Cuaderno de trabajo:

Algoritmo Perceptrón

Albert Sanchis

DSIC

Departamento de Sistemas
Informáticos y Computación

Objetivos formativos

- Aplicar el algoritmo Perceptrón a un problema de clasificación

Algoritmo Perceptrón

Entrada: $\{(\mathbf{x}_n, c_n)\}_{n=1}^{N}$, $\{\mathbf{w}_c\}_{c=0}^{C}$, $\alpha \in \mathbb{R}^{>0}$ **y** $b \in \mathbb{R}$

Salida: $\{\mathbf{w}_c\}^* = \arg\min\limits_{\{\mathbf{w}_c\}} \sum\limits_{n} \left[\max\limits_{c \neq c_n} \mathbf{w}_c^t \mathbf{x}_n + b > \mathbf{w}_{c_n}^t \mathbf{x}_n \right]$

$$[\,P\,] = \begin{cases} 1 & \text{si} \quad P = \text{verdadero} \\ 0 & \text{si} \quad P = \text{falso} \end{cases}$$

Método:

repetir

para todo dato $\mathbf{x}_n$

$err = \texttt{falso}$

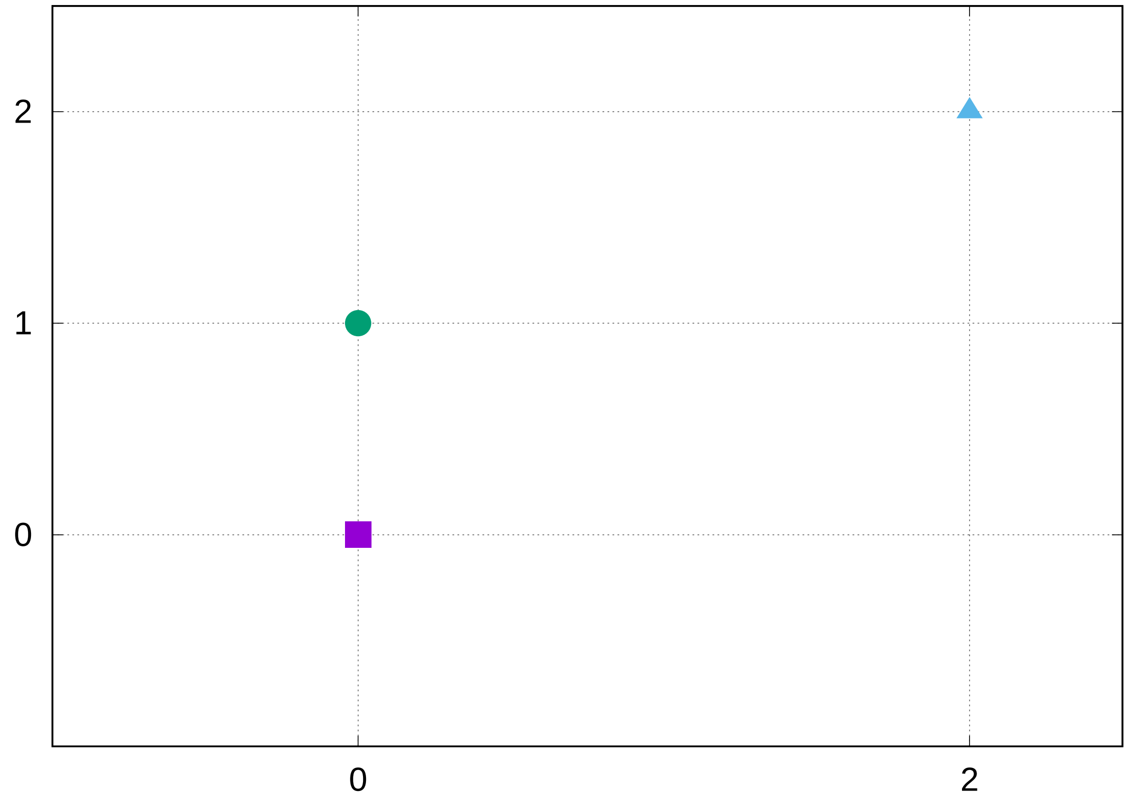
para toda clase c distinta de c_n

si $\mathbf{w}_c^t \mathbf{x}_n + b > \mathbf{w}_{c_n}^t \mathbf{x}_n$: $\mathbf{w}_c = \mathbf{w}_c - \alpha \cdot \mathbf{x}_n$; $err = \texttt{verdadero}$

si err: $\mathbf{w}_{c_n} = \mathbf{w}_{c_n} + \alpha \cdot \mathbf{x}_n$

hasta que no quedan muestras mal clasificadas

- ***Cuestión 1***: Sea un problema de clasificación en 3 clases ($c = 1, 2, 3$), para objetos representados mediante vectores de características bidimensionales ($\boldsymbol{x} = (x_1, x_2)^t$). Supóngase que se dispone de 3 muestras de entrenamiento $\boldsymbol{x_1} = (0, 0)^t$ de la clase $c_1 = 1$; $\boldsymbol{x_2} = (0, 1)^t$ de la clase $c_2 = 2$; y $\boldsymbol{x_3} = (2, 2)^t$ de la clase $c_3 = 3$ tal como se muestra en la siguiente figura:



Encuentra un clasificador lineal de mínimo error mediante el algoritmo Perceptrón, con vectores de pesos iniciales de las clases nulos, factor de aprendizaje $\alpha = 1$ y margen $b = 0.1$. Presenta una traza de ejecución que incluya las sucesivas actualizaciones de los vectores de pesos de las clases.

- ***Vectores de pesos iniciales para cada clase (en notación homogénea):***

$$\boldsymbol{w}_1 = (0, 0, 0)^t$$
$$\boldsymbol{w}_2 = (0, 0, 0)^t$$
$$\boldsymbol{w}_3 = (0, 0, 0)^t$$

- ***Iteración 1:***
  - ***Muestra*** $\boldsymbol{x_1} = (1, 0, 0)^t$ ***(en notación homogénea),*** $c_1 = 1$***:***

$$g_1(\boldsymbol{x_1}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_1} = 0 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 = 0$$

$$g_2(\boldsymbol{x_1}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_1} = 0 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 = 0$$

$g_2(\boldsymbol{x_1}) + b > g_1(\boldsymbol{x_1})?\ 0 + 0{,}1 > 0? \rightarrow$ ***Sí (Error)***

$$\boldsymbol{w_2} = \boldsymbol{w_2} - \alpha \cdot \boldsymbol{x_1} = (0, 0, 0)^t - 1 \cdot (1, 0, 0)^t = (-1, 0, 0)^t$$

$$g_3(\boldsymbol{x_1}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_1} = 0 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 = 0$$

$g_3(\boldsymbol{x_1}) + b > g_1(\boldsymbol{x_1})?\ 0 + 0{,}1 > 0? \rightarrow$ ***Sí (Error)***

$$\boldsymbol{w_3} = \boldsymbol{w_3} - \alpha \cdot \boldsymbol{x_1} = (0, 0, 0)^t - 1 \cdot (1, 0, 0)^t = (-1, 0, 0)^t$$

Error? $\rightarrow$ ***Sí***

$$\boldsymbol{w_1} = \boldsymbol{w_1} + \alpha \cdot \boldsymbol{x_1} = (0, 0, 0)^t + 1 \cdot (1, 0, 0)^t = (1, 0, 0)^t$$

- ***Muestra*** $\boldsymbol{x_2} = (1,0,1)^t$***,*** $c_2 = 2$***:***

$$g_2(\boldsymbol{x_2}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_2} = -1 \cdot 1 + 0 \cdot 0 + 0 \cdot 1 = -1$$

$$g_1(\boldsymbol{x_2}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_2} = 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 1 = 1$$

$g_1(\boldsymbol{x_2}) + b > g_2(\boldsymbol{x_2})?\ 1 + 0{,}1 > -1? \rightarrow$ ***Sí (Error)***

$$\boldsymbol{w_1} = \boldsymbol{w_1} - \alpha \cdot \boldsymbol{x_2} = (1,0,0)^t - 1 \cdot (1,0,1)^t = (0,0,-1)^t$$

$$g_3(\boldsymbol{x_2}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_2} = -1 \cdot 1 + 0 \cdot 0 + 0 \cdot 1 = -1$$

$g_3(\boldsymbol{x_2}) + b > g_2(\boldsymbol{x_2})?\ -1 + 0{,}1 > -1? \rightarrow$ ***Sí (Error)***

$$\boldsymbol{w_3} = \boldsymbol{w_3} - \alpha \cdot \boldsymbol{x_2} = (-1,0,0)^t - 1 \cdot (1,0,1)^t = (-2,0,-1)^t$$

Error? $\rightarrow$ ***Sí***

$$\boldsymbol{w_2} = \boldsymbol{w_2} + \alpha \cdot \boldsymbol{x_2} = (-1,0,0)^t + 1 \cdot (1,0,1)^t = (0,0,1)^t$$

- ***Muestra*** $\boldsymbol{x_3} = (1, 2, 2)^t$***,*** $c_3 = 3$***:***

$$g_3(\boldsymbol{x_3}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_3} = -2 \cdot 1 + 0 \cdot 2 + (-1) \cdot 2 = -4$$

$$g_1(\boldsymbol{x_3}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_3} = 0 \cdot 1 + 0 \cdot 2 + (-1) \cdot 2 = -2$$

$g_1(\boldsymbol{x_3}) + b > g_3(\boldsymbol{x_3})? \; -2 + 0{,}1 > -4? \rightarrow$ ***Sí (Error)***

$$\boldsymbol{w_1} = \boldsymbol{w_1} - \alpha \cdot \boldsymbol{x_3} = (0, 0, -1)^t - 1 \cdot (1, 2, 2)^t = (-1, -2, -3)^t$$

$$g_2(\boldsymbol{x_3}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_3} = 0 \cdot 1 + 0 \cdot 2 + 1 \cdot 2 = 2$$

$g_2(\boldsymbol{x_3}) + b > g_3(\boldsymbol{x_3})? \; 2 + 0{,}1 > -4? \rightarrow$ ***Sí (Error)***

$$\boldsymbol{w_2} = \boldsymbol{w_2} - \alpha \cdot \boldsymbol{x_3} = (0, 0, 1)^t - 1 \cdot (1, 2, 2)^t = (-1, -2, -1)^t$$

Error? $\rightarrow$ ***Sí***

$$\boldsymbol{w_3} = \boldsymbol{w_3} + \alpha \cdot \boldsymbol{x_3} = (-2, 0, -1)^t + 1 \cdot (1, 2, 2)^t = (-1, 2, 1)^t$$

- ***Iteración 2:***
  - ***Muestra*** $\boldsymbol{x_1} = (1,0,0)^t$***,*** $c_1 = 1$***:***

$$g_1(\boldsymbol{x_1}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_1} = -1 \cdot 1 + (-2) \cdot 0 + (-3) \cdot 0 = -1$$

$$g_2(\boldsymbol{x_1}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_1} = -1 \cdot 1 + (-2) \cdot 0 + (-1) \cdot 0 = -1$$

$$g_2(\boldsymbol{x_1}) + b > g_1(\boldsymbol{x_1})? \; -1 + 0{,}1 > -1? \rightarrow \textbf{\textit{Sí (Error)}}$$

$$\boldsymbol{w_2} = \boldsymbol{w_2} - \alpha \cdot \boldsymbol{x_1} = (-1,-2,-1)^t - 1 \cdot (1,0,0)^t = (-2,-2,-1)$$

$$g_3(\boldsymbol{x_1}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_1} = -1 \cdot 1 + 2 \cdot 0 + 1 \cdot 0 = -1$$

$$g_3(\boldsymbol{x_1}) + b > g_1(\boldsymbol{x_1})? \; -1 + 0{,}1 > -1? \rightarrow \textbf{\textit{Sí (Error)}}$$

$$\boldsymbol{w_3} = \boldsymbol{w_3} - \alpha \cdot \boldsymbol{x_1} = (-1,2,1)^t - 1 \cdot (1,0,0)^t = (-2,2,1)^t$$

Error? $\rightarrow$ ***Sí***

$$\boldsymbol{w_1} = \boldsymbol{w_1} + \alpha \cdot \boldsymbol{x_1} = (-1,-2,-3)^t + 1 \cdot (1,0,0)^t = (0,-2,-3)^t$$

- ***Muestra*** $\boldsymbol{x_2} = (1,0,1)^t$***,*** $c_2 = 2$***:***

$$g_2(\boldsymbol{x_2}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_2} = -2 \cdot 1 + (-2) \cdot 0 + (-1) \cdot 1 = -3$$

$$g_1(\boldsymbol{x_2}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_2} = 0 \cdot 1 + (-2) \cdot 0 + (-3) \cdot 1 = -3$$

$$g_1(\boldsymbol{x_2}) + b > g_2(\boldsymbol{x_2})? \;\; -3 + 0{,}1 > -3? \rightarrow \textbf{\textit{Sí (Error)}}$$

$$\boldsymbol{w_1} = \boldsymbol{w_1} - \alpha \cdot \boldsymbol{x_2} = (0,-2,-3)^t - 1 \cdot (1,0,1)^t = (-1,-2,-4)^t$$

$$g_3(\boldsymbol{x_2}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_2} = -2 \cdot 1 + 2 \cdot 0 + 1 \cdot 1 = -1$$

$$g_3(\boldsymbol{x_2}) + b > g_2(\boldsymbol{x_2})? \;\; -1 + 0{,}1 > -3? \rightarrow \textbf{\textit{Sí (Error)}}$$

$$\boldsymbol{w_3} = \boldsymbol{w_3} - \alpha \cdot \boldsymbol{x_2} = (-2,2,1)^t - 1 \cdot (1,0,1)^t = (-3,2,0)^t$$

Error? $\rightarrow$ ***Sí***

$$\boldsymbol{w_2} = \boldsymbol{w_2} + \alpha \cdot \boldsymbol{x_2} = (-2,-2,-1)^t + 1 \cdot (1,0,1)^t = (-1,-2,0)^t$$

◦ ***Muestra*** $\boldsymbol{x_3} = (1,2,2)^t$***,*** $c_3 = 3$***:***

$$g_3(\boldsymbol{x_3}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_3} = -3 \cdot 1 + 2 \cdot 2 + 0 \cdot 2 = 1$$

$$g_1(\boldsymbol{x_3}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_3} = -1 \cdot 1 + (-2) \cdot 2 + (-4) \cdot 2 = -13$$

$g_1(\boldsymbol{x_3}) + b > g_3(\boldsymbol{x_3})? \; -13 + 0{,}1 > 1? \rightarrow$ ***No***

$$g_2(\boldsymbol{x_3}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_3} = -1 \cdot 1 + (-2) \cdot 2 + 0 \cdot 2 = -5$$

$g_2(\boldsymbol{x_3}) + b > g_3(\boldsymbol{x_3})? \; -5 + 0{,}1 > 1? \rightarrow$ ***No***

Error? $\rightarrow$ ***No***

- **Iteración 3:**
  - **Muestra $\boldsymbol{x_1} = (1,0,0)^t$, $c_1 = 1$:**

$$g_1(\boldsymbol{x_1}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_1} = -1 \cdot 1 + (-2) \cdot 0 + -4 \cdot 0 = -1$$

$$g_2(\boldsymbol{x_1}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_1} = -1 \cdot 1 + (-2) \cdot 0 + 0 \cdot 0 = -1$$

$$g_2(\boldsymbol{x_1}) + b > g_1(\boldsymbol{x_1})? \ -1 + 0{,}1 > -1? \rightarrow \textbf{\textit{Sí (Error)}}$$

$$\boldsymbol{w_2} = \boldsymbol{w_2} - \alpha \cdot \boldsymbol{x_1} = (-1,-2,0)^t - 1 \cdot (1,0,0)^t = (-2,-2,0)^t$$

$$g_3(\boldsymbol{x_1}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_1} = -3 \cdot 1 + 2 \cdot 0 + 0 \cdot 0 = -3$$

$$g_3(\boldsymbol{x_1}) + b > g_1(\boldsymbol{x_1})? \ -3 + 0{,}1 > -1? \rightarrow \textbf{\textit{No}}$$

Error? $\rightarrow$ **Sí**

$$\boldsymbol{w_1} = \boldsymbol{w_1} + \alpha \cdot \boldsymbol{x_1} = (-1,-2,-4)^t + 1 \cdot (1,0,0)^t = (0,-2,-4)^t$$

◦ ***Muestra** $\boldsymbol{x_2} = (1,0,1)^t$**,** $c_2 = 2$**:***

$$g_2(\boldsymbol{x_2}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_2} = -2 \cdot 1 + (-2) \cdot 0 + 0 \cdot 1 = -2$$

$$g_1(\boldsymbol{x_2}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_2} = 0 \cdot 1 + (-2) \cdot 0 + (-4) \cdot 1 = -4$$

$g_1(\boldsymbol{x_2}) + b > g_2(\boldsymbol{x_2})? \; -4 + 0{,}1 > -2? \rightarrow$ ***No***

$$g_3(\boldsymbol{x_2}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_2} = -3 \cdot 1 + 2 \cdot 0 + 0 \cdot 1 = -3$$

$g_3(\boldsymbol{x_2}) + b > g_2(\boldsymbol{x_2})? \; -3 + 0{,}1 > -2? \rightarrow$ ***No***

