Vector calculus and applications

Cris Lanting

Clinical Physicist, Audiologist University Medical Center Groningen
Dept. of Otorhinolaryngology

c.p.lanting@umcg.nl
https://github.com/Crisly/Lectures

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Overview

Take home messages

This session is part of a broader course where you are given a hand-picked selection of topics. These vary quite a bit but there is common theme: Make sense of the mathematics we see (but don't quite grasp) in papers. This specific session is about vectors and stuff you can do with vectors.

- Know what vectors are and how to use them
- Know how to add and subtract and visualise vectors
- Know and use the dot or inner product
- Know and use the cross or outer product

Background

Q

What is actually a vector and what a scalar?

Q

What is actually a vector and what a scalar?

A

A (Euclidian) vector describes a geometric entity that has both a *direction* and *magnitude*. In contrast, a scalar only represents a magnitude but *not* a direction. The components $(a_1, a_2, \ldots, a_n)$ of $\mathbf{a}$ are the lengths of the projection of $\mathbf{a}$ along the n coordinate axes.

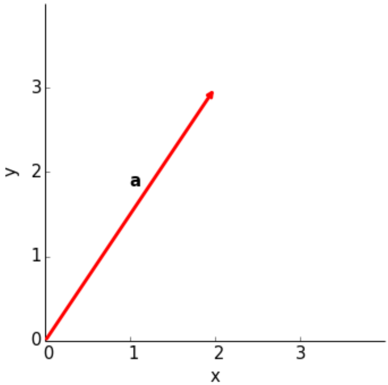


Figure: $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Q

Scalar or vector?

examples

Speed, Weight, Velocity, Force?

Given two vectors

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\mathbf{w} = \begin{pmatrix} c \\ d \end{pmatrix}$$

Vector addition is then simply

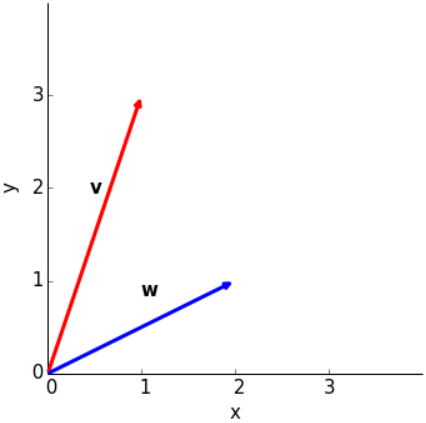
$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$$

similarly:

$$\mathbf{v} - \mathbf{w} = \begin{pmatrix} a - c \\ b - d \end{pmatrix}$$

The geometry of vector addition

If $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, draw $\mathbf{v} + \mathbf{w}$



$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} 1+2 \\ 3+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

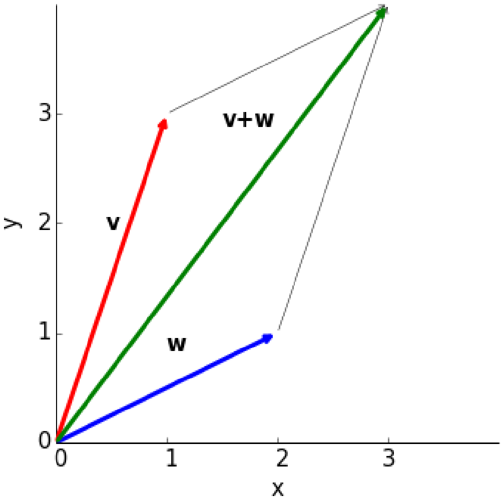


Figure:

Vector addition

Question number 2 of the preparatory questions:

Let $P = (-2, -1)$, $Q = (-3, 3)$, and $R = (-1, -4)$ in the xy-plane.

- Draw these vectors: $\mathbf{v}$ joining P to Q; $\mathbf{w}$ joining Q to R, and $\mathbf{u}$ joining R to P
- What are the component vectors of $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$?
- What is $\mathbf{v} + \mathbf{w} + \mathbf{u}$?

[ex 9. p. 13]

Scalar multiplication

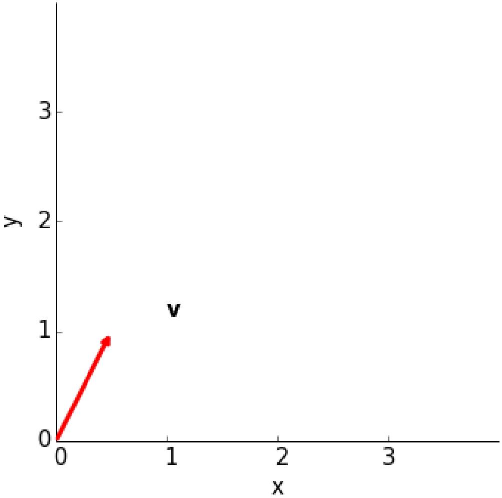
if we have a vector

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$

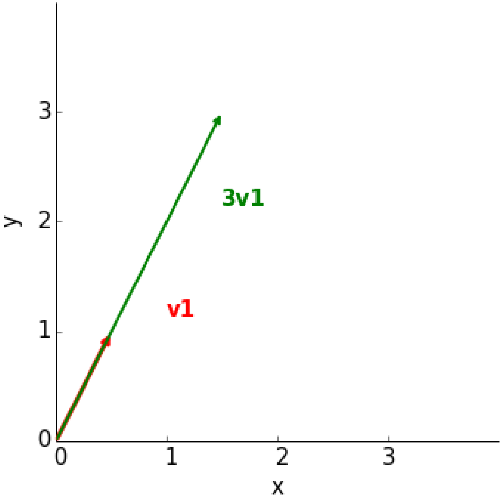
then the scalar multiplication (with scalar s) equals:

$$s\mathbf{v} = \begin{pmatrix} sa \\ sb \end{pmatrix}$$

The geometry of scalar multiplication



scalar multiplication



Vectors: the inner product or dot product

The inner product has some nice features – it is often used to quantify the angle between two vectors. First let's define the inner product of **v** and **w**.

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

The inner product of two vectors is thus a scalar quantity

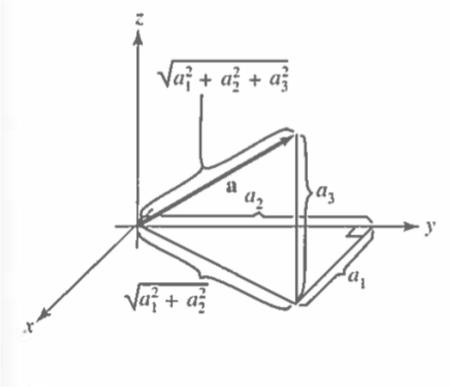
Important concepts in vector calculus are the length of a vector. From the Pythagorean theorem if easily follows that the length of vector

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

is

$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

This quality is often called the norm of a. This can also be used to calculate the distance between the endpoints of e.g., a and b as $||\mathbf{a} - \mathbf{b}||$.



So, the length –or norm– of a vector a is equal to

$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Since $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2$, it follows that

$$||\mathbf{a}|| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = (\mathbf{a} \cdot \mathbf{a})^{1/2}$$

Vectors with norm 1 are called unit vectors. It follows that for any (nonzero) vector, the unit vector can be obtained by dividing a by $||\mathbf{a}||$ and is also called a normalised a.

One application of the inner product is that it can be used to determine the angle between two vector. How? Let's see. Remember the cosine rule?

$$c^2 = a^2 + b^2 - 2ab\cos C$$

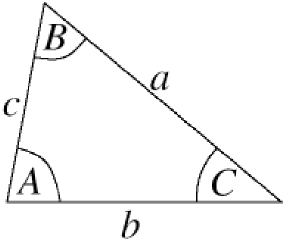


Figure:

So, by knowing the length of the three sides of a triangle we can obtain the angle between each pair of vectors; we know the length of the two vectors **a** and **b**. We also know the distance between them: $||\mathbf{b} - \mathbf{a}||$. Plugging these values into the cosine rule:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$||\mathbf{b} - \mathbf{a}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2||\mathbf{a}||||\mathbf{b}||\cos\theta$$

Since $||\mathbf{b} - \mathbf{a}||^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$, $||\mathbf{a}||^2 = \mathbf{a} \cdot \mathbf{a}$ and $||\mathbf{b}||^2 = \mathbf{b} \cdot \mathbf{b}$, we can rewrite this as:

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2||\mathbf{a}||||\mathbf{b}||\cos\theta$$

Now

$$
\begin{aligned}
(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) &= \mathbf{b}\cdot(\mathbf{b}-\mathbf{a})-\mathbf{a}\cdot(\mathbf{b}-\mathbf{a}) \\
&= \mathbf{b}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{a} \\
&= \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\mathbf{ab}
\end{aligned}
$$

Thus

$$
= \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\mathbf{ab} = \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2||\mathbf{a}||||\mathbf{b}||\cos\theta
$$

That is,

$$
= \mathbf{a}\cdot\mathbf{b} = ||\mathbf{a}||||\mathbf{b}||\cos\theta
$$

So, the two vector define the angle θ between them by

$$\theta = cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right)$$

Exercise [ex 4, p 29]:

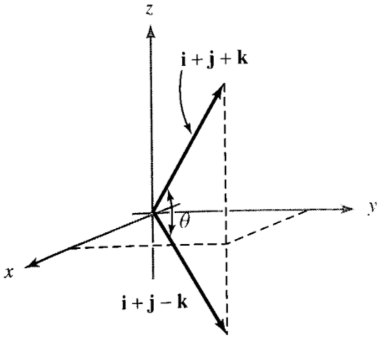


Figure:

Practical application: physics

Q

A bird is flying in a straight line with velocity vector 10 i + 6 j + k (in km/h). Suppose that (x, y) are its coordinates on the ground and z its height above the ground. [ex 11, p 35]

- If the bird is at $(1, 2, 3)$ at a certain moment, where is it 1h later?
- How many second does it take the bird to climb 10m?

Orthogonal projection

The orthogonal projection **p** of **w** on **v** is the vector whose tip is obtained by dropping a perpendicular line to the line *l* (along v) from the top of **w**. We can obtain the projection as:

$$\mathbf{p} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2}\mathbf{v}$$

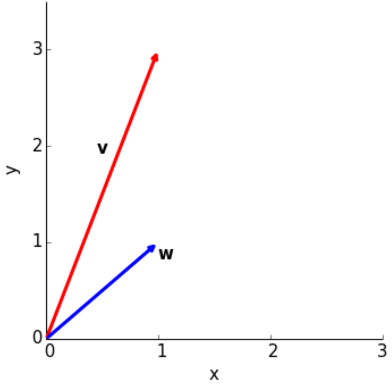


Figure:

So, $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

and $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\mathbf{v} \cdot \mathbf{w} = 1 * 1 + 3 * 1 = 4$$

$$\|\mathbf{v}\|^2 = 1^2 + 3^2 = 10$$

$$\mathbf{p} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{4}{10}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 1.2 \end{pmatrix}$$

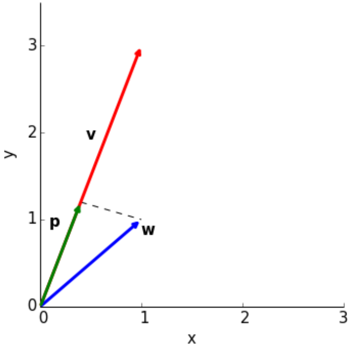


Figure:

Vectors: determinants and the cross-product

So far we've encountered

- vector addition
- scalar-product
- the inner-product closely related to the length of a vector and used to calculate the angle between vectors.

In what follows, we'll discuss another property of vector calculus: the outer product

Previously, we've encountered the inner product of two vectors that generated a scalar. In what follows we define a product of vectors that is a vector. Given two vectors **v** and **w**, we can obtain a third vector $\mathbf{v} \times \mathbf{w}$, the cross-product of **v** and **w**. This vector will have the (pleasing) geometrical property that is it perpendicular to the plane spanned by **v** and **w**.

First, let's define a 2 x 2 matrix **M** as

$$\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the determinant is then defined as:

$$|\mathbf{M}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

or, with a 3 x 3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or, with a 3 x 3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

the determinant is then defined as:

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or, with a 3 x 3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

This looks like a lot of work to get another vector, but there is some history and application to it:

- Linear equations and Cramer's rule (remember)
- 2D: The area of the parallelogram defined by two vectors
- 3D: The volume of a parallelepiped

Cramer's law to linear equations (1750!)

Say we have a system of equations:

$$\begin{aligned} 2x + y + z &= 3 \\ x - y - z &= 0 \\ x + 2y + z &= 0 \end{aligned}$$

Or rewritten like $Ax = b$,

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

This system can be solved (i.e., finding the values of x, y, z for which all three equations make sense) by using *determinants*.

Cramer's law : step 1

First find the determinant of the equations:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix} =$$

$$2\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = \ldots$$

Cramer's law : step 2

The next step is to plug the answer vector (the vector describing the results) into the three separate determinants – these are going to be used to determine the solution to the system of equations. So,

$$\Delta_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 3$$

$$\Delta_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -6$$

$$\Delta_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 9$$

Cramer's law : step 3

The final step to the solution is Cramer's rule:

$$x = \frac{\Delta_x}{\Delta} = \frac{3}{3} = 1$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-6}{3} = -2$$

$$z = \frac{\Delta_z}{\Delta} = \frac{9}{3} = 3$$

Exercise 1: check whether the above solution is a valid one.

Exercise 2: check whether this method works for the following

$$\begin{aligned} 2x + y + z &= 3 \\ x + y/2 + z/2 &= 3/2 \\ x + 2y + z &= 0 \end{aligned}$$

Geometry of 2x2 determinants

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ be two vectors in the (i,j) plane. Then the cross-product can be described as a determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

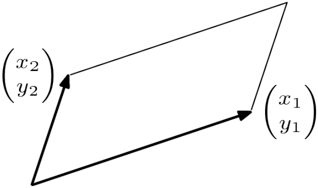
This is a very interