff}^{D\leq 5} = \frac{1}{2}(\partial_\mu a)(\partial^\mu a) - \frac{m_{a,o}^2}{2}a^2 - \frac{a}{f}\left(\bar{Q}\Phi\tilde{\boldsymbol{Y}}_d d_R + \bar{Q}\tilde{\Phi}\tilde{\boldsymbol{Y}}_u u_R + \bar{L}\Phi\tilde{\boldsymbol{Y}}_e e_R + h.c.\right)\\ + \tilde{c}_{GG}\frac{\alpha_s}{4\pi}\frac{a}{f}G_{\mu\nu}^a\tilde{G}^{\mu\nu,a} + \tilde{c}_{WW}\frac{\alpha_s}{4\pi}\frac{a}{f}W_{\mu\nu}^a\tilde{W}^{a\mu\nu} + \tilde{c}_{BB}\frac{\alpha_s}{4\pi}\frac{a}{f}B_{\mu\nu}\tilde{B}^{\mu\nu}.\end{aligned} \tag{4.18}$$

where

$$\begin{aligned}\tilde{\boldsymbol{Y}}_d &= i(\boldsymbol{Y}_d\boldsymbol{c}_d - \boldsymbol{c}_Q\boldsymbol{Y}_d)\\ \tilde{\boldsymbol{Y}}_u &= i(\boldsymbol{Y}_u\boldsymbol{c}_u - \boldsymbol{c}_Q\boldsymbol{Y}_u)\\ \tilde{\boldsymbol{Y}}_e &= i(\boldsymbol{Y}_e\boldsymbol{c}_e - \boldsymbol{c}_L\boldsymbol{Y}_e)\end{aligned} \tag{4.19}$$

and the ALP-gauge boson couplings also got redefined.

$$\begin{aligned}\tilde{c}_{GG} &= c_{GG} + \frac{1}{2}\,\mathrm{Tr}\,(\boldsymbol{c}_u + \boldsymbol{c}_d - N_L\boldsymbol{c}_Q),\\ \tilde{c}_{WW} &= c_{WW} - \frac{1}{2}\,\mathrm{Tr}\,(N_c\boldsymbol{c}_Q + \boldsymbol{c}_L),\\ \tilde{c}_{BB} &= c_{BB} + \ \mathrm{Tr}\,\left[N_c\left(\frac{4}{9}\boldsymbol{c}_u + \frac{1}{9}\boldsymbol{c}_d - 2\frac{1}{36}\boldsymbol{c}_Q\right) + \boldsymbol{c}_e - 2\frac{1}{4}\boldsymbol{c}_L\right].\end{aligned} \tag{4.20}$$

4.3 Flavour-changing probes of ALPs

In this section we will review some work on ALPs-quark couplings that are flavour-changing [27–29]. Rare meson decays provide valuable probes for exploring new physics due to their sensitivity to new particles and interactions that might not be accessible at higher energy scales directly. Thus, ALPs can be produced in two-body rare meson decays: $M_1 \longrightarrow M_2\ a$. By momentum conservation we can find that the energy of the final state meson M_1 is

$$E_2 = \frac{m_1^2 + m_2^2 - m_a^2}{2m_1}, \tag{4.21}$$

in the M_1 rest frame. The decay rate of a two-body decay is given by

$$\Gamma(M_1 \longrightarrow M_2\ a) = \frac{1}{16\pi m_1}|\mathcal{M}(M_1 \longrightarrow M_2\ a)|^2 \lambda^{1/2}\left(\frac{m_2^2}{m_1^2}, \frac{m_a^2}{m_1^2}\right), \tag{4.22}$$

where:

$$\lambda(u,v) = 1 - 2(u+v) + (u-v)^2. \tag{4.23}$$

We borrow the result from [13] and write the decay rate of B and D meson into light mesons in function of the ALP flavour-changing in the quark sectors.

$$\begin{aligned}
\Gamma(B^- \longrightarrow \pi^-\ a) &= \frac{m_B^3}{64\pi f^2}|[\kappa_D + \kappa_d]_{13}|^2 |F_o^{B\longrightarrow\pi}(m_a^2)|^2 \left(1 - \frac{m_\pi^2}{m_B^2}\right)\lambda^{1/2}\left(\frac{m_\pi^2}{m_B^2}, \frac{m_a^2}{m_B^2}\right), \\
\Gamma(B^- \longrightarrow K^-\ a) &= \frac{m_B^3}{64\pi f^2}|[\kappa_D + \kappa_d]_{23}|^2 |F_o^{B\longrightarrow K}(m_a^2)|^2 \left(1 - \frac{m_K^2}{m_B^2}\right)\lambda^{1/2}\left(\frac{m_K^2}{m_B^2}, \frac{m_a^2}{m_B^2}\right), \\
\Gamma(D_s^+ \longrightarrow K^+\ a) &= \frac{m_{D_s}^3}{64\pi f^2}|[\kappa_U + \kappa_u]_{12}|^2 |F_o^{D_s\longrightarrow K}(m_a^2)|^2 \left(1 - \frac{m_K^2}{m_{D_s}^2}\right)\lambda^{1/2}\left(\frac{m_K^2}{m_{D_s}^2}, \frac{m_a^2}{m_{D_s}^2}\right), \\
\Gamma(D^+ \longrightarrow \pi^+\ a) &= \frac{m_D^3}{64\pi f^2}|[\kappa_U + \kappa_u]_{12}|^2 |F_o^{D\longrightarrow\pi}(m_a^2)|^2 \left(1 - \frac{m_\pi^2}{m_D^2}\right)\lambda^{1/2}\left(\frac{m_\pi^2}{m_D^2}, \frac{m_a^2}{m_D^2}\right).
\end{aligned} \tag{4.24}$$

We can see the heavy- into light-meson decays can constrain the off-diagonal components of the ALP-coupling matrices: $\kappa_U,\ \kappa_u,\ \kappa_D, \kappa_d$ for a widely ALP mass range: $0 < m_a <$ fewGeV. It is of our interest to investigate the constrains for the diagonal components $[\kappa_f]_{ii}$ of the ALP-quark coupling for the same transitions but in QCD Factorization for non-leptonic framework.

There has been a recent work where the diagonal components of the ALP-coupling are constrained in Kaon decays into pion and ALPs (see [12]) using consistent implementation of weak decays involving ALPs in the context of an effective chiral Lagrangian.

$$i\mathcal{M}(K^- \longrightarrow \pi^- a) \approx \frac{im_K^2}{f_a}\left[N_8\left(1 + \frac{c_{uu} + c_{dd}}{2c_{GG}}\right) - \frac{[\kappa_q + \kappa_Q]^{23}}{2C_{GG}}\right]. \tag{4.25}$$

Where $N_8 \approx 1.53 \cdot 10^{-7}$ and the ALP couplings are defined at the low scale $\mu_o = 2\text{GeV}$. In addition, for ALP flavour-conserving couplings: c_{ff} see Eq. 4.17. As long as the off-diagonal ALP term is small enough, implies

$$\frac{1}{f_a}\left|1+\frac{c_{uu}+c_{dd}}{2c_{GG}}\right| < \frac{1}{31.9\text{TeV}}. \tag{4.26}$$

for light ALP masses below the chiral symmetry breaking $\mu_\chi = 4\pi f_\pi$.

5

QCD FACTORIZATION

In this chapter we will introduce the framework of Quantum Chromodynamics (QCD) Factorization in order to study processes involving hadrons. In particular, we focus on heavy meson weak decays into light meson. We introduce the tools that are necessary such as Light-Cone Distribution Amplitudes and Soft-Collinear Effective Theory.

5.1 The idea of Factorization

The QCD Factorization in exclusive decays aims to parameterize the hadronic matrix element into non-perturbative objects (form factors and/or light-cone distribution amplitudes) and perturbative objects (Wilson coefficients and/or partonic amplitude).

In the present chapter, we will focus on two approaches discussed in [14, 15]. The first approach is called the "naive factorization" where the hadronic matrix elements is parameterized as product of the form factor of the final state and the partonic amplitude for a given current operator. It yields that

$$\langle K\ a| \mathcal{H}_{eff} |B\rangle = \mathcal{T}^{I}_{\mu}(q^2) \cdot \langle K\ a| \bar{s}\Gamma^{\mu} b |B\rangle . \tag{5.1}$$

This assumes that the exchange of hard gluons with virtuality below $\mu \sim m_b$ are neglected. The second approach to study non-leptonic decays is called the hard-scattering approach. Here we assume what we have neglected in the last approach. We consider that the hadronic amplitude is dominated by an hard exchange gluon. Then the hadronic amplitude is parameterized as a convolution of light-cone distribution amplitudes[1] of the mesons with the partonic amplitude of the process. It yields that,

$$\langle K\ a| \mathcal{H}_{eff} |B\rangle = \langle K| \bar{s}_{\beta} q_{\alpha} |0\rangle \otimes \mathcal{T}^{II}_{\beta\alpha\rho\eta} \otimes \langle 0| \bar{q}_{\rho} b_{\eta} |B\rangle . \tag{5.2}$$

Summarizing these two approaches into the hadronic matrix element, we have the factorization formula at large recoil ($E, m_B >> \Lambda_{QCD}, m_K$) and at leading order in $\mathcal{O}(1/m_B)$ for heavy- to light-mesons. [30].

$$\langle K\ a| \mathcal{H}_{eff} |B\rangle = \mathcal{T}^{I}_{\mu}(q^2)\ F^{\mu}_{B\to K}(q^2) + \iint du\ d\omega\ \Phi^{K}(u)_{\beta\alpha} \mathcal{T}^{II}_{\beta\alpha\rho\eta}(q^2, u, \omega) \Phi^{B}_{\rho\eta}(\omega), \tag{5.3}$$

1 we will describe it in details in the next section.

where $F^{\mu}_{B\to K}$ represents the form factors for generic Dirac structures, the $\Phi^{B,K}$ represent the Light-Cone Distribution Amplitudes (LCDA) and the variables μ, ω parameterize the distribution of the meson momentum into the constituent quarks. The form factor and the LCDA are non-perturbative quantities and describes long-distances effects. Fortunately, these quantities are universal and they can be investigated by experiments of many other decays. On the other hand, the $\mathcal{T}^{I}_{\mu}$ and $\mathcal{T}^{II}$ are the partonic scattering amplitudes. They encode short-distances dynamics and can be calculated perturbatively for each process.

The first term in the factorization formula are the contributions where the light quark in the meson is really an spectator and it is disconnected from the rest of the graph. The second term that involve LCDA and partonic amplitude are usually the contributions when the spectator quark actively participates due to an exchange of gluons.

5.2 Light-cone Distribution Amplitudes

Light-cone distribution amplitudes describe the non-perturbative effects contained in the matrix element of a bi-local current in between a single meson state and the vacuum. Such matrix elements are typically encountered when the spectator quark actively participates in a process.

The light-cone distribution amplitude (LCDA) for mesons denoted by $\Phi_M(u,\mu)$ describes the probability amplitude for a meson M to have a particular configuration of quarks and gluons, in terms of their momentum fractions (u) of the constituents.

5.2.1 Light-cone distribution amplitudes for light mesons

Analogously to the parton distributions for inclusive processes, light-cone distribution amplitudes play the same role for hard exclusive processes. We define the leading-twist amplitude for a light pseudoscalar meson M and assume that the fraction momenta carried by the constituents are of order 1.

$$\langle M(p)|\, \bar{q}(y)\gamma\mu\gamma_5 q'(x)\, |0\rangle = -if_M p_\mu \int_0^1 du e^{i(\bar{u}p\cdot x + uq\cdot y)}\Phi(u), \tag{5.4}$$

where f_M is the decay constant and $\bar{u} = 1 - u$. Higher-twist light-cone distribution amplitudes for the light meson are power suppressed in the heavy quark limit; however, this subleading terms can sometimes be large if they appear together with the chiral enhancement factors [15]. In our case, this subleading terms might give an relevant contribution when the partonic amplitude by itself is subleading. The expressions of the light-cone distribution amplitudes from higher-twist operators of a pseudoscalar meson M in terms of bi-local operators matrix elements are

$$\langle M(p)|\,\bar{q}(y) i\gamma_5 q'(x)\,|0\rangle = \mu_M f_M \int_0^1 du e^{i(\bar{u}p\cdot x + uq\cdot y)}\Phi(u)_p,$$
$$\langle M(p)|\,\bar{q}(y)\sigma_{\mu\nu}\gamma_5 q'(x)\,|0\rangle = \mu_M f_M\,[p_\mu, z_\nu] \int_0^1 du e^{i(\bar{u}p\cdot x + uq\cdot y)}\frac{\Phi_\sigma(u)}{6}, \tag{5.5}$$

where

$$\mu_M = \frac{m_M^2}{m_1 + m_2} \sim \Lambda_{QCD} \tag{5.6}$$

Where m_M is the mass of the pseudoscalar meson and $m_1,\ m_2$ the masses of the partons inside the meson.