Error? $\rightarrow$ ***No***

○ ***Muestra*** $\boldsymbol{x_3} = (1, 2, 2)^t$***,*** $c_3 = 3$***:***

$$g_3(\boldsymbol{x_3}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_3} = -3 \cdot 1 + 2 \cdot 2 + 0 \cdot 2 = 1$$

$$g_1(\boldsymbol{x_3}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_3} = 0 \cdot 1 + (-2) \cdot 2 + (-4) \cdot 2 = -12$$

$$g_1(\boldsymbol{x_3}) + b > g_3(\boldsymbol{x_3})? \; -12 + 0{,}1 > 1? \rightarrow \textbf{\textit{No}}$$

$$g_2(\boldsymbol{x_3}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_3} = -2 \cdot 1 + (-2) \cdot 2 + 0 \cdot 2 = -6$$

$$g_2(\boldsymbol{x_3}) + b > g_3(\boldsymbol{x_3})? \; -6 + 0{,}1 > 1? \rightarrow \textbf{\textit{No}}$$

Error? $\rightarrow$ ***No***

- ***Iteración 4:***
  - ***Muestra*** $\boldsymbol{x_1} = (1,0,0)^t$***,*** $c_1 = 1$***:***

$$g_1(\boldsymbol{x_1}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_1} = 0 \cdot 1 + (-2) \cdot 0 + (-4) \cdot 0 = 0$$

$$g_2(\boldsymbol{x_1}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_1} = -2 \cdot 1 + (-2) \cdot 0 + 0 \cdot 0 = -2$$

$$g_2(\boldsymbol{x_1}) + b > g_1(\boldsymbol{x_1})? \; -2 + 0{,}1 > 0? \rightarrow \textbf{\textit{No}}$$

$$g_3(\boldsymbol{x_1}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_1} = -3 \cdot 1 + 2 \cdot 0 + 0 \cdot 0 = -3$$

$$g_3(\boldsymbol{x_1}) + b > g_1(\boldsymbol{x_1})? \; -3 + 0{,}1 > 0? \rightarrow \textbf{\textit{No}}$$

Error? $\rightarrow$ ***No***

◦ ***Muestra*** $\boldsymbol{x_2} = (1,0,1)^t$***,*** $c_2 = 2$***:***

$$g_2(\boldsymbol{x_2}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_2} = -2 \cdot 1 + (-2) \cdot 0 + 0 \cdot 1 = -2$$

$$g_1(\boldsymbol{x_2}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_2} = 0 \cdot 1 + (-2) \cdot 0 + (-4) \cdot 1 = -4$$

$$g_1(\boldsymbol{x_2}) + b > g_2(\boldsymbol{x_2})? \; -4 + 0{,}1 > -2? \rightarrow \textbf{\textit{No}}$$

$$g_3(\boldsymbol{x_2}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_2} = -3 \cdot 1 + 2 \cdot 0 + 0 \cdot 1 = -3$$

$$g_3(\boldsymbol{x_2}) + b > g_2(\boldsymbol{x_2})? \; -3 + 0{,}1 > -2? \rightarrow \textbf{\textit{No}}$$

Error? $\rightarrow$ ***No***

- *Muestra* $\boldsymbol{x_3} = (1, 2, 2)^t$, $c_3 = 3$:

$$g_3(\boldsymbol{x_3}) = \boldsymbol{w_3}^t \cdot \boldsymbol{x_3} = -3 \cdot 1 + 2 \cdot 2 + 0 \cdot 2 = 1$$

$$g_1(\boldsymbol{x_3}) = \boldsymbol{w_1}^t \cdot \boldsymbol{x_3} = 0 \cdot 1 + (-2) \cdot 2 + (-4) \cdot 2 = -12$$

$$g_1(\boldsymbol{x_3}) + b > g_3(\boldsymbol{x_3})? \; -12 + 0{,}1 > 1? \rightarrow \textbf{\textit{No}}$$

$$g_2(\boldsymbol{x_3}) = \boldsymbol{w_2}^t \cdot \boldsymbol{x_3} = -2 \cdot 1 + (-2) \cdot 2 + 0 \cdot 2 = -6$$

$$g_2(\boldsymbol{x_3}) + b > g_3(\boldsymbol{x_3})? \; -6 + 0{,}1 > 1? \rightarrow \textbf{\textit{No}}$$

Error? $\rightarrow$ ***No***

- ***Cuestión 2***: Indica cómo han quedado definidas las funciones discriminantes una vez finalizado el algoritmo Perceptrón

  - ***Vectores de pesos finales para cada clase (en notación homogénea):***

$$\boldsymbol{w}_1 = (0, -2, -4)^t$$
$$\boldsymbol{w}_2 = (-2, -2, 0)^t$$
$$\boldsymbol{w}_3 = (-3, 2, 0)^t$$

  - ***Funciones discriminantes:***

$$g_1(\boldsymbol{x}) = -2x_1 - 4x_2$$
$$g_2(\boldsymbol{x}) = -2 - 2x_1$$
$$g_3(\boldsymbol{x}) = -3 + 2x_1$$