Backtracking (2)

By: Aminul Islam

Based on Chapter 5 of Foundations of Algorithms

Objectives

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- Describe the backtrack programming technique
- Determine when the backtracking technique is an appropriate approach to solving a problem
- Define a state space tree for a given problem
- Define when a node in a state space tree for a given problem is promising/non-promising
- Create an algorithm to prune a state space tree
- Create an algorithm to apply the backtracking technique to solve a given problem

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- The goal is to find all subsets of integers that sum to W.

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- Order of complexity $= 2^n$

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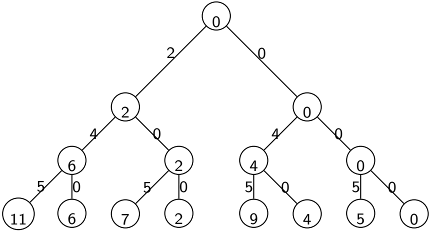

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- A node is promising if *weight* $+$ *total* $>= W$

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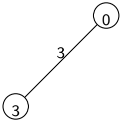

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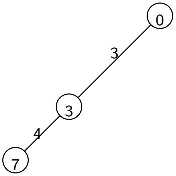

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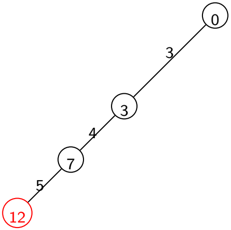

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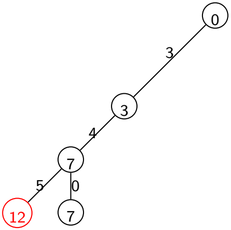

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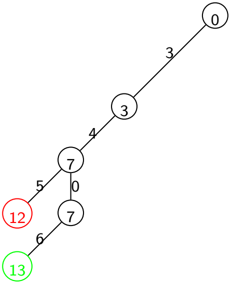

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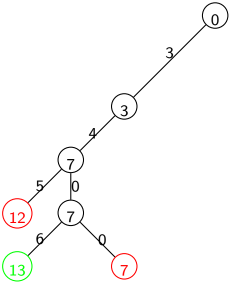

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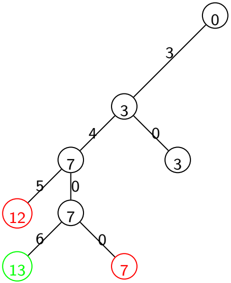

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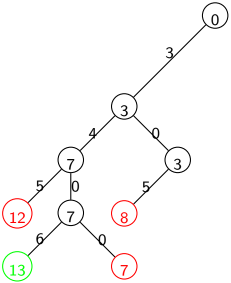

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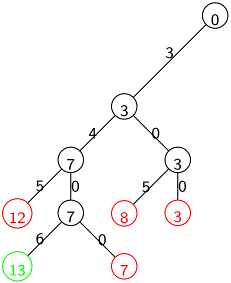

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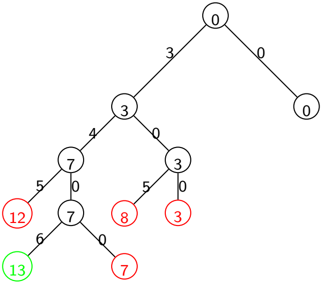

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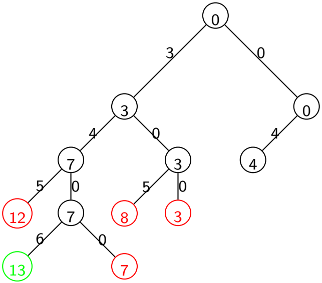

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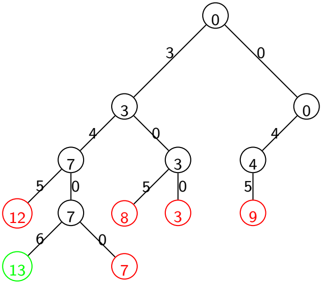

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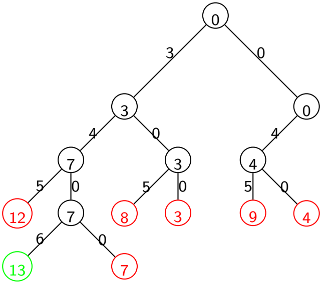

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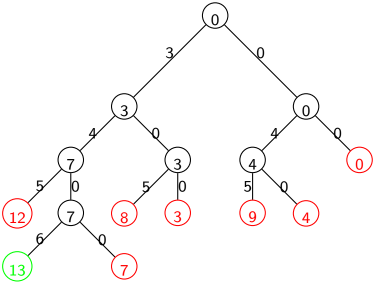

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Backtracking Alg. for Sum of Subsets Problem