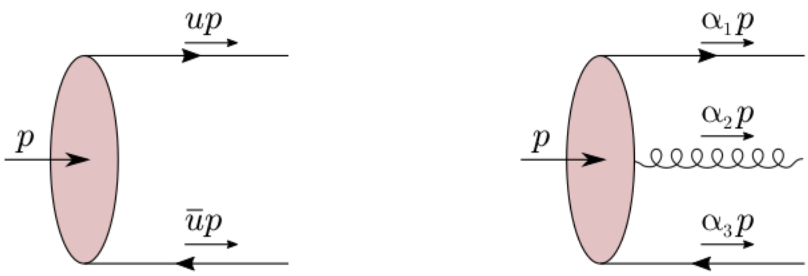


Figure 5.1: Illustrative representation of quark-antiquark particle contribution to the distribution amplitude on the left and quark-antiquark-gluon particle contribution to the distribution amplitude. Image was taken from [31]

$\Phi(u)$ is the leading-twist (twist-2) distribution amplitude, and $\Phi_p(u)$ together with $\Phi_\sigma(u)$ have subleading twist (twist-3). It is important to stress that the subleading twist and subleading in Λ_{QCD} are different scale of subleading but in this case the twist-3 contribution is also subleading because is proportional to μ_M. Combining the contributions from leading and subleading twist-operators, we find

$$\langle M(p)|\,\bar{q}(y) q'(x)\,|0\rangle = \frac{if_M}{4}\int_0^1 du e^{i(\bar{u}p\cdot x + uq\cdot y)}\left\{\not{p}\Phi(u) - \mu_M\gamma_5\left(\Phi_p(u) - \sigma_{\mu\nu}p^\mu z^\nu \frac{\Phi_\sigma(u)}{6}\right)\right\}. \tag{5.7}$$

We have implicitly assumed that the matrix elements encapsulates the Wilson line between the partons so as to make the definitions of the light-cone distribution amplitudes gauge invariant. In order to have the expression of the LCDA in our Factorization formula in Eq. 5.3, we should obtain the distribution amplitude in the momentum space. We assign the following momentum's to the constituents in the collinear approximation [2].

$$p_1^\mu = up^\mu + k_\perp^\mu + \frac{\boldsymbol{p}_\perp^2}{2up\cdot\bar{p}}\bar{p}^\mu \quad , \quad p_2^\mu = \bar{u}p^\mu - k_\perp^\mu + \frac{\boldsymbol{p}_\perp^2}{2\bar{u}p\cdot\bar{p}}\bar{p}^\mu. \tag{5.8}$$

2 where it is assumed that the partons inside a collinear meson carry a fraction of the meson momentum in the collinear direction.

Where $p_1 + p_2 = p + \mathcal{O}(\Lambda^2_{QCD})$. In order to transform to the momentum space, we need to perform the integrals over the coordinate z,

$$z^\nu \longrightarrow -i\frac{\partial}{\partial_p} = (-i)\left(\frac{\bar{p}^\nu}{p\cdot\bar{p}}\frac{\partial}{\partial u} + \frac{\partial}{\partial k_{\nu\perp}} + ...\right). \tag{5.9}$$

Where the derivative over the perpendicular component applies to the partonic amplitude in the momentum space representation. Then, the term that is proportional to the commutator will give to a new term proportional to the derivative over the perpendicular component of the constituent inside the meson.

$$\Phi^M = \frac{if_M}{4}\left\{\not{p}\gamma_5\Phi(u) - \mu_M\gamma_5\left(\Phi_p - i\sigma_{\mu\nu}\frac{p^\mu\bar{n}^\nu}{p\cdot\bar{n}}\frac{\Phi'_\sigma(u)}{6} + i\sigma_{\mu\nu}p^\mu\frac{\Phi_\sigma}{6}\frac{\partial}{\partial k_{\perp\nu}}\right)\right\} \tag{5.10}$$

The leading-twist contribution in the LCDA of the meson is usually expanded in terms of *Gegenbauer polynomials* [15].

$$\Phi(u,\mu) = 6u\bar{u}\left[1 + \sum_{n=1}^{\infty}\alpha_n^M(\mu)C_n^{(3/2)}(2u-1)\right], \tag{5.11}$$

where μ is the renormalization scale where the LCDA is evaluated but for the asymptotic limit $\mu \longrightarrow \infty$ is $\Phi(u) = 6u(1-u)$. The expression of the coefficients in the series are: $C_1^{(3/2)}(u) = 3u$ and $C_2^{3/2}(u) = \frac{3}{2}(5u^2 - 1)$. The uncertainties in the polynomial momentum give an uncertainty in the theoretical prediction.

The twist-three two-particle are related between them via the equations of motion.

$$\frac{u}{2}\left(\Phi_p(u) + \frac{\Phi'_\sigma}{6}\right) = \frac{\Phi_\sigma}{6}\quad,\quad \frac{1-u}{2}\left(\Phi_p(u) - \frac{\Phi'_\sigma}{6}\right) = \frac{\Phi_\sigma}{6}. \tag{5.12}$$

Solving this equation of motion and expressing the two-particle amplitudes as a function of one of them, we find

$$\begin{aligned}\Phi'_\sigma &= 6(1-2u)\Phi_p = 6(\bar{u}-u)\\ \Phi_\sigma &= 6u(1-u)\Phi_p = 6u\bar{u},\end{aligned} \tag{5.13}$$

where in the asymptotic limit $\Phi_p \longrightarrow 1$.

5.2.2 Light-cone distribution amplitudes for Heavy mesons

The B meson LCDA is defined in the coordinate space for bi-local matrix element. [30]

$$\begin{aligned}\tilde{\Phi}^B(z) &= \langle 0|\,\bar{q}(z)P(z,0)b(0)\,|\bar{B}(p)\rangle\\ &= -\frac{if_B}{4}m_B\left(\frac{1+\not{v}}{2}\right)\left[2\tilde{\Phi}_+^B(t) + \frac{\tilde{\Phi}_-^B(t) - \tilde{\Phi}_+^B(t)}{t}\not{z}\right]\gamma_5\end{aligned} \tag{5.14}$$

where $z^2 = 0$ and $t = v \cdot z$ and $p^\mu = m_B v^\mu$ and the Wilson lines

$$P(z_2, z_1) = P \exp\left(i g_s \int_{z_2}^{z_1} dz^\mu A_\mu(z) \right) \tag{5.15}$$

in order to ensure the gauge invariance. Then, we are interested in the projection of the LCDA of heavy meson (see Eq. 5.14) into the partonic amplitude in momentum space. We multiply these two in the coordinate space and transform to the momentum space by Fourier transformation.

$$\int d^4 z M(z) A(z) = \int \frac{d^4 l}{(2\pi)^4} A(l) \underbrace{\int d^4 z M(z) e^{-ilz}}_{M^B} = \int_0^\infty dl_+ M^B A(l), \tag{5.16}$$

where in the second equality we wrote the amplitude in the momentum space and the exponential introduced through the Fourier transform brings the LCDA to the momentum space. Writing the LCDA in the coordinate space as functions in the momentum space and the coordinate z can be replaced by a derivative that acts in the partonic amplitude

$$\int d^4 z M(z) A(z) = -\frac{i f_B m_B}{4} \left[\frac{1+\not{v}}{2} \int_0^\infty d\omega \left\{ 2\Phi_+^B(\omega) - \int_0^\omega d\xi (\Phi_-^B(\xi) - \Phi_+^B(\xi)) \gamma^\mu \frac{\partial}{\partial l_\mu} \right\} \gamma_5 \right] A(l)|_{l=\omega v} \tag{5.17}$$

The partonic amplitude in a decay of a heavy- to light-meson is independent of the l_- component. Therefore the derivative over l^μ has components over perpendicular and l_+. Then, the projector of the B meson LCDA is

$$\Phi^B = -\frac{i f_B m_B}{4} \left[\frac{1+\not{v}}{2} \left\{ \Phi_+^B(\omega) \not{\bar{n}} + \Phi_-^B(\omega) \not{n} - \int_0^\omega d\xi (\Phi_-^B(\xi) - \Phi_+^B(\xi)) \gamma^\mu \frac{\partial}{\partial l_{\mu\perp}} \right\} \gamma_5 \right] \tag{5.18}$$

Where after projecting the LCDA in to the partonic amplitude, the soft momentum is set to $l = \omega \frac{\bar{n}}{2}$. In comparison to the light meson LCDA, the heavy meson LCDA is parameterized by two functions only. The final expression can be even more simplified when we use the models for: Φ_+^B and Φ_-^B.

$$\Phi_+^B(\omega) = \frac{\omega}{\omega_o^2} e^{-\frac{\omega}{\omega_o}} \quad , \quad \Phi_-^B = \frac{1}{\omega_o} e^{-\omega/\omega_o} \tag{5.19}$$

Then, when we perform the integral that is proportional to the $l_\perp$ we find that is equal to the Φ_-^B. Then, the B meson LCDA projector can be simplified to

$$\Phi^B = -\frac{i f_B m_B}{4} \left[\frac{1+\not{v}}{2} \left\{ \Phi_+^B(\omega) \not{\bar{n}} + \Phi_-^B(\omega) \left(\not{n} - \omega \gamma^\mu \frac{\partial}{\partial l_{\mu\perp}} \right) \right\} \gamma_5 \right], \tag{5.20}$$

6

PHENOMENOLOGY OF $B \to K\ a$

During the last chapters we have introduced the tools and the recipe in order to calculate the hadronic matrix element of $B \longrightarrow K\ a$. We have seen in the Chapter 4 that there are limits in the ALP-quark couplings for both: flavor- and conserving-changing for light masses, $m_a \sim 300$MeV obtained by the Kaon decays into pion and ALPs [12]. Besides, for heavier ALP masses, a heavy meson decay is needed as B and D meson. Heavy- into light-mesons can constrain the ALP-quark flavor-changing couplings [13] for heavy ALPs, $m_a \leq 4GeV$. Therefore, there is a window that has not been explored for heavier ALP masses with flavor-conserving couplings and it is our goal to describe it here.

6.1 QCD Factorization of B $\to$ K a

Decays of B^- to K^- requires flavor-changing: $b \longrightarrow s$ which is mediated by the effective weak interaction. Recalling the effective operators in the Section 3.2 and the effective ALP-quark coupling below the electroweak scale (see Eq. 4.2.1).

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} \lambda_p \left(C_1(\mu)\mathcal{Q}_1^{(p)} + C_2\mathcal{Q}_2^{(p)} + \sum_{i=3,\ldots,8} C_i\mathcal{Q}_i \right) + \frac{\partial^\mu a}{f} \left[\bar{q}_L \kappa_Q \gamma^\mu q_L + \bar{q}_R \kappa_q \gamma^\mu q_R\right]. \tag{6.1}$$

Where $q = u,\ d$, $\lambda_p = V_{pb}V_{ps}^*$ and the effective operators $\mathcal{Q}_i$ are given in Eqs. 3.26 and 3.27. Given the hamiltionian, we can see the possible diagrams to compute at tree-level in Fig. 6.1 and at loop level in Fig. 6.2. The theoretical challenge associated with these exclusive decay modes is typically expressed in terms of factorizable and non-factorizable contributions. Recalling our QCD Factorization theorem, we have:

$$\langle K\ a|\, \mathcal{H}_{eff}\, |B\rangle = \mathcal{T}_\mu^I(q^2)\ F^\mu_{B\to K}(q^2) + \iint du\ d\omega\ \Phi^K(u)_{\beta\alpha}\mathcal{T}^{II}_{\beta\alpha\rho\eta}(q^2, u, \omega)\Phi^B_{\rho\eta}(\omega), \tag{6.2}$$

Where the second term represents non-factorizable contributions (by "non-factorizable" we mean all those corrections that are not contained in the definition of the QCD form factors for heavy-to-light transitions). For instance, diagrams as in Fig. 6.1 (a) and (b) and Fig. 6.2 (a) where the light quark in the B meson interacts through a hard-collinear gluon exchange. Whereas, the first term represents diagrams as in Fig. 6.2 (b) where the spectator quark participates through soft interactions inside the B meson.

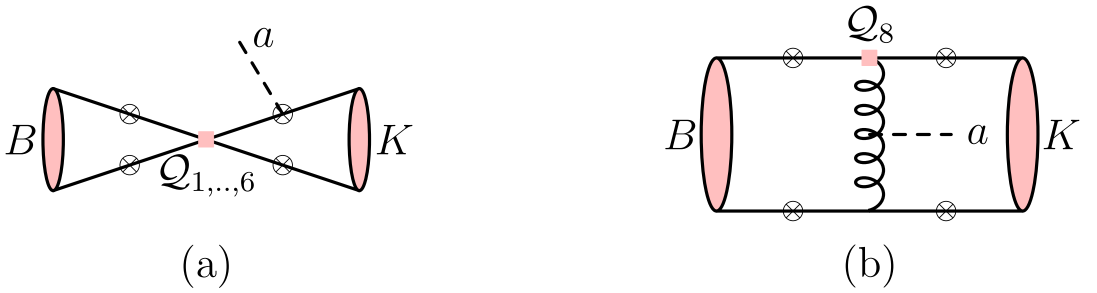


Figure 6.1: Tree-level contributions of the decay: $B \longrightarrow K\ a$. The circled cross represent the possible insertions of the ALP in the final state and in the gluon exchange in (b).

6.1.1 Kinematic

For the $B \longrightarrow K + a$ decay, we work in the rest frame of the B meson,

$$p_B^\mu = m_B v^\mu = m_B \frac{n^\mu}{2} + m_B \frac{\bar{n}^\mu}{2}, \tag{6.3}$$

where m_B is the hard scale. The heavy meson decays into two particles: a Kaon that is approximately collinear due to its small mass in comparison to the b-quark, and an Axion-like particle, which, as we will see could be anti-collinear, hard, or hard-anti-collinear. Choosing the decay axis to be the z-axis, the Kaon and ALP momentum reads

$$p_K^\mu = \left(E, 0, 0, \sqrt{E^2 - m_K^2}\right) \quad , \quad p_a^\mu = \left(E_a, 0, 0, -\sqrt{E^2 - m_K^2}\right). \tag{6.4}$$

From momentum conservation we have,

$$E_a = m_B - E \quad , \tag{6.5}$$

We can write the Kaon energy in function of the mass of the particles.

$$(m_B - E)^2 - (E^2 - m_K^2) = m_a^2 \longrightarrow E = \frac{m_B^2 + m_K^2 - m_a^2}{2m_B} \approx \frac{m_B^2 - m_a^2}{2m_B} \tag{6.6}$$

Then, we can decompose the momentum of our particles into these components. Let us start with the Kaon momenta.

$$\begin{aligned} p_K^\mu &= \left(E + \sqrt{E^2 - m_K^2}\right) \frac{n^\mu}{2} + \left(E - \sqrt{E^2 - m_K^2}\right) \frac{\bar{n}^\mu}{2}, \\ &= \left[E + E\left(1 - \frac{m_K^2}{2E^2} + \ldots\right)\right] \frac{n^\mu}{2} + \left[E - E\left(1 - \frac{m_K^2}{2E^2} + \ldots\right)\right] \frac{\bar{n}^\mu}{2}, \end{aligned} \tag{6.7}$$

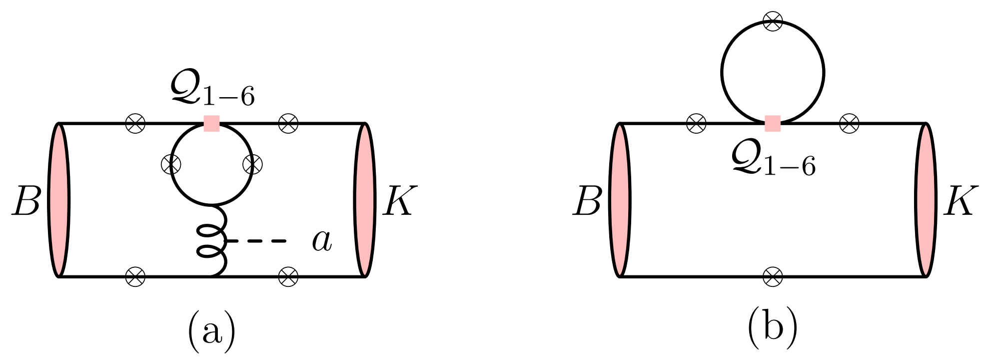


Figure 6.2: Loop-level contributions of the decay: $B \longrightarrow K\ a$. The circled cross represent the possible insertions of the ALP in the final state and in the gluon exchange in (a).

Since the energy is much larger than the Kaon mass, we can expand in powers of $\lambda \sim \frac{m_K}{E^2}$. The B meson is roughly ten times heavier than the Kaon, so we can see that the anti-collinear component is a small one ($\mathcal{O}(\lambda^2)$) and the collinear component is hard.

$$p_K^\mu \approx\ 2E\frac{n^\mu}{2} + \frac{m_K^2}{2E}\frac{\bar{n}^\mu}{2} \tag{6.8}$$

However, if the ALP is too heavy as the B meson then the collinear component could be also small (this is an interesting behavior). If the ALP is small enough in comparison to the B meson, then the collinear component of the Kaon is hard, therefore we might conclude that the Kaon is collinear. Let us see the ALP momentum.

$$\begin{aligned}
p_a^\mu &= \left(m_B - E - \sqrt{E^2 - m_K^2}\right)\frac{n^\mu}{2} + \left(m_B - E + \sqrt{E^2 - m_K^2}\right)\frac{\bar{n}^\mu}{2}, \\
&\approx \left(m_B - E - E\left[1 - \frac{m_K^2}{2E^2}\right]\right)\frac{n^\mu}{2} + \left(m_B - E + E\left[1 - \frac{m_K^2}{2E^2}\right]\right)\frac{\bar{n}^\mu}{2}, \\
&= (m_B - 2E)\frac{n^\mu}{2} + m_B\frac{\bar{n}^\mu}{2} + \mathcal{O}(m_K^2), \\
&= \left(\frac{m_a^2}{m_B}\right)\frac{n^\mu}{2} + m_B\frac{\bar{n}^\mu}{2},
\end{aligned} \tag{6.9}$$

Then, we can see that the hard component of the ALP is anti-collinear, then if the ALP mass heavy or not will determine if the collinear component is hard or soft. Summarizing the kinematic of our process.

$$p_K^\mu = 2E\frac{n^\mu}{2} + \frac{m_K^2}{2E}\frac{\bar{n}^\mu}{2} \tag{6.10}$$

$$p_a^\mu = m_B\left(\frac{m_a^2}{m_B^2}\right)\frac{n^\mu}{2} + m_B\frac{\bar{n}^\mu}{2}, \tag{6.11}$$

$$p_B^\mu = m_B\frac{n^\mu}{2} + m_B\frac{\bar{n}^\mu}{2}. \tag{6.12}$$

where we have defined the Kaon energy in the final state as

$$E = \frac{m_B^2 - m_a^2}{2m_B} \tag{6.13}$$

In addition, the light-cone components of each momentum are

$$n \cdot p_B = p_b^+ = m_B \tag{6.14}$$

$$n \cdot p_B = p_b^- = m_B \tag{6.15}$$

$$n \cdot p_a = p_a^+ = m_B \tag{6.16}$$

$$\bar{n} \cdot p_a = p_a^- = m_B \left(\frac{m_a^2}{m_B^2} \right) \tag{6.17}$$

$$n \cdot p_K = p_K^+ \approx \frac{m_K^2}{2E} \tag{6.18}$$

$$\bar{n} \cdot p_K = p_K^- \approx 2E \tag{6.19}$$

$$p_a \cdot p_K = 2E \cdot m_B + \frac{m_K^2}{2E} \frac{m_a^2}{2m_B} \approx 2m_B E = m_B^2 - m_a^2. \tag{6.20}$$

So far we have described the kinematic of the process at the hadronic level; however, the partonic particles inside the mesons play a role in hard-scattering amplitude. Partons inside the meson carry a fraction of the total momentum. In case of the Kaon, the strange and up quark carry a fraction of the momentum in the collinear direction. The Kaon perpendicular component is null but the partons can have perpendicular components such as way the total component vanishes.

Kaon:

$$p_s = uEn^\mu + k_\perp^\mu \quad , \quad p_u = \bar{u}En^\mu - k_\perp^\mu \quad , \quad p = p_s + p_u = En^\mu + ... \tag{6.21}$$

where $E = \frac{m_B^2 - m_a^2}{2m_B}$. The decomposition of the momentum in the B meson is different because the partons inside are a hard- and soft- particles. In the heavy-quark limit, we decompose the B meson as the b-quark carrying the whole momenta with an off-shellness given to the soft-quark.

B meson:

$$p_b = m_b v^\mu - l^\mu \qquad , \qquad l = l_+ \frac{\bar{n}^\mu}{2} + l_- \frac{n^\mu}{2} + l_\perp^\mu \qquad , \qquad \omega = n \cdot l \tag{6.22}$$

6.2 Partonic Amplitude

Let us start calculating the partonic amplitude of the tree-level contributions in the Fig. 6.1. The systematic procedure is to start writing the amplitude with the combination of effective weak interactions, effective ALP interaction and also QCD interaction. Then, Soft-collinear effective theory (SCET) comes to play and we use it as a tool to systematically disentangle the various relevant scales in processes involving soft (light-quark in B meson) and collinear partons (constituents in the Kaon) [32]. Thus, we decompose the full-QCD spinor for the partons inside the Kaon into collinear SCET spinors. After keeping the leading contribution for each diagram, we project into the light-cone distribution amplitudes or contract with the form factors. This last step will depend on the factorizable nature of the diagram.

6.2.1 Tree-level contributions

We start with the tree-level diagrams of the Fig. 6.1.

Four-fermion operator

We start with the Fig. 6.1 (a) in the case where the ALP couples to the strange quark in the final state. Since, there exist six operators to take into account, let us start with the operators $\mathcal{Q}_{1,2}$. The partonic amplitude for the four-fermion operator with ALP-strange interaction is given by

$$\begin{aligned}
\mathbb{A}^s_{1,2} &= -\lambda_u C_{1,2}\bar{s}\not{p}_a\frac{(P_L\kappa_S + P_R\kappa_s)}{f}\frac{\not{p}_x + m_s}{p_x^2 - m_s^2}\gamma_\mu(1-\gamma_5)v\;\bar{v}\gamma_\mu(1-\gamma_5)b \\
&= -\lambda_u\frac{C_{1,2}}{f}\bar{s}\frac{\not{p}_a(\not{p}_a + \not{p}_s)\kappa_S + m_s\not{p}_a\kappa_s}{m_a^2 + 2p_a p_s}\gamma_\mu(1-\gamma_5)v\;\bar{v}\gamma_\mu(1-\gamma_5)b \\
&= -\lambda_u\frac{C_{1,2}}{f}\bar{s}\frac{(m_a^2 + 2p_a\cdot p_s)\kappa_S + m_s\not{p}_a(\kappa_s - \kappa_S)}{m_a^2 + 2p_a p_s}\gamma_\mu(1-\gamma_5)v\;\bar{v}\gamma_\mu(1-\gamma_5)b \\
&= -\lambda_u\frac{C_{1,2}}{f}\bar{s}\left(\kappa_S + \frac{m_s}{m_a^2 + 2p_a\cdot p_s}\not{p}_a(\kappa_s - \kappa_S)\right)\gamma_\mu(1-\gamma_5)v\;\bar{v}\gamma_\mu(1-\gamma_5)b.
\end{aligned} \tag{6.23}$$

At this point, the leading contribution is of the order 1 and comes from the first term. Recalling the decomposition of the full-spinor into collinear spinors (see Eq. 3.66).

$$v = \left[\frac{\not{n}\not{\bar{n}}}{4} - \frac{\not{\bar{n}}}{2}\frac{\not{p}^{\perp} - m_v}{p_s^-}\right]v_\xi \quad , \quad \bar{s} = \bar{s}_\xi\left[\frac{\not{\bar{n}}\not{n}}{4} - \frac{\not{p}^{\perp} - m_s}{p_s^-}\frac{\not{\bar{n}}}{2}\right] \tag{6.24}$$

where the first term represents the large components, meanwhile the second one, the small components. Inserting the collinear spinors in the amplitude, we find

$$\mathbb{A}^s_{1,2} = -\,\lambda_u C_{1,2}\frac{\kappa_S}{f}\bar{s}^i_\xi\frac{\not{\bar{n}}}{2}(1-\gamma_5)v^j_\xi\;\bar{v}^l\not{n}(1-\gamma_5)b^k + \mathcal{O}(\lambda), \tag{6.25}$$

We have used in the last step the property that the perpendicular component of gamma matrices between two collinear spinors vanish because of $\{\not{n}, \gamma^\mu_\perp\} = 0$. In addition, in our convention $\kappa_S = [\kappa_D]_{22}$ and $\kappa_s = [\kappa_d]_{22}$. Now we calculate the contribution of the remaining four-fermion operators $\mathcal{Q}_{3,..,6}$

$$\begin{aligned}
\mathbb{A}^s_{3,\ldots,6} &= (\mathbf{-1})(-\lambda_t)(-C_{3,\ldots,6})\ \bar{s}\not{p}_a \frac{(P_L\kappa_S + P_R\kappa_s)}{f}\frac{\not{p}_x + m_s}{p_x^2 - m_s^2}\gamma_\mu(1-\gamma_5)b\ \bar{v}\gamma_\mu(1\pm\gamma_5)v \\
&= -\lambda_t \frac{C_{3,\ldots,6}}{f}\bar{s}\frac{\not{p}_a(\not{p}_a + \not{p}_s)\kappa_S + m_s\not{p}_a\kappa_s}{m_a^2 + 2p_a p_s}\gamma_\mu(1-\gamma_5)b\ \bar{v}\gamma_\mu(1\pm\gamma_5)v \\
&= -\lambda_t \frac{C_{3,\ldots,6}}{f}\bar{s}\frac{(m_a^2 + 2p_a\cdot p_s)\kappa_S + m_s\not{p}_a(\kappa_s - \kappa_S)}{m_a^2 + 2p_a p_s}\gamma_\mu(1-\gamma_5)v\ \bar{v}\gamma_\mu(1\pm\gamma_5)v \\
&= -\lambda_t \frac{C_{3,\ldots,6}}{f}\bar{s}\left(\kappa_S + \frac{m_s}{m_a^2 + 2p_a\cdot p_s}\not{p}_a(\kappa_s - \kappa_S)\right)\gamma_\mu(1-\gamma_5)b\ \bar{v}\gamma_\mu(1\pm\gamma_5)v
\end{aligned} \tag{6.26}$$

There is a relative minus sign between operators $\mathcal{Q}_{1,2}$ and $\mathcal{Q}_{3,\ldots,6}$ because of the antisymmetry in fermions between the s and t channel [17]. Similarly, the leading order of this amplitude is order 1, so after inserting the collinear spinors we find

$$\mathbb{A}^s_{3,\ldots,6} = -\lambda_t C_{3,\ldots,6}\frac{\kappa_S}{f}\bar{s}_\xi\gamma_\mu^\perp(1-\gamma_5)b\ \bar{v}_\xi\gamma_\perp^\mu(1\pm\gamma_5)v. \tag{6.27}$$

Then, we can apply the Fierz transformation to have the same structure as the operators $\mathcal{Q}_{1,2}$,

$$\begin{aligned}
2\,(\bar{u}_\xi\mathbb{M}u_h)\,(\bar{v}_q\mathbb{N}v_\xi) = {}& \left(\bar{u}_\xi\frac{\bar{\not{n}}}{2}v_\xi\right)\left(\bar{v}_q\mathbb{N}\frac{\not{n}}{2}\mathbb{M}u_h\right) + \left(\bar{u}_\xi\frac{\bar{\not{n}}}{2}\gamma_5 v_\xi\right)\left(\bar{v}_q\mathbb{N}\gamma_5\frac{\not{n}}{2}\mathbb{M}u_h\right) \\
&+ \left(\bar{u}_\xi\frac{\bar{\not{n}}}{2}\gamma_\alpha^\perp v_\xi\right)\left(\bar{v}_q\mathbb{N}\gamma_\perp^\alpha\frac{\not{n}}{2}\mathbb{M}u_h\right).
\end{aligned} \tag{6.28}$$

Where "h" represents the hard spinors which is the b quark in our case, the subscript ξ represents collinear spinors which are the strange and anti-up quark inside the Kaon. Then, the last spinor represents to the soft component in the B meson. Our Dirac structure is given by,

$$\mathbb{M} = \gamma_{\mu\perp}(1-\gamma_5) \quad , \quad \mathbb{N} = \gamma_{\mu\perp}(1\pm\gamma_5) \tag{6.29}$$

Where we have V-A and V+A structure in the $\mathbb{N}$ matrix.

