U1.1 Probabilistic reasoning: Representation and inference

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1 The qualification problem

Qualification problem: Practical impossibility of knowing and checking all the **qualifications** (conditions) that would need to be guaranteed in order to ensure the fulfillment of an action

- Example: Leaving the airport 90 minutes before the flight allows me to arrive on time IF there are no traffic jams AND no flat tyres AND ...
- Example: A boat allows us to cross a river IF it is a rowing boat AND it has oars and rowlocks AND they are not broken AND they fit AND ...

Uncertainty: Current intelligent systems include **uncertainty** as part of knowledge and represent it by **probabilities** associated with the events (propositions) of interest

2 Probabilistic representation

Joint Probability Distribution: of the random variables of interest in order to represent probabilistic knowledge

Example of the dentist: Knowledge to diagnose caries

Random variables of interest:

$$\begin{array}{ll} \text{Toothache:} & T \in \{0,1\} \\ \text{Caries:} & C \in \{0,1\} \\ \text{Hole:} & H \in \{0,1\} \end{array}$$

Representation: table below with

$$\boxed{P(T=t, C=c, H=h)} \quad \text{for all } t, c, h \in \{0,1\}$$

t	c	h	P
0	0	0	0.576
0	0	1	0.008
0	1	0	0.144
0	1	1	0.072
1	0	0	0.064
1	0	1	0.012
1	1	0	0.016
1	1	1	0.108
			1.000

3 Probabilistic inference

$P(x,y,z) = P(x \mid y,z) \cdot P(y,z) = P(x \mid y,z) \cdot P(z \mid y) \cdot P(y)$

$= P(x,y \mid z) \cdot P(z) = P(x \mid y,z) \cdot P(y \mid z) \cdot P(z)$

Sum and product rules: Basic rules for calculating the probability of any **event** (**proposition**) of interest from the joint distribution

$$\boxed{P(x) = \sum_y P(x,y)} \quad \text{and} \quad \boxed{P(x,y) = P(x)\,P(y \mid x)} = P(y) \cdot P(x \mid y)$$

Important note: In general it is not necessary to know the full table of joint probabilities to calculate the probability of a given event using the sum and product rules

Dentist example: Calculation of the probability of observing...

$= P(t=0, c=1, h=1) + P(t=1, c=1, h=1)$ (0.072, 0.108)

- Caries and observing a hole (at the same time): $P(c=1, h=1) = \sum_{t=0,1} P(t, c=1, h=1) = 0.180$
- Hole: $P(h=1) = \sum_{t=0,1} \sum_{c=0,1} P(t, c, h=1) = 0.200$
- Caries
  Decay after observing a hole: $P(c=1 \mid h=1) = \boxed{\dfrac{P(c=1, h=1)}{P(h=1)}} = \dfrac{0.180}{0.200} = 0.900$

$P(c=1, h=1) = P(c=1 \mid h=1) \cdot P(h=1)$

$P(h=1)$

In [2]:
```
import numpy as np
T = np.array([[0,0,0,.576], [0,0,1,.008], [0,1,0,.144], [0,1,1,.072],
              [1,0,0,.064], [1,0,1,.012], [1,1,0,.016], [1,1,1,.108]])
Pc1b1 = np.sum(T[(T[:,1]==1) & (T[:,2]==1),-1])
Pb1 = np.sum(T[T[:,2]==1,-1])
Pc1Db1 = Pc1b1/Pb1
print(f"Pc1b1 = {Pc1b1:.3f}  Pb1 = {Pb1:.3f}  Pc1Db1 = {Pc1Db1:.3f}")
```

```
Pc1b1 = 0.180  Pb1 = 0.200  Pc1Db1 = 0.900
```

4 Independence

Independent variables: Two variables x and y are **independent** if

$$P(x,y) = P(x)\,P(y) \quad \text{or} \quad P(x \mid y) = P(x) \quad \text{or} \quad P(y \mid x) = P(y)$$

Expert knowledge: independence can be established by expert knowledge and convenience

Dentist example:

- We consider a new variable the weather when the patient visits the dentist

$$W \in \{\text{sun}, \text{clouds}, \text{rain}, \text{snow}\}$$

- Let's assume that the three variables we already had are independent of the weather

$$\boxed{P(t,c,h,w)} = P(w)\,P(t,c,h \mid w) = P(w)\,P(t,c,h)$$

- This is how we reduce the number of probabilities to store: 32 vs $4+8$

5 Bayes' theorem

Bayes' theorem: It allows us to update our knowledge about a hypothesis y after observing new evidence x

$p(x,y) = p(y|x) \cdot p(x)$

$$\boxed{P(y \mid x) \;=\; \frac{P(x,y)}{P(x)}} \;=\; P(y)\,\frac{P(x \mid y)}{P(x)}$$

$p(x,y) = p(x|y) \cdot p(y)$

- In other words: $P(y \mid x)$ is the probability that the effect y will occur after observing that the cause x has occurred

Dentist example:

- We know that the probability of caries is: $P(c=1)=0.34$
- We know that the probability of toothache is: $P(t=1)=0.20$
- We know that the probability of toothache after observing caries is: $P(t=1 \mid c=1)=0.36$
- What is the probability of caries after observing toothache, $P(c=1 \mid t=1)$?

$$P(c=1 \mid t=1) = P(c=1)\boxed{\frac{P(t=1 \mid c=1)}{P(t=1)}} = 0.34\,\frac{0.36}{0.20} = 0.61$$

```
In [3]: Pc1 = 0.34; Pd1 = 0.20; Pd1c1 = 0.36; Pc1Dd1 = Pc1 * Pd1c1 / Pb1; print(f"Pc1Dd1 = {Pc1Dd1:.2f}")
```

```
Pc1Dd1 = 0.61
```

T1.1 Probabilistic reasoning: representation and inference

2023_01_26_Question 1: Given the following table of joint probabilities for three variables of interest:

A	0	0	0	0	1	1	1	1
B	0	0	1	1	0	0	1	1
C	0	1	0	1	0	1	0	1
$P(A, B, C)$	0.093	0.100	0.133	0.163	0.157	0.150	0.117	0.087

What is the value of $P(A = 1, B = 1 \mid C = 1)$?

$$P(A,B,C) = P(A,B \mid C) \cdot P(C)$$

1. $P(A = 1, B = 1 \mid C = 1) \leq 0.25$
2. $0.25 < P(A = 1, B = 1 \mid C = 1) \leq 0.50$
3. $0.50 < P(A = 1, B = 1 \mid C = 1) \leq 0.75$
4. $0.75 < P(A = 1, B = 1 \mid C = 1)$

$$P(A=1, B=1 \mid C=1) = \frac{P(A=1, B=1, C=1)}{P(C=1)} = \frac{P(A=1, B=1, C=1)}{\sum_A \sum_B P(A, B, C=1)} = \frac{0{,}087}{0.100 + 0.163 + 0.150 + 0.087}$$

$$= 0.174$$

Solution: Option 1; $P(A = 1, B = 1 \mid C = 1) = 0.174$

2023_01_17_Question 1: Suppose we have two boxes with 40 oranges in the first and 80 in the second. The first box contains 9 Navelina and 31 Caracara oranges. The second box contains three times more Navelina oranges than Caracara. Now suppose a box is chosen at random, and then an orange is chosen at random from the chosen box. If the orange chosen is Navelina, the probability P that it comes from the first box is:

1. $0/4 \leq P < 1/4$.
2. $1/4 \leq P < 2/4$.
3. $2/4 \leq P < 3/4$.
4. $3/4 \leq P \leq 4/4$.

Random variables $B = \{1, 2\}$

$O = \{N, C\}$

$P(B=1) = \frac{40}{40+80}$ $\quad P(B=2) = \frac{80}{40+80}$

$P(O=N \mid B=1) = \frac{9}{40}$ $\quad P(O=C \mid B=1) = \frac{31}{40}$

$P(O=N \mid B=2) = 3 \cdot x = \frac{3}{4}$ $\quad P(O=C \mid B=2) = x = \frac{1}{4}$

$3x + x = 1 \rightarrow x = \frac{1}{4}$

$4x = 1$

$$\sum_B P(B, O=N) = P(B=1, O=N) + P(B=2, O=N)$$
$$= P(O=N \mid B=1) \cdot P(B=1) + P(O=N \mid B=2) \cdot P(B=2)$$
$$= \frac{9}{40} \cdot \frac{40}{120} + \frac{3}{4} \cdot \frac{80}{120}$$

$$P(B=1 \mid O=N) = \frac{P(B=1, O=N)}{P(O=N)} = \frac{P(O=N \mid B=1) \cdot P(B=1)}{\sum_B P(B, O=N)} = \frac{\frac{9}{40} \cdot \frac{40}{120}}{\frac{9}{40} \cdot \frac{40}{120} + \frac{3}{4} \cdot \frac{80}{120}} = 0.13$$

Solution:

$$
\begin{aligned}
P = P(C=1 \mid T=N) &= \frac{P(C=1)P(T=N \mid C=1)}{P(C=1)P(T=N \mid C=1) + P(C=2)P(T=N \mid C=2)} \\
&= \frac{40/120 \cdot 9/40}{40/120 \cdot 9/40 + 80/120 \cdot 3/4} = 0.13
\end{aligned}
$$

2022_01_27_Question 4: Given the following table of joint frequencies of three variables of interest:

A	0	0	0	0	1	1	1	1
B	0	0	1	1	0	0	1	1
C	0	1	0	1	0	1	0	1
$N(A,B,C)$	124	28	227	175	126	222	23	75

What is the value of $P(A = 1 \mid B = 1, C = 0)$?

1. 0.023
2. 0.250
3. 0.092
4. 0.446

Solution: Option 3.

2022_01_13_Question 7: Follow a probabilistic reasoning problem about road trips, with the random variables of interest:

- Climatology (C): {clear (CLE), cloudy (CLO), rain (RAI)}
- Brightness (B): {day (DAY), night (NIGHT)}
- Safety (S): {safe (SAF), accident (ACC)}

The joint probability of the three variables is given in the table:

	DAY			NIGHT		
$P(S,B,C)$	CLE	CLO	RAI	CLE	CLO	RAI
SAF	0.27	0.23	0.07	0.16	0.07	0.06
ACC	0.02	0.01	0.02	0.02	0.03	0.04

The conditional probability $P(S = \mathrm{ACC} \mid B = \mathrm{DAY}, C = \mathrm{CLO})$ is:

1. 0.042
2. 0.010
3. 0.240
4. 0.140

Solution: Option 1.

U1.2 Continuous variables and Bayes' rule

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1 Continuous variables

Probability density function: Usual characterization of continuous variables for the representation of probabilistic knowledge

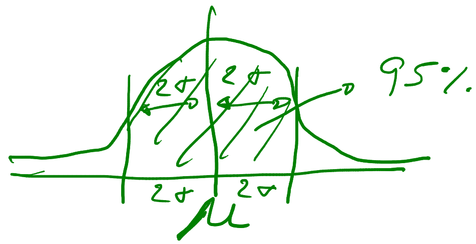
$x \in \mathbb{R}$

$$p(x) \geq 0 \quad \text{for all } x \qquad \text{and} \qquad \int_{x \in \mathbb{R}} p(x)\, dx = 1$$

$$\sum_x p(x) = 1$$

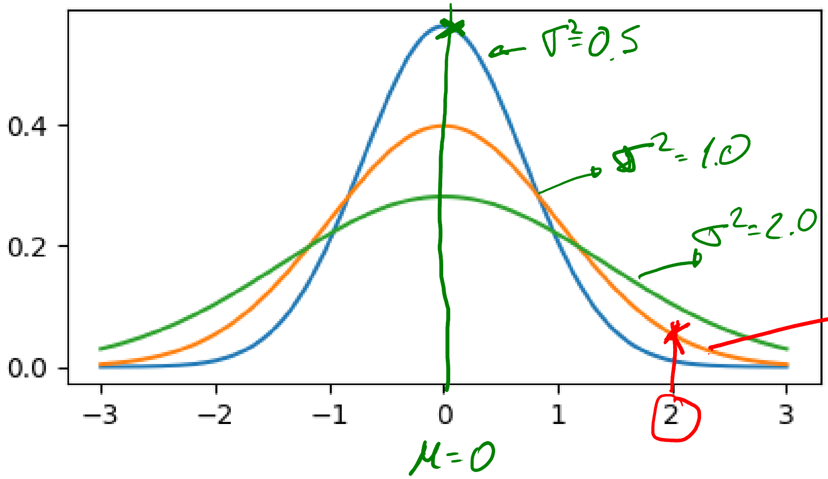
The normal density: $p(x) \sim \mathcal{N}(\mu, \sigma^2)$

$$\boxed{p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\,\sigma^2}\right)} \qquad P(x \in [\mu \pm 1.96\sigma]) = 0.95$$



Example: normal densities with $\mu = 0$ and $\sigma^2 = 0.5, 1, 2$

In [1]:
```
import numpy as np; from scipy.stats import norm; import matplotlib.pyplot as plt
x = np.linspace(-3, 3, 200)
plt.figure(figsize=(5, 2.5))
plt.plot(x, norm.pdf(x, 0, np.sqrt(0.5)), x, norm.pdf(x, 0, 1), x, norm.pdf(x, 0, np.sqrt(2)));
```



$p(x=2 \mid \sigma^2 = 0.5)$

$p(x=2 \mid \sigma^2 = 1.0)$

$\mu = 2 \quad x = 2$

2 Bayes theorem in the continuous case

Bayes theorem in the continuous case: Probability of a hypothesis y after observing (new) evidence x

$$P(y \mid x) = P(y)\,\frac{p(x \mid y)}{p(x)}$$

Example: $x =$ result of a saliva test for caries diagnosis

- Without caries, $c = 0$, $\boxed{p(x \mid c = 0)} \sim \boxed{\mathcal{N}(\mu = 0, \sigma^2 = 1)}$
- With caries, $c = 1$, $p(x \mid c = 1) \sim \mathcal{N}(\mu = 2, \sigma^2 = 0.5)$
- We know that the probability (a priori) of caries is: $P(c = 1) = 0.34$
- **If the test gives $x = 2$, what is the (posterior) probability of caries?**

$$P(c = 1 \mid x = 2) = P(c = 1)\,\frac{p(x = 2 \mid c = 1)}{p(x = 2)} = 0.340\,\frac{0.227}{0.307} = 0.843$$

$$p(x=2) = \sum_c p(x=2, c)$$
$$= \sum_c p(x=2 \mid c) \cdot p(c) \quad 1 - 0.34$$
$$= p(x=2 \mid c=0) \cdot p(c=0) + p(x=2 \mid c=1) \cdot p(c=1)$$
$$0.34$$

- Note that first it was necessary to find the (density of) probability (a priori) of test $x = 2$.

$$\begin{aligned} p(x = 2) &= P(c = 0)p(x = 2 \mid c = 0) + P(c = 1)p(x = 2 \mid c = 1) \\ &= (1 - 0.34) \cdot 0.054 + 0.34 \cdot 0.564 = 0.227 \end{aligned}$$

$$p(x=2 \mid c=0)$$

```
In [2]: Pc1 = 0.34; px2Dc0 = norm.pdf(2, 0, 1); px2Dc1 = norm.pdf(2, 2, np.sqrt(0.5))
px2 = (1-Pc1) * px2Dc0 + Pc1 * px2Dc1; Pc1Dx2 = Pc1 * px2Dc1 / px2
print(f"px2Dc0 = {px2Dc0:.3f}  px2Dc1 = {px2Dc1:.3f}  px2 = {px2:.3f}  Pc1Dx2 = {Pc1Dx2:.3f}")
```

```
px2Dc0 = 0.054  px2Dc1 = 0.564  px2 = 0.227  Pc1Dx2 = 0.843
```

c = Number 6 or 9 $\qquad C = \{6, 9\}$

x = Image

3 Bayes' decision rule

Bayes' decision rule: Predicts a hypothesis after observing some evidence x by choosing, from a set of possible hypotheses $\mathcal{C}$, a hypothesis of maximum **posterior probability** (from the observation of the evidence)

$$\boxed{c^*(x) = \operatorname*{argmax}_{c \in \mathcal{C}} P(c \mid x)}$$

Probability of error: That is, the probability that the predicted hypothesis is different from the correct one

$$P(\text{error} \mid x) = 1 - P(c^*(x) \mid x)$$

$P(c=6 \mid x) = 0.3$

$P(c=9 \mid x) = 0.7$

Bayes' rule optimality: No other choice would improve this error probability!

Dentist example:

$x=2$

$c=6 \rightarrow P(\text{err}) = 0.7$

$c=9 \rightarrow P(\text{err}) = 0.3$

$$c^*(x=2) = \operatorname*{argmax}_{c} \begin{pmatrix} P(c=0 \mid x=2) = 0.116 \\ P(c=1 \mid x=2) = \boxed{0.884} \end{pmatrix} = 1$$

Bayes' rule based on prior probabilities and conditional (densities) of classes: Instead of (arg-)maximize $P(c \mid x)$ in c, we do it as a function of $P(c)\,p(x \mid c)$ since the result is the same

$$c^*(x) = \operatorname*{argmax}_{c \in \mathcal{C}} P(c \mid x) = \operatorname*{argmax}_{c \in \mathcal{C}} \boxed{P(c)\,\frac{p(x \mid c)}{p(x)}} = \operatorname*{argmax}_{c \in \mathcal{C}} P(c)\,p(x \mid c)$$

Dentist example:

$$c^*(x=2) = \operatorname*{argmax}_{c} \begin{pmatrix} P(c=0)\,p(x=2 \mid c=0) = 0.036 \\ P(c=1)\,p(x=2 \mid c=1) = 0.271 \end{pmatrix} = 1$$

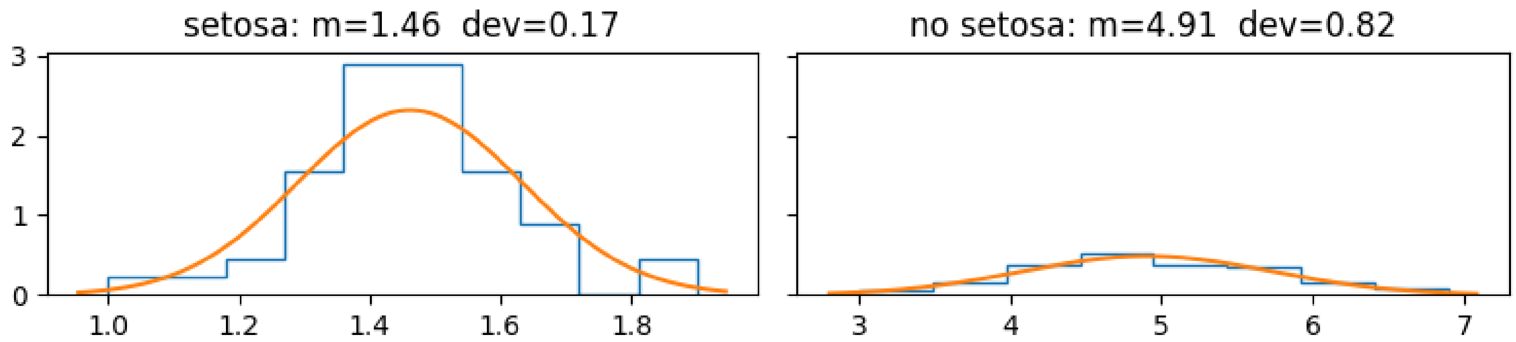
U1.2 Continuous variables and Bayes' rule

Problem: Consider the classification of iris flowers as setosa or non-setosa based on petal length, x. It can be shown that distributions of x for setosa and non-setosa can be approximated by normal distributions of means and standard deviations:

$$p(x \mid c = \text{set}) \sim \mathcal{N}(\mu_{\text{set}} = 1.46, \sigma_{\text{set}} = 0.17) \qquad \text{and} \qquad p(x \mid c = \text{nos}) \sim \mathcal{N}(\mu_{\text{nos}} = 4.91, \sigma_{\text{nos}} = 0.82)$$

In [3]:

```
import numpy as np; import matplotlib.pyplot as plt
from sklearn.datasets import load_iris; from scipy.stats import norm
iris = load_iris(); X = iris.data.astype(np.float16); y = iris.target.astype(np.uint)
x_set = np.squeeze(X[np.where(y==0), 2]); x_nos = np.squeeze(X[np.where(y!=0), 2])
fig, axs = plt.subplots(1, 2, figsize=(8, 2), sharey=True, tight_layout=True)
axs[0].hist(x_set, bins='auto', density=True, histtype='step')
x_set_range = np.arange(*axs[0].get_xlim(), .01)
x_set_mean = x_set.mean(); x_set_dev = np.sqrt(x_set.var())
axs[0].set_title(f'setosa: m={x_set_mean:.2f}  dev={x_set_dev:.2f}')
axs[0].plot(x_set_range, norm.pdf(x_set_range, x_set_mean, x_set_dev))
axs[1].hist(x_nos, bins='auto', density=True, histtype='step')
x_nos_range = np.arange(*axs[1].get_xlim(), .01)
x_nos_mean = x_nos.mean(); x_nos_dev = np.sqrt(x_nos.var())
axs[1].set_title(f'no setosa: m={x_nos_mean:.2f}  dev={x_nos_dev:.2f}')
axs[1].plot(x_nos_range, norm.pdf(x_nos_range, x_nos_mean, x_nos_dev));
```



If the estimated normal densities are true and the prior probability of setosa is $1/3$, what is the posterior probability that a flower of petal length 2 is setosa?

Solution:

$$
\begin{aligned}
P(c=\text{set}\mid x=2) &= \frac{P(c=\text{set})\,p(x=2\mid c=\text{set})}{p(x=2)}\\
&= \frac{P(c=\text{set})\,p(x=2\mid c=\text{set})}{P(c=\text{set})\,p(x=2\mid c=\text{set})+P(c=\text{nos})\,p(x=2\mid c=\text{nos})}\\
&= \frac{1/3\cdot\mathcal{N}(x=2\mid \mu_{\text{set}}=1.46,\sigma_{\text{set}}=0.17)}{1/3\cdot\mathcal{N}(x=2\mid \mu_{\text{set}}=1.46,\sigma_{\text{set}}=0.17)+2/3\cdot\mathcal{N}(x=2\mid \mu_{\text{nos}}=4.91,\sigma_{\text{nos}}=0.82)}\\
&= \frac{\frac{1}{0.17}\exp\left(-\frac{(2-1.46)^2}{2\cdot 0.17^2}\right)}{\frac{1}{0.17}\exp\left(-\frac{(2-1.46)^2}{2\cdot 0.17^2}\right)+\frac{2}{0.82}\exp\left(-\frac{(2-4.91)^2}{2\cdot 0.82^2}\right)} = \frac{0.0379}{0.0379+0.0045} = 0.89
\end{aligned}
$$