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TEMPERATURE SYNCHRONIZATION, PHASE DYNAMICS AND OSCILLATION DEATH IN A RING OF THERMALLY-COUPLED ROOMS

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ABSTRACT

Synchronization of coupled, self-excited oscillators in complex systems is a common occurance. This report examines the effects of thermal coupling through the walls of a building on temperature oscillations due to hysteretic thermostatic control. The specific case of three rooms is studied. A system of differential equations models the dynamics of each room temperature, accounting for on-off heating, heat loss to the environment, and heat exchange between rooms. Three types of solutions are observed: one in which all room temperatures oscillate in phase, another with the oscillations equidistant in phase, and a third that is time-independent. The existence and linear stability of each solution type is investigated as a function of a parameter k that represents the thermal interaction between neighboring rooms. The in-phase behavior exists and is linearly stable for all k, the out-of-phase oscillations exist in a band of k and are stable in a smaller band, and the time-independent solution exists above a certain k where they are stable.

NOMENCLATURE

k nondimensional thermal coupling parameter between rooms
Q_i nondimensional heat input in room i
Q_T $= Q_1 + Q_2 + Q_3$
t nondimensional time
t_s nondimensional switching time
T_i nondimensional temperature in room i
T_∞ nondimensional exterior temperature = 0
T_i^0 nondimensional initial temperature in room i
T_T $= T_1 + T_2 + T_3$
T_{max} nondimensional maximum temperature
T_{min} nondimensional minimum temperature
ε_i perturbation to temperature of room i
ρ spectral radius
τ nondimensional period of oscillation

1 Introduction

The dynamics of weakly-coupled self-excited oscillators has been a topic of considerable interest in a wide variety of fields; such systems can exhibit synchronous, asynchronous and chaotic behavior [1–3]. By synchronization is meant that at least two or more of the oscillators that comprise the system have the same frequency with a constant, possibly zero phase difference between them. Synchronization in systems in which the oscillations are a result of switching is also fairly common. It has been shown that, even in weakly coupled systems, the coupling can have substantial effects on global behavior producing synchronization, asynchronization, and other unexpected phenomena. Synchronous behavior has been experimentally observed in a wide variety of oscillatory systems. Mechanical systems include the swinging of pendulums [4, 5], sound from organ pipes [6], the motion of vibro-exciters [7], and the operation of mechanical respirators [8]. Mathematical models of prediction have been developed for many of these systems.

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Our interest here is in thermal systems with temperature oscillations [9]. Synchronization has been experimentally found in temperature oscillations in a flow loop facility composed of three secondary loops with heat exchangers that exchange heat with a primary heating loop on one side and a primary cooling loop on the other [10]. A hysteretic thermostatic controller (also called on-off, two-position, or bang-bang) sensed the temperature at the outlet of a heat exchanger and modulated the flow rate in that loop. The resulting temperature oscillations showed frequency locking, phase synchronization as well as phase slips. Another thermal system of great practical interest are chambers with temperature control and with thermal interaction between them. Measurements in a supermarket refrigeration system have shown that control strategies operating on flow valves can lead to synchronization of temperature oscillations and an increase of energy consumption [11, 12]. The synchronization dynamics of the refrigeration system was also analyzed from a dynamical systems perspective and it was suggested that controlled desynchronization would be beneficial [13, 14]. Numerical solutions for thermostatic temperature control for a set of five chambers in a ring have been obtained [15, 16], and synchronized behavior has been found.

Though thermal systems without fluid instability or control generally do not oscillate by themselves, self-excited temperature oscillations can be obtained by switching [17]. A hysteretic thermostat is a device that switches to a higher or a lower heat rate depending on whether a measured temperature falls below or rises above certain prescribed limits; as a result the temperature oscillates between these two limits. This paper investigates synchronized oscillations in thermostatically-controlled heating of multiple rooms. In large, multi-room buildings such as apartments, office buildings and student residence halls, the temperature of each room is usually controlled by its individual thermostat. Each room acts independently, and each is a self-excited oscillator. The effect of heat conduction through the walls between adjacent rooms is usually ignored in the design of the control system. Heat transfer through the common wall between neighboring rooms means that there is thermal coupling between them. Understanding the global, dynamic behavior of the coupled system may help develop a better means of temperature control in buildings.

The specific case of the temperature dynamics of a three-room ring, in which each of the rooms is thermally coupled to its neighbors, is considered. Three rooms are chosen because it is the smallest number that permits both in-phase and wave-like solutions. A system of differential equations, accounting for thermostatically-controlled heating, heat loss to the environment, and the coupling effects of the walls between rooms, is used to model the temperature of each room.

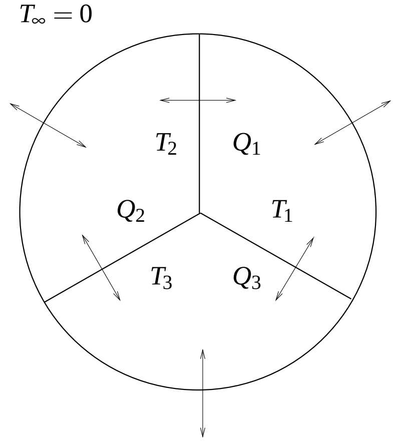


FIGURE 1. Schematic of three rooms $i = 1,2,3$ with thermal coupling; T_i are the temperatures and Q_i the heating rate; the arrows indicate possible heat exchange.

