MAT 1400 Calcul I

Suite - Sucesión (succession) → lista ordenada de números

Séries - serie → suma de una lista de números

Série convergent (serie convergente) → cuando sumas parciales son sucesiones convergentes

Contenu du cours

- calcul différentiel à plusieur variables
- dérivées partielles
- intégrales multiples
- Résoudre des problèmes d'optimisation simples

Suite $= \{a_n\} = \{a_1, a_2, a_3, \dots\} = \{a_n\}_{n=1}^{\infty}$

$\lim_{n\to\infty} a_n = L$ → Terms of succession $\{a_n\}$ approach L when n becomes large enough

→ Formal Definition: $\boxed{\lim_{n\to\infty} a_n = L \iff \forall \epsilon > 0, \text{ there is a } N \text{ such that if } n > N \Rightarrow |a_n - L| < \epsilon}$

Limits properties:

$$\lim_{n\to\infty} a_n \pm b_n = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \text{ if } \lim_{n\to\infty} b_n \neq \emptyset$$

$$\lim_{n\to\infty} (a_n)^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{For } p>0 \text{ and } a_n > 0$$

$$\text{if } \lim_{n\to\infty} |a_n| = \emptyset \Rightarrow \lim_{n\to\infty} a_n = \emptyset$$

compression:

if $a_n \leq b_n \leq c_n$, for $n > n_0$

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \Rightarrow \lim_{n\to\infty} b_n = L$

$\boxed{\text{if } \lim_{n\to\infty} a_n = L}$ an $f(x)$ is a continuous function in L

$$\Rightarrow \boxed{\lim_{n\to\infty} f(a_n) = f(L)}$$

$$\Rightarrow \boxed{\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)}$$ For f continuous in

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L'hôpital $\longrightarrow$ $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$ with $x \in \mathbb{R}$

Notice that L'hopital applies to continuous functions of real variables, not to series or successions

But we can use this to apply L'hopital to series:

$$\boxed{\text{if } \lim_{x \to \infty} f(x) = L \text{ and } f(n) = a_n \text{ with } n \text{ an integer} \Rightarrow \lim_{n \to \infty} a_n = L}$$

succession $\{a_n\}$ is
- increasing $\longrightarrow$ if $a_n < a_{n+1}$ $\forall n \geq 1$
- decreasing $\longrightarrow$ if $a_n > a_{n+1}$ $\forall n \geq 1$
- monotonic (monotone) $\longrightarrow$ if it increasing or decreasing

Any bounded monotonic succession is convergent

[bounded $\to$ acotada $\to$ délimité]

bounded $\to$ for $\{a_n\}$, there is a

Note: mathematical induction
1) demonstrate for $n=1$
2) Assume for $n=k$
3) Demonstrate for $n=k+1$

Interesting:

$a_n = (-1)^n$ is divergent!

$n \geq 0 \Rightarrow \{a_n\} = \{1, -1, 1, -1, 1, -1, \dots\}$

$\rightarrow$ it never converges to a single value but oscillates bt. -1 and 1

$\Rightarrow$ it is divergent

Series - Series

Given A series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$.

we define its <u>nth partial sum</u> as: $\boxed{S_n = \sum_{i=1}^{n} a_i} = a_1 + a_2 + a_3 + \dots + a_n$

if $S_n = \sum_{i=1}^{n} a_i$ <u>is convergent</u> $\Rightarrow$ $\boxed{\lim_{n \to \infty} S_n = S = \sum_{n=1}^{\infty} a_n}$

otherwise, if the succession $\{S_n\}$ is divergent, then the series is divergent

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$\boxed{\text{Limit of a geometric series}}$ $\Rightarrow$ n is on the <u>exponent</u>.

$\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + ar^3 + \dots \Rightarrow$ if $|r| < 1 \Rightarrow \boxed{\lim_{n \to \infty} \sum_{i=1}^{n} a r^{i-1} = \frac{a}{1-r}}$

THEOREM:

$\boxed{\text{if } \sum_{n=1}^{\infty} a_n \text{ is convergent} \Longrightarrow \lim_{n \to \infty} a_n = \emptyset}$

Note that it doesn't apply both ways $\rightarrow$ $\boxed{\lim_{n \to \infty} a_n = \emptyset \not\Rightarrow \sum^{\infty} a_n \text{ is convergent}}$

But we can say that if $\lim_{n \to \infty} a_n \neq \emptyset$, then $\sum^{\infty} a_n$ won't converge

Divergency test:

$\boxed{\text{if } \lim_{n \to \infty} a_n \nexists \text{ or } \lim_{n \to \infty} a_n \neq \emptyset \text{ then } \sum_{n=1}^{\infty} a_n \text{ is divergent}}$

Some useful exercises:

① For series $\{a_n\} = \{a_1, a_2, a_3, \dots, a_n\}$

the partial sum is $S_n = a_1 + a_2 + a_3 + \dots + a_n = \boxed{\frac{2n}{3n+5} = S_n}$

Determine $\sum_{n=1}^{\infty} a_n \longrightarrow \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{2n}{3n+5}\right] =$ // trick: Divide each section by "n"

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}\left[\frac{2n/n}{3n/n + 5/n}\right] = \lim_{n\to\infty}\left[\frac{2}{3 + \underbrace{5/n}_{\to \phi}}\right] = \lim_{n\to\infty}\left[\frac{2}{3}\right] = \frac{2}{3}$$

$$\boxed{\sum_{n=1}^{\infty} a_n = \frac{2}{3}}$$

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② $\lim_{n\to\infty} \sin(\pi/n) = ? = \phi$

Use of $\boxed{\lim_{n\to\infty} f(a_n) = f(L)}$ if $\lim_{n\to\infty} a_n = L$ and $f()$ is continuous at L

$$\Rightarrow \lim_{n\to\infty} \sin(\pi/n) = \sin\left[\cancel{\lim_{n\to\infty}\left(\pi/n\right)}^{\,0}\right] = \sin(\phi) = \phi$$

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③ $\lim_{n\to\infty} \frac{n!}{n^n} = ? = \phi$

Use of property $\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

$$\frac{n!}{n^n} = \cancel{\frac{1}{n}}^{\,\phi_{n\to\infty}} \cdot \cancel{\frac{2}{n}}^{\,\phi_{n\to\infty}} \cdot \frac{3}{n} \cdot \dots \cdot \frac{n}{n}$$

$$\lim_{n\to\infty} \frac{n!}{n^n} = \cancel{\lim_{n\to\infty}\left(\frac{1}{n}\right)}^{\,0} \cdot \cancel{\lim_{n\to\infty}\left(\frac{2}{n}\right)}^{\,0} \cdot \dots \lim_{n\to\infty}\left(\frac{n}{n}\right)$$

$$= 0$$

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④ **Geometric series:**

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a r^{n-1} = ?, \quad a \neq 0$$

1) Define $S_n = \sum_{i=1}^{n} a_n$ to evaluate $\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$

2) "trick" for geometric series $\rightarrow rS_n = \sum_{i=1}^{\infty} a r^n$

3) realise that $S_n = a + \sum_{i=1}^{n} a r^n = a + rS_n$

4) solve for $\boxed{S_n = \frac{a}{1-r}} \Rightarrow \boxed{\lim_{n\to\infty} S_n = \frac{a}{1-r}}$

$$\rightarrow \boxed{\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}}$$

, for $|r|<1$ ((for the series to be convergent)), $a \neq \phi$

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⑤. Use of L'Hôpital Determine $\lim_{n\to\infty} \frac{\ln(n)}{n}$

L'hopital definition

$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$] Now, this applies only to functions, <u>not series</u>

But, we can say: if $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$] → then $\lim_{n\to\infty} a_n = L$

So: 1) we determine the limit for continuous functions with $x \in \mathbb{R}$:

$f(x) = \ln(x)$, $g(x) = x$] by L'hôpital: $\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{\ln'(x)}{x'} = \frac{1/x}{1} = \frac{1}{x} \to \emptyset$