$$\begin{aligned}
\mathbb{N}\frac{\not{n}}{2}\mathbb{M} &= \gamma_{\mu\perp}(1\pm\gamma_5)\frac{\not{n}}{2}\gamma_{\mu\perp}(1-\gamma_5) = \gamma_{\mu\perp}\frac{\not{n}}{2}\gamma_{\mu\perp}2(1-\gamma_5) = -2\not{n}(1-\gamma_5) \\
\mathbb{N}\gamma_5\frac{\not{n}}{2}\mathbb{M} &= \gamma_{\mu\perp}(1\pm\gamma_5)\gamma_5\frac{\not{n}}{2}\gamma_{\mu\perp}(1-\gamma_5) = -\gamma_{\mu\perp}\frac{\not{n}}{2}\gamma_{\mu\perp}2(1-\gamma_5) = 2\not{n}(1-\gamma_5) \\
\mathbb{N}\gamma_\alpha^\perp\frac{\not{n}}{2}\mathbb{M} &= \gamma_{\mu\perp}(1\pm\gamma_5)\gamma_\alpha^\perp\frac{\not{n}}{2}\gamma_{\mu\perp}(1-\gamma_5) \propto \gamma_{\mu\perp}\gamma_\alpha^\perp\gamma_\perp^\mu = 0.
\end{aligned} \tag{6.30}$$

The V+A structure that belongs to the operators $\mathcal{Q}_{5,6}$ vanish. Then, there are only contributions from Wilson Coefficients $C_{3,4}$.

$$\mathbb{A}^s_{3,\ldots,6} = \lambda_t C_{3,4}\frac{\kappa_S}{f}\ \bar{s}_\xi\ \frac{\bar{\not{n}}}{2}(1-\gamma_5)\ v_\xi\ \bar{v}\ \not{n}(1-\gamma_5)\ b \tag{6.31}$$

Then, combining the two contributions.

$$\mathbb{A}^s_{1,\dots,6} = -(\lambda_u C_{1,2} - \lambda_t C_{3,4})\frac{\kappa_S}{f}\ \bar{s}_\xi\ \frac{\not{n}}{2}(1-\gamma_5)\ v_\xi\ \bar{v}\ \not{n}(1-\gamma_5)\ b \tag{6.32}$$

The Wilson coefficients $C_{1,2,3,4}$ carry the color indices.

$$\begin{aligned}
\mathcal{Q}_1 &= (\bar{s}_i v_i)_{V-A}\,(\bar{v}_j b_j)_{V-A} \to \delta_{ij}\delta_{lk}\ \ \bar{s}_i v_j \bar{v}_l b_k \\
\mathcal{Q}_2 &= (\bar{s}_i v_j)_{V-A}\,(\bar{v}_j b_i)_{V-A} \to \delta_{ik}\delta_{lj}\ \ \bar{s}_i v_j \bar{v}_l b_k \\
\mathcal{Q}_3 &= (\bar{s}_i b_i)_{V-A}\,(\bar{v}_j v_j)_{V-A} \to \delta_{ik}\delta_{lj}\ \ \bar{s}_i v_j \bar{v}_l b_k \\
\mathcal{Q}_4 &= (\bar{s}_i b_j)_{V-A}\,(\bar{v}_j v_i)_{V-A} \to \delta_{ij}\delta_{lk}\ \ \bar{s}_i v_j \bar{v}_l b_k
\end{aligned} \tag{6.33}$$

The partonic amplitude of the four-operator for the ALP-strange interaction scenario is given by

$$\mathbb{A}^s_{1,\dots,6} = -\left[\delta_{ij}\delta_{lk}(\lambda_u C_1 - \lambda_t C_4) + (\lambda_u C_2 - \lambda_t C_3)\delta_{ik}\delta_{lj}\right]\frac{\kappa_S}{f}\ \bar{s}^i_\xi\ \frac{\not{n}}{2}(1-\gamma_5)\ v^j_\xi\ \bar{v}^l\ \not{n}(1-\gamma_5)\ b^k \tag{6.34}$$

We expect that the contribution of the Feynman graph where the ALP couples to the b-quark has a similar Dirac structure but proportional to the ALP-bottom flavor-conserving coupling. Let us start with the operators $\mathcal{Q}_{1,2}$.

$$\mathbb{A}^b_{1,2} = \lambda_u \frac{C_{1,2}}{f}\ \bar{s}_\xi\ \frac{\not{n}}{2}(1-\gamma_5)\ v_\xi\ \bar{v}\ \not{n}(1-\gamma_5)\left[\kappa_B + \frac{\not{p}_a}{m_b}(\kappa_b - \kappa_B)\right]\ b \tag{6.35}$$

This amplitude is similar to the amplitude in Eq. 6.25 except just for a factor (-1) because of the direction of the propagator. In this case, we can not neglect the mass term the ALP momentum has hard components. Now continuing with the left operators $\mathcal{Q}_{3,\dots,6}$. Also, we expect to have a similar Dirac structure as the amplitude in Eq. 6.27.

$$\mathbb{A}^b_{3,\dots,6} = \lambda_t \frac{C_{3,\dots,6}}{f}\bar{s}_\xi\ \gamma^\perp_\mu(1-\gamma_5)\left[\kappa_B + \frac{\not{p}_a}{m_b}(\kappa_b - \kappa_B)\right]\ b\ \bar{v}_\xi\ \gamma^\perp_\mu(1\pm\gamma_5)v. \tag{6.36}$$

Where the (-1) factor sign relative the $\mathcal{Q}_{1,2}$ has been included. Then, in order to have contraction between collinear spinors, we have to use the Fierz Transformation given by Eq. 6.28. Our matrices are

$$\mathbb{M} = \gamma_{\mu\perp}(1-\gamma_5)\left[\kappa_B + \frac{\not{p}_a}{m_b}(\kappa_b - \kappa_B)\right] \quad , \quad \mathbb{N} = \gamma_{\mu\perp}(1\pm\gamma_5) \tag{6.37}$$

The Dirac structure is very similar as the ALP-s case except for the ALP coupling that is in the very right. We are able to use the result of Eq. 6.30.

$$\mathbb{A}^b_{3,\dots,6} = -\ \lambda_t \frac{C_{3,4}}{f}\ \bar{s}_\xi\ \frac{\not{n}}{2}(1-\gamma_5)\ v_\xi\ \bar{v}\ \not{n}(1-\gamma_5)\left[\kappa_B + \frac{\not{p}_a}{m_b}(\kappa_b - \kappa_B)\right]\ b \tag{6.38}$$

Where again the V+A structure operators do not contribute. Then, superposing the solutions we have

$$\mathbb{A}^b_{1,\ldots,6} = \frac{\lambda_u C_{1,2} - \lambda_t C_{3,4}}{f}\ \bar{s}_\xi\ \frac{\not{\bar{n}}}{2}(1-\gamma_5)\ v_\xi\ \bar{v}\ \not{n}(1-\gamma_5)\left[\kappa_B + \frac{\not{p}_a}{m_b}(\kappa_b - \kappa_B)\right]\ b \tag{6.39}$$

Where the Wilson coefficients contain the color factors given by the color structure in Eq. 6.33. In principle, we can decompose the ALP momentum on its light-like components.

$$\mathbb{A}^b_{1,\ldots,6} = \frac{C_{1,2,3,4}}{f}\ \bar{s}^i_\xi\ \frac{\not{\bar{n}}}{2}(1-\gamma_5)\ v^j_\xi\ \bar{v}^l\ \not{n}(1-\gamma_5)\left[\kappa_B + \frac{\not{\bar{n}}}{2}(\kappa_b - \kappa_B)\right]\ b^k, \tag{6.40}$$

where

$$C_{1,2,3,4} = \delta_{ij}\delta_{lk}(\lambda_u C_1 - \lambda_t C_4) + (\lambda_u C_2 - \lambda_t C_3)\delta_{ik}\delta_{lj}. \tag{6.41}$$

Since $\not{n}$ anticommutes with γ_5, then the anti-collinear component of the ALP momentum survives only. The contribution from the ALP-up coupling in the initial state is equal in module but opposite sign than the amplitude of the ALP-up coupling in the final state.

$$\mathbb{A}^{u,f}_{1,\ldots,6} = -\mathbb{A}^{u,o}_{1,\ldots,6} \tag{6.42}$$

Summing up the four contributions of this type

$$\begin{aligned}
&\mathbb{A} = \mathbb{A}^s + \mathbb{A}^b + \mathbb{A}^{u,o} + \mathbb{A}^{u,f} \\
&\mathbb{A} = \frac{C_{1,2,3,4}}{f}\ \bar{s}^i_\xi\ \frac{\not{\bar{n}}}{2}(1-\gamma_5)\ v^j_\xi\ \bar{v}^l\ \not{n}(1-\gamma_5)\left[\kappa_B - \kappa_S + \frac{\not{\bar{n}}}{2}(\kappa_b - \kappa_B)\right]\ b^k
\end{aligned} \tag{6.43}$$

Chromo-magnetic dipole operators

The next diagram to calculate is the type (b) in Fig. 6.1. The light quark in the initial state is soft, meanwhile the light quark in the final state is collinear. Therefore, the exchange gluon propagator must be hard-collinear. In other words, the leading contribution in the DENOMINATOR of this amplitude is subleading. We will see that when the ALP couples to the external legs, the main contribution in the NUM is subleading as well which means that the partonic amplitude is order 1. On the other hand, when the ALP interacts with the gluon propagator, the leading contribution of the NUMERATOR is order 1. Therefore, the amplitude will scale as $\frac{1}{\lambda}$.

ALP couples to the gluon propagator

The partonic amplitude is given by

$$\mathbb{C} = i\left(\frac{-g_s^2 m_b}{4\pi^2}\right)\left(\frac{C_{GG}}{f}\frac{\alpha_s}{\pi}\right)(-\lambda_t)\ C_8\ \bar{s}^i\,[p_x, \gamma_\mu]\, P_R t^a_{ij} b_j\ \bar{v}_l \gamma_\nu t^a_{lk} v_k \cdot \epsilon_{\alpha\mu\beta\nu}\frac{p_x^\alpha p_y^\beta}{p_x^2 p_y^2} \tag{6.44}$$

Where $p_x = p_b - p_s$ and $p_y = p_u - l$. We can focus in the commutator that one component contributes only.

$$\text{NUM} \propto \left[\not{p}_x, \gamma_\mu\right] \epsilon_{\alpha\mu\beta\nu} p_x^\alpha = (2\not{p}_x \gamma_\mu - 2p_{\mu x}) = 2\not{p}_x \gamma_\mu \epsilon_{\alpha\mu\beta\nu} p_x^\alpha \tag{6.45}$$

where the second term vanishes because the product of symmetric times anti-symmetric terms under $\mu \longleftrightarrow \alpha$. We can also focus on the term that is proportional to the Levi-Civita tensor which involves the two propagator and one of them is hard-collinear. Therefore, the leading contribution in the DENOMINATOR is order of λ.

$$\begin{aligned}
\epsilon_{\alpha\mu\beta\nu} \frac{p_x^\alpha p_y^\beta}{p_x^2 p^y} &= \frac{\epsilon_{\alpha\mu\beta\nu}}{p_x^2 p_y^2} \left(\frac{n^\alpha}{2} p_x^- + \frac{\bar{n}^\alpha}{2} p_x^+ + p_x^\perp\right) \left(\frac{n^\alpha}{2} p_y^- + \frac{\bar{n}^\alpha}{2} p_y^+ + p_y^\perp\right) \\
&= \frac{\epsilon_{\alpha\mu\beta\nu}}{p_x^2 p_y^2} \frac{n^\alpha \bar{n}^\beta}{4} \left(p_x^- \ p_y^+ - p_y^- p_x^+\right) + \mathcal{O}(p^\perp) \\
&= \epsilon_{\alpha\mu\beta\nu} \frac{n^\alpha \bar{n}^\beta}{4} \left(\frac{1}{p_y^- p_x^+} - \frac{1}{p_x^- \ p_y^+}\right) + \ldots \\
&= \epsilon_{\alpha\mu\beta\nu} \frac{n^\alpha \bar{n}^\beta}{4} \left(\frac{1}{2\bar{u} E m_B} - \frac{1}{m_b - 2uE} \frac{1}{(-\omega)}\right) \\
&= \epsilon_{\alpha\mu\beta\nu} \frac{n^\alpha \bar{n}^\beta}{4} \left(\frac{1}{m_b - 2uE} \frac{1}{\omega}\right) + \ldots
\end{aligned} \tag{6.46}$$

We can see that the omega factor in the denominator after the LCDA's projection will give a big contribution to the amplitude. Therefore, this term is by far the leading contribution and we neglect terms that are order of 1 or λ. Plugging the leading contribution into the amplitude and forgetting for a while the color factors and other constants, we get.

$$\begin{aligned}
\mathbb{C} &= -i\lambda_t C_8 \left(\frac{-g_s^2 m_b}{4\pi^2}\right) \left(\frac{C_g}{f} \frac{\alpha_s}{\pi}\right) \ \bar{s}^i 2\not{p}_x \gamma_\perp^\mu P_R b_j \ \bar{v}_l \gamma_\nu^\perp v_k \cdot \epsilon_{\alpha\mu\beta\nu} \frac{n^\alpha \bar{n}^\beta}{4} \left(\frac{1}{(m_b - 2uE)} \frac{1}{\omega}\right) (t^a)^2 \\
&= -i\lambda_t C_8 \left(\frac{-g_s^2 m_b}{4\pi^2}\right) \left(\frac{C_g}{f} \frac{\alpha_s}{\pi}\right) \ \bar{s}^i \frac{\not{\bar{n}}}{2} \gamma_\perp^\mu P_R b_j \ \bar{v}_l \ \gamma_\perp^\nu \ v_k \cdot \epsilon_{\mu\nu}^\perp \left(\frac{m_b}{(m_b - 2uE)} \frac{1}{\omega}\right) (t^a)^2.
\end{aligned} \tag{6.47}$$

Since we are taking the order 1 terms in the NUMERATOR, then the only term that has survived from the gamma matrices contraction is the perpendicular one. Then, we can use Fierz Transformation: $\mathbb{M} = \frac{\not{\bar{n}}}{2} \gamma_\perp^\mu P_R \ , \ \mathbb{N} = \gamma_\nu^\perp \epsilon_{\mu\nu}^\perp$.

$$\begin{aligned}
\mathbb{N} \frac{\not{n}}{2} \mathbb{M} &= i \frac{\not{n}\not{\bar{n}}}{4} 2P_R \\
\mathbb{N} \gamma_5 \frac{\not{n}}{2} \mathbb{M} &= -N \frac{\not{n}}{2} \mathbb{M} \\
\mathbb{N} \gamma_\alpha^\perp \frac{\not{n}}{2} \mathbb{M} &= 0.
\end{aligned} \tag{6.48}$$

Then the amplitude after Fierz Transformation.

$$\mathbb{C} = C_8 \frac{\lambda_t}{2} \left(\frac{-g_s^2 m_b}{4\pi^2}\right) \left(\frac{C_g}{f} \frac{\alpha_s}{\pi}\right) \left[\frac{m_B}{m_B - 2uE} \frac{1}{\omega}\right] (t^a)^2 \ \bar{s}_\xi^i \frac{\not{\bar{n}}}{2} (1 - \gamma_5) v_\xi^k \bar{v}^l \frac{\not{n}\not{\bar{n}}}{4} (1 + \gamma_5) b^j. \tag{6.49}$$

6.2.1.1 ALP couples to the external states

Again, the ALP can couple in the external legs and as expected the ALP-up quark amplitudes are equal in modulo but opposite sign. Therefore, we have the contributions from the ALP-strange and ALP-bottom interaction. Let us start with the first one.

$$\mathbb{B}^s = \frac{(-\lambda_t)}{f}\left(\frac{g_s^2 m_B}{4\pi^2}\right) C_8 \cdot \bar{s}^i \not{p}_a (P_L \kappa_S + P_R \kappa_s) \frac{\not{p}_h + m_s}{p_h^2 - m_s^2} \left[\not{p}_{hc}, \gamma_\mu\right] P_R\, t^a_{ij}\, b^j \frac{1}{p_{hc}^2} \bar{v}^l \gamma_\mu t^a_{lk} v_k. \tag{6.50}$$

where $p_h = p_a + p_s$ and $p_{hc} = p_u - l$. The propagators are given by

$$\begin{aligned} p_h = p_a + p_s &\longrightarrow p_h^2 - m_s^2 = 2p_a \cdot p_s + p_a^2 = p_a^+ \cdot p_s^+ + p_a^- \cdot p_s^+ + m_a^2 = 2m_b u E + m_a^2 \\ p_{hc} = p_u - l &\longrightarrow p_{hc}^2 = -2p_u \cdot l + p_u^2 + l^2 \approx -2p_u \cdot l = -p_u^- \cdot l^+ = -2\bar{u}E\omega. \end{aligned} \tag{6.51}$$

Following the same steps as in Eq. 6.23, we find

$$\mathbb{B}^s = \frac{(-\lambda_t)}{f}\left(\frac{g_s^2 m_B}{4\pi^2}\right) \frac{C_8}{-2\bar{u}E\omega} (t^a)^2 \cdot \bar{s}^i \left[\kappa_S + \frac{m_s \not{p}_a}{m_a^2 + 2m_B u E}(\kappa_s - \kappa_S)\right] \left[\not{p}_{hc}, \gamma_\mu\right] P_R\, b^j \bar{v}^l \gamma_\mu v_k. \tag{6.52}$$

Similarly for the case when the ALP couples to the bottom quark. We find

$$\mathbb{B}^b = -\frac{(-\lambda_t)}{f}\left(\frac{g_s^2 m_B}{4\pi^2}\right) \frac{C_8}{-2\bar{u}E\omega} (t^a)^2 \cdot \bar{s}^i \left[\not{p}_{hc}, \gamma_\mu\right] P_R \left[\kappa_b + \frac{\not{p}_a}{m_B}(\kappa_B - \kappa_b)\right] b^j \bar{v}^l \gamma_\mu v_k. \tag{6.53}$$

We see that in the two amplitudes above there is common factor $\frac{1}{\omega}$ that carry the subleading scale in the DENOMINATOR. Writing the amplitude in function of collinear fields will explicitly show that the NUMERATOR is also subleading because a hard-collinear momentum next to a collinear field $\sim \lambda$. In order to exhibit this in $\mathbb{B}^s$ we use the property of gamma matrices: $[\not{p}, \gamma_\mu]\, \not{p}_a = \not{p}_a\, [\not{p}, \gamma_\mu] + 4(p_{a\mu}\not{p} - p_a \cdot p_{hc}\gamma_\mu)$ to bring the commutator next to the strange spinor. For simplicity, we neglect the factors.

$$\begin{aligned} \mathbb{B}^s \propto \bar{s}\, \left[\not{p}_{hc}, \gamma_\mu\right] \left[\kappa_S + \frac{m_s \not{p}_a}{m_a^2 + 2p_a \cdot p_s}(\kappa_s - \kappa_S)\right] P_R\, b\, \frac{1}{\omega}\, \bar{v}\, \gamma_\mu\, v_k + \\ + 4\frac{m_s}{m_a^2 + 2m_B u E}(\kappa_s - \kappa_S)\bar{s}\, \left[(p_a \cdot p_{hc})\gamma_\mu - \not{p}_{hc} p_{a\mu}\right] P_R\, b\, \frac{1}{\omega}\, \bar{v}\, \gamma_\mu\, v_k. \end{aligned} \tag{6.54}$$

We intent to show that a hard-collinear momentum next to a collinear spinor is subleading.

$$\begin{aligned} \bar{s}\not{p}_{hc} &= \bar{s}_\xi \left[\frac{\not{\bar{n}}\not{n}}{4} - \frac{\not{p}^\perp - m_s}{p_s^-}\frac{\not{\bar{n}}}{2}\right] \not{p}_{hc} \\ &= \bar{s}_\xi \left[\frac{\not{n}}{2}p_{hc}^+ + \frac{\bar{n}}{2}\not{p}_{hc}^\perp - \left(\frac{\not{p}^\perp - m_s}{p_s^-}\right)\frac{\bar{n}\not{n}}{4}p_{hc}^- + \mathcal{O}(\lambda^2)\right]. \end{aligned} \tag{6.55}$$

We can see that the large components of the collinear spinors kills the hard component of the momentum and takes the subleading one. Then, combining the expression of these two amplitudes in matter and keeping terms up to $\mathcal{O}(\lambda)$ we find

$$\mathbb{B} = \mathbb{B}^s + \mathbb{B}^b \tag{6.56}$$

$$\propto \bar{s}\,\left[\not{p}_{hc}, \gamma_\mu\right] P_R \left[\kappa_S - \kappa_b - \frac{\not{p}_a}{m_b}(\kappa_B - \kappa_b)\right]\; b\,\frac{1}{\omega}\,\bar{v}\,\gamma_\mu\,v_k \tag{6.57}$$

$$+\, 2\frac{m_s}{m_a^2 + 2p_a \cdot p_s} p_a^+ \cdot p_{hc}^-\; \bar{s}\gamma_\mu\; P_R(\kappa_s - \kappa_S)\; b\frac{1}{\omega}\;\bar{v}\gamma_\mu v + \mathcal{O}(\lambda^2) \tag{6.58}$$

The standard procedure to follow as the past diagrams would be to apply Fierz Transformation in order to have the collinear partons contracted between them and the soft- and hard-parton contracted between them. Using SCET will allows us to factorize the soft and collinear scales indeed; however, Fierz transformation is an extra step in the calculation that will extend it by far and might bring potential sources of mistakes in the calculation. Therefore, we opt to use the old fashioned QCD Factorization for this diagram. We write the amplitude with the full-QCD spinors as we start, replace the kinematic and considering terms up to $\mathcal{O}(\lambda)$ since as we have seen there are subleading contributions and finally project the amplitude into the LCDA's.

$$\begin{aligned}\mathbb{B} = K \cdot \bar{s}^i \left[\not{p}_{hc}, \gamma_\mu\right] P_R \left[\kappa_S - \kappa_b + \frac{m_s \not{p}_a}{m_a^2 + 2m_B u E}(\kappa_s - \kappa_S) + \frac{\not{p}_a}{m_B}(\kappa_b - \kappa_B)\right] b^j \bar{v}^l \gamma_\mu v_k \\ + K \cdot 4 \frac{m_s}{m_a^2 + 2u m_B E}\bar{s}^i \left(p_a \cdot p_{hc}\gamma_\mu - \not{p}_{hc} p_{a\mu}\right) P_R(\kappa_s - \kappa_S) b^j\; \bar{v}^l \gamma_\mu v^k.\end{aligned} \tag{6.59}$$

where $K = \frac{(-\lambda_t)}{f}\left(\frac{g_s^2 m_B}{4\pi^2}\right)\frac{C_8}{-2\bar{u}E\omega}(t^a)^2$. The kinematic for this process is given by

$$\begin{aligned} p_s &= \bar{u}En^\mu + k^\perp \\ p_u &= \bar{u}En^\mu - k^\perp \\ l &= \omega\frac{\bar{n}^\mu}{2} + l^\perp \\ p_{hc} &= p_u - l = \bar{u}En^\mu - \frac{\bar{n}^\mu}{2}\omega - k^\perp - l^\perp. \end{aligned} \tag{6.60}$$

6.3 LCDA's projection

At tree-level diagrams in Fig. 6.1, we have only "non-factorizable" terms, therefore the partonic amplitude has to be convoluted with LCDA's.

Four-fermion operator

Recalling the amplitude in Eq. 6.34. Thus, we have to identify the Dirac structure between the spinors in order to project them into the LCDA's.

$$\Gamma^K = \frac{\bar{\not{n}}}{2}(1-\gamma_5) \quad , \quad \Gamma^B = \not{n}(1-\gamma_5)\left[\kappa_B - \kappa_S + \frac{\bar{\not{n}}}{2}(\kappa_b - \kappa_B)\right] \tag{6.61}$$

Projecting into the LCDA Kaon we have

$$\begin{aligned}\Phi^K_{ij}\Gamma^K &= \frac{\delta_{ij}}{N_c} i\frac{f_K}{4}\left[\not{p}\gamma_5\Phi^K(u) - \mu_K\gamma_5\left(\Phi^K_p(u) - i\sigma_{\mu\nu}\frac{p^\mu\bar{p}^\nu}{p\cdot\bar{p}}\frac{\Phi'_\sigma}{6} + i\sigma_{\mu\nu}p^\mu\frac{\Phi_\sigma}{6}\frac{\partial}{\partial k^\nu_\perp}\right)\right]\frac{\bar{\not{n}}}{2}(1-\gamma_5) \\ \mathrm{Tr}\left\{\Phi^k\Gamma^K\right\} &= \frac{\delta_{ij}}{N_c} i\frac{f_K}{4}\left[2\bar{n}\cdot p\ \Phi^K\right] = \frac{\delta_{ij}}{N_c} i\frac{f_K}{4}\left[4E\right]\ \Phi^K.\end{aligned} \tag{6.62}$$

Now, projecting the Dirac structure into the B meson LCDA.

$$\begin{aligned}\Phi^B_{lk}\Gamma^B &= -i\frac{\delta_{lk}}{N_c}\frac{f_B}{4}m_B\left(\frac{1+\not{v}}{2}\left[\Phi^B_+\bar{\not{n}} + \Phi^B_-\not{n} - \omega\Phi^B_-\gamma^\nu_\perp\frac{\partial}{\partial l^\nu_\perp}\right]\gamma_5\right)\not{n}(1-\gamma_5) \\ &\qquad \left[\kappa_B - \kappa_S + \frac{\bar{\not{n}}}{2}(\kappa_b - \kappa_B)\right], \\ \mathrm{Tr}\{\Phi^B_{lk}\Gamma^B\} &= -i\frac{\delta_{lk}}{N_c}\frac{f_B}{4}m_B\left[\mathrm{Tr}\left\{\frac{1+\not{v}}{2}\bar{\not{n}}\not{n}(1-\gamma_5)\right\}(\kappa_B-\kappa_S)\Phi^B_+ \ + \right. \\ &\qquad \left.\mathrm{Tr}\left\{\frac{1+\not{v}}{2}\bar{\not{n}}\not{n}(1-\gamma_5)\frac{\bar{\not{n}}}{2}\right\}(\kappa_b-\kappa_B)\Phi^B_+\right] \\ &= -i\frac{\delta_{lk}}{N_c}\frac{f_B}{4}m_B\left[(2\bar{n}\cdot n)(\kappa_B-\kappa_S)\Phi^B_+ + 2(\bar{n}\cdot n)(\bar{n}\cdot v)(\kappa_b-\kappa_B)\Phi^B_+\right] \\ &= -i\frac{\delta_{lk}}{N_c}\frac{f_B}{4}m_B\left[4(\kappa_B-\kappa_S)\Phi^B_+ + 4(\kappa_b-\kappa_B)\Phi^B_+\right] \\ &= -i\frac{\delta_{lk}}{N_c}\frac{f_B}{4}4m_B(\kappa_b-\kappa_S)\Phi^B_+.\end{aligned} \tag{6.63}$$

Then, we have to integrate over the momentum fractions.

$$\begin{aligned}\mathbb{A} &\propto \int d\omega\,\mathrm{Tr}\{\Phi^B\Gamma^B\}\int du\,\mathrm{Tr}\{\Phi^K\Gamma^K\} \\ \mathbb{A} &= \frac{C_{1,2,3,4}}{f}\frac{\delta_{ij}\delta_{lk}}{N_c^2}\frac{-i^2 f_K f_B}{16}(4E)(4m_B)(\kappa_b-\kappa_S)\int d\omega\Phi^B_+\int du\Phi^K \\ \mathbb{A} &= \frac{G_F}{\sqrt{2}}\frac{\lambda_u(N_cC_1+C_2)-\lambda_t(N_cC_4+C_3)}{N_c}(f_Kf_B)\frac{(m_B^2-m_a^2)}{2}\frac{(\kappa_b-\kappa_S)}{f},\end{aligned} \tag{6.64}$$

where $C_{1,2,3,4} = \delta_{ij}\delta_{lk}(\lambda_uC_1 - \lambda_tC_4) + (\lambda_uC_2 - \lambda_tC_3)\delta_{ik}\delta_{lj}$.

Chromo-magnetic dipole operators

ALP-gluon coupling

Recalling the amplitude in Eq. 6.49. Identifying the Dirac structures, we have

$$\Gamma^K = \frac{\not{\bar{n}}}{2}(1-\gamma_5) \quad , \quad \Gamma^B = \frac{\not{n}\not{\bar{n}}}{4}(1+\gamma_5). \tag{6.65}$$

The Kaon Dirac structure is the same as Eq. 6.61. Therefore the trace over the projection.

$$\mathrm{Tr}\Big\{\Phi^k\Gamma^K\Big\} = \frac{\delta_{ik}}{N_c} i\frac{f_K}{4}\left[2\bar{n}\cdot p\ \Phi^K\right] = \frac{\delta_{ij}}{N_c} i\frac{f_K}{4}\left[4E\right]\ \Phi^K. \tag{6.66}$$

Projecting into B meson LCDA we find

$$\begin{aligned}
\Phi^B_{lj}\Gamma^B &= -i\frac{\delta_{lj}}{N_c}\frac{f_B}{4}m_B\left(\frac{1+\not{v}}{2}\left[\Phi^B_+\not{\bar{n}} + \Phi^B_-\not{n} - \omega\Phi^B_-\gamma^\nu_\perp\frac{\partial}{\partial l^\nu_\perp}\right]\gamma_5\right)\frac{\not{n}\not{\bar{n}}}{4}(1+\gamma_5),\\
\mathrm{Tr}\big(\Phi^B_{lj}\Gamma^B\big) &= -\frac{\delta_{lj}}{N_c}\frac{if_B}{4}m_B\left\{\mathrm{Tr}\left[\frac{1+\not{v}}{2}\not{\bar{n}}\gamma_5\frac{\not{n}\not{\bar{n}}}{4}(1+\gamma_5)\right]\right\}\Phi^B_+,\\
&= -\frac{\delta_{lj}}{N_c}\frac{if_B}{4}(2m_B)\Phi^B_+.
\end{aligned} \tag{6.67}$$

Inserting the traces into the amplitude, we find

$$\begin{aligned}
\mathbb{C} = C_8\frac{\lambda_t}{2}\left(\frac{-g_s^2 m_B}{4\pi^2}\right)\left(\frac{C_g}{f}\frac{\alpha_s}{\pi}\right)\left(\frac{\delta_{ik}\delta_{lj}(t^a)^2}{N_c^2}\right)\left(\frac{-i^2 f_B f_K}{16}\right)(2m_B^2)(4E)\int\frac{\Phi^B_+}{\omega}\\
\int du\Phi^K\frac{1}{m_B-2uE}.
\end{aligned} \tag{6.68}$$

We solve the integral for the Kaon LCDA for a general ALP mass.

$$\int du\frac{\Phi_K}{m_B-2uE} = \frac{3}{m_B}\left(\frac{r^2-2r\log(r)-1}{(r-1)^3}\right) = \frac{3}{m_B}\mathbb{L}(r),\quad r=\frac{m_a^2}{m_B^2} \tag{6.69}$$

Where we have used the asymptotic limit of the kaon LCDA: $\Phi_K = 6u(1-u)$.

$$\mathbb{C} = C_8\frac{\lambda_t}{2}\left(\frac{-g_s^2}{4\pi^2}\right)\left(\frac{N_c^2-1}{2N_c^2}\right)\left(\frac{f_B f_K}{4}\right)\left(\frac{C_g}{f}\frac{\alpha_s}{\pi}\right)\frac{3m_B}{\omega_o}(m_B^2-m_a^2)\mathbb{L}(r)\ . \tag{6.70}$$

For the particular case when the ALP is anti-collinear, in other words: $\frac{m_a^2}{m_B^2}\sim\lambda^2$

$$\mathbb{C} = -\lambda_t C_8\left(\frac{g_s^2}{4\pi^2}\right)\left(\frac{N_c^2-1}{2N_c^2}\right)\left(\frac{f_B f_K}{4}\right)\left(\frac{C_g}{f}\frac{\alpha_s}{\pi}\right)(m_B^3)\frac{3}{2\omega_o} \tag{6.71}$$

ALP couples to the external legs

Recalling the expression in Eq. 6.59.

$$\begin{aligned}\mathbb{B} = K \cdot \bar{s}^i \left[\not{p}_{hc}, \gamma_\mu\right] P_R \left[\kappa_S - \kappa_b + \frac{m_s \not{p}_a}{m_a^2 + 2m_B uE}(\kappa_s - \kappa_S) + \frac{\not{p}_a}{m_B}(\kappa_b - \kappa_B)\right] b^j \bar{v}^l \gamma_\mu v_k \\ + K \cdot 4\frac{m_s}{m_a^2 + 2um_B E}\bar{s}^i \left(p_a \cdot p_{hc}\gamma_\mu - \not{p}_{hc} p_{a\mu}\right) P_R(\kappa_s - \kappa_S) b^j\ \bar{v}^l \gamma_\mu v^k.\end{aligned} \tag{6.72}$$

where $K = \frac{(-\lambda_t)}{f}\left(\frac{g_s^2 m_B}{4\pi^2}\right)\frac{C_8}{-2\bar{u}E\omega}(t^a)^2$. We divide the amplitude in three pieces.

$$\begin{aligned}\mathbb{B}_1 &= 2K\bar{s}\not{p}_{hc}\gamma_\mu P_R \left[\kappa_S - \kappa_b + \frac{m_s \not{p}_a}{m_a^2 + 2m_B uE}(\kappa_s - \kappa_S) + \frac{\not{p}_a}{m_B}(\kappa_b - \kappa_B)\right] b^j\ \bar{v}^l \gamma_\mu v_k, \\ \mathbb{B}_2 &= -2K\bar{s} P_R \left[\kappa_S - \kappa_b + \frac{m_s \not{p}_a}{m_a^2 + 2m_B uE}(\kappa_s - \kappa_S) + \frac{\not{p}_a}{m_B}(\kappa_b - \kappa_B)\right] b^j\ \bar{v}^l \not{p}_{hc} v_k, \\ \mathbb{B}_3 &= K \cdot 4\frac{m_s}{m_a^2 + 2um_B E}\bar{s}^i \left(p_a \cdot p_{hc}\gamma_\mu - \not{p}_{hc} p_{a\mu}\right) P_R(\kappa_s - \kappa_S) b^j\ \bar{v}^l \gamma_\mu v^k,\end{aligned} \tag{6.73}$$

The approach is to plug the kinematic of the momentum's taking up to subleading terms (see Eq. 6.60).

$$\begin{aligned}\mathbb{B}_1 &= 2K\bar{s}\left(\bar{u}E\not{n} - \omega\frac{\not{\bar{n}}}{2} - \not{l}^\perp - \not{k}^\perp\right)\gamma_\mu P_R\,[\ldots]\, b^j\ \bar{v}^l \gamma_\mu v_k, \\ \mathbb{B}_2 &= -2K\bar{s} P_R\,[\ldots]\, b^j\ \bar{v}^l \left(\bar{u}E\not{n} - \omega\frac{\not{\bar{n}}}{2} - \not{l}^\perp - \not{k}^\perp\right) v_k, \\ \mathbb{B}_3 &= K \cdot 4\frac{m_s(\kappa_s - \kappa_S)}{m_a^2 + 2um_B E}\left[\bar{s}^i\left(\bar{u}m_B E\right)\gamma_\mu P_R b^j \gamma^\mu \bar{v}^l\right.\end{aligned} \tag{6.74}$$

$$\left. -\bar{s}^i\left(\bar{u}E\not{n} - \omega\frac{\not{\bar{n}}}{2} - \not{l}^\perp - \not{k}^\perp\right) P_R b^j\ \bar{v}^l \not{p}_a v^k\right], \tag{6.75}$$

Besides, since we are not using Fierz transformation, we will have long but only trace for the LCDA's projection.

$$\mathrm{tr}(\Phi_K \Gamma_K) \cdot \mathrm{tr}\big(\Phi^B \Gamma_B\big) \longrightarrow \mathrm{tr}(\Phi_K \mathbb{M} \Phi_B \mathbb{N}). \tag{6.76}$$

where

$$\mathbb{B} = \bar{s}\ \mathbb{M}\ b\ \bar{v}\ \mathbb{N}\ v. \tag{6.77}$$

At this point, the long traces are not longer doable by hand. We opt to use FeynCal.

$$\begin{aligned}
\mathrm{tr}(\Phi_K \mathbb{M}_1 \Phi_B \mathbb{N}_1) = &\frac{\delta_{ik}\delta_{lj}}{N_c^2}\frac{f_K f_B}{16}\cdot K\left[-8\mu_K \bar{u} E \Phi_+^B \Phi_p\left[(\kappa_b-\kappa_B)p_a^+ + (\kappa_S-\kappa_b)m_B\right]-\right.\\
&\left.4\omega(2E)m_B\Phi_K\left(\Phi_-^B\left[(\kappa_b-\kappa_B)\frac{p_a^+}{m_B}+\kappa_S-\kappa_b\right]+\Phi_+^B\left[(\kappa_b-\kappa_B)\frac{p_a^-}{m_B}+\kappa_S-\kappa_b\right]\right)\right],\\
=&\frac{\delta_{ik}\delta_{lj}}{N_c^2}\frac{f_K f_B}{16}\cdot K(-8Em_B)\left\{\mu_K\bar{u}\Phi_+^B\Phi_p(\kappa_S-\kappa_B)+\right.\\
&\left.\omega\Phi_K\left(\Phi_-^B(\kappa_S-\kappa_B)+\Phi_+^B\left[(\kappa_b-\kappa_B)\frac{m_a^2}{m_B^2}+\kappa_S-\kappa_b\right]\right)\right\}, \qquad (6.78)
\end{aligned}$$

As expected, the leading contribution is of order λ. The scale of this term is carried by μ_K and ω. Then, we perform the integral over the LCDA's

$$\begin{aligned}
\iint du d\omega\, \mathrm{tr}(\Phi_K \mathbb{M}_1 \Phi_B \mathbb{N}_1) = &\frac{(-\lambda_t)}{f}\left(\frac{g_s^2 m_B}{4\pi^2}\right)C_8\frac{\delta_{ik}\delta_{lj}(t^a)^2}{N_c^2}\frac{f_K f_B}{4}m_B\left\{(\kappa_S-\kappa_B)\mu_K\int d\omega\frac{\Phi_+^B}{\omega}\int du\Phi_p\right.\\
&\left.+\int du\frac{\Phi_K}{\bar{u}}\int d\omega\left(\Phi_-^B(\kappa_S-\kappa_B)+\Phi_+^B\left[(\kappa_b-\kappa_B)\frac{m_a^2}{m_B^2}+\kappa_S-\kappa_b\right]\right)\right\}\\
=&\frac{(-\lambda_t)C_8}{f}\left(\frac{g_s^2 m_B}{4\pi^2}\right)\left(\frac{N_c^2-1}{2N_c^2}\right)\frac{f_K f_B}{4}m_B\left\{(\kappa_S-\kappa_B)\frac{\mu_K}{\lambda_o}\right.\\
&\left.+3(2\kappa_S-\kappa_B-\kappa_b)+\frac{3m_a^2}{m_B^2}(\kappa_b-\kappa_B)\right\}. \qquad (6.79)
\end{aligned}$$

Where we have used the asymptotic limit for $\Phi_p \longrightarrow 1$ and $\mu_K \sim \Lambda_{QCD}$ and $\omega_o \sim 460MeV$ [33]. The next term is given by

$$\begin{aligned}
\mathrm{tr}(\Phi_K \mathbb{M}_2 \Phi_B \mathbb{N}_2) =&\frac{\delta_{ik}\delta_{lj}}{N_c^2}\frac{f_K f_B}{16}m_B\cdot K\mu_K(E)\left[-4(\Phi_+^B\Phi_p\bar{u})(\kappa_S-\kappa_B)\right.\\
&\left.+\frac{2}{3}\Phi_+^B\Phi_\sigma'\bar{u}(\kappa_S-\kappa_B)+\frac{4}{3}\Phi_+^B\Phi_\sigma(\kappa_S-\kappa_B)\right]\\
=&\frac{\delta_{ik}\delta_{lj}}{N_c^2}\frac{f_K f_B}{16}m_B\cdot K\mu_K(4E)(\kappa_S-\kappa_B)\Phi_+^B\,\bar{u}\left[\frac{\Phi_\sigma}{3\bar{u}}-\Phi_p+\frac{\Phi_\sigma'}{6}\right]\\
=&\,0 \qquad (6.80)
\end{aligned}$$

Where we have used the equation of motion of the twist-3 amplitudes (see Eq. 5.12).

$$\Phi_p - \frac{\Phi_\sigma'}{6} = \frac{\Phi_\sigma}{3\bar{u}}. \qquad (6.81)$$

The last piece to calculate has the following expression

$$\begin{aligned}
\mathrm{tr}(\Phi_K \mathbb{M}_3 \Phi_B \mathbb{N}_3) &= \frac{\delta_{ik}\delta_{lj}}{N_c^2}\frac{f_K f_B}{16} m_B \cdot K \frac{4m_s}{m_a^2 + 2u m_B E}\left[-2E(\kappa_s - \kappa_S) m_B \Phi_+^B \Phi_K \bar{u}(2E)\right] \\
&= \frac{\delta_{ik}\delta_{lj}}{N_c^2}\frac{f_K f_B}{16} m_B \cdot K \frac{4m_s \bar{u}}{m_a^2 + 2u m_B E}(-4E^2)(\kappa_s - \kappa_S) m_B \Phi_+^B \Phi_K \\
&= \frac{-\lambda_t C_8}{2}\left(\frac{g_s^2}{4\pi^2}\right)\left(\frac{\delta_{ik}\delta_{lj}(t^a)^2}{N_c^2}\right)\frac{f_K f_B}{16} m_B^3 \cdot (4m_s)(4E) \\
&\quad \frac{(\kappa_s - \kappa_S)}{f}\int d\omega \frac{\Phi_+^B}{\omega}\int du \frac{\Phi_K}{m_a^2 + 2u m_B E} \\
&= -\lambda_t C_8 \left(\frac{g_s^2}{4\pi^2}\right)\left(\frac{N_c^2 - 1}{2N_c^2}\right)\frac{f_K f_B}{4} m_B^2 \cdot \left(m_B^2 - m_a^2\right)\frac{(\kappa_s - \kappa_S)}{f}\frac{m_s}{\omega_o}\frac{\mathcal{I}}{m_B^2}
\end{aligned} \tag{6.82}$$

where the integral is defined as

$$\mathcal{I} = \int du \frac{6u(1-u)}{u + \bar{u}\cdot r} = \frac{3}{m_B^2}\left[\frac{r^2 - 2r\log(r) - 1}{(r-1)^3}\right] = \frac{3\mathbb{L}(r)}{m_B^2}, \qquad r = \frac{m_a^2}{m_B^2} \ . \tag{6.83}$$