2 Governing equations

Consider a set of rooms in the form of a ring, as shown in Fig. 1. Though the mathematical model is, of course, applicable both to heating and cooling, for simplicity we will talk only about heating. Each room has a different nondimensional temperature $T_i(t)$, $i = 1,2,3$, which is thermostatically controlled by a heater that has a nondimensional value of $Q_i(t)$. Each room exchanges heat with the exterior that is nondimensionally at zero temperature, and adjacent rooms exchange heat between them by conduction through their common wall. A straightforward, lumped mathematical model was previously developed for this [16].

The dimensionless equations governing the temperature of each room are

$$\frac{dT_1}{dt} + T_1 = Q_1 + k(T_2 - T_1) + k(T_3 - T_1), \tag{1a}$$
$$\frac{dT_2}{dt} + T_2 = Q_2 + k(T_3 - T_2) + k(T_1 - T_2), \tag{1b}$$
$$\frac{dT_3}{dt} + T_3 = Q_3 + k(T_1 - T_3) + k(T_2 - T_3), \tag{1c}$$

where t is dimensionless time, and

$$Q_i(t) = \begin{cases} 0 & \text{if heater } i \text{ is } \textit{off}, \\ 1 & \text{if heater } i \text{ is } \textit{on}. \end{cases}$$

For each room the first term on the left represents the rate of accumulation of thermal energy, and the second is the heat loss by convection to the surroundings. On the right, the first term is heat input due to the heater in the room and the other two terms are the heat gain from adjacent rooms. The heater is thermostatically controlled; it goes *on* if the room temperature falls below

T_{min}, and goes *off* if it rises above T_{max}. T_{min} and T_{max} have been arbitrarily fixed at 0.25 and 0.75, respectively. The parameter $k \geq 0$ is the thermal resistance between a room and the exterior compared to that between adjacent rooms. It represents the thermal coupling between rooms, so that there is no heat exchange between them if $k = 0$.

Summing the three equations eliminates the coupling terms and produces a single differential equation

$$\frac{dT_T}{dt} + T_T = Q_T, \tag{2}$$

that relates the sum of the temperatures to the sum of the heating values, where

$$Q_T(t) = Q_1 + Q_2 + Q_3, \tag{3}$$
$$T_T(t) = T_1 + T_2 + T_3. \tag{4}$$

Thermal coupling does not add to or take heat out of the three-room system.

At the instants at which $\max\{T_1, T_2, T_3\}$ or $\min\{T_1, T_2, T_3\}$ reaches T_{max} or T_{min}, respectively, Q_1, Q_2 or Q_3 changes; these instants will be referred to as switching. The system of equations is non-linear since the moment at which switching occurs is not known *a priori* but depends on the dependent variables T_i. Between switches, however, Eqs. (1) are easy to solve as

$$T_i(t) = f(T_1^0, T_2^0, T_3^0, Q_1, Q_2, Q_3, t), \tag{5a}$$

where

$$f = \frac{1}{3}\left\{Q_T - \frac{3}{3k+1}(Q_T - 3Q_i) - \left(Q_T - T_T^0\right)e^{-t} + \left(3T_i^0 - T_T^0 + 3Q_i - \frac{Q_T}{3k+1}\right)e^{-(3k+1)t}\right\}. \tag{5b}$$

The initial conditions are $T_i^0 = T_i(0)$, where t resets to 0 each time there is a switch. If the parameter k is given, and if the type of switch is known (i.e. which room changes its heating and whether the temperature there hits T_{max} or T_{min}), the time $t = t_s$ at which the next switch occurs can be calculated by substituting $T_i = T_{max}$ or T_{min} for the appropriate room and solving the transcendental equation above numerically using Newton's method. By taking the final temperatures of this interval as the initial temperatures, the system can be integrated for the next time interval to the following switch, and so on.

The state of the system governed by Eqs. (1) at any given time is defined by 6 scalar variables: the temperatures of the rooms, $T_{min} \leq T_i \leq T_{max}$ as well as the heating states of each of the rooms, $Q_i = 0$ or 1. Geometrically, the solution lies within a temperature cube and simultaneously belongs to one of the corners of a heating cube, both of which are shown in Fig. 2. The two cubes together constitute the phase space for this problem. The path of the solution in time is continuous (though not differentiable) within the temperature cube but jumps from corner to corner in the heating cube. A jump occurs as the temperature bounces off any of the walls of its cube; a specific jump corresponds to a bounce off a specific wall.

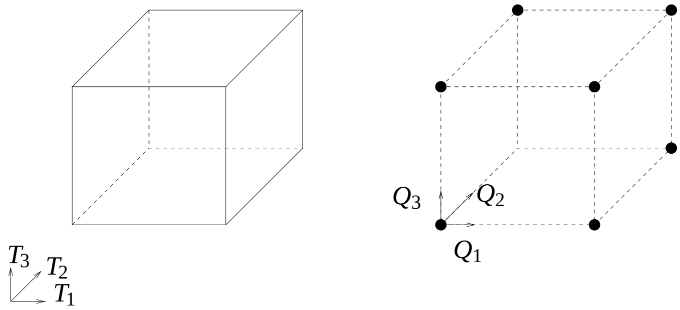


FIGURE 2. Phase space of the dynamical system; at any instant the solution must be a point inside or on the wall of the temperature cube (left), while the heating should belong to one of the corners of the heating cube (right).

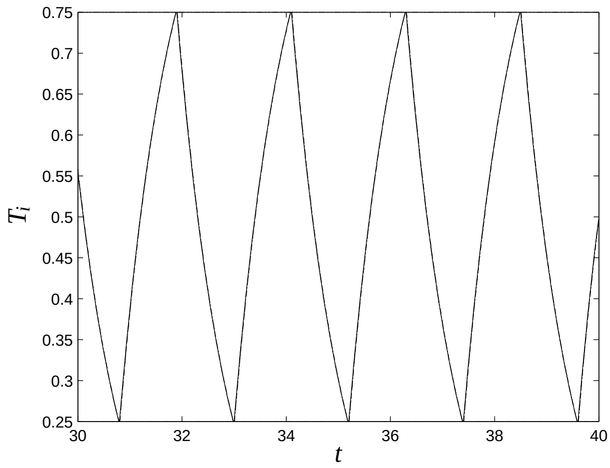
3 Existence of dynamic solutions

Periodic behavior is one in which, starting from an initial point in phase space, the system undergoes a series of motions and eventually returns to the same point. A search for a periodic solution begins with a postulate of a heating protocol, i.e. a sequence of Q_i's that the system undergoes. The generality of this approach lies in being able to postulate all possible sequences. Equating the initial to the final conditions over a period gives a set of transcendental equations that can be solved for the time intervals t_s between switches. The period is then a sum of these intervals, $\tau = \sum_i t_{s,i}$. This can be done for different thermal coupling parameter k.

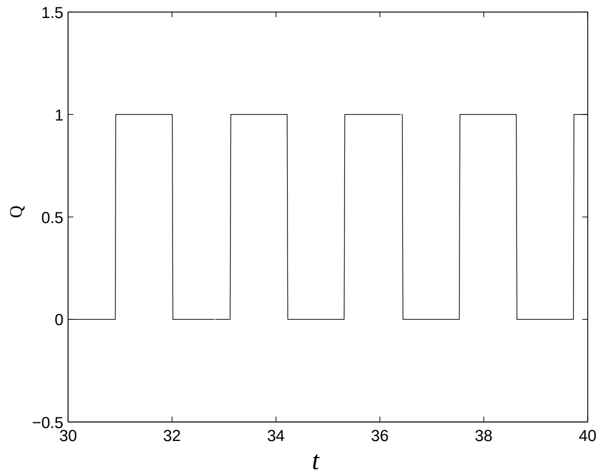
In-phase solution

This is the simplest periodic behavior possible; the three T_i's and the three Q_i's are synchronized both in frequency and phase. The rooms occupy the same point and they follow the same path in phase space. An example of the temperature-time behavior is shown in Fig. 3(a), and the corresponding heating protocol is shown in Fig. 3(b). The path of the solution in phase space is shown in Fig. 4.

