CMPS 327: Introduction to Video Game Design and Development

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Fall 2020

Lecture 7: Math Concepts I

*Adapted from Dunn and Parberry

Announcements

- Project 3 is available on Moodle
  - Due: Wednesday, September 30, 2020 11:00 PM
  - Creating a game environment in Unity
  - Make sure to use the correct version of Unity 3D
    - 2020.1.3f1
- Midterm Exam 1
  - Wednesday, September 23, 2020
  - ODS: Schedule your exam with ODS by September 18, 2020

About Today's Lecture

- Cartesian coordinate system
  - 2D cartesian space
  - 3D cartesian space
- Vectors

Cartesian Coordinate System

- 3D math is all about measuring locations, distances, and angles precisely and mathematically in 3D space.

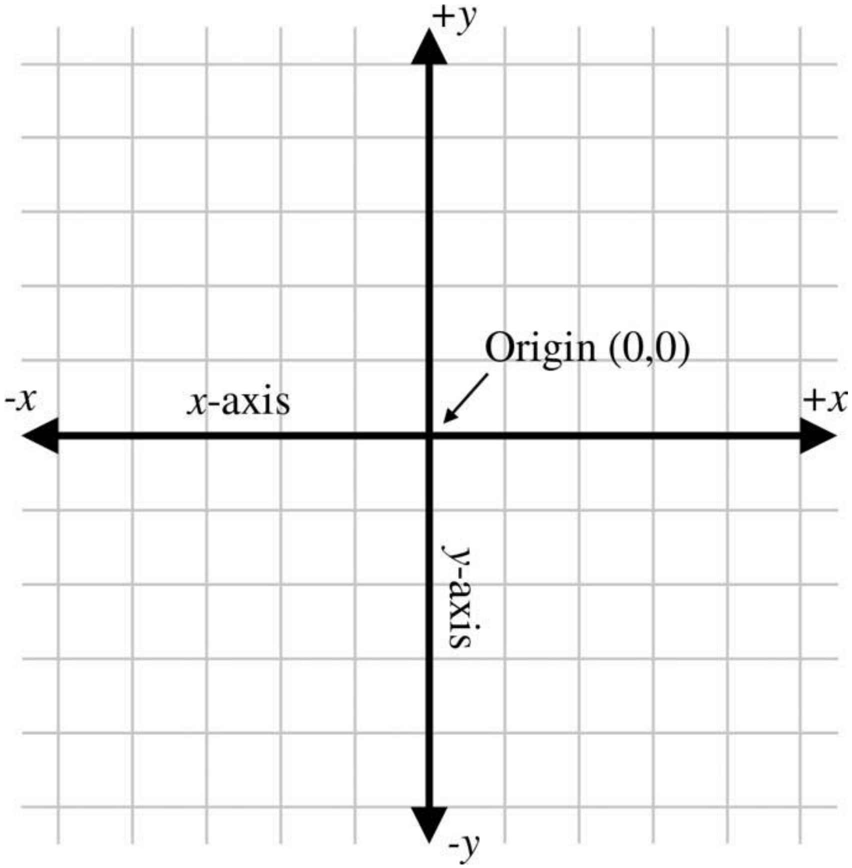
- The most frequently used framework to perform such calculations using a computer is called the Cartesian coordinate system.

2D Cartesian Coordinate System

2D Math Concepts

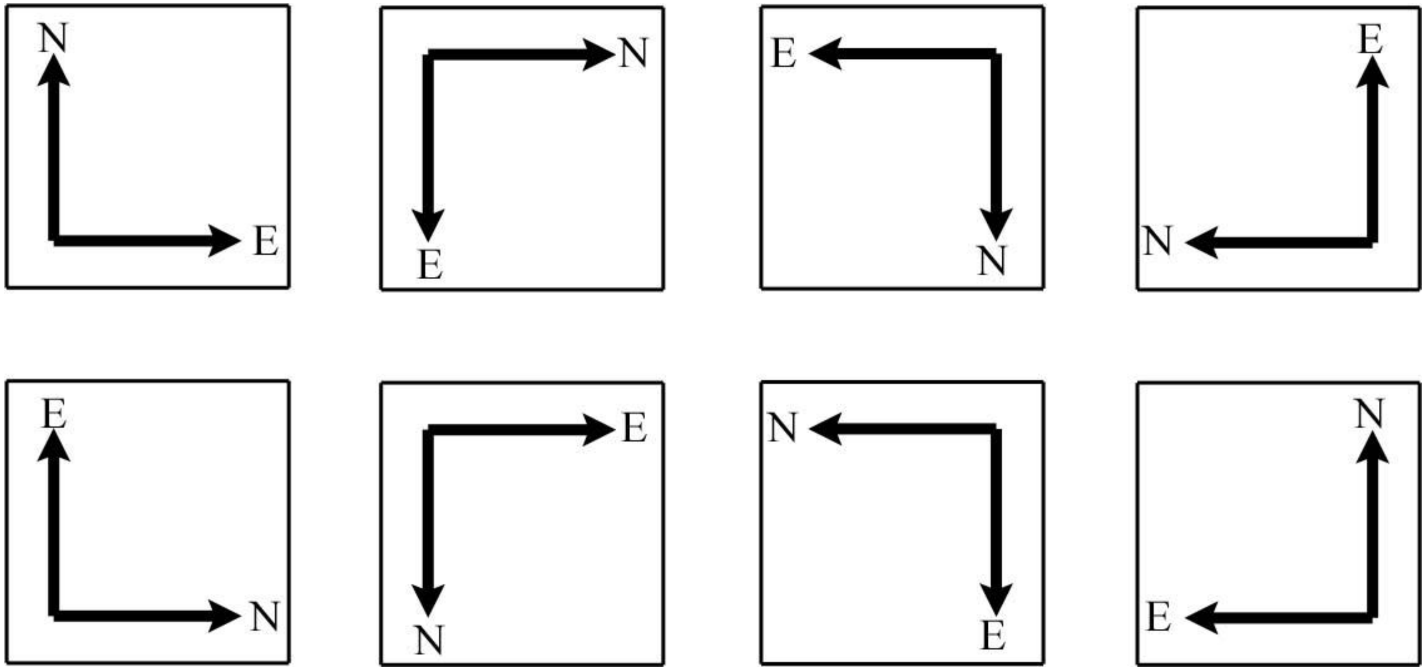
2D Coordinate Space

- Origin: (0,0)
  - The center of the coordinate system
- Axes: X and Y
  - X: Horizontal Axes
  - Y: Vertical Axes