2) now: $f(n) = a_n = \ln(n)$, $g(n) = b_n = n$] with n an integer

3) so $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(n) = \emptyset$, $\lim_{n\to\infty} b_n = \lim_{x\to\infty} g(n) = 1$] $\boxed{\lim_{n\to\infty} \frac{\ln(n)}{n} = \emptyset}$

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⑥ using geometric series (n as exponent)

i) $0.\overline{8} = \sum_{i=1}^{\infty} 8 \cdot \left(\frac{1}{10}\right)^i$ → now multiply by $\frac{1}{10} \cdot 10$ to get the form r^{i-1} of a geometric series] → $0.\overline{8} = \sum_{i=1}^{\infty} 8 \cdot \left(\frac{1}{10}\right)^i \cdot 10 \cdot \frac{1}{10} = \sum_{i=1}^{\infty} \frac{8}{10} \cdot \left(\frac{1}{10}\right)^{i-1}$

$\Rightarrow a = 8/10$, $r = 1/10$

$\Rightarrow 0.\overline{8} = \lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{8/10}{1-1/10} = \frac{8/10}{9/10} = \boxed{\ }$

ii) $2.\overline{516} = 2 + 516 \cdot \sum_{i=1}^{\infty} \left(\frac{1}{10^3}\right)^i \cdot \overbrace{10^3 \cdot \frac{1}{10^3}}^{\text{same "trick"}} = 2 + \frac{516}{10^3} \sum_{i=1}^{\infty} \left(\frac{1}{10^3}\right)^{i-1}$

same "trick"

$a = 516/10^3$, $r = 1/10^3$] $\Rightarrow \lim_{n\to\infty} S_n = 2 + \frac{516/10^3}{1 - \frac{1}{10^3}} = 2 + \frac{516}{999} =$

$= \frac{1998 + 516}{999} = \frac{2514}{999} = \boxed{\frac{838}{333}}$

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⑦ Use of partial fractions decomposition

Demonstrate convergence and calculate: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

i) partial fraction decomposition $\rightarrow$ $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)} = \frac{\cancel{n}+1-\cancel{n}}{n(n+1)} = \frac{1}{n\cdot(n+1)}$ ✓

ii) now, expand: $S_n = \sum_{i=1}^{n} \frac{1}{i(1+i)} = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{(i+1)} = \left[\frac{1}{1} - \cancel{\frac{1}{2}}\right] + \left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right] + \left[\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right] + \dots + \left[\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right] + \left[\cancel{\frac{1}{n}} - \frac{1}{n+1}\right]$

elimination "trick"

$S_n = 1 - \frac{1}{n+1} \Rightarrow \lim_{n\to\infty} S_n = \boxed{1 = \sum_{n=1} \frac{1}{n(n+1)}}$ convergent to value $\underline{\underline{1}}$

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⑧ Recursive calculation:

$\{a_n\} = \begin{cases} a_1 = 2, & \text{for } n=1 \\ a_{n+1} = \frac{1}{2}(a_n+6), & \text{for all other } n \end{cases}$

Hint: Remember that $\boxed{\text{Any bounded monotonic series is convergent}}$

$\{a_n\} = \{2, \overbrace{\frac{1}{2}(2+6)}^{4}, \overbrace{\frac{1}{2}(4+6)}^{5}, \overbrace{\frac{1}{2}(5+6)}^{5.5} + \overbrace{\frac{1}{2}(5.5+6)}^{5.75} + \overbrace{\frac{1}{2}(5.75+6)}^{5.85} + \overbrace{\frac{1}{2}(5.85+6)}^{5.925} + \dots$

$\{a_n\}$ appears to be upper bounded at $\underline{6}$
and always increasing $\rightarrow$ monotonic
$\Rightarrow$ convergent.

Let's demonstrate: ⓘ $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left[\frac{1}{2}(a_n+6)\right] = \frac{1}{2}\left[\lim_{n\to\infty}(a_n) + 6\right]$

If a_n is bounded, at $n\to\infty$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L \Rightarrow \underbrace{\lim_{n\to\infty} a_{n+1}}_{L} = \frac{1}{2}\left[\lim_{n\to\infty} a_n + 6\right]$

$= \frac{1}{2}[L+6]$

$\Rightarrow \boxed{L = 6}$

ⓘⓘ Using induction, we need to prove

a) the succession is monotonically increasing

i) validate for $n=1$

$a_1 = 2 < a_2 = 4$ ✓

ii) assume for $n=k$ and validate for $n=k+1$

if: $a_k < a_{k+1} \quad / +6$

$a_k + 6 < a_{k+1} + 6 \quad / :2$

$\underbrace{\frac{1}{2}(a_k+6)}_{a_{k+1}} < \underbrace{\frac{1}{2}(a_{k+1}+6)}_{a_{k+2}}$ ✓✓

b) bounded at 6

i) $n=1 \Rightarrow a_1 = 2 < 6$ ✓

ii) assume $a_k < 6$ and demonstrate for $k+1$

$a_k < 6 \quad / +6$

$a_k + 6 < 12 \quad / :2$

$\underbrace{\frac{1}{2}(a_k+6)}_{\boxed{a_{k+1} < 6}} < 6$ ✓✓

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⑧ more on geometric series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ converges? if so, to what value?

First, we try to frame this into a geometric form:

$$a_n = \sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} 4^n \cdot \frac{3}{3^n} = 3 \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n \cdot \overbrace{\frac{3}{4} \cdot \frac{4}{3}}^{\text{to get } (r^{n-1}) \text{ form}} = \cancel{3} \sum_{n=1}^{\infty} \frac{4}{\cancel{3}} \cdot \left(\frac{4}{3}\right)^{n-1} = \sum_{n=1}^{\infty} 4 \left(\frac{4}{3}\right)^{n-1}$$

$a = 4$

$r = \frac{4}{3} \rightarrow$ but $|r| > 1 \Rightarrow$ series diverges

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limits and use of exponentials for continuous functions:

if $\boxed{y = f(x)^{g(x)}}$ then we can calculate $\lim_{x \to c} y(x)$ like this

i) take ln $\Rightarrow$ $y = f(x)^{g(x)}$ $/\ln()$

$\ln(y) = g(x) \ln[f(x)]$

$\Rightarrow$ notice that this means $\rightarrow$ $y = e^{g(x) \ln[f(x)]}$

ii) this means

$$\lim_{x \to c} y(x) = \lim_{x \to c} e^{g(x) \ln[f(x)]} = e^{\lim_{x \to c} g(x) \ln(f(x))}$$

if $\lim_{x \to c} g(x) \ln[f(x)] = L$ (exists and is equal to L)

$$\Rightarrow \boxed{\lim_{x \to c} y(x) = \lim_{x \to c} f(x)^{g(x)} = e^L, \text{ if } \lim_{x \to c} g(x) \ln(f(x)) = L}$$

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up until now we've used some function that defines the sum of a succession/serie $\longrightarrow$ $S_n = \sum_{n=1}^{\infty} a_n = f(a_n)$

IF <u>no function is easily defined</u> we can use an approximation with the functions integral:

$$\longrightarrow \boxed{\begin{array}{l} \sum_{n=1}^{\infty} a_n \text{ converges } \underline{\underline{\text{if}}} \\ \int_1^{\infty} f(x)\,dx \text{ converges, where } f(n) = a_n \end{array}}$$

and diverges if $\int_1^{\infty} f(x)\,dx$ diverges.

notice that:

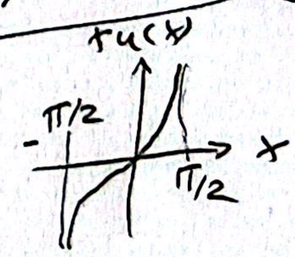
- in general $\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx$ $\rightarrow$ this only evaluates convergence
- the start of the sum need not be at $\emptyset$
- $f(x)$ need not be always diminishing (it's enough that it diminishes for $x > N$ for some N)

<u>Exercise</u>:

$f(x) = \frac{1}{x^2+1}$ $\longrightarrow$ decreasing, continuous, positive

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1}(x)\Big|_1^t = \lim_{t\to\infty} \left[\cancel{\tan^{-1}(t)}^{\,\pi/2} - \cancel{\tan^{-1}(1)}^{\,\pi/4}\right] = \frac{\pi}{4}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ converges}}$$



p-functions:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

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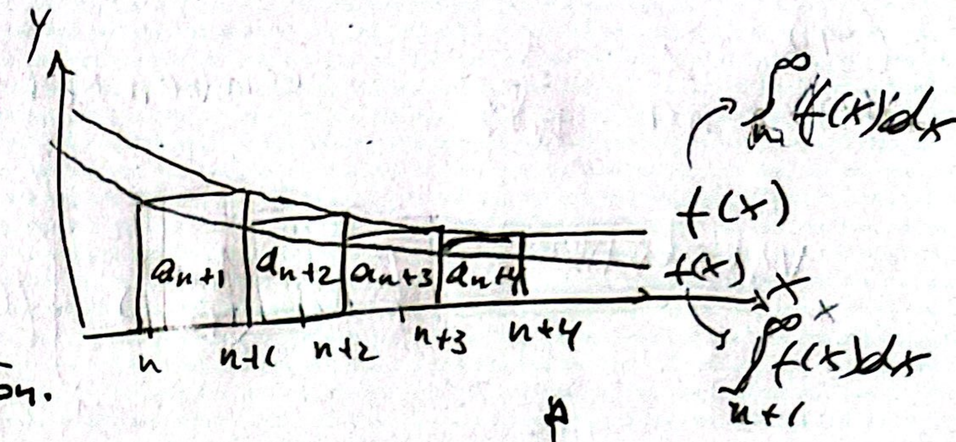
Estimation of <u>error</u> or deviation from actual value

we've defined $S_n = \sum_{i=1}^{n} a_i$

and thus $S = \lim_{n\to\infty} S_n$ is <u>always</u> an <u>approximation</u>.



since we will use $n < \infty$

we define the error:

$$\boxed{R_n = S - S_n} = \sum_{i=1}^{\infty} a_i - \sum_{i=1}^{n} a_i$$

and we know that we'll always have:

$$\boxed{\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx}$$

where $f(n) = a_n$

Ex: For $\sum 1/n^3$, obtain the error for the first 10 elements $(n=10)$

$$f(x) = \frac{1}{x^3} \rightarrow \int_{n+1}^{\infty} f(x)dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

$$\int_n^\infty f(x)dx = \int_n^\infty \frac{1}{x^3}dx = \left(-\frac{x^{-2}}{2}\right)\Big|_n^\infty = \frac{1}{2}\left(\cancel{-\frac{1}{x^2}}^{\,0} + \frac{1}{n^2}\right) = \frac{1}{2n^2}$$

$$\int_{n+1}^\infty f(x)dx = \int_{n+1}^\infty \frac{1}{x^3}dx = \left(-\frac{x^2}{2}\right)\Big|_{n+1}^\infty = \frac{1}{2}\left(\cancel{-\frac{1}{x^2}}^{\,\infty} + \frac{1}{(n+1)^2}\right) = \frac{1}{2(n+1)^2}$$

limits for R_n

now $S_n = S_{10} = \sum_{i=1}^{10} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots \frac{1}{10^3} = \boxed{1.197 = S_n} = S_{10}$

which has an error of: $\frac{1}{2n^2}\Big|_{n=10} = 0.005$ and $\frac{1}{2(n+1)^2}\Big|_{n=10} = 0.00413$

$$\Rightarrow \boxed{0.00413 \le R_n \le 0.005}$$ <u>For $S_{10} = 1.197$</u>

How many values (n) does it take to achieve a 0.0005 level of error?

$$0.0005 = \frac{1}{2n^2} \Rightarrow n = \sqrt{\frac{1}{0.0005 \times 2}} = 31.6 \Rightarrow \boxed{n \ge 32}$$

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other exercises:

(9) $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ converges?

notice that this is a p-series with $p = \sqrt{2} = 1.41.. > 1 \Rightarrow$ it should converge

validation: $f(x) = \frac{1}{x^{\sqrt{2}}} \Rightarrow \int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^{\sqrt{2}}}\,dx = \lim_{t\to\infty} \frac{x^{-\sqrt{2}+1}}{(-\sqrt{2}+1)}\Big|_1^t = \frac{1}{1-\sqrt{2}}\left[\cancel{\frac{1}{t^{1-\sqrt{2}}}}^{\,0} - \cancel{\frac{1}{1^{1-\sqrt{2}}}}^{\,1}\right] = \frac{1}{1-\sqrt{2}}$

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx = -2.414 \checkmark$$

$\Rightarrow$ converges ✓

(10) $\sum_{n=3}^{\infty} n^{-0.9999}$ converges? NO, it diverges

again, this is a p-series with $p \leq 1$
$\Rightarrow$ it should diverge

$f(x) = \frac{1}{x^{0.9999}} \rightarrow \int_3^{\infty} f(x)\,dx = \int_3^{\infty} x^{-0.9999}\,dx = \lim_{t\to\infty} \frac{x^{-0.9999+1}}{1-0.9999}\Big|_3^t = \lim_{t\to\infty}\left[\cancel{\frac{t^{0.0001}}{0.0001}}^{\,\infty} - \cancel{\frac{3^{0.0001}}{0.0001}}^{\,10.001.0987}\right]$

(14) $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \dots \rightarrow$ converges?

the series can be represented by $\sum_{n=1}^{\infty} \frac{1}{n^3} \Rightarrow f(x) = \frac{1}{x^3}$

again, p-series with $p > 1$ ✓ it will converge

convergence test $\int_1^{\infty} f(x) = \lim_{t\to\infty} \int_1^t \frac{1}{x^3}\,dx = \lim_{t\to\infty} \frac{1}{-2x^2}\Big|_1^t = \lim_{t\to\infty}\left[\cancel{\frac{1}{-2t^2}}^{\,0} - \frac{1}{2\cdot 1^2}\right] = -\frac{1}{2} \checkmark$

it converges

Alternating series

These are series where the sign is alternating (+) and (−) (not just positive terms as seen so far)

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \dots \quad , \quad b_n > 0$$

$$\boxed{\text{if } b_{n+1} \leq b_n \quad \forall n \quad \text{and } \lim_{n\to\infty} b_n = \emptyset} \longrightarrow \text{the series } \underline{\text{converges}}.$$

ERROR boundaries for alternating series:

$$\left.\begin{array}{l} \text{if } S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n \\ \text{with } b_{n+1} \leq b_n \\ \text{and } \lim_{n\to\infty} b_n = 0 \end{array}\right\} \longrightarrow \boxed{|R_n| = |S - S_n| \leq b_{n+1}}$$

when b_{n+1} is the 1st element not considered to calculate $S_n = \sum_{i=1}^{n} b_n$

Ex: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ calculate with 3 decimals of exactitud.

1) converges? $\longrightarrow$ i) $\frac{(-1)^{n+1}}{(n+1)!} = \frac{(-1)^n}{n!(n+1)} = b_n \cdot \frac{1}{n+1} \leq b_n \Rightarrow \boxed{b_{n+1} \leq b_n \ \forall n > 0}$ ✓

it converges ✓ $\Leftarrow$

ii) $\lim_{n\to\infty} b_n = \cancel{\frac{1}{n!}} \to 0$ ✓

2) $|R_n| = |S - S_n| \leq 0.0009$ $\longrightarrow$

For $n = 7$

$b_0 = 1/0! = 1$

$b_1 = 1/1! = 1$

$b_2 = 1/2! = 0.5$

$b_3 = 1/3! = 0.16\overline{6}$

$b_4 = 1/4! = 0.041\overline{6}$

$b_5 = 1/5! = 0.008\overline{3}$

$b_6 = 1/6! = 0.00138\overline{8}$

$b_7 = 1/7! = 0.00019\overline{8} < 0.0009$ ✓✓

$$\boxed{S_{n=7} = \sum_{n=0}^{7} \frac{(-1)^n}{n!} = 0.36786}$$

with an error ≤ 0.00019

~~and~~ $\cancel{\tan^{-1}(x) = \frac{1}{1+x^2}}$

$\cancel{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}$

Note:

$$\left(1 + \frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e$$

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Convergence Radius

For a series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ we can have:

i) it converges for any value of x (convergence radius $R = \infty$)

ii) it converges for $x = a$ ($R = a$)

iii) it converges only if $\boxed{R > |x-a|}$ where $R \in \mathbb{R}$ and is defined as <u>convergence radius</u>