Putting $T_1 = T_2 = T_3 = T$, $Q_1 = Q_2 = Q_3 = Q$, the analysis

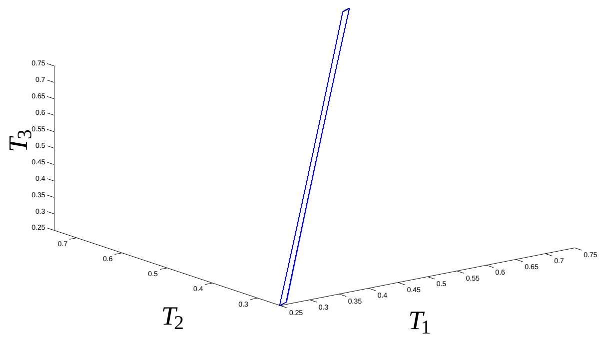


(a) Temperature dynamics, $T_1(t), T_2(t), T_3(t)$

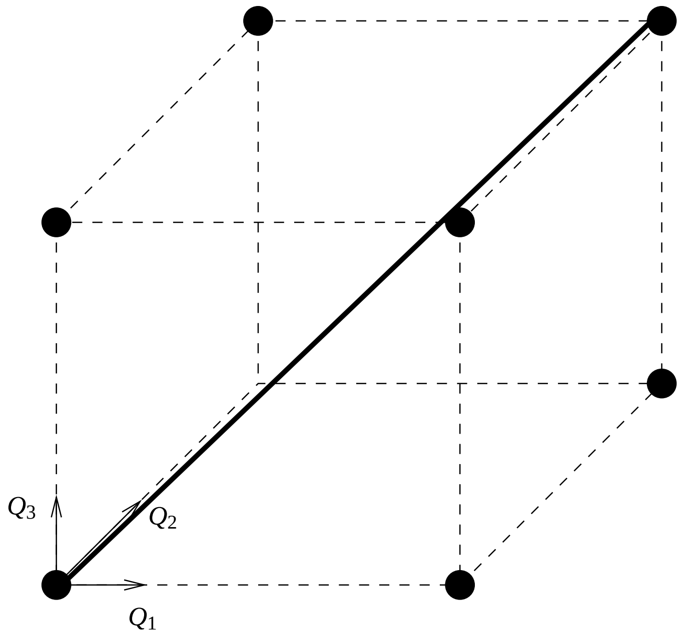


(b) Heating protocol, $Q_1(t), Q_2(t), Q_3(t)$

FIGURE 3. In-phase solution, any k



(a) Temperature cube



(b) Heating cube

FIGURE 4. Phase-space path for in-phase solution, any k

is very simple. Eq. (1) reduces to

$$\frac{dT}{dt} + T = \begin{cases} 0 & \text{if heaters are all } \textit{off} \\ 1 & \text{if heaters are all } \textit{on} \end{cases}. \tag{6}$$

This is easily solved for the *off* and *on* intervals, and the total period of the oscillation is found to be

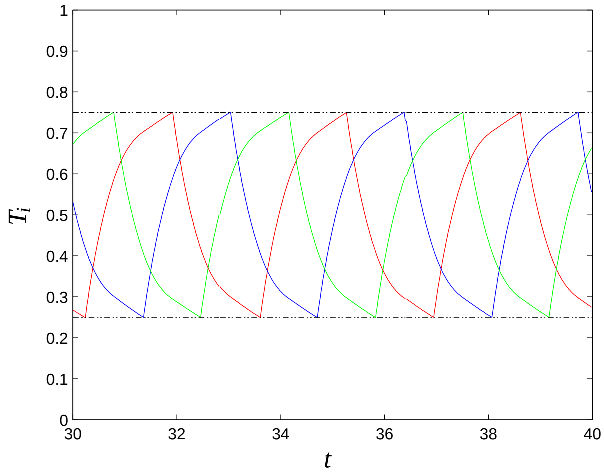
$$\tau = \ln\frac{1-T_{min}}{1-T_{max}} + \ln\frac{T_{max}}{T_{min}}. \tag{7}$$

Eq. (6) is independent of k, indicating that the same dynamics exist for all k.

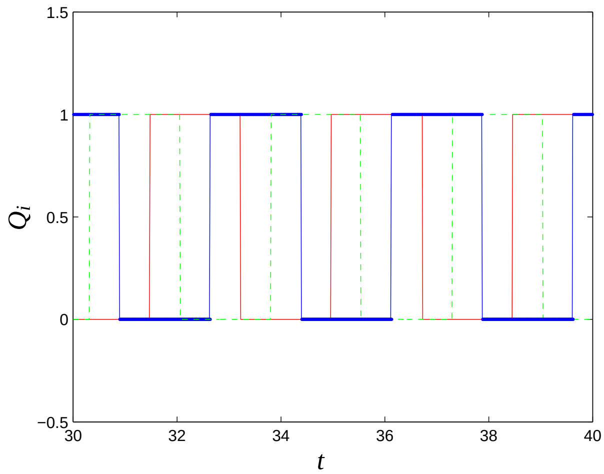
Out-of-phase solution

Another interesting dynamics occurs when the room temperatures are periodic at the same frequency, but their phases are equally separated from each other. Each of the three temperature curves follow an identical path in phase space, but they always maintain a constant and equal time separation between them. The temperature difference between the three is not necessarily constant, but the time difference between them is. This is thus a wave that goes around the ring. There are two ways this can happen, with the rooms following each other either in a clockwise or a counter-clockwise fashion. Since the two are essentially the same, only the latter will be discussed.

Examples of the out-of-phase solution are shown in Figs. 5 and 6 for $k = 0.29$ and $k = 0.54$ respectively. Figs. 5(a) and