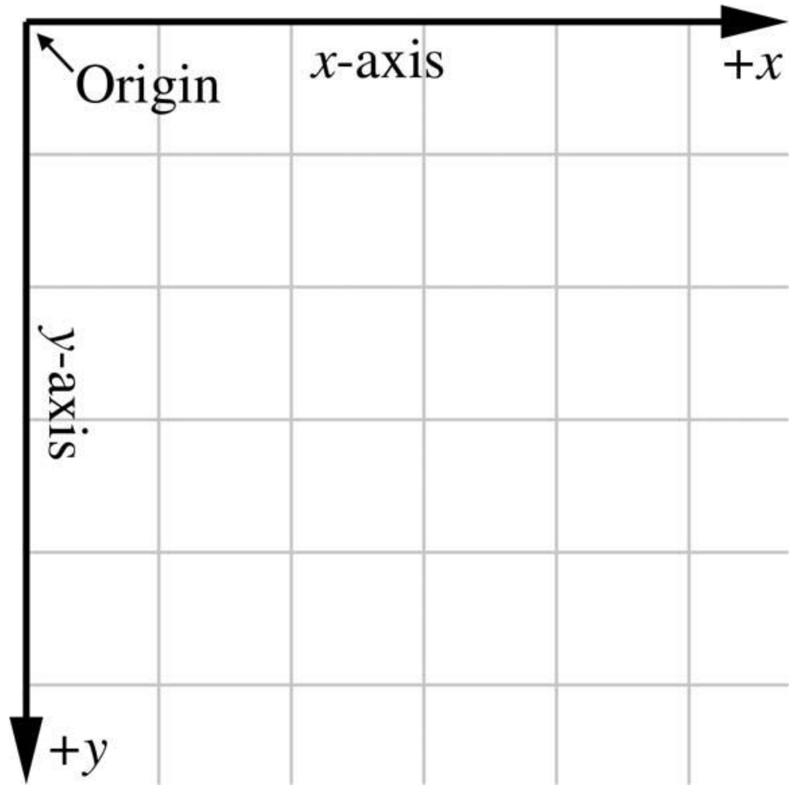
Axis Orientation

There are 8 possible ways of orienting the Cartesian axes.



Screen Space

- Screen space is how you measure on a computer screen, with the origin at the top left corner.
- But it doesn't have to be this way. It's only a convention.
- In screen space, for example, +y points down.
- Unity 3D GUI coordinates use this convention

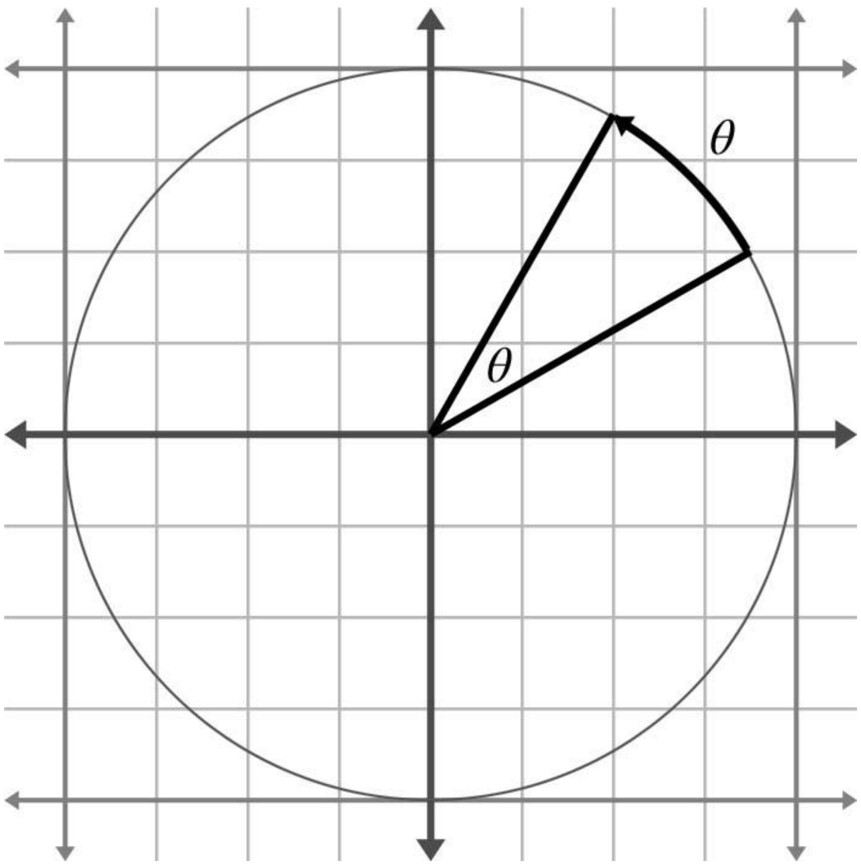


Angles and Trigonometry

Degrees and Radians

Angle Measurement

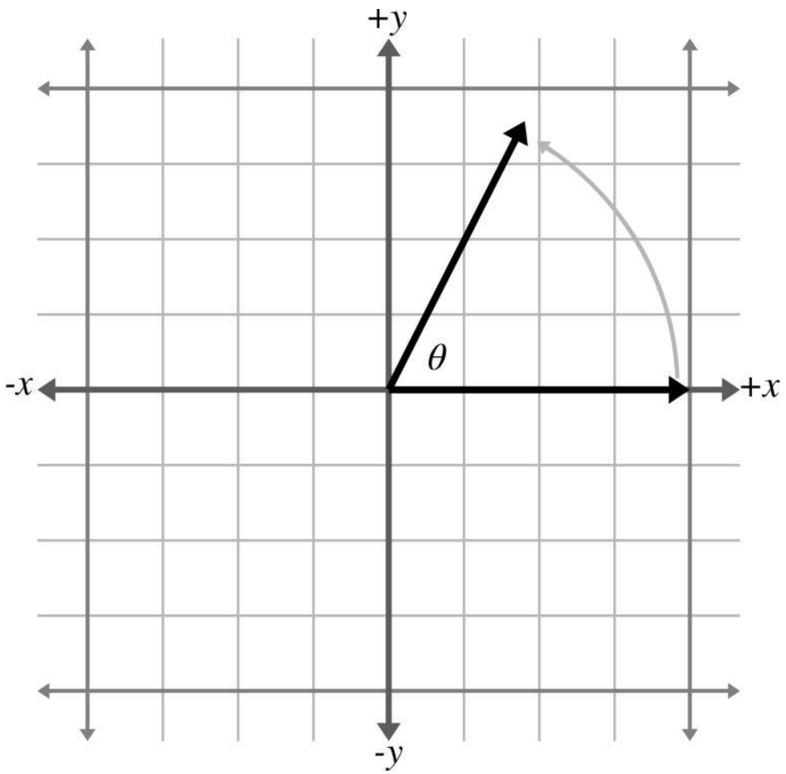
- The most important units of measure are degrees (°) and radians (rad).
- Angle between two rays in radians
  - The length of the intercepted arc of a unit circle
- To convert an angle from radians to degrees
  - multiply by $180/\pi \approx 57.29578$
- To convert an angle from degrees to radians
  - multiply by $\pi/180 \approx 0.01745329$.