This integral is the same integral as for the case of ALP-gluon case (see Eq. 6.69).

$$\mathrm{tr}(\Phi_K \mathbb{M}_3 \Phi_B \mathbb{N}_3) = -\lambda_t C_8 \left(\frac{g_s^2}{4\pi^2}\right)\left(\frac{N_c^2 - 1}{2N_c^2}\right)\frac{f_K f_B}{4}\left(m_B^2 - m_a^2\right)\frac{(\kappa_s - \kappa_S)}{f}\frac{3m_s}{\omega_o}\left[\frac{r^2 - 2r\log(r) - 1}{(r-1)^3}\right] \tag{6.84}$$

Then, summing up the three pieces.

$$\begin{aligned}
\mathbb{B} = -\lambda_t C_8 \left(\frac{\alpha_s}{\pi}\right)\left(\frac{N_c^2 - 1}{2N_c^2}\right)\frac{f_K f_B}{4}\Big\{ & m_B^2 \frac{\mu_K}{\lambda_o}\frac{(\kappa_S - \kappa_B)}{f} + 3m_B^2 \frac{(2\kappa_S - \kappa_B - \kappa_b)}{f} \\
& + 3m_a^2 \frac{(\kappa_b - \kappa_B)}{f} + 3(m_B^2 - m_a^2)\frac{m_s}{\lambda_o}\frac{(\kappa_s - \kappa_S)}{f}\mathbb{L}\left(\frac{m_a^2}{m_B^2}\right)\Big\} .
\end{aligned} \tag{6.85}$$

The amplitude $\mathbb{B}$ at the end is of order one since the order λ terms in the NUMERATOR are compensated by the order λ term in the DENOMINATOR. To illustrate this, we see in Eq. 6.85 that in the first term the λ scale is canceled by μ_K and λ_o. The second and third term already simplified this scaling and in the fourth term the λ_o compensates the scaling of m_s.

6.4 Phenomenology

Once we have calculated the three contributions at tree-level from the Fig. 6.1, we should superpose them as a whole amplitude. The decay rate expression includes the integration of the phase space of a two body decay and is given by

$$\Gamma(B \longrightarrow K\ a) = \frac{|\mathbb{A}+\mathbb{B}+\mathbb{C}|^2}{16\pi m_B}\lambda^{1/2}\left(\frac{m_K^2}{m_B^2}, \frac{m_a^2}{m_B^2}\right), \tag{6.86}$$

where:

$$\lambda(u, v) = 1 - 2(u + v) + (u - v)^2. \tag{6.87}$$

where summarizing each contributions, we have

$$\begin{aligned}
\mathbb{A} &= \frac{G_F}{\sqrt{2}}\frac{\lambda_u(N_cC_1 + C_2) + (-\lambda_t)(N_cC_4 + C_3)}{N_c}(f_K f_B)\frac{(m_B^2 - m_a^2)}{2}\frac{(\kappa_b - \kappa_S)}{f}, \\
\mathbb{B} &= (-\lambda_t)\frac{G_F}{\sqrt{2}}C_8\left(\frac{\alpha_s}{\pi}\right)\left(\frac{N_c^2-1}{2N_c^2}\right)\frac{f_K f_B}{4}\left\{m_B^2\frac{\mu_K}{\omega_o}\frac{(\kappa_S - \kappa_B)}{f} + 3m_B^2\frac{(2\kappa_S - \kappa_B - \kappa_b)}{f}\right. \\
&\left. +3m_a^2\frac{(\kappa_b - \kappa_B)}{f} + 3(m_B^2 - m_a^2)\frac{m_s}{\omega_o}\frac{(\kappa_s - \kappa_S)}{f}\mathbb{L}(r)\right\}, \\
\mathbb{C} &= (-\lambda_t)\frac{G_F}{\sqrt{2}}C_8\left(\frac{g_s^2}{4\pi^2}\right)\left(\frac{N_c^2-1}{2N_c^2}\right)\left(\frac{f_B f_K}{4}\right)\left(\frac{C_g}{f}\frac{\alpha_s}{\pi}\right)(m_B)\frac{3}{2\omega_o}(m_B^2 - m_a^2)\mathbb{L}(r).
\end{aligned} \tag{6.88}$$

where during the calculation of every graph, we have omitted for simplicity the common factor $\frac{G_f}{\sqrt{2}}$ which we have brought it back now. The amplitude $\mathbb{A}$ represents the contribution from the four-fermion operators. Whereas, the amplitudes $\mathbb{B}$ and $\mathbb{C}$ represent the contribution from the dipole operator which in principle arise from integration out of the W boson in loops (see Fig. 3.2).

We can see in Fig. 6.3 that the biggest contribution is given by the four-fermion operator (see Fig. 6.1 (a)). On the other hand, the contribution of the amplitudes given by the dipole operator are around two order of magnitude and this can be explained because the effective dipole operator is suppressed by the α_s.

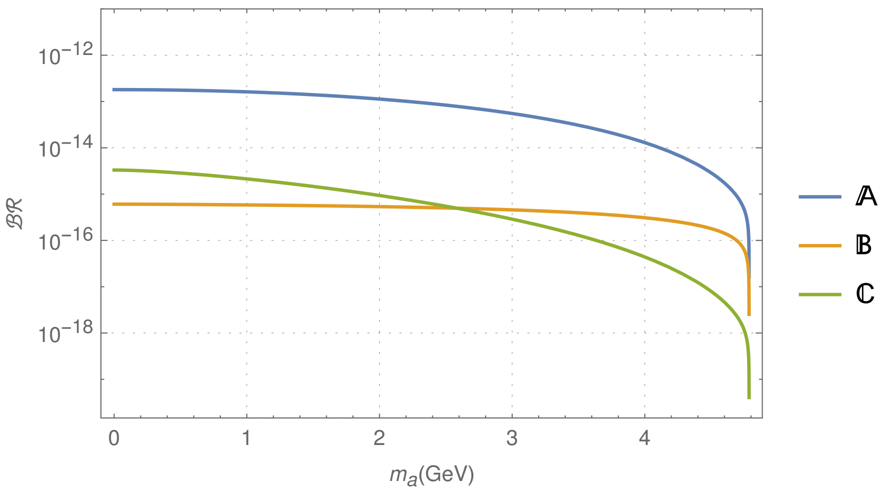


Figure 6.3: Contribution to the Branching Ratio of each graph at tree-level (see Fig. 6.1) in function on the ALP-mass. The assumption of the ALP-couplings being naturally order $\mathcal{O}(1)$ has been taken.

ALP couplings	m_a = 0 GeV	m_a = 1 GeV	m_a = 4 GeV
κ_S	$(1.07^{+0.15}_{-0.18}) \cdot 10^4$	$(1.07^{+0.22}_{-0.13}) \cdot 10^4$	$(3.70^{+0.53}_{-0.62}) \cdot 10^4$
κ_s	$(2.92^{+0.42}_{-0.49}) \cdot 10^6$	$(3.65^{+0.52}_{-0.61}) \cdot 10^6$	$(2.54^{+0.36}_{-0.43}) \cdot 10^7$
κ_B	$(2.08^{+0.30}_{-0.35}) \cdot 10^5$	$(2.10^{+0.30}_{-0.35}) \cdot 10^5$	$(2.76^{+0.39}_{-0.46}) \cdot 10^5$
κ_b	$(1.11^{+0.16}_{-0.19}) \cdot 10^4$	$(1.12^{+0.17}_{-0.20}) \cdot 10^4$	$(4.15^{+0.59}_{-0.70}) \cdot 10^4$
C_g	$(8.33^{+1.19}_{-1.40}) \cdot 10^4$	$(10.39^{+1.49}_{-1.74}) \cdot 10^4$	$(7.24^{+1.04}_{-1.21}) \cdot 10^5$

Table 6.1: Fitting values of the ALP-couplings at 1σ to the $\mathcal{BR}(B^+ \to K^+\nu\bar{\nu}) = (2.3 \pm 0.5(stat)^{+0.5}_{-0.4}(syst)) \times 10^{-5}$ [34].

Besides, one will expect that the amplitude $\mathbb{C}$ is notably suppressed by the orders in α_s since the ALP-gluon coupling carries itself one order of magnitude in α_s. However, due to the scale of this amplitude is inversely proportional to ω_o, at the end of the day it has bigger contributions in comparison to $\mathbb{B}$ for light ALPs. Clearly, we can see in Fig. 6.3 that the Branching Ratio ($\mathcal{BR}$) has an inverse dependence on the ALP-mass. In other words, heavy ALPs close to the threshold, given by the kinematic, offer small values of $\mathcal{BR}$ which will lead to large values to the ALP-coupling. On the other hand, light ALPs give larger $\mathcal{BR}$ which will lead smaller[1] ALP-couplings.

In order to calculate the decay rate of our process, we have to compute the square of the amplitude which bring itself terms proportional to ALP-couplings mixing. Obviously these mixing terms will bring complexity to the phenomenology and due to the several ALP-couplings in the game, we proceed to work in the ideal scenario where the ALPs couple to one quark only. It is relevant to stress that the assumption of this ideal scenario does not imply that the ALP will not couple to more quarks at other energy scales. On the other hand, due to the RG evolution of the ALP-couplings, non-zero ALP couplings will be introduced [24].

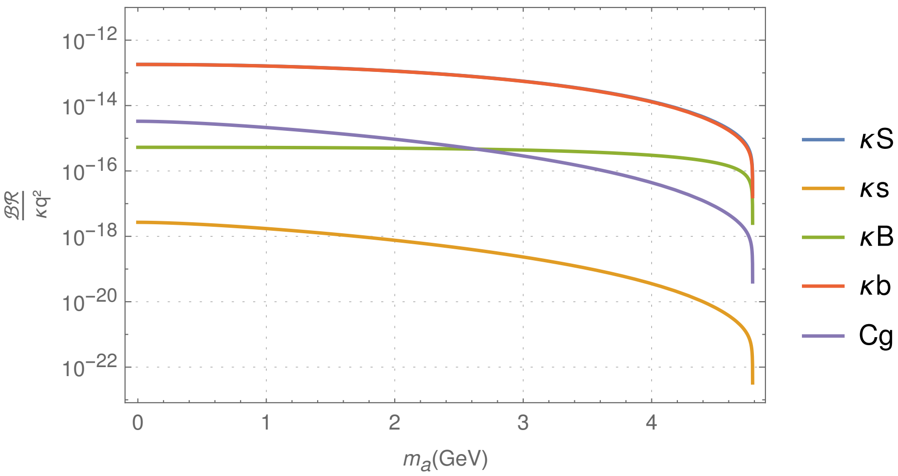


Figure 6.4: Dependency of the ratio between Branching ratio and ALP-coupling on the ALP-mass in the scenario where the ALPs couple to one quark at a time.

1 Still large but smaller in comparison to heavy ALPs

We can see in Fig. 6.4 that the ratio between the theoretical prediction of $\mathcal{BR}$ and the ALP-coupling in the one-coupling scenario. As expected the $\mathcal{BR}$ has an inverse dependency on the ALP-mass. In addition, the smallest ratio is given by the coupling between the ALP and the right-handed strange quark which will lead to a large κ_s. Besides, we can see that the contribution from κ_b and κ_S has unexpectedly the same dependency on the ALP-mass. In addition, the biggest ratio in Fig. 6.4 is given by these two ALP-couplings: κ_b and κ_S which will lead to the smallest ALP-couplings.

<table>
<tr><td colspan="6">Wilson Coefficients [15]</td></tr>
<tr><td>C_1</td><td>C_2</td><td>C_3</td><td>C_4</td><td colspan="2">C_8</td></tr>
<tr><td>1.08</td><td>-0.19</td><td>0.01</td><td>-0.04</td><td colspan="2">0.01</td></tr>
<tr><td colspan="6">Masses [GeV] [20]</td></tr>
<tr><td colspan="2">m_B</td><td>m_s</td><td colspan="2">m_K</td><td>m_u</td></tr>
<tr><td colspan="2">5.279</td><td>0.093</td><td colspan="2">0.493</td><td>0.002</td></tr>
<tr><td colspan="6">LCDA's parameters [GeV]</td></tr>
<tr><td colspan="2">f_K [35]</td><td>ω_o [33]</td><td>μ_K [20]</td><td colspan="2">f_B [36]</td></tr>
<tr><td colspan="2">0.160</td><td>0.46</td><td>2.55</td><td colspan="2">0.194</td></tr>
<tr><td colspan="6">Other parameters</td></tr>
<tr><td>λ_u [20]</td><td>λ_t [20]</td><td>f</td><td>α_s (at 5GeV) [20]</td><td>τ_{B^-} [GeV^{-1}] [20]</td><td>G_f[GeV^{-2}] [20]</td></tr>
<tr><td>(3.4 - i 7.4) $\cdot 10^{-4}$</td><td>-0.04</td><td>1 TeV</td><td>0.21 GeV</td><td>2.49 10^{12}</td><td>1.16 10^{-5}</td></tr>
</table>

Table 6.2: Input parameters used to calculate the Branching Ratio of the decay.

7 CONCLUSIONS AND OUTLOOK

In our study, we have illustrated the practical applications of effective field theories, including Effective Weak Interactions, Soft-Collinear Effective Theory, Heavy-Quark Effective Theory, and the formulation of an Effective Axion-Like Particle (ALP) Lagrangian. Our investigation has revealed that ALPs hold promise for addressing several pressing issues concerning the naturalness of Standard Model parameters, notably the resolution of the strong CP problem and the nature of dark matter. As a result, ALPs have been a well motivated search both in particle colliders and cosmological and astrophysical observations.

We have observed significant constraints on heavy ALP masses, particularly concerning flavor-changing ALP-quark interactions, as evidenced by studies of heavy-into-light meson decays such as $B \longrightarrow Ka$. Notably, we aim to find constrains, on the contrary, on flavor-conserving ALP interaction that might be derived by examining $B \longrightarrow K\ a$ within the QCD factorization framework.

Moreover, our investigation yielded intriguing insights into these processes. For example, we discovered that among the three types of diagrams at the tree level (see Fig. 6.1), the ALP-gluon interaction unexpectedly dominates due to its amplitude's proportionality to $\frac{1}{\omega_o}$, leading to a significant contribution. Additionally, we encountered complexity in analyzing type (b) diagrams in Fig. 6.1, necessitating the incorporation of subleading contributions and twist-2 terms in the Light-Cone Distribution Amplitudes of the Kaon and B meson. While Fierz transformation facilitated the separation of collinear- and soft-spinors, it also introduced additional, unnecessary calculations. Consequently, we reverted to the traditional QCD Factorization method for handling these intricate graphs.

The future trajectory of this project is rich with numerous objectives awaiting exploration. For instance, we aim to incorporate the contributions stemming from loop diagrams (refer to Fig. 6.2), which may exhibit dependence on the ALP mass. Additionally, our focus extends to elucidating constraints on flavor-conserving ALP couplings across multiple scales. We intend to incorporate the Evolution of the Running ALP-quark coupling to refine these constraints, and examine their bounds at high and low energy scales.

Furthermore, our interests encompass delving into alternative models for the Light-Cone Distribution Amplitudes governing the behavior of the Kaon and B meson, alongside the consideration of terms extending beyond the asymptotic limit. These endeavors promise to deepen our understanding and refine our analyses as we continue to probe the intricacies of this fascinating area of research.

A

FEYNMAN RULES

In this section we are going to describe the Feynman rules for every theory we use during the thesis.

A.1 QCD Lagrangian

The Lagrangian for a non-Abelian gauge theory.

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar{\Psi}_i\left(i\not{D}_{ij} - m\delta_{ij}\right)\Psi_j + \left(D^\mu\Phi\right)^\dagger\left(D^\mu\Phi\right) - M^2\Phi^\dagger\Phi - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + \ldots \quad \text{(A.1)}$$

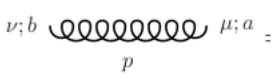
The first term is the kinetic term of the gauge boson. The second term is the kinetic term of the fermions and since the covariant derivative is involved, it also describes the interaction between the gauge boson and fermions. The third and fourth term are the kinetic term and the mass term of the scalar field respectively. The dots are the end represent the terms that contains the Faddeev-Popov ghosts. The explicit expression of covariant derivative and gauge tensor are

$$i\not{D}_{ij} = i\delta_{ij}\not{\partial} + g\not{A}^a T^a_{ij} \quad \text{(A.2)}$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \quad \text{(A.3)}$$

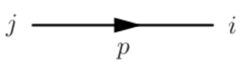
The propagators of the fields are given by the kinetic terms of the Lagrangian.

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - \frac{1}{2\xi}\left(\partial_\mu A^a_\mu\right)^2 + \bar{\Psi}_i\left(i\not{\partial}_{ij} - m\delta_{ij}\right)\Psi_j - \Phi^\dagger\left(\Box + M^2\right) \quad \text{(A.4)}$$

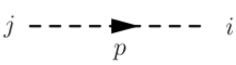
We can derive the Feynman rules from this Lagrangian.



$$\propto \quad i\frac{-\eta^{\mu\nu} + (1-\xi)p^\mu p^\nu}{p^2 + i\epsilon}\delta^{ab}$$

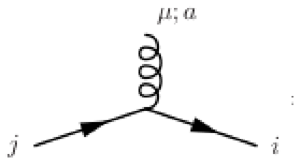


$$\propto \quad \frac{i\delta^{ij}}{\not{p} - m + i\epsilon}$$



$$\propto \quad \frac{i}{p^2 - M^2 + i\epsilon}$$

Then, we have the interacting terms between fermions and bosons



$$\propto ig\gamma_\mu T^a_{ij}$$

There are more interacting terms such as a vertex of three gauge bosons and fourth gauge bosons, etc. However, these interactions do not appear in our decay process [18].

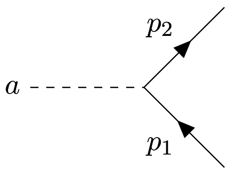
A.2 Axion-like Particles

The Feynman rules of the ALP-fermion and -boson coupling are.

First, let us study the decays at tree-level of ALPs going to two fermions. The interaction between the ALPs and fermions comes from the derivative coupling.

$$\mathcal{L} \supset \frac{\partial_\mu a}{f_a}\bar{\Psi}\boldsymbol{c}\gamma^\mu\Psi. \tag{A.5}$$

We can see from the coupling that the Feynman rule is going to be proportional to the momentum of the ALP.



$$\propto \quad (P_L \cdot \kappa_\Psi + P_R \cdot \kappa_\psi)\frac{(\not{p}_1 + \not{p}_2)}{f} \tag{A.6}$$

Last but not least, the last Feynman rule is from ALP going to two gauge bosons. The interaction between ALPs and gauge bosons are given by

$$\mathcal{L} \supset C_{GG}\frac{\alpha_s}{4\pi}\frac{a}{f_a}G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}, \tag{A.7}$$

where

$$\tilde{G}^{a,\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G^a_{\rho\sigma} \tag{A.8}$$

This interaction can be simplified

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G_{\mu\nu}G_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\partial_\mu\left[G^a_\nu\left(G^a_{\rho\sigma} - \frac{g}{3}f^{abc}G^b_\rho G^c_\sigma\right)\right] \tag{A.9}$$

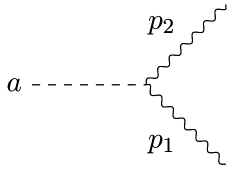
We are interested only in 2 gauge-bosons interactions.

$$G^a\tilde{G}^a \supset \partial_\mu\left[\epsilon^{\mu\nu\rho\sigma}G^a_\nu\left(\partial_\rho G^a_\sigma - \partial_\sigma G^a_\rho\right)\right] \tag{A.10}$$

$$\supset 2\partial_\mu\left[\epsilon^{\mu\nu\rho\sigma}G^a_\nu\partial_\rho G^a_\sigma\right] \tag{A.11}$$

$$\supset 2\epsilon^{\mu\nu\rho\sigma}\partial_\mu G^a_\nu\partial_\rho G^a_\sigma \tag{A.12}$$

There is an extra factor of two after the contraction of the two gluons. Each derivative comes with an "i", and there is another i coming from the expansion of $\exp\{i\mathcal{L}_{\text{int}}\}$, so we get $i^2(i) = -i$. Finally,



$$\propto \qquad -i\frac{C_\Psi}{f}\frac{\alpha_s}{4\pi f}\cdot 4\epsilon^{\mu\nu\rho\sigma}p_1^\mu p_2^\rho \tag{A.13}$$

A.3 Effective Weak Theory

A.3.0.1 Feynman Rules

Let us derive the Feynman rule for operators $\mathcal{O}_{7,8}$.

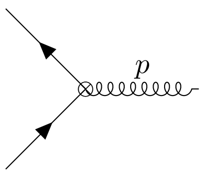
$$\mathcal{O}_{7,8} = \frac{gm_b}{8\pi^2}\bar{\boldsymbol{s}}\,\sigma_{\mu\nu}t^a(1+\gamma_5)\,\boldsymbol{b}\,G^a_{\mu\nu},\ \text{where}\ \sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu,\gamma_\nu\right] \tag{A.14}$$

The term that contributes to this operator of the gauge tensor is only the kinetic term

$$\sigma_{\mu\nu}t^aP_R\,*\,G^a_{\mu\nu} \supset \frac{i}{2}\left[\gamma_\mu,\gamma_\nu\right]t^a(1+\gamma_5)\,*\,\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) \tag{A.15}$$

$$i\left[\gamma_\mu,\gamma_\nu\right]t^a(1+\gamma_5)\,*\,\partial_\mu A^a_\nu \tag{A.16}$$

The derivative of the gauge boson gives a term prop to its momentum and also comes with a factor "i". There is another factor of "i" from the expansion of $\exp\{i\mathcal{L}_{int}\}$. The operator comes with an "i" from the anticonmutator, therefore, we have $i^3 = -i$.



$$\propto \qquad -i\frac{g_sm_b}{8\pi^2}\left[\not{p},\gamma_\mu\right](1+\gamma_5)t^a \tag{A.17}$$

B

OPERATOR BASIS OF WET

As we can see in the chapter of Electroweak Effective Weak Operators, we have been working with the following operator basis

$$\mathcal{Q}_1^{(p)} = (\bar{s}_i p_i)_{V-A} \, (\bar{p}_j b_j)_{V-A} \tag{B.1}$$

$$\mathcal{Q}_2^{(p)} = (\bar{s}_i p_j)_{V-A} \, (\bar{p}_j b_i)_{V-A} \tag{B.2}$$

The "QCD penguin operators".

$$\mathcal{Q}_3 = (\bar{s}_i b_i)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_j q_j)_{V-A} \tag{B.3}$$

$$\mathcal{Q}_4 = (\bar{s}_i b_j)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_j q_i)_{V-A} \tag{B.4}$$

$$\mathcal{Q}_5 = (\bar{s}_i b_i)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_j q_j)_{V+A} \tag{B.5}$$

$$\mathcal{Q}_6 = (\bar{s} i b_j)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_j q_i)_{V+A} \tag{B.6}$$

$$\mathcal{Q}_{7\gamma} = \frac{e m_b}{8\pi^2} \bar{s}_L \sigma^{\mu\nu} t^a (1+\gamma_5) b F_{\mu\nu} \tag{B.7}$$

$$\mathcal{Q}_{8g} = \frac{g_s m_b}{8\pi^2} \bar{s}_L \sigma^{\mu\nu} t^a (1+\gamma_5) b G^a_{\mu\nu} \tag{B.8}$$

Where we have used the short-hand convention that

$$(\bar{q}_1 q_2)_{V\pm A} \equiv \bar{q}_1 \gamma_\mu (1 \pm \gamma_5) q_2 \tag{B.9}$$

However, another operator basis can be found in references such as [37]

$$P_1 = (\bar{s}\gamma_\mu t^a P_L p)\,(\bar{p}\gamma_\mu t^a P_L b) \tag{B.10}$$
$$P_2 = (\bar{s}\gamma_\mu P_L p)\,(\bar{p}\gamma_\mu P_L b) \tag{B.11}$$
$$P_3 = (\bar{s}\gamma_\mu P_L b)\sum_q (\bar{q}\gamma_\mu q) \tag{B.12}$$
$$P_4 = (\bar{s}\gamma_\mu t^a P_L b)\sum_q (\bar{q}\gamma_\mu t^a q) \tag{B.13}$$
$$P_5 = (\bar{s}\gamma_\mu\gamma_\nu\gamma_\sigma P_L b)\sum_q (\bar{q}\gamma^\mu\gamma^\nu\gamma^\sigma b) \tag{B.14}$$
$$P_6 = (\bar{s}\gamma_\mu\gamma_\nu\gamma_\sigma P_L t^a b)\sum_q (\bar{q}\gamma^\mu\gamma^\nu\gamma^\sigma t^a b) \tag{B.15}$$
$$P_{7\gamma} = -\frac{e m_b}{8\pi^2}\bar{s}_L\sigma^{\mu\nu}t^a b_R F_{\mu\nu} \tag{B.16}$$
$$P_{8g} = -\frac{g_s m_b}{8\pi^2}\bar{s}_L\sigma^{\mu\nu}t^a b_R G^a_{\mu\nu} \tag{B.17}$$

It is straightforward to see that the operators $P_{1,2,3,4,7,8}$ have the same structure as the other basis; however the operators $P_{5,6}$ have three gamma matrices. Therefore, we have to prove that these two operators can be written as a linear combination of some of the operators of the other basis.

$$\gamma_\mu\gamma_\nu\gamma_\sigma = \eta_{\mu\nu}\gamma_\sigma + \eta_{\nu\sigma}\gamma_\mu - \eta_{\mu\sigma}\gamma_\nu - i\epsilon_{\alpha\mu\nu\sigma}\gamma^\alpha\gamma^5, \tag{B.18}$$

We use this identity in the operator P_5

$$\begin{aligned}\gamma_\mu\gamma_\nu\gamma_\sigma \;*\; \gamma^\mu\gamma^\nu\gamma^\sigma &\propto (\gamma_\sigma \;*\; \gamma_\mu\gamma^\mu\gamma^\sigma) + (\gamma_\mu \;*\; \gamma^\mu\gamma_\nu\gamma^\nu) - (\gamma_\nu \;*\; \gamma^\mu\gamma_\nu\gamma^\mu) + \\ &\quad \left(\gamma^\alpha\gamma^5 \;*\; i\epsilon_{\alpha\mu\nu\sigma}\gamma^\mu\gamma^\nu\gamma^\sigma\right) \end{aligned} \tag{B.19}$$
$$\propto (\gamma_\sigma \;*\; \gamma_\mu\gamma^\mu\gamma^\sigma) + \left(\gamma^\alpha \;*\; \epsilon_{\alpha\mu\nu\sigma}\epsilon^{\beta\mu\nu\sigma}\gamma_\beta\gamma_5\right) \tag{B.20}$$
$$\propto (\gamma_\sigma \;*\; \gamma^\sigma) + \left(\gamma^\alpha \;*\; \delta^\beta_\alpha\gamma_\beta\gamma_5\right) \tag{B.21}$$

Where before the contraction of the two levi-civita tensors, we use the same identity of the product of three gamma matrices but only the anti-symmetric term survives which is the levi-civita term.

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