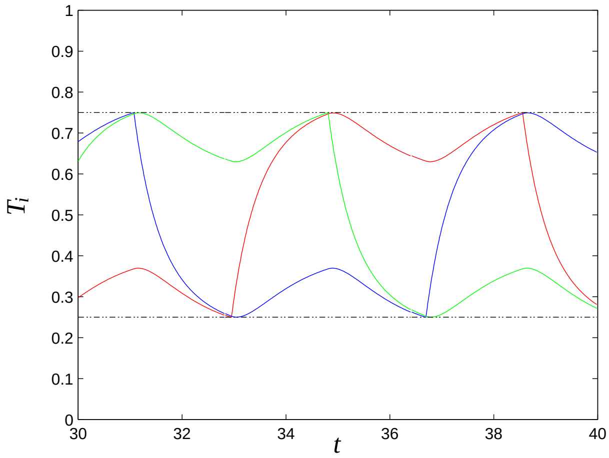


(a) Temperature dynamics, $T_1(t), T_2(t), T_3(t)$

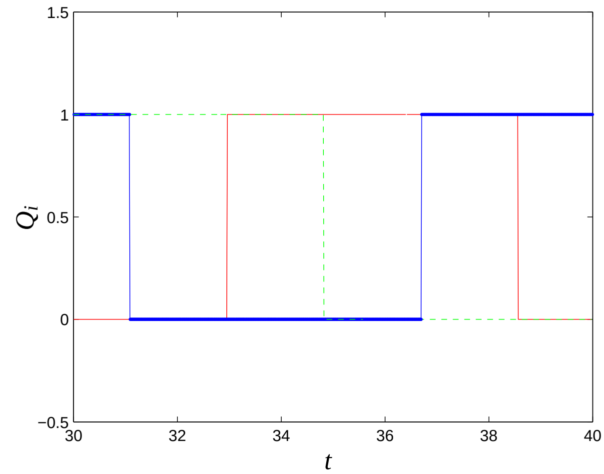


(b) Heating protocol, $Q_1(t), Q_2(t), Q_3(t)$

FIGURE 5. Out-of-phase solution, $k = 0.29$



(a) Temperature dynamics, $T_1(t), T_2(t), T_3(t)$



(b) Heating protocol, $Q_1(t), Q_2(t), Q_3(t)$

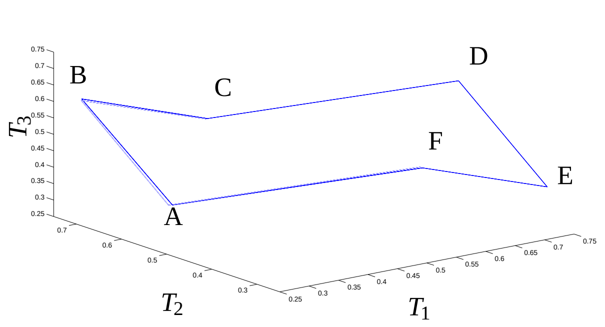
FIGURE 6. Out-of-phase solution, $k = 0.54$

6(a) are the time variation of the temperature, while Figs. 5(b) and 6(b) are the heating protocol. The respective phase paths are shown in Figs. 7 and 8.

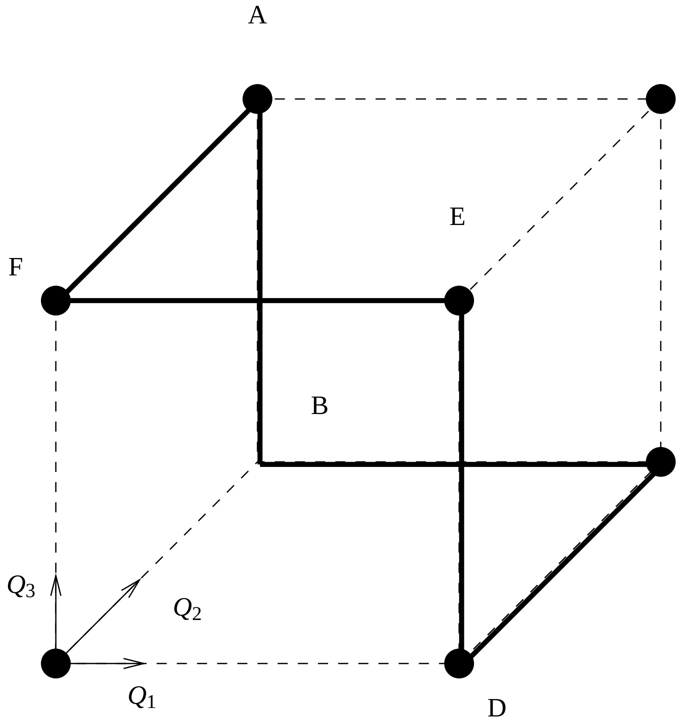
Though the period τ consists of six switches, it is just a two-switch pattern of intervals $t_{s,1}$ and $t_{s,2}$ repeated three times, with their initial conditions each time in this subcycle being repeated. Accordingly, the following three sets of three equations are obtained. The first set goes from T_{min}, T_2^0, T_3^0 to T_1^1, T_{max}, T_3^1 in a time interval of $t_{s,1}$ so that

$$T_1^1 = f(T_{min}, T_2^0, T_3^0, 1, 1, 0, t_{s,1}), \tag{8a}$$
$$T_{max} = f(T_{min}, T_2^0, T_3^0, 1, 1, 0, t_{s,1}), \tag{8b}$$
$$T_3^1 = f(T_{min}, T_2^0, T_3^0, 1, 1, 0, t_{s,1}), \tag{8c}$$