$x \in (a-R, a+R)$

Some examples:

$\sum_{n=0}^{\infty} x^n$ (geometric series) $\longrightarrow R = 1$ [converges for $-1 < x < 1$] $(-1, 1)$

$\sum_{n=0}^{\infty} n! x^n \longrightarrow R = 0$ $\{0\}$ $\Rightarrow$ it diverges

$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n} \longrightarrow R = 1 \Rightarrow [2, 4)$ converges for $2 \leq x < 4$

$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2} \longrightarrow R = \infty \Rightarrow$ converges for any x $(-\infty, \infty)$

How to <u>assess convergence</u>? $\longrightarrow$ $\boxed{\text{evaluate for } \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1}$

example convergence radius:

$$f(x) = \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} \Rightarrow a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$$

to assess convergence, we need to evaluate:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1} x^{n+1}}{\sqrt{(n+1)+1}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = \left|(-3)x \frac{\sqrt{n+1}}{\sqrt{n+2}}\right| = \left|(-3)x \frac{\sqrt{1+\cancel{1/n}}}{\sqrt{1+\cancel{2/n}}}\right| \xrightarrow[n\to\infty]{} |3x|$$

$\Rightarrow f(x)$ converges when $|3x| < 1 \Rightarrow \boxed{-\frac{1}{3} < x \leq \frac{1}{3}}$ $\quad R = \frac{1}{3}$

now we need to evaluate the borders of the interval

i) $x = -\frac{1}{3} \longrightarrow f(-\frac{1}{3}) = \sum_{n=0}^{\infty} \frac{(-3)^n (-1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} (\cancel{-3})(\cancel{-\frac{1}{3}})^n \cdot \frac{1}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, for $m = n+1$

$\Downarrow$

$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}}$ is a p-series with $p = 0.5 < 1$

$\Rightarrow$ it diverges

$f(-1/3)$ diverges

ii) $x = \frac{1}{3} \Rightarrow f(\frac{1}{3}) = \sum_{n=0}^{\infty} \frac{(-3)^n (1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

now, this is an alternating series, it's easy to see that

$b_{n+1} = \frac{1}{\sqrt{n+2}} \leq b_n = \frac{1}{\sqrt{n+1}}$ ✓

now we have to evaluate $\lim_{n\to\infty} b_n \Rightarrow \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ ✓

so $f(1/3)$ converges!

$\Rightarrow$ $\boxed{\text{the convergence interval is } (-1/3, 1/3]}$

Taylor Series

For $\boxed{f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n}$, with $|x-a| < R$ (convergence radius)

then its coefficients are $\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$ with $f^{(n)}(a)$ the n-th derivative of $f(x)$ at $x=a$.

Replacing c_n in $f(x)$, we get the <u>taylor series</u>

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n}$$

Taylor series

Notice this allows us to represent functions in terms of sums <u>or series!!</u>

when $a=0 \rightarrow$ we get the $\boxed{\text{maclaurin series}}$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n} = \frac{f(0)}{0!} + \frac{f'(1)x}{1!} + \frac{f''(a)x^2}{2!} + \dots$$

Notice that we use <u>Taylor</u> as an <u>approximation</u> for $f(x)$, it's useful to evaluate the <u>error</u> we accumulate at a given "n" value

$$|R_n(x)| = |f(x) - T_n(x)|$$, with $\boxed{T_n(x) = \sum_{i=0}^{n} \frac{f^{(n)}(a)(x-a)^n}{n!}}$ Taylor

to determine convergence, we still evaluate $\boxed{\left|\frac{a_{n+1}}{a_n}\right| < 1}$

ex: Represent maclaurin series for $f(x) = e^x$ [note: $f'(x) = f^{(n)}(x) = e^x$]

we know: by maclaurin

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{(n)0} x^n}{n!} = \frac{e^0 x^0}{0!} + \frac{e^0 x^1}{1!} + \frac{e^0 x^2}{2!} + \dots$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

evaluation of convergence radius $\longrightarrow \left| \frac{a_{n+1}}{a_n} \right| < 1$

$$\Rightarrow \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right| \xrightarrow[n \to \infty]{} 0 \Rightarrow f(x) \text{ converges for all values of } (x)$$

Vectors and Space geometry

Distance of 2 points P_1 and P_2 → $\boxed{|P_1P_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}$

Equation for a sphere centered at $O=(0,0,0)$ and radius R → $\boxed{\sqrt{x^2+y^2+z^2} = R}$

As for a sphere centered at point $p = (p_x, p_y, p_z)$ → $\sqrt{(x-p_x)^2+(y-p_y)^2+(z-p_z)^2} = R$

$\boxed{\text{vector } \vec{v}}$ → quantity with a magnitude and Direction

vector longitud $\vec{v}$ (a_1, a_2, a_3) in $\mathbb{R}^3$ → $\boxed{|v| = \sqrt{a_1^2+a_2^2+a_3^2}}$

$\updownarrow$

$\vec{v} = \hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3$

vector sum:

$\vec{v_1} = (a_1, a_2, a_3)$

$\vec{v_2} = (b_1, b_2, b_3)$

$\Rightarrow \vec{v_1}+\vec{v_2} = (a_1+b_1, a_2+b_2, a_3+b_3)$

Cross product

$$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1) = \vec{c}$$

or $\vec{a}\times\vec{b} = \|\vec{a}\|\|\vec{b}\| \sin\theta\, \hat{c}$ where $\|\hat{c}\| = 1$ and $\vec{a}\cdot\hat{c} = \phi$, $\vec{b}\cdot\hat{c} = \phi$ → $\hat{c}$ is orthogonal to both $\vec{a}$ and $\vec{b}$ (or to the plane defined by $\vec{a}$ and $\vec{b}$)

$\Rightarrow$ $\boxed{\vec{a}\times\vec{b} = \phi \text{ if } \vec{a} \text{ and } \vec{b} \text{ are parallels}}$

Defined in matrix form:

$\vec{a}\times\vec{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

$$= \hat{i}(a_2b_3 - a_3b_2) + \hat{j}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - a_2b_1)$$

dot product

$\vec{v_1}\cdot\vec{v_2} = a_1b_1 + a_2b_2 + a_3b_3$

$\vec{v_1}\cdot\vec{v_2} = |v_1||v_2| \cos\theta$

$\theta \measuredangle$ of $\vec{v_1}$ and $\vec{v_2}$

$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 = 2|a||b|\cos\theta$

$\vec{a}, \vec{b}$ are orthogonal if $\vec{a}\cdot\vec{b} = \phi$ | $\theta = 90°$

scale projection of $\vec{b}$ over $\vec{a}$ → $\boxed{|b|\cos\theta}$, where $\theta \measuredangle$ $\vec{a}$ and $\vec{b}$

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Les equations des droites (rectas) et des planes

↳ straight lines

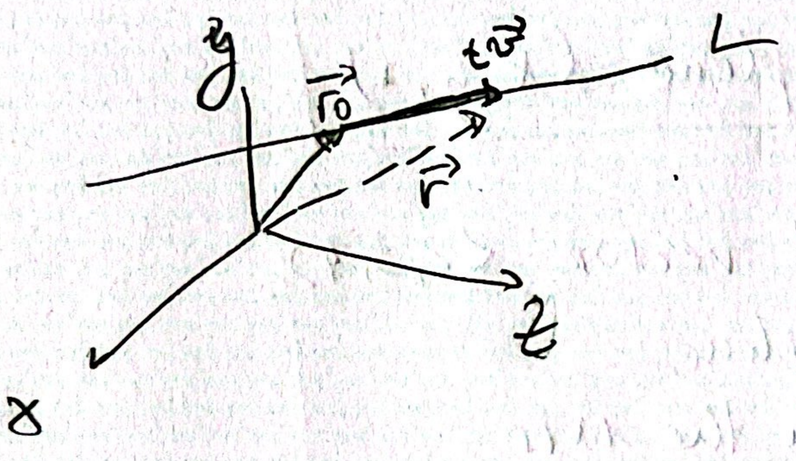
Droite → is determined by $\boxed{\text{a point}}$ on the straight line and a $\boxed{\text{direction}}$

⇒ equation for Droite (straight line) ⇒ $\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$

where $\vec{r_0}$ is a vector from the origin (0,0,0) to a point on the straight line.

$\vec{v}$: direction of the line and

t : a scalar



$$\vec{r} = (x_0 + at,\ y_0 + bt,\ z_0 + ct)$$

with $\vec{v} = (a, b, c)$

now, since "t" exists for all coordinates and defines the straight line, we can say:

$$\left.\begin{array}{l} x = x_0 + at \Rightarrow t = (x - x_0)/a \\ y = y_0 + bt \Rightarrow t = (y - y_0)/b \\ z = z_0 + ct \Rightarrow t = (z - z_0)/c \end{array}\right] \Rightarrow \text{we can represent the line equation as:}$$

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}} = t$$

symmetric equation

notice that if one of the dimensions of the $\vec{v}$ vector is zero → the straight line L lies on the corresponding plane

$$\Rightarrow \boxed{\underset{a=0}{x = x_0}, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c} = t}$$

⇓ the line is on plane $x = x_0$

Plane Definition

→ a plane is fully defined by a point on the plain and a perpendicular vector

$$\boxed{\vec{n}\cdot(\vec{r}-\vec{r_0})=0}$$

with:

$\vec{n}$: vector orthogonal to the plane

$\vec{r_0} = \langle x_0, y_0, z_0\rangle$ a position vector from the origin $(0,0,0)$ to a point on the plane (x_0, y_0, z_0)

$\boxed{\vec{r}: \text{vector representing the plane } (x,y,z)}$

Ex: ① given the plane with point $(2,4,-1)$ on it and vector $\langle 2,3,4\rangle$ orthogonal to it, determine → equation of the plane
→ intersection with the axis

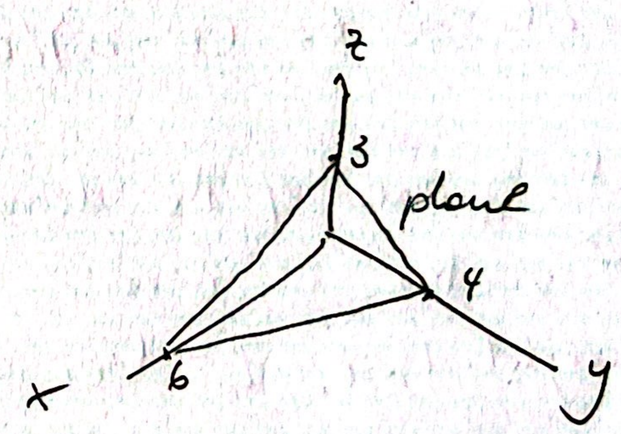
$$\Rightarrow \left.\begin{array}{l}\vec{n} = \langle 2,3,4\rangle \\ \vec{r_0} = (2,4,-1)\end{array}\right] \Rightarrow \quad \vec{n}\cdot(\vec{r}-\vec{r_0}) = 0$$

$$(2,3,4)\cdot(x-2, y-4, z+1) = 2(x-2)+3(y-4)+4(z+1) = \emptyset$$

$$\Rightarrow \boxed{2x+3y+4z = 12} \text{ equation for the plane}$$

intersections with the axis are given by:

$$\left.\begin{array}{l} x \Rightarrow y=z=\emptyset \Rightarrow x=6 \\ y \Rightarrow x=z=\emptyset \Rightarrow y=4 \\ z \Rightarrow x=y=\emptyset \Rightarrow z=3 \end{array}\right] \Rightarrow$$



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② Determine the eq. for a plane that contains these 3 points,

$P_1 = (1,3,2)$
$P_2 = (3,-1,6)$
$P_3 = (5,2,0)$

⇒ we need a vector ⊥ to the plane.
↳ to get it, we can use the cross product of 2 vectors on the plane
↳ vectors on the plane can be obtained with the points on the plane

$$\vec{v_1} = P_1 - P_2 = \langle 1-3, 3--1, 2-6\rangle = \langle -2, 4, -4\rangle$$

$$\vec{v_2} = P_2 - P_3 = \langle 3-5, -1-2, 6-0\rangle = \langle -2, -3, 6\rangle$$

$$\vec{n} = \vec{v_1}\times\vec{v_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 4 & -4 \\ -2 & -3 & 6 \end{vmatrix} = \hat{i}(24--12) - \hat{j}(-12-8) + \hat{k}(6--8) = \boxed{\langle 12, 20, 14\rangle = \vec{n}}$$

taking $\vec{P_1} = \vec{r_0} = (1,3,2)$ → $\vec{n}\cdot(\vec{r}-\vec{r_0}) = \langle 12, 20, 14\rangle\cdot\langle x-1, y-3, z-2\rangle = \emptyset$

$$\Rightarrow 12x - 12 + 20y - 60 + 14z - 28 = \emptyset$$

$$12x + 20y + 14z = 100 \Rightarrow \boxed{6x + 10y + 7z = 50} \text{ equation from the plane}$$

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we can validate this equation using all 3 points

$P_1 = (1, 3, 2) \longrightarrow 6 \cdot 1 + 10 \cdot 3 + 7 \cdot 2 = 50$ ✓

$P_2 = (3, -1, 6) \longrightarrow 6 \cdot 3 + 10 \cdot (-1) + 7 \cdot 6 = 18 - 10 + 42 = 50$ ✓

$P_3 = (5, 2, 0) \longrightarrow 6 \cdot 5 + 10 \cdot 2 + 7 \cdot \phi = 50$ ✓

Another important concept:

Distance $\underline{D}$ from a plane ($ax + by + cz + d = \phi$) to a point $P_1(x_1, y_1, z_1)$

$$\Rightarrow \boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Vector Functions

These functions have <u>vectors as their domain</u> (instead of values in $\mathbb{R}$)

→ they are typically used to define movement on space

$$r(t) = \langle f(t), g(t), h(t) \rangle = \boxed{f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} = r(t)}$$

For each "t" in r, there is a <u>unique V_3 defined by $r(t)$</u>

> notice the use of "t" instead of "x", because these functions are normally used to represent <u>movement</u> as a function of time

ex: $r(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$

$\Rightarrow f(t) = t^3$, $g(t) = \ln(3-t)$, $h(t) = \sqrt{t}$

The domain of r is defined as "t" for which the <u>codomain r(t) exists</u> / is defined → in the case above:

$f(t) = t^3 \rightarrow \exists \ \forall t$

$g(t) = \ln(3-t) \rightarrow \exists$ for $t<3$

$h(t) = \sqrt{t} \rightarrow \exists$ for $t \geq 0$

$\Rightarrow$ Domain for $r(t)$ is t in $[0,3)$

[notice that $r(t)$ can be seen as the equation for the positional vector of point $P(f(t), g(t), h(t))$ as that point moves through a 3D space]

The limit for a vector function is:

$$\boxed{\lim_{t\to a} r(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle \quad \text{where } r(t) = \langle f(t), g(t), h(t) \rangle}$$

And $r(t)$ is <u>continuous</u> in "a" if $\lim_{t\to a} r(t) = r(a)$

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Derivatives and integrals for vector functions

Derivatives $\longrightarrow$ $\boxed{\dfrac{dr}{dt} = r'(t) = \lim_{h \to \emptyset} \dfrac{r(t+h) - r(t)}{h}} = \langle f'(t), g'(t), h'(t) \rangle$

Also defined as <u>tangent vector</u> when made unit vector

$$\boxed{T(t) = \frac{r'(t)}{|r'(t)|}} \Longrightarrow |T(t)| = 1$$

Notice that the definition works the same way as that in $\mathbb{R}$
the rules also work the same way.

Ex: for $r(t) = (1+t^3)\hat{\imath} + (te^{-t})\hat{\jmath} + \sin(2t)\hat{k}$

Determine a) its derivative
b) its tangent vector on $t = \emptyset$.

$\Rightarrow$ a) $\boxed{r'(t) = \langle 3t^2, e^{-t} + te^{-t}, 2\cos(2t) \rangle}$

b) $r'(0) = \langle 0, 1, 2 \rangle \Rightarrow |r'(0)| = \sqrt{0^2 + 1^2 + 2^2} = \sqrt{5}$

$\Rightarrow \boxed{T(0) = \langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle}$

notice that this vector should "begin" on $r(\emptyset) = \langle \underset{f(0)}{1}, \underset{g(0)}{\emptyset}, \underset{h(0)}{\emptyset} \rangle$

Some derivative rules on vector functions.

$\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$ [Dot product]

$\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$ [cross product]

$\frac{d}{dt}[u(f(t))] = f'(t)\, u'(f(t))$ [chain derivative]

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INTEGRALS OF VECTOR FUNCTIONS

$$\boxed{\int_a^b r(t)\,dt = \left(\int_a^b f(t)\,dt\right)\hat{\imath} + \left(\int_a^b g(t)\,dt\right)\hat{\jmath} + \left(\int_a^b h(t)\,dt\right)\hat{k}}$$

as within $\mathbb{R}$ $\rightarrow$ $\displaystyle\int_1^\infty r(t)\,dt = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(t_i^*)\hat{\imath} + \sum_{i=1}^{n} g(t_i^*)\hat{\jmath} + \sum_{i=1}^{n} h(t_i^*)\hat{k}\right]$

ex: $r(t) = 2\cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + 2t\hat{k}$

$\rightarrow \int r(t)\,dt = 2\sin(t)\hat{\imath} - \cos(t)\hat{\jmath} + t^2\hat{k} + C$ → notice that we must include the constant "C" when we are not evaluating the integral

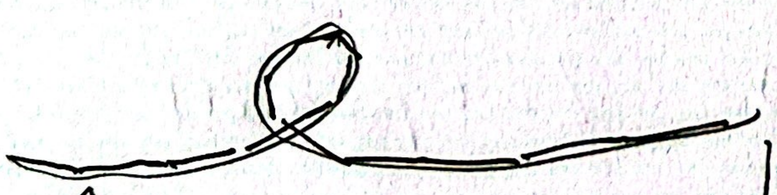
[the constant is eliminated when we evaluate the integral]

$$\int_0^{\pi/2} r(t)\,dt = \left[\cancel{2\sin\left(\tfrac{\pi}{2}\right)}^{1}\hat{\imath} - \cancel{\cos\left(\tfrac{\pi}{2}\right)}^{0}\hat{\jmath} + \left(\tfrac{\pi}{2}\right)^2\hat{k}\right] - \left[\cancel{2\sin(0)}^{0}\hat{\imath} - \cancel{\cos(0)}^{1}\hat{\jmath} + \cancel{0\hat{k}}^{0}\right] + \cancel{C} - \cancel{C} = 2\hat{\imath} + 1\hat{\jmath} + \left(\tfrac{\pi}{2}\right)^2\hat{k}$$

The length L of a curve is defined as the length of inscribed polygons

↳ if $f'(t)$, $g'(t)$ and $h'(t)$ are continuous,

we can say: $$\boxed{L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt}$$



$$\boxed{L = \int_a^b |r'(t)|\,dt}$$ length of $r(t)$ from "a" to "b"

↑ this is basically approximating the sum of length as several straight tangents with very small length.

CURVATURE $\longrightarrow$ the way the tangent vector changes over the length of its arch (magnitude of change in the tangent vector)

remember $\frac{d}{dx}\frac{f}{g} = \frac{f'g - g'f}{g^2}$

$$K(t) = \frac{|T'(t)|}{|r'(t)|} = \boxed{\frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = K(t)}$$ curvature

since $T(t) = \dfrac{r'(t)}{|r'(t)|}$

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Functions de plusieur variables

The limit for multivariate functions

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y)}$$ only exists if

$$\lim_{x\to a} f(x,y) = \lim_{y\to b} f(x,y)$$

if we approach the limit from different directions the limit value must be the same

⇓

if not, the limit doesn't exist!

ex: ① $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ ⟶ · approach from $x\to\emptyset$

$$\lim_{x\to 0} \frac{\cancel{x^2}-y^2}{\cancel{x^2}+y^2} = \frac{-y^2}{y^2} = -1$$

· approach from $y\to\emptyset$

$$\lim_{y\to 0} \frac{x^2-\cancel{y^2}}{x^2+\cancel{y^2}} = \frac{x^2}{x^2} = 1$$

Different values for the limit ⇓ it doesn't exist!

[notice that this is the same for the 1variable case ⟶ $\lim_{x\to a} f(x) \Rightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$] ⇒ limit must be the same if we approach it from "left" or "righ.

② $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{xy}{x^2+y^2}$ ⟶ Approaching from $x\to\emptyset$

$$\lim_{x\to\emptyset} \frac{\cancel{x}^{0}y}{\cancel{x^2}+y^2} = \emptyset$$

Approaching from $y\to\emptyset$

$$\lim_{y\to\emptyset} \frac{xy}{x^2+y^2} = \emptyset$$

same value!!

BUT what if we approach from $x=y$?

$$\lim_{x\to y} f(x,y) = \frac{\cancel{y^2}}{2\cancel{y^2}} = \frac{1}{2}$$

different value!

⇒ limit ⟶ $(0,0)$ for $f(x,y) = \frac{xy}{x^2+y^2}$ doesn't exist.

clearly, the evaluation of the limit this way is not easy ... we might easily miss one direction where the limit is different and assume it exists when it doesn't.

we must use this approach to estimate limits for multivariate functions.

For $f(x,y)$ in D

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \quad \text{exists}$$

if for any $\varepsilon > \phi$, there is a $\delta > \phi$

that $\longrightarrow$ if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

then $\rightarrow$ $|f(x,y) - L| < \varepsilon$ (for any $\varepsilon > \phi$)

notice this is the distance bt (x,y) and (a,b)

ex: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = L$ exists?

1) we can suspect $L = \phi$ by approaching from $x \to \phi$, $y \to \phi$

2) take $\varepsilon > \phi$, we need to find $\delta > \phi$ that for $(a,b) = (\phi,\phi)$

if $0 < \sqrt{(x-\phi)^2 + (y-\phi)^2} < \delta$

then $|f(x,y) - L| < \varepsilon$ for any $\varepsilon > \phi$

$\longrightarrow |f(x,y) - L| = \left| \frac{3x^2y}{x^2+y^2} - \cancel{L}^{\,0} \right| = \frac{3x^2}{x^2+y^2}|y|$

since $y^2 \geq 0 \Rightarrow x^2+y^2 \geq x^2 \Rightarrow \frac{x^2}{x^2+y^2} \leq 1$

$\frac{3x^2|y|}{x^2+y^2} \leq 3 \cdot 1 \cdot |y| = 3|y| = 3\sqrt{y^2}$

we also have that $3\sqrt{y^2} \leq 3\sqrt{x^2+y^2}$, for $x \geq \phi$

So, we HAVE:

$$|f(x,y)-L| = \frac{3x^2|y|}{x^2+y^2} \leq 3\sqrt{x^2+y^2} < \varepsilon$$

and we need $0 < \sqrt{x^2+y^2} < \delta$

if we take $\delta = \varepsilon/3$

$\Downarrow$

$3\sqrt{x^2+y^2} < 3\delta$

$\Downarrow$

$\boxed{\sqrt{x^2+y^2} < \delta}$

$\rightarrow \boxed{\lim_{(x,y)\to(0,0)} f(x,y) = \phi}$

since we have $\varepsilon > \phi$ and $\delta > \phi$ $\left(\delta = \frac{\varepsilon}{3}\right)$

where $|f(x,y)-L| < \varepsilon$ when $0 < \sqrt{(x-\phi)^2+(y-\phi)^2} < \delta$

notice that we had to get some preliminary value FOR L and then analise overall existance

Notice that all these are <u>also valid FOR 3 OR more variables</u>

$\Downarrow$

$$\boxed{\begin{array}{l} \lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L \\ \text{if } 0 < \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < \delta \\ \text{then } |f(x,y,z)-L| < \varepsilon, \text{ FOR any } \varepsilon > 0 \end{array}}$$

Partial Derivatives for 2-variable functions

multivariate functions

For $f(x,y)$ it partial derivatives would be:

$$\boxed{\begin{aligned} f_x(x,y) &= \lim_{h\to\phi} \frac{f(x+h,y)-f(x,y)}{h} \\ f_y(x,y) &= \lim_{h\to\phi} \frac{f(x,y+h)-f(x,y)}{h} \end{aligned}}$$

Notice that this is basically deriving over 1 variable while keeping the other constant

⇓

Notice also that not derived items can be eliminated when deriving for the other variable

↳ this also written as:

$$z = f(x,y) \Rightarrow f_x(x,y) = \frac{\partial f(x,y)}{\partial x}$$

Tangent For a curve defined as $\boxed{z = f(x,y)}$

... the **tangent plane** at point $P(x_0, y_0, z_0)$ is:

$$\boxed{z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

where $f_x(x_0,y_0)$ is the derivative of $f(x,y)$ over x and evaluated at point $P(x_0,y_0,z_0)$.

ex: For eliptical parabole $\boxed{z = 2x^2+y^2}$, Find the tangent plane on $P(1,1,3)$ (x_0,y_0,z_0)

$\Rightarrow f_x(x,y) = 4x \Longrightarrow f_x(1,1) = f_x(x_0,y_0) = 4$

$f_y(x,y) = 2y \Longrightarrow f_y(1,1) = 2$

$$\Rightarrow z - \overset{(z_0)}{3} = \overbrace{f_x(x_0,y_0)}^{4}(x-\cancel{x_0})^{1} + \overbrace{f_y(x_0,y_0)}^{2}(y-\cancel{y_0})^{1} = 4(x-1) + 2(y-1) = 4x+2y-6$$

$$\Rightarrow \boxed{z = 4x+2y-3}$$

Total differential for $z = f(x,y)$

$$\rightarrow \boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy}$$

↳ Same concept, but $dx = x - x_0$, $dy = y - y_0$

and $x_0 \to \phi$, $y_0 \to \phi$

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Chain Rule for composite functions:

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

for $z = f(t)$, $x = g(t)$, $y = h(t)$

---

ex: $z = x^2 y + 3xy^4$ and $x = \sin(2t)$, $y = \cos(t)$

$$\Rightarrow \frac{dz}{dt} = \overbrace{[2xy + 3y^4]}^{\partial z/\partial x} \cdot \underbrace{2\cos(2t)}_{dx/dt} + \overbrace{[x^2 + 12xy^3]}^{\partial z/\partial y} \cdot \underbrace{(-\sin(t))}_{dy/dt}$$

now, evaluating at $t = \phi$:

$\rightarrow x = \sin(2t)\big|_{t=\phi} = \phi$ $\qquad \frac{dx}{dt} = 2\cdot\cos(2t)\big|_{t=\phi} = 2$

$y = \cos(t)\big|_{t=\phi} = 1$ $\qquad \frac{dy}{dt} = -\sin(t)\big|_{t=\phi} = \phi$

$$\Rightarrow \frac{dz}{dt} = [\overbrace{2\cdot\phi\cdot 1}^{2\cdot x\cdot y} + \overbrace{3\cdot(1)^4}^{3\cdot y^4}]\cdot \overbrace{2}^{2\cos(2t)} + [\overbrace{\phi^2}^{x^2} + \overbrace{12\phi(1)^3}^{x\cdot y^3}]\overbrace{(-\phi)}^{-\sin(t)}$$

$$= 3\cdot 2 + \phi = \boxed{6 = \frac{dz}{dt}\Big|_{t=\phi}}$$

Chain Rule, general case:

$u = f(x_1, x_2, x_3, \dots, x_n)$

with $x_i(t)$ = derivative function of $x(t)$ over "t"

$$\boxed{\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_i}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}}$$

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DIRECTIONAL DERIVATIVE maximisation

↳ what's the direction for change that maximises the increase (or decrease) of the function??

$f(\vec{x})$ has the maximum directional derivative value on the direction of the $\boxed{\text{Gradient vector } \nabla f(\vec{x})}$

$$\boxed{\nabla f(\vec{x}) = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}}$$

this 3D vector is orthogonal to the surface of $f(\vec{x})$

$$\boxed{\text{max change rate} = |\nabla f(x,y)| = \sqrt{x^2+y^2}}$$

ex: Determine the max change direction for $f(x,y) = xe^y$ at $P(2,0)$

$$\rightarrow \left.\begin{aligned} \frac{\partial f}{\partial x} &= e^y \\ \frac{\partial f}{\partial y} &= xe^y \end{aligned}\right] \quad \nabla f(\vec{x}) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\Big|_{P(2,0)} = e^y\hat{i} + xe^y\hat{j} = e^0\hat{i} + 2e^0\hat{j}$$

$$\boxed{\nabla f(\vec{x}) = \hat{i} + 2\hat{j} = \langle 1, 2 \rangle}$$

↳ the max change rate is $|\nabla f(x,y)| = \sqrt{1^2+2^2} = \underline{\underline{\sqrt{5}}}$

max or min (local or global/absolut) must fulfill this condition:

$$\boxed{\text{For } f(x,y) \longrightarrow \left.\begin{aligned} f_x(a,b) &= 0 \\ f_y(a,b) &= 0 \end{aligned}\right] \Rightarrow (a,b) \text{ represents a max/min (local or absolut)}}$$

(a,b) is called a $\boxed{\text{critical point}}$

ex: $f(x,y) = x^2+y^2-2x-6y+14$

$$\Rightarrow \left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x-2 \\ \frac{\partial f}{\partial y} &= 2y-6 \end{aligned}\right] \quad \left.\begin{aligned} 2x-2=0 &\Rightarrow x=1 \\ 2y-6=0 &\Rightarrow y=3 \end{aligned}\right] \quad \boxed{\langle 1,3 \rangle \text{ represents the critical point}}$$

Evaluating: $f(1,3) = 1^2+3^2-2\cdot1-6\cdot3+14 = 4$

Note that $f(1,4) = 5$] → both larger, so $\langle 1,3 \rangle$ is a minimum.

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Now, to evaluate if we have a max. or a min, we can evaluate the 2ND derivative to review the trend or direction of the change

$$\boxed{D = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2}$$

assess whether it's max or min

if $D>0$ and $f_{xx}(a,b)>0 \Longrightarrow f(a,b)$ is a local minimum

$D>0$ and $f_{xx}(a,b)<0 \Longrightarrow f(a,b)$ is a local maximum

$D<0 \Rightarrow f(a,b)$ is neither a max nor a min $\Rightarrow f(a,b)$ is a saddle point

Determinant FORM FOR D $\longrightarrow$ $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$

EX: Find local max/min for $f(x,y) = x^4 + y^4 - 4xy + 1$

1) Find critical points $\rightarrow$ $\frac{\partial f}{\partial x} = 4x^3 - 4y = \emptyset \Rightarrow x^3 = y$

$\frac{\partial f}{\partial y} = 4y^3 - 4x \Rightarrow y^3 = x$

$x = y^3$

$\Rightarrow y = (y^3)^3 = y^9$

$y - y^9 = \emptyset \Rightarrow y(1-y^8) = y(1+y^4)(1-y$

$\Rightarrow$ Solutions: $y = \emptyset,\ y = 1,\ y = -1$

$x = \emptyset,\ x = 1,\ x = -1$

$\boxed{\text{critical points}: \langle 0,0\rangle, \langle 1,1\rangle, \langle -1,-1\rangle}$

2) Now, let's validate if the points are max/min:

$f_{xx} = \frac{\partial^2 f}{\partial x} = 12x^2$ $\quad f_{xy} = -4$

$f_{yy} = \frac{\partial^2 f}{\partial y} = 12y^2$ $\quad f_{yx} = -4$ ✓

$\Rightarrow D = f_{xx}(a,b)\, f_{yy}(a,b - [f_{xy}(a,b)]^2$

$\boxed{D = 12x^2 \cdot 12y^2 - 16}$

$D|_{\langle 0,0\rangle} = -16 \longrightarrow$ saddle point

$D|_{\langle 1,1\rangle} = 144 - 16 = 128$; $f_{xx}(1,1) = 12 \longrightarrow$ local minimum point

$D|_{\langle -1,-1\rangle} = 144 - 16 = 128$; $f_{xx}(-1,-1) = 12 \longrightarrow$ local minimum point

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Taylor approximation for multivariate functions

Remember we had:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}}$$

notice we can make n large to reduce error or small to simplify calculation

now, for $f(x,y)$, we have:

$$\boxed{f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = L(x,y)}$$

(1st degree taylor's derivative (since we only use the 1st derivative)

2nd derivative taylor's:

$$\boxed{\begin{aligned} Q(x,y) = {} & f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ & + \tfrac{1}{2} f_{xx}(a,b)(x-a)^2 + \underbrace{f_{xy}(a,b)(x-a)(y-b)}_{\text{notice this element!}} + \tfrac{1}{2} f_{yy}(a,b)(y-b)^2 \end{aligned}}$$

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Les intégrales multiples

L'integrale double :

$$\boxed{\iint_R f(x,y)\,dA \approx \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A}$$

$\Delta A = \Delta x \times \Delta y$ area

(x_{ij}^*, y_{ij}^*) is any point on the rectangular section ΔA, which is the projection of $f(x,y)$ on plane xy

Reimann double sum as approximation to $\iint$

Ex : CONSIDER

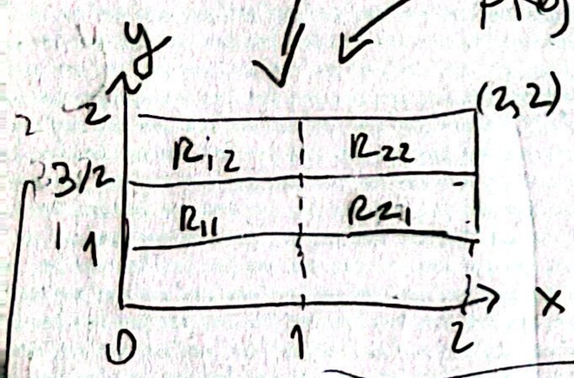
$m = 2$
$n = 2$

to evaluate $\iint_R (x-3y^2)\,dA$

where $R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$

using the middle point (x_{ij}^*, y_{ij}^*) as the middle point for each subrectangle

Projection of $f(x,y)$ on plane xy



middle point on x : $\frac{(2-0)}{2} + 0 = 1$

y : $\frac{(2-1)}{2} + 1 = 3/2$

$$\Rightarrow \iint_R (x-3y^2)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_i)\,dA \Rightarrow$$

Let's determine dA:

$dA = \Delta x\,\Delta y$ FOR any subrectangle R_{ij}

Lets' use R_{22}

$$dA = R_{22} = \underbrace{(2-1)}_{\Delta x}\underbrace{(2-3/2)}_{\Delta y} = \boxed{\frac{1}{2} = \Delta A}$$

SO :

$$\sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_i)\Delta A = f(\bar{x}_1, \bar{y}_1)\Delta A + f(\bar{x}_1, \bar{y}_2)\Delta A + f(\bar{x}_2, \bar{y}_1)\Delta A + f(\bar{x}_2, \bar{y}_2)\Delta A$$

NOW, Let's determine the middle points FOR each subrectangle:

$\bar{x}_1 = \frac{1-0}{2} + 0 = \frac{1}{2}$ $\qquad \bar{y}_1 = \frac{3/2 - 1}{2} + 1 = 5/4$

$\bar{x}_2 = \frac{2-1}{2} + 1 = 3/2$ $\qquad \bar{y}_2 = \frac{2 - 3/2}{2} + \frac{3}{2} = 7/4$

$$\Rightarrow \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_i)\Delta A = (\bar{x}_1 - 3\bar{y}_1^2)\Delta A + (\bar{x}_1 - 3\bar{y}_2^2)\Delta A + (\bar{x}_2 - 3\bar{y}_1^2)\Delta A + (\bar{x}_2 - 3\bar{y}_2)\Delta A$$

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then: $\iint_R f(x,y)\,dA = \iint_R (x-3y^2)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i,\bar{y}_i)\,\Delta A$

$$\sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i,\bar{y}_i)\Delta A = (\bar{x}_1-3\bar{y}_1^2)\Delta A + (\bar{x}_1-3\bar{y}_2^2)\Delta A + (\bar{x}_2-3\bar{y}_1^2)\Delta A + (\bar{x}_2-3\bar{y}_2^2)\Delta A$$

$$= \left(\tfrac{1}{2}-3\cdot\left(\tfrac{5}{4}\right)^2\right)\tfrac{1}{2} + \left(\tfrac{1}{2}-3\left(\tfrac{7}{4}\right)^2\right)\tfrac{1}{2} + \left(\tfrac{3}{2}-3\cdot\left(\tfrac{5}{4}\right)^2\right)\tfrac{1}{2} + \left(\tfrac{3}{2}-3\left(\tfrac{7}{4}\right)^2\right)\tfrac{1}{2}$$

$$= \frac{-67}{16}\cdot\frac{1}{2} + \left(\frac{-139}{16}\right)\cdot\frac{1}{2} + \left(\frac{-51}{16}\right)\cdot\frac{1}{2} + \left(\frac{-123}{16}\right)\cdot\frac{1}{2} = \boxed{-11.875 \approx \iint_R f(x,y)\,dA}$$

Now, let's use $\boxed{\text{Fubini's theorem}}$ to assess how close the approximation was

$$\boxed{\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy}$$

For $R=\{(x,y) \mid a\le x\le b,\ c\le y\le d\}$

Let's take the same example:

$$\boxed{\begin{array}{l} f(x,y) = x-3y^2 \\ 0\le x\le 2 \\ 1\le y\le 2 \end{array}} \Rightarrow \iint_R f(x,y)\,dA = \int_0^2\int_1^2 (x-3y^2)\,dy\,dx = \int_0^2 \left(xy - y^3\right)\Big|_1^2 dx$$

$$= \int_0^2 \left(x\cdot 2 - 2^3 - x\cdot(1) - -(1)^3\right)dx = \int_0^2 (x-7)\,dx = \frac{x^2}{2} - 7x\Big|_0^2$$

$$= \frac{2^2}{2} - 7\cdot 2 - \cancel{\frac{\emptyset^2}{2} - 7\cdot\emptyset}^{\,0} = 2-14 = \boxed{-12 = \iint_R f(x,y)\,dA}$$

$\Rightarrow$ the approximation error was $-12 - -11.875 = \underline{\underline{0.025}}$

we can also define the integration not over a rectangular zone (like the xy plane) but over a region defined by a function D

$$\boxed{\begin{array}{l} D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\} \\ \Downarrow \\ \iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \end{array}}$$

Les intégrales triples

Fubini's theoreme

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz}$$

with $dV = dx\,dy\,dz$ (volume) and $B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$

ex: $f(x,y,z) = xyz^2$

$B = \{(x,y,z) \mid 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}$

$$\Rightarrow \iiint_B f(x,y,z)\,dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,dx\,dy\,dz = \int_0^3 \int_{-1}^2 \frac{x^2}{2} y z^2 \Big|_0^1 dy\,dz = \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\,dy\,dz$$

$$= \int_0^3 \frac{y^2 z^2}{4}\Big|_{-1}^2 dz = \int_0^3 \frac{1}{4}\left((2)^2 z^2 - (-1)^2 z^2\right) dz = \int_0^3 \frac{3z^2}{4}\,dz = \frac{z^3}{4}\Big|_0^3$$

$$= \frac{3^3}{4} - \cancel{\frac{0^3}{4}} = \boxed{\frac{27}{4} = \iiint_B f(x,y,z)}$$

Notice that, while double integrals define the volume under a surface, triple integrals define the volume within a 3 dimensional region (solid)