Cosine & Sine

$$\cos\theta = x$$
$$\sin\theta = y$$

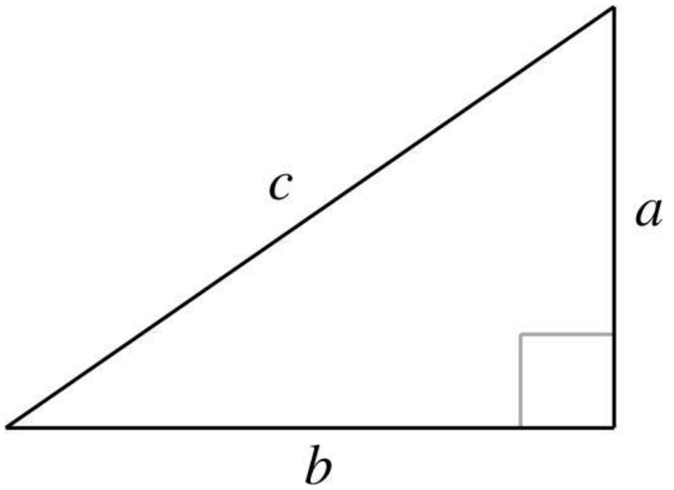
- You can easily remember which is which because they are in alphabetical order
  - x comes before y
  - cos comes before sin



The Pythagorean Theorem

- Theorem
  - $a^2 + b^2 = c^2$

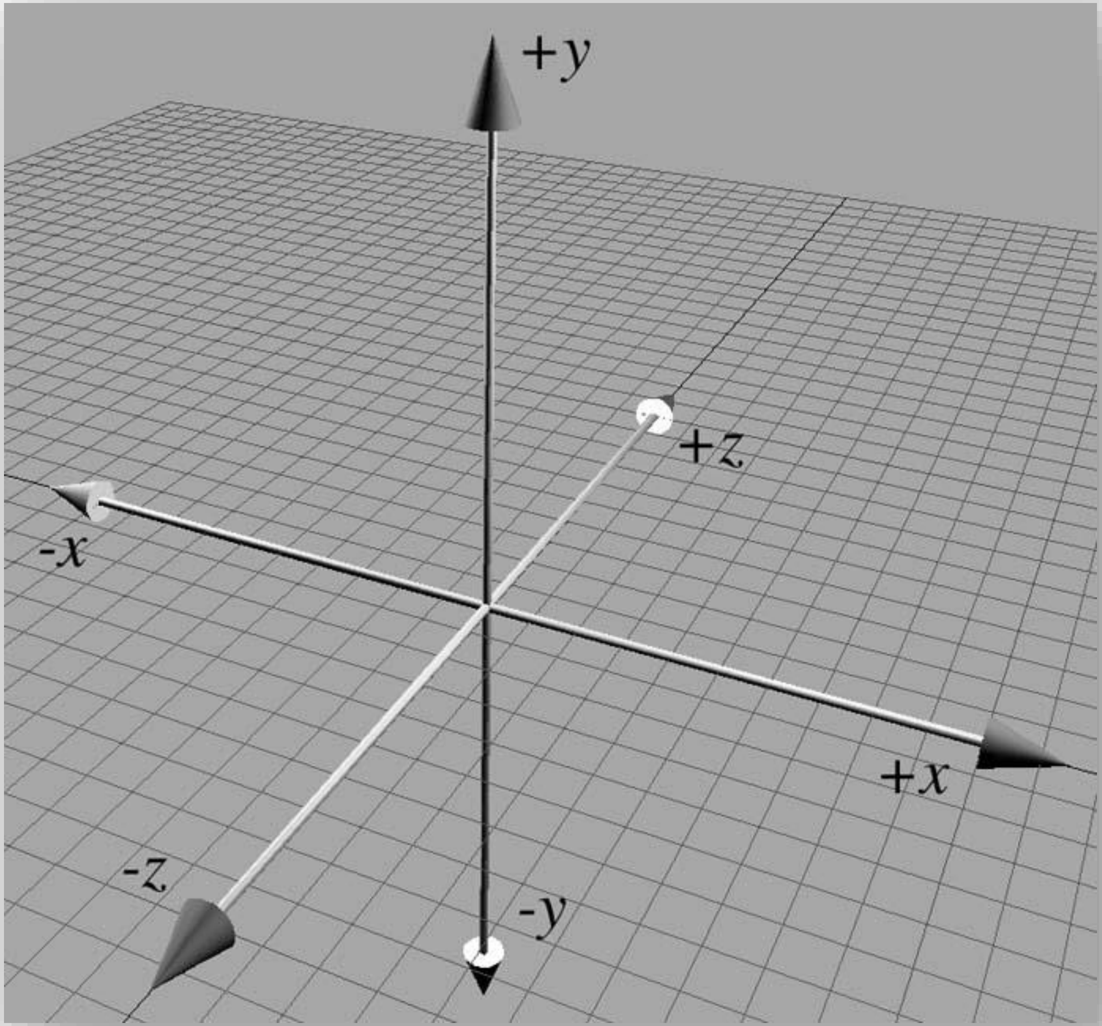
- Other Identities
  - $\sin^2 \theta + \cos^2 \theta = 1$
  - $1 + \tan^2 \theta = \sec^2 \theta$
  - $1 + \cot^2 \theta = \csc^2 \theta$



3D Cartesian Coordinate System

3D Math Concepts

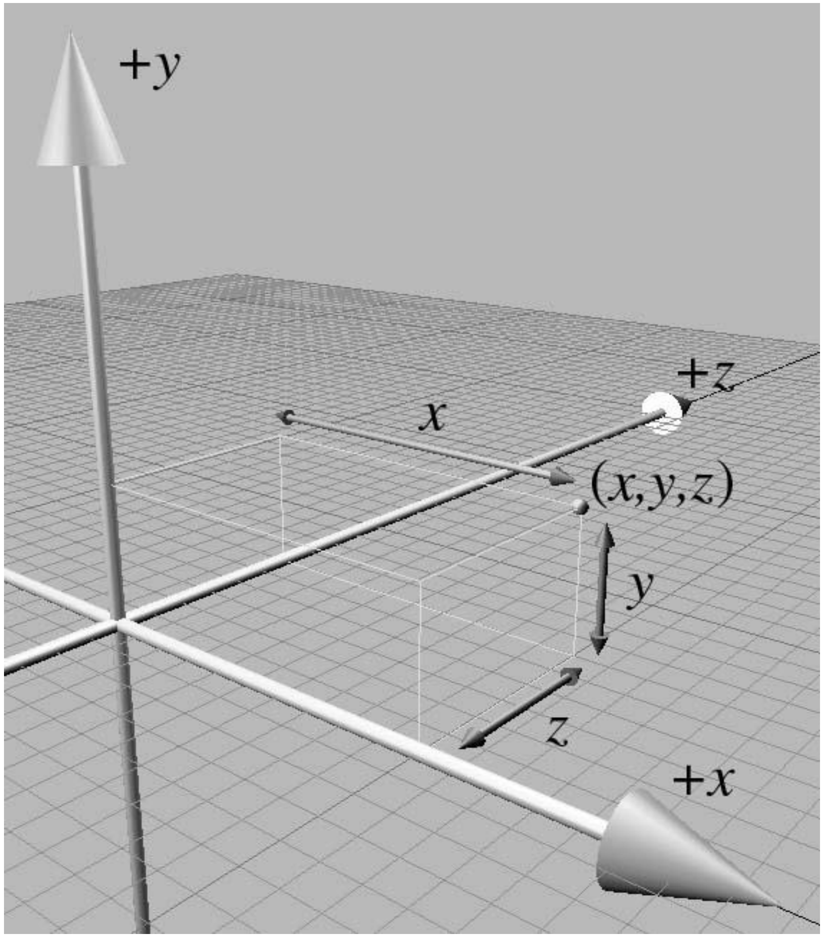
3D Cartesian Space



Locating Points in 3D

Point (x, y, z) is located

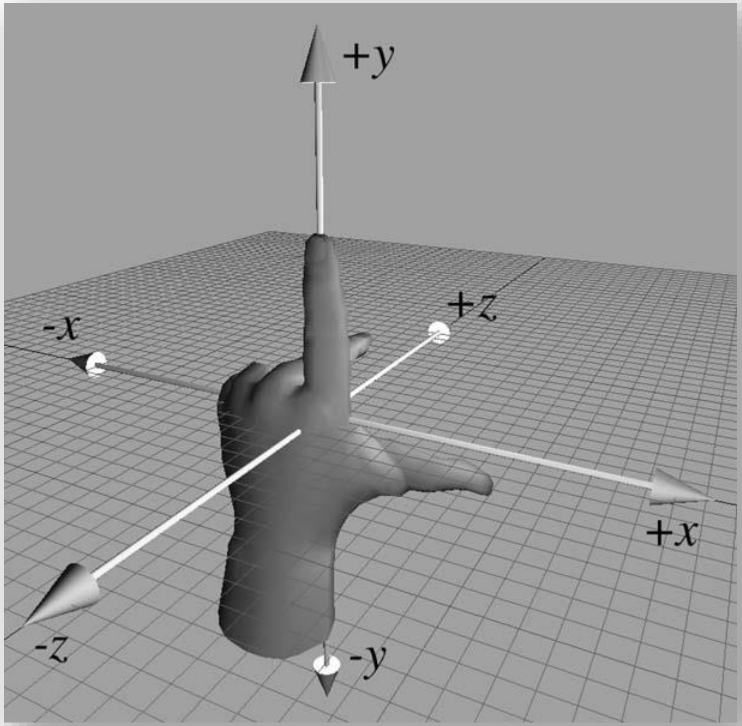
- *x* units along the *x*-axis
- *y* units along the *y*-axis
- *z* units along the *z*-axis
- All distances from the origin.



Visualizing 3D Space

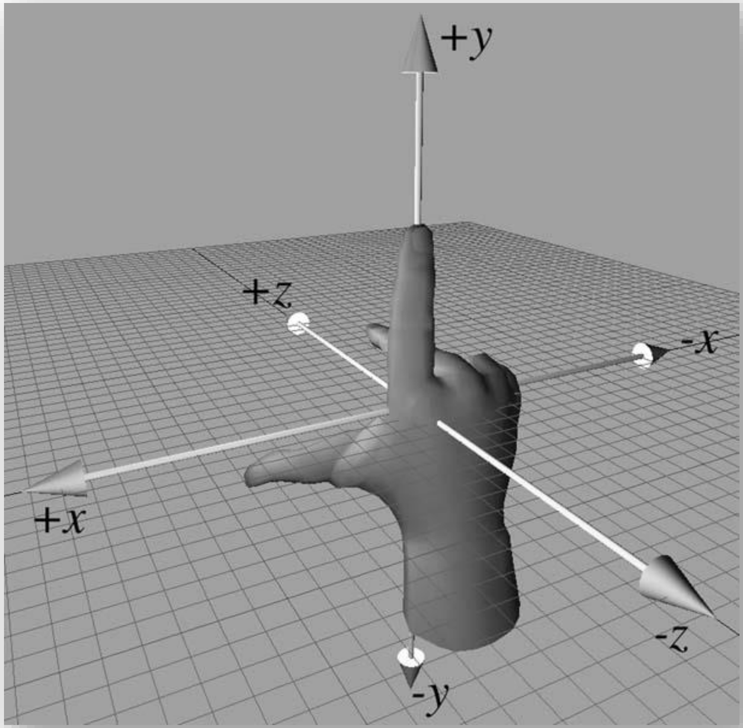
- The usual convention
  - the *x*-axis is horizontal and positive is right
  - the *y*-axis is vertical and positive is up
- The *z*-axis is depth
  - but should the positive direction go forwards “into” the screen or backwards “out from” the screen?

Left-handed Coordinates



- +*z* goes "into" screen
- Use your left hand
- Thumb is +*x*
- Index finger is +*y*
- Second finger is +*z*

Right-handed Coordinates



- + *z* goes "out from" screen
- Use your right hand
- Thumb is +*x*
- Index finger is +*y*
- Second finger is +*z*
- (Same fingers, different hand)

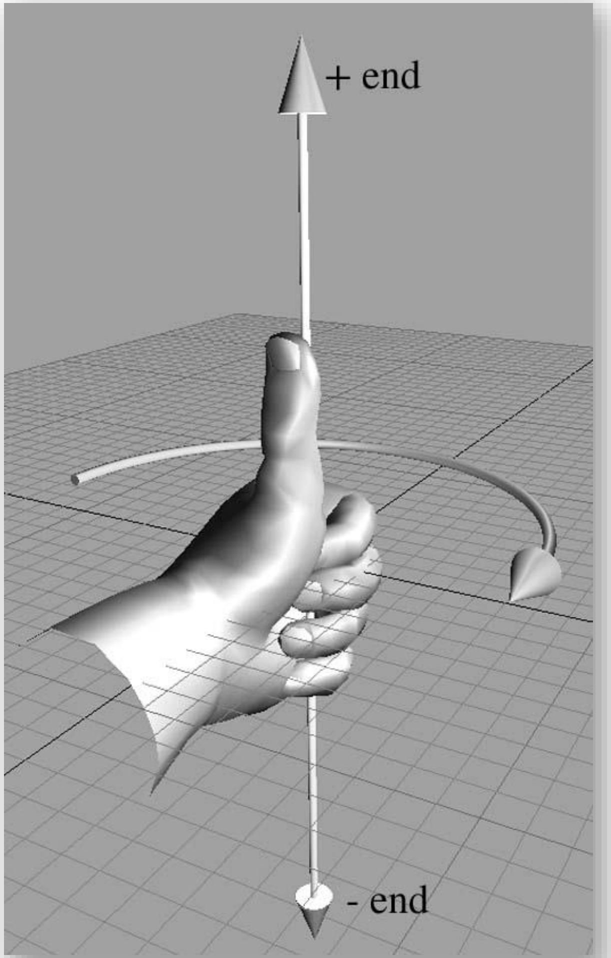
Changing Conventions

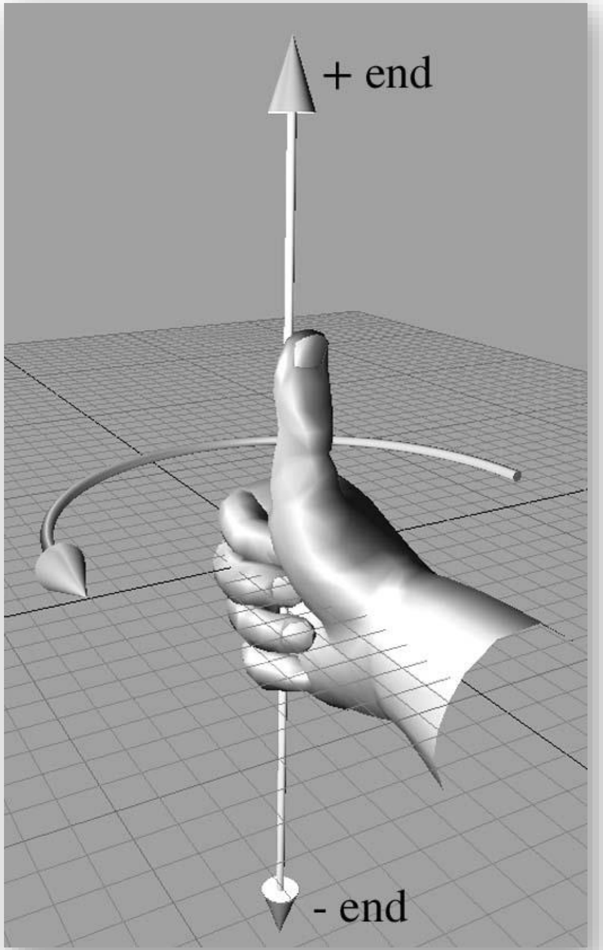
- To swap between left and right-handed coordinate systems, negate the *z.*
- Linear algebra books usually use right-handed.
- Graphics books usually use left-handed.
- We'll use left-handed.
  - Unity 3D uses left-handed

Positive Rotation

- Use your left hand for a left-handed coordinate space, and your right hand for a right-handed coordinate space.
- Point your thumb in the positive direction of the axis of rotation (which may not be one of the principal axes).
- Your fingers curl in the direction of positive rotation.

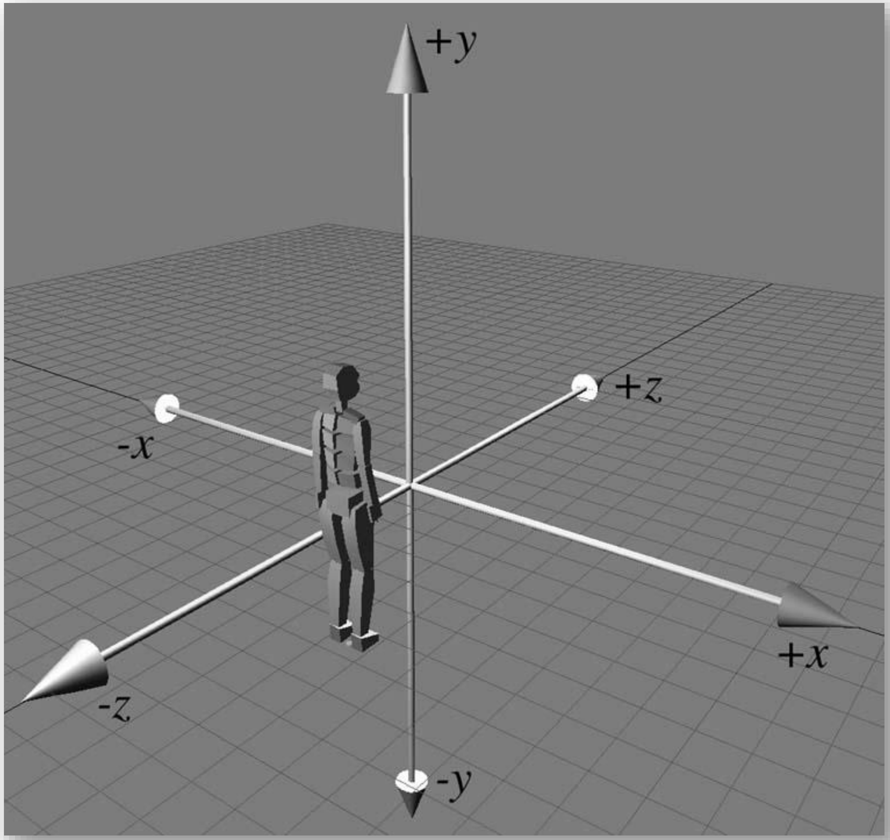
Positive Rotation





Our Convention

- We will use a left-handed coordinate system.
- Unity uses left-handed system



Vectors

Vector Related Math

Vectors and Scalars

- An "ordinary number" is called a *scalar.*
- Algebraic definition of a vector: a list of scalars (with parenthesis around them)
  - Example: [1, 2, 3].
- Vector *dimension* is the number of numbers in the list.
- Typically we use dimension 2 for 2D work, dimension 3 for 3D work.

Row vs. Column Vectors

- Vectors can be written in one of two different ways: horizontally or vertically.
- Row vector: [1, 2, 3]
- Column vector: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
- Scalar parts of a vector are called components
  - **v** = [6, 19, 42]
  - Components: $\mathbf{v}_1 = 6$, $\mathbf{v}_2 = 19$, $\mathbf{v}_3 = 42$

More Notation

- Can also use *x*, *y*, *z* for subscripts.
- 2D vectors: $[\mathbf{v}_x, \mathbf{v}_y]$.
- 3D vectors: $[\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z]$.

The Zero Vector

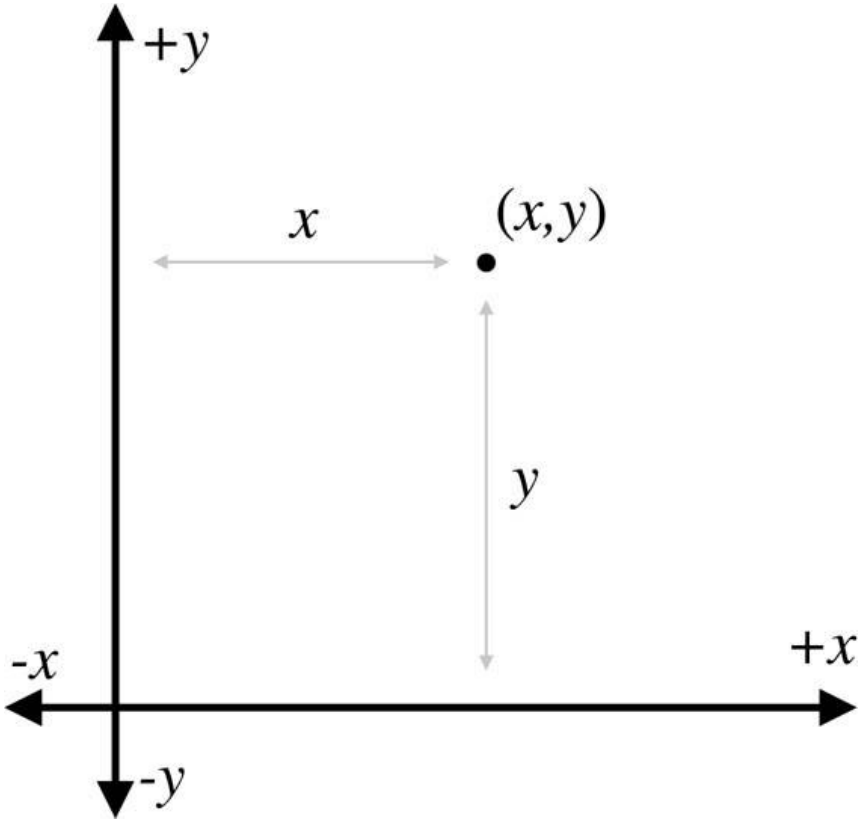
- The zero vector **0** is the additive identity
  - meaning that for all vectors **v**, **v** + **0** = **0** + **v** = **v**.

- **0** = [0, 0,…, 0]

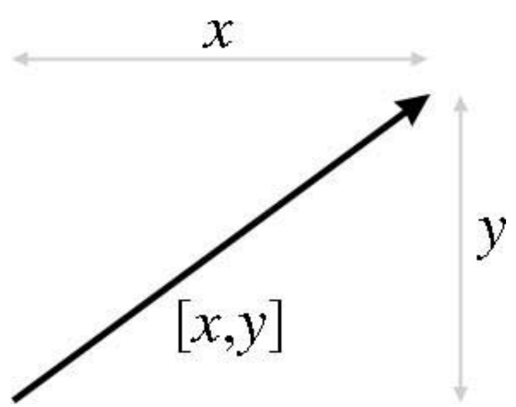
- The zero vector is unique: It’s the only vector that doesn’t have a direction

Vectors vs Points

- Points are measured relative to the origin.
- A vector can be used to represent a point.
- The point (x,y) is the point at the head of the vector [x,y] when its tail is placed at the origin.
- But vectors don’t have a location

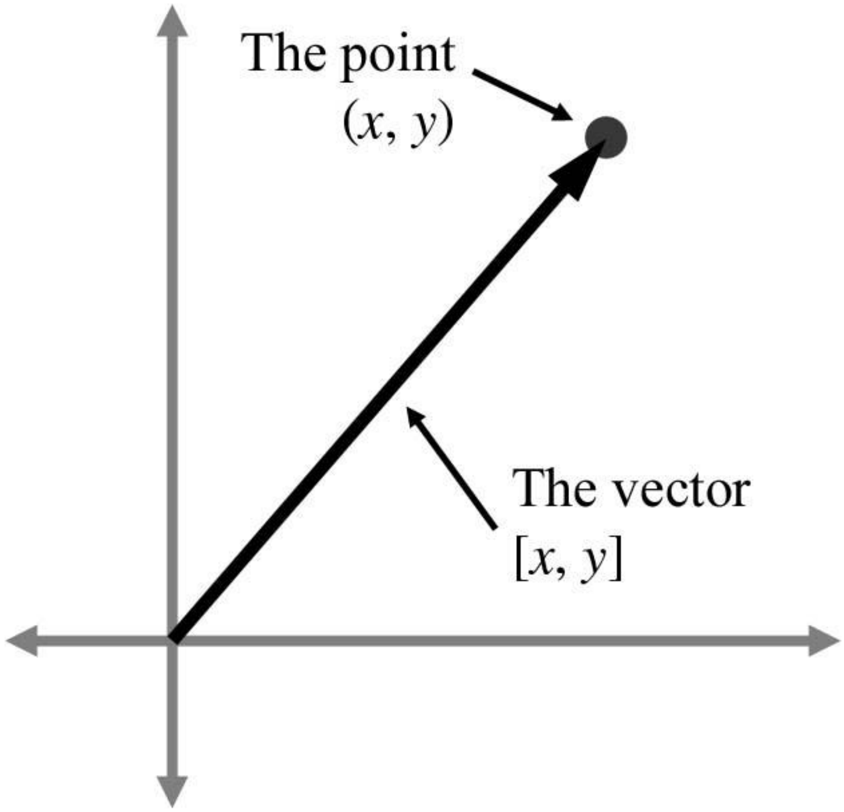


+y
x
(x,y)
y
-x
+x
-y



x
y
[x,y]

Vectors vs Points



Vector Operations

- Negation
- Multiplication by a scalar
- Addition and Subtraction
- Displacement
- Magnitude
- Normalization
- Dot product
- Cross product

Vector Negation: Algebra

- Negation is the additive inverse:

  v + -**v** = -**v** + **v** = **0**

- To negate a vector, negate all of its components.

$$-\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_{n-1} \\ -a_n \end{bmatrix}$$

Examples

$$-\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} -x & -y \end{bmatrix}$$
$$-\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} -x & -y & -z \end{bmatrix}$$
$$-\begin{bmatrix} x & y & z & w \end{bmatrix} = \begin{bmatrix} -x & -y & -z & -w \end{bmatrix}$$

$$-\begin{bmatrix} 4 & -5 \end{bmatrix} = \begin{bmatrix} -4 & 5 \end{bmatrix}$$
$$-\begin{bmatrix} -1 & 0 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sqrt{3} \end{bmatrix}$$
$$-\begin{bmatrix} 1.34 & -3/4 & -5 & \pi \end{bmatrix} = \begin{bmatrix} -1.34 & 3/4 & 5 & -\pi \end{bmatrix}$$

Multiplication by a Scalar: Algebra

- Can multiply a vector by a scalar.
- Result is a vector of the same dimension.
- To multiply a vector by a scalar, multiply each component by the scalar.
- For example, if $k\mathbf{a} = \mathbf{b}$, then $\mathbf{b}_1 = k\mathbf{a}_1$, etc.

$$k \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} k = \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_{n-1} \\ ka_n \end{bmatrix}$$

- So vector negation is the same as multiplying by the scalar –1.
- Division by a scalar same as multiplication by the scalar multiplicative inverse.

Vector Addition: Algebra

- Can add two vectors of the same dimension.
- Result is a vector of the same dimension.
- To add two vectors, add their components.
- For example, if $\mathbf{a} + \mathbf{b} = \mathbf{c}$, then $\mathbf{c}_1 = \mathbf{a}_1 + \mathbf{b}_1$, etc.
- Subtract vectors by adding the negative of the second vector, so $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

Vector Addition: Algebra

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_{n-1} + b_{n-1} \\ a_n + b_n \end{bmatrix}$$

Vector Subtraction: Algebra

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} + \left(- \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} \right) = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_{n-1} - b_{n-1} \\ a_n - b_n \end{bmatrix}$$

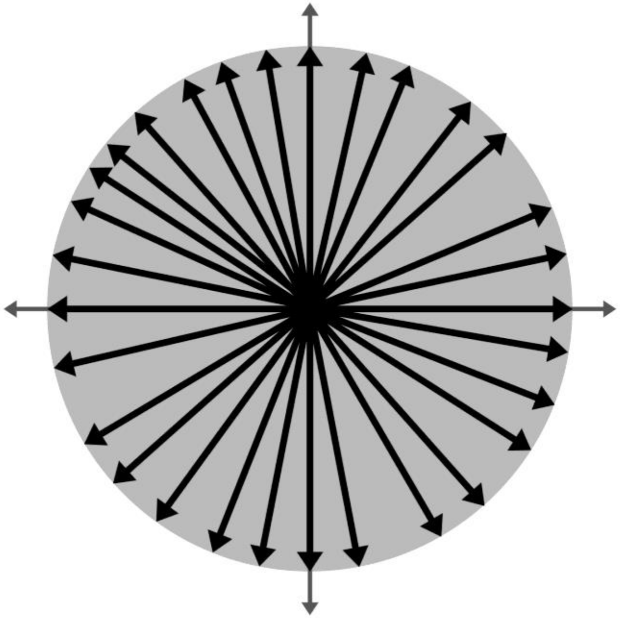
Vector Magnitude

- The magnitude of a vector is a scalar.
- Also called the "norm".
- It is always positive
- Magnitude of a vector is its length.

$$||\mathbf{v}|| = \sqrt{\sum_{i=1}^{n} {v_i}^2} = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_{n-1}}^2 + {v_n}^2}$$

Observations

- The zero vector has zero magnitude.
- There are an infinite number of vectors of each magnitude (except zero).



Normalized Vector

- A *normalized* vector always has unit length.
- To normalize a nonzero vector, divide by its magnitude.

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Example

Normalize [12, -5]:

$$\frac{\begin{bmatrix}12 & -5\end{bmatrix}}{\left\|\begin{bmatrix}12 & -5\end{bmatrix}\right\|} = \frac{\begin{bmatrix}12 & -5\end{bmatrix}}{\sqrt{12^2+5^2}} = \frac{\begin{bmatrix}12 & -5\end{bmatrix}}{\sqrt{169}} = \frac{\begin{bmatrix}12 & -5\end{bmatrix}}{13} = \begin{bmatrix}\frac{12}{13} & \frac{-5}{13}\end{bmatrix}$$

$$\approx \begin{bmatrix}0.923 & -0.385\end{bmatrix}$$

Computing Distance

- To find the geometric distance between two points *a* and *b*.

- Compute the vector **d** from **a** to **b**.
  - **d** = **b** – **a**

- Compute the magnitude of **d**.

Dot Product

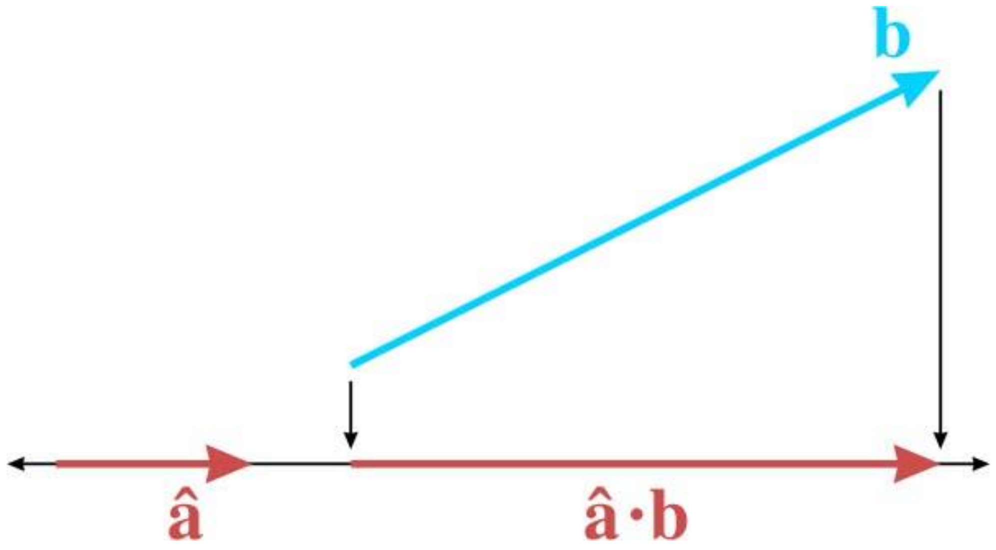
Can take the dot product of two vectors of the same dimension. The result is a scalar.

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i$$

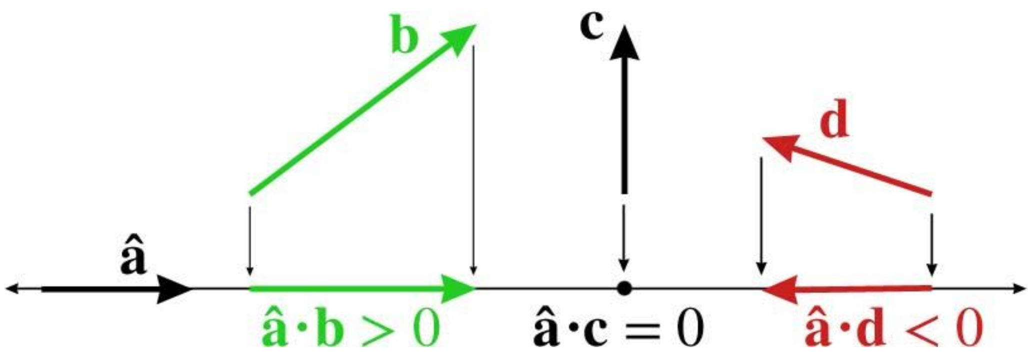
$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_{n-1} b_{n-1} + a_n b_n$$

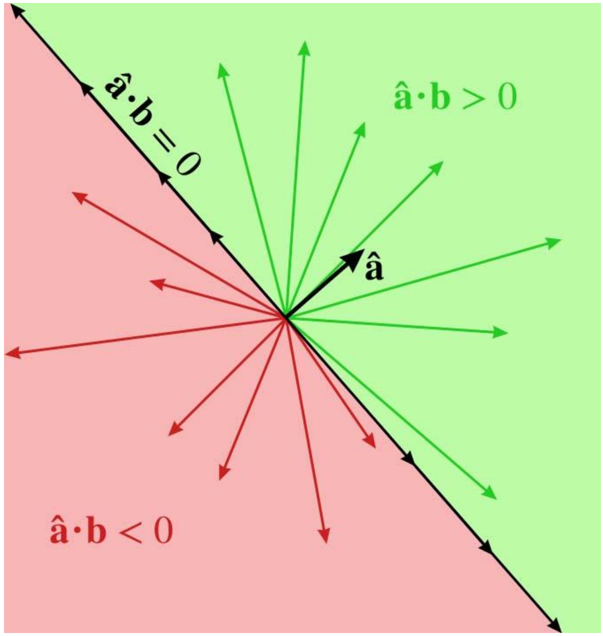
Dot Product: Geometry

Dot product is the magnitude of the projection of one vector onto another.



Sign of Dot Product





Sign of Dot Product

$\mathbf{a} \cdot \mathbf{b}$	θ	**Angle is**	**a and b are**
> 0	$0^\circ \leqslant \theta < 90^\circ$	acute	pointing mostly in the same direction
0	$\theta = 90^\circ$	right	perpendicular
< 0	$90^\circ < \theta \leqslant 180^\circ$	obtuse	pointing mostly in the opposite direction

Cross Product

- Can take the cross product of two vectors of the same dimension.
- Result is a vector of the same dimension.

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

Cross Pattern

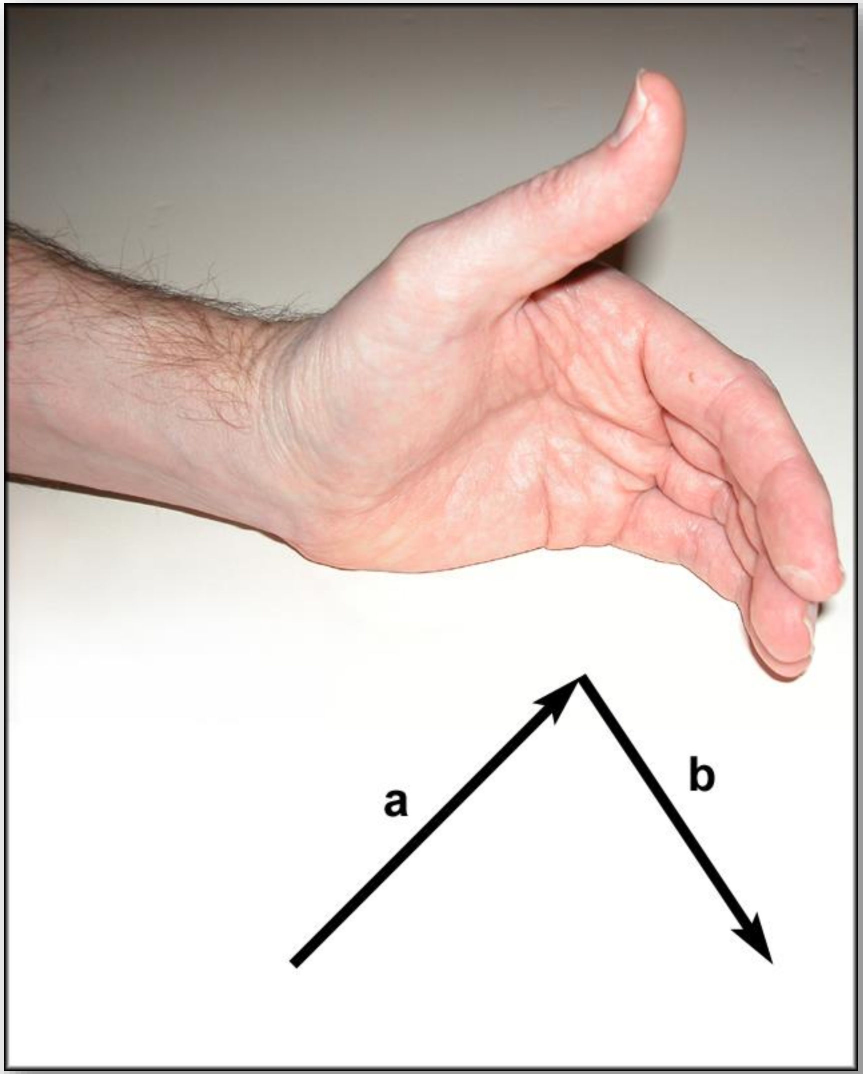
$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

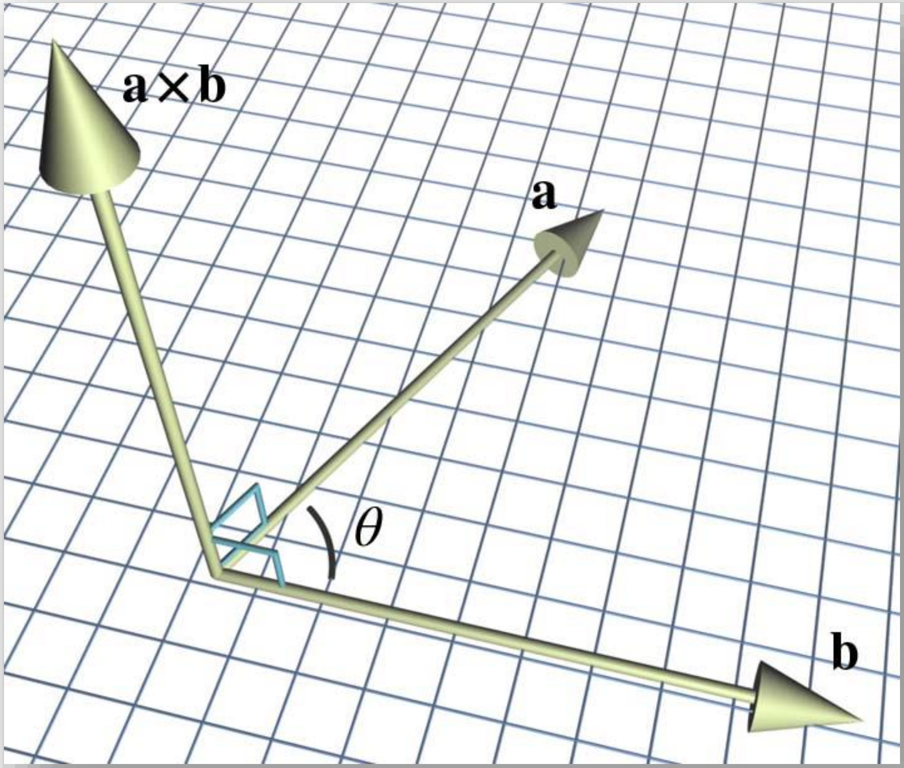
Cross Product: Geometry

- Given 2 nonzero vectors **a**, **b**.
  - They are (must be) coplanar.
- The cross product of **a** and **b** is a vector perpendicular to the plane of **a** and **b**.
- The magnitude is related to the magnitude of **a** and **b** and the angle between **a** and **b**.
- The magnitude is equal to the area of a parallelogram with sides **a** and **b**.



- In a left-handed coordinate system, use your left hand.
- Curl fingers in direction of vectors
- Thumb points in direction of **a** x **b**

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta$$



Summary

- 2D Cartesian coordinates
- 3D Cartesian coordinates
  - Left-handed vs right-handed
- Vectors
  - Operations with vectors
- Next class
  - Continue this discussion on math concepts