where f is defined in Eq. (5b). The second goes from T_1^1, T_{max}, T_3^1



(a) Temperature cube

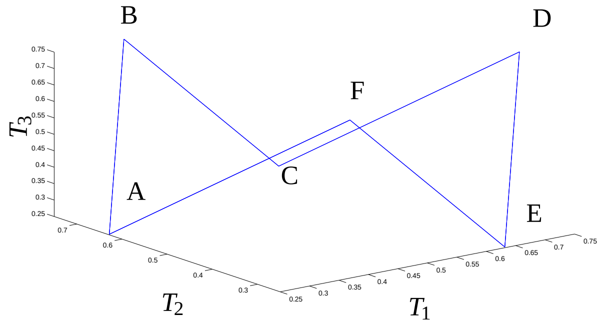


(b) Heating cube

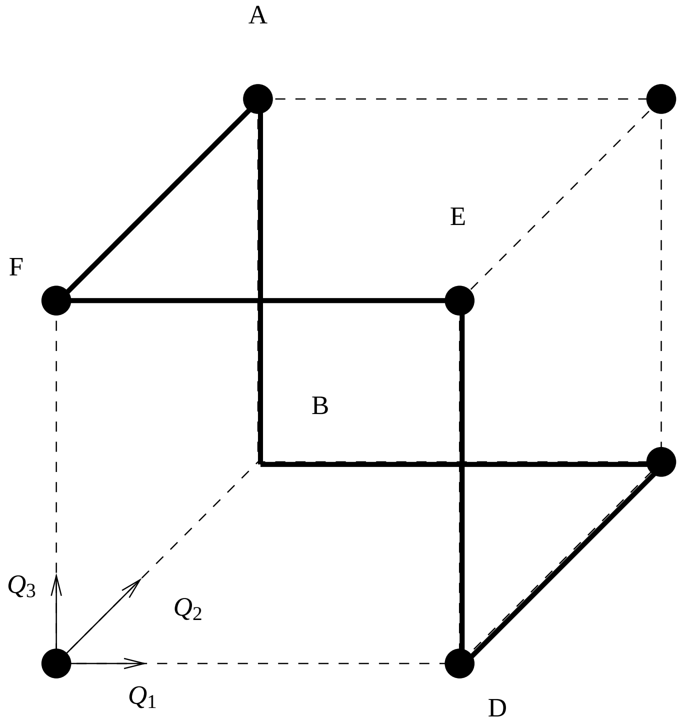
FIGURE 7. Phase-space path for out-of-phase solution, $k = 0.29$

to T_1^2, T_2^2, T_{min} in a time $t_{s,2}$ for which

$$T_1^2 = f(T_1^1, T_{max}, T_3^1, 1, 0, 0, t_{s,2}), \tag{8d}$$
$$T_2^2 = f(T_1^1, T_{max}, T_3^1, 1, 0, 0, t_{s,2}), \tag{8e}$$
$$T_{min} = f(T_1^1, T_{max}, T_3^1, 1, 0, 0, t_{s,2}). \tag{8f}$$



(a) Temperature cube



(b) Heating cube

FIGURE 8. Phase-space path for out-of-phase solution, $k = 0.54$

The third maps the temperatures

$$T_1^0 = T_3^2, \tag{8g}$$
$$T_2^0 = T_1^3, \tag{8h}$$
$$T_3^0 = T_3^3. \tag{8i}$$

After eliminating the intermediate temperatures, two coupled transcendental equations in $t_{s,1}$ and $t_{s,2}$ are obtained. These two time intervals can be found by solving the equations numerically, and it so happens that for the T_{min} and T_{max} chosen

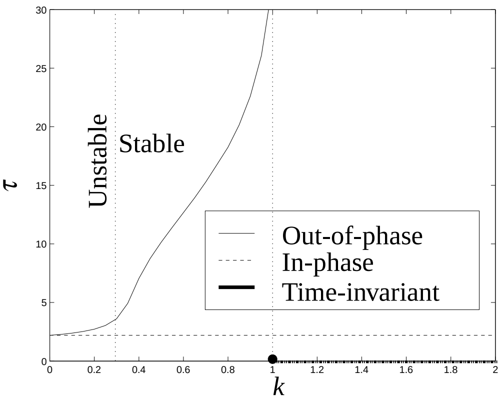


FIGURE 9. Periods of oscillation for different k.

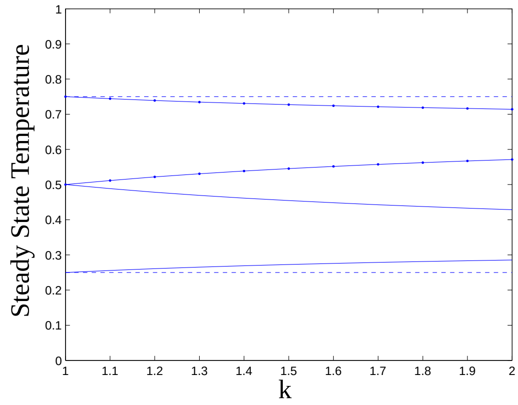


FIGURE 10. Variation of time-invariant solution with k: solid lines are with one heater *on*, while dotted solid lines are two heaters *on*. Dashed lines show T_{min} and T_{max}.

here, $t_{s,1} = t_{s,2}$. The period of oscillation is thus $\tau = 3(t_{s,1} + t_{s,2})$, shown in Fig. 9. At low k the switching time remains fairly constant, demonstrating that weak coupling does not significantly affect the period.

The out-of-phase solution for $k < 0.35$ is different from that for $k \geq 0.35$. This difference is seen by comparing Figs. 5(a) and 6(a). The behavior slowly morphs from one to the other as k is increased. The dip in Fig. 6(a) corresponds to $dT_i/dt = 0$, which first appears around $k = 0.35$. The out-of-phase solution exists only for $k < 1$.

Time invariant solution

In addition to the dynamic solutions discussed above, a time invariant solution is also possible for which $dT_i/dt = 0$ in Eq. (1). There are two types of solutions, one in which only one room heater is *on* and one in which two heaters are *on*.

(a) If one room heater is *on*, e.g. $Q_1 = 1, Q_2 = 0, Q_3 = 0$, the time invariant solution is

$$T_1 = \frac{k+1}{3k+1}, \quad T_2 = T_3 = \frac{k}{3k+1}. \tag{9}$$

Since T_1, T_2, T_3 must all be between T_{min} and T_{max}, this exists only for $k \geq 1$.

(b) If, however, two room heaters are *on*, e.g. $Q_1 = 1, Q_2 = 1, Q_3 = 0$, then

$$T_1 = T_2 = \frac{2k+1}{3k+1}, \quad T_3 = \frac{2k}{3k+1} \tag{10}$$

which also exists only for $k \geq 1$.

Figure 10 graphically depicts these two solution types. The time-independent solutions correspond to a substantial heat transfer between individual rooms. The coupling parameter is high and the rooms behave less as individual entities so that the heaters serve to warm the entire system rather than just their individual room. Thus it is possible, as k increases through unity, for an oscillatory solution described in the previous sections to die.

4 Linear stability of dynamic solutions

After having identified the existence of these solution types, their linear stability will be examined. Only linearly stable solutions will be observed in practice.

In-phase solution

The initial conditions are changed from $[T_{min}, T_{min}, T_{min}]^T$ to $[T_{min}, T_{min} + \varepsilon_2, T_{min} + \varepsilon_3]^T$ where ε_2 and ε_3 are small. Six switching intervals carry these perturbed initial conditions into $[T_{min}, T_{min} + \varepsilon_2', T_{min} + \varepsilon_3']^T$, each interval corresponding to the time between switches. The transcendental functions in time intervals that are generated are expanded in a Taylor series, keeping only the first-order terms. This produces, after some algebra, a mapping from the original perturbations, ε_2 and ε_3, to the final, ε_2' and ε_3'. The deviations from the in-phase periodic solution at the beginning and at the end of one period are related by

$$\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = \begin{bmatrix} R^{3k} & 0 \\ 0 & R^{3k} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \tag{11}$$

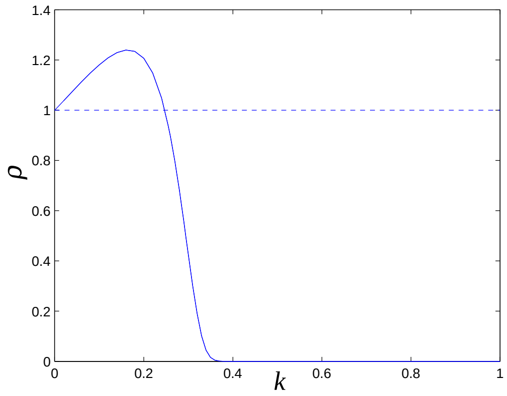


FIGURE 11. Spectral radius of out-of-phase perturbation matrix vs. coupling parameter k.

where

$$R = \frac{T_{min}}{T_{max}} \frac{1 - T_{max}}{1 - T_{min}}. \tag{12}$$

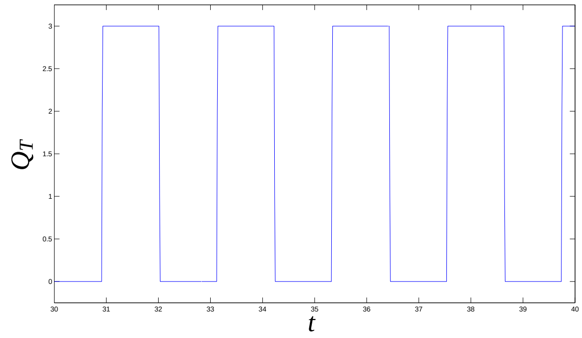
Since $0 \leq T_{min} < T_{max} \leq 1$, we have $R < 1$. The spectral radius of the map is less than unity, so that the perturbation will tend toward 0. The in-phase solutions are thus linearly stable for all k.