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```
void sumOfsubset(index i, int weight, int total){
    if (promising(i))
       if (weight==W)
          cout<< include[1] through include[i];
       else {
          include[i+1]=‘‘yes’’;
          sumOfsubset (i+1, weight+w[i+1], total-w[i+1]);
          include[i+1]=‘‘no’’;
          sumOfsubset (i+1, weight, total-w[i+1]);
       }
}
                // Call the function with sumOfsubset(0,0, total)
                //total:  initially sum of weights of all items

bool promising (index i){
    return (weight + total ≥ W) && (weight + w[i+1]≤W);
}
```

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- Their state space trees are also similar
  - Go to the left from root to include first item and to the right to exclude the first item and so on ...
- The difference is that this is an "optimization" problem!
  - We do not know the optimal solution until the search is over

General Backtracking Algorithm for Optimization Problems

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```
void checkNode (node v)
{
    node u;
    if (value(v) is better than best)
      best = value(v);
    if (promising(v))
      for (each child u of v)
      checkNode(u);
}
```

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- A node at level i is promising if:
  (upper bound)$_i >$ maxprofit

Example of 0-1 Knapsack Problem using backtracking

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$n = 4,\ W = 16$

Item i	P_i	W_i	P_i / W_i
1	40	2	20
2	30	5	6
3	50	10	5
4	10	5	2

Promising:

- (current node) bound $>$ maxprofit
- (current node) weight $< W$

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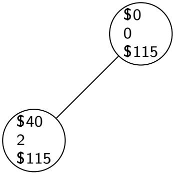
maxprofit $= \$40$



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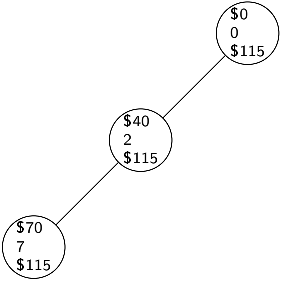
maxprofit $= \$70$



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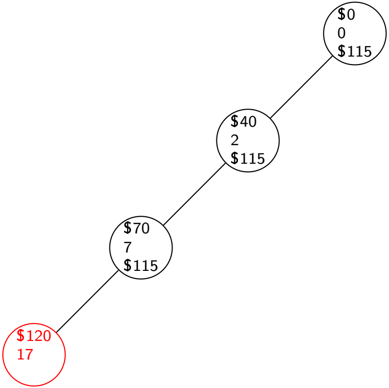

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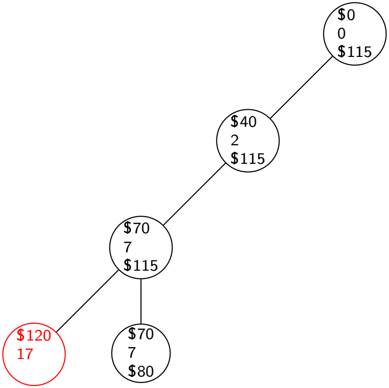

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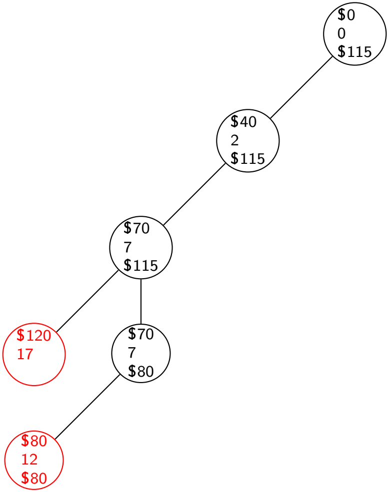
maxprofit $=$ \$80



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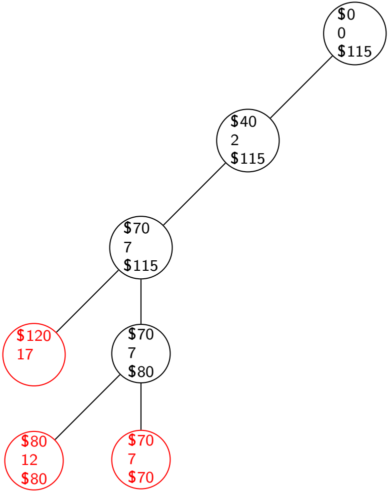

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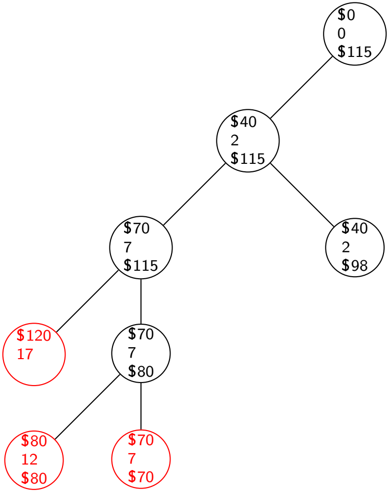
Item i	P_i	W_i	P_i/W_i
1	40	2	20
2	30	5	6
3	50	10	5
4	10	5	2