Out-of-phase solutions

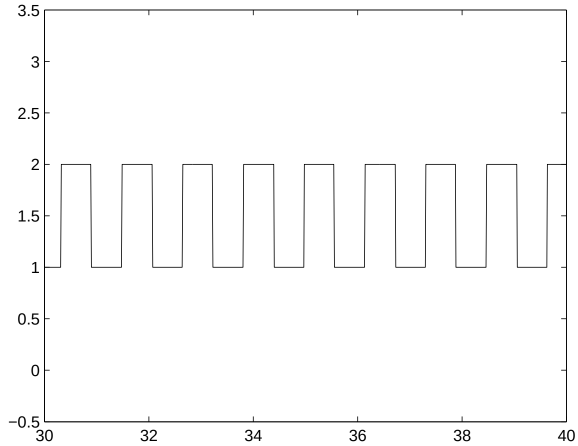
A similar procedure is used to test the linear stability of the out-of-phase solutions. To prevent the algebra from becoming prohibitively complex, however, numerical values are used for the parameters instead of symbolic variables. The stability of the out-of-phase solution is evaluated for $k \in [0,1]$, the region in which it exists. Figure 11 shows the spectral radius as a function of coupling parameter k. The out-of-phase solution is linearly stable for $0.2466 \leq k \leq 1$ and unstable below that.

Time-independent solutions

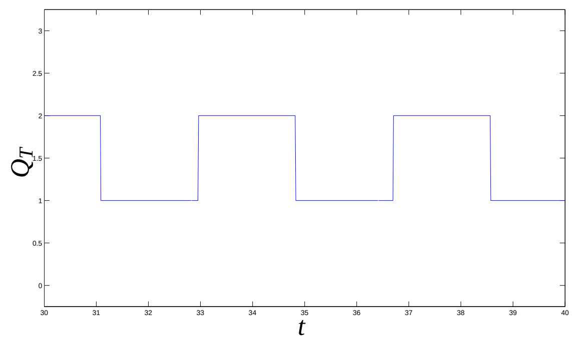
Since there is no switching in the time-invariant solutions, the stability analysis is very straightforward. Both solutions (with one heater *on* and with two heaters *on*) are found to be linearly stable for $k \geq 1$, the region in which they exist.



(a) Total heating, in-phase solution, any k



(b) Total heating, out-of-phase solution, $k = 0.29$



(c) Total heating, out-of-phase solution, $k = 0.54$

FIGURE 12. Total heating, $Q_T(t)$

5 Discussion

With the existence of different, linearly stable, dynamic solutions, the relative importance of the oscillatory solution types in a practical application must be evaluated. Fig. 12(a) shows graphically the total heating Q_T required for in-phase periodic solutions, and Figs. 12(b) and 12(c) the same for out-of-phase periodic solutions.

Though the same average heat rate is required for the same average temperature, the total number of switches is significantly less in the out-of-phase solutions. Fig. 9 shows that the oscillation frequency for the out-of-phase solution is smaller than that of the in-phase. As Wisniewski et al. [13] have pointed out, reducing the number of switches per unit time is desirable in thermal control systems because doing so can significantly increase the lifetime of heating or cooling equipment by reducing mechanical wear. It is generally more energy-efficient to run thermal equipment nearly at a constant load than to force them to repeatedly turn on and off; the initial transient of a motor or furnace after start is its most inefficient state. Additionally, the out-of-phase solution places a much smaller total heating load on the system at any instant of time: a maximum of 2 heaters are on at any given time, as opposed to 3 for the in-phase behavior. The capital cost of the required heating system installed is thus lower for the out-of-phase dynamics.

It should be noted that there is virtually no performance difference between the two periodic solution types. The mean temperatures of both are identical to within 1%. The standard deviation of the out-of-phase solution differs from that of the in-phase by less than 10%.

The time-invariant solutions are different from the periodic solutions in that they do not all require the same amount of total heating and there is no switching involved. The one heater *on* solution is especially desirable in that it requires the least amount of heating to maintain room temperatures in the desired range of all the solution types, periodic or not. However, the mean temperature is lower than that of the periodic solutions. Similarly, the two heaters *on* solution is the least desirable since it requires the largest total amount of heating of all the solution types, while the mean temperature is higher than the periodic solutions.

6 Conclusions

The behavior of a differential model of thermostatically-controlled heating in a ring of three thermally-coupled rooms has been studied. Three different types of solutions, in-phase periodic, out-of-phase periodic, and time-independent, are found. The first exists for all k, the second for $k < 1$, and the third for $k \geq 1$. Linear stability analysis shows that, wherever they exist, the first and third are always stable, and the second is conditionally stable. As the coupling parameter k increases from zero, the global response of the system goes from synchronized oscillations with in-phase or out-of-phase oscillations to in-phase oscillations or an eventual oscillation death.

The results are important for the design of controllers for buildings and systems with multiple chambers. There are several advantages to the out-of-phase solution in terms of life expectancy and capital cost of installation of heating systems, which could be worth exploiting. It is stable under some circumstances, and in others can be stabilized through feedback control.

The investigation can be further expanded by considering the behavior of a large number of rooms. While it is easy to see that in-phase periodic solutions will exist for any number of rooms, it is less clear what other dynamic behaviors can be expected. Out-of-phase, clustering, and metachronal waves are all possible.

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