Promising:
- (current node) bound $>$ maxprofit
- (current node) weight $<$ W

maxprofit $=$ \$80



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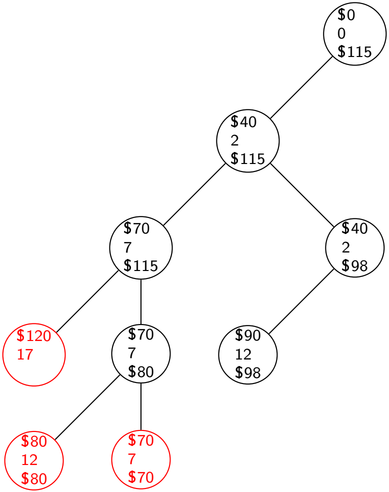

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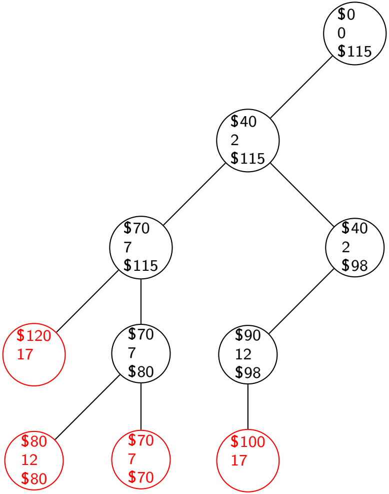

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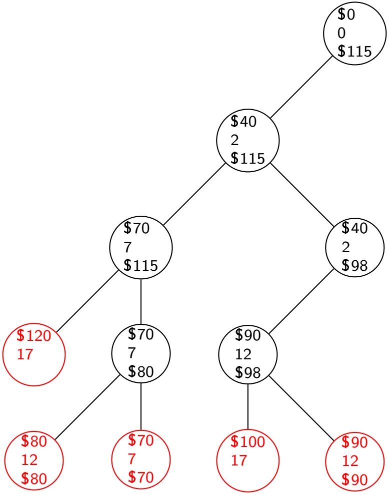

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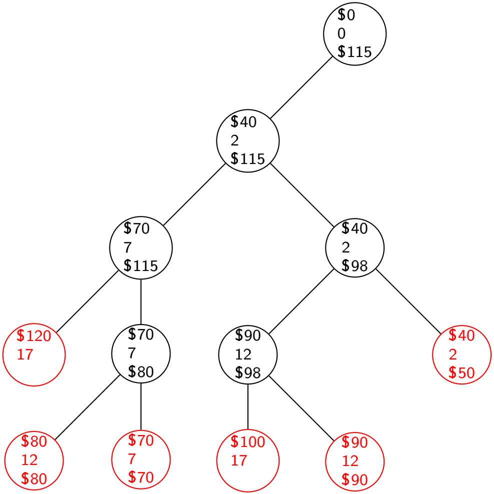

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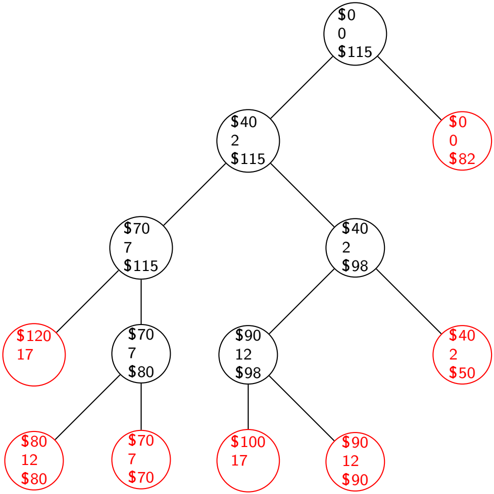

0-1 Knapsack Problem using backtracking

```
void knapsack (index i,
               int profit, int weight)
{
  if (weight <= W && profit > maxprofit){
     maxprofit = profit;
     numbest = i;
     bestset = include;
  }


  if (promising(i)){
     include[i + 1] = "yes";
     knapsack(i + 1, profit + p[i + 1], weight + w[i + 1]);
     include[i + 1] = "no";
     knapsack(i + 1, profit, weight);
  }
}

bool promising (index i)
{
  index j, k;
  int totweight;
  float bound;

  if (weight >= W)
     return false;
  else{
     j = i + 1;
     bound = profit;
     totweight = weight;
     while (j <= n && totweight + w[j] < = W){

        totweight = totweight + w[j];
        bound = bound + p[j];
        j++;
     }
     k = j;
     if (k <=n)
        bound = bound + (W - totweight) * p[k]/w[k];

     return bound > maxprofit;
  }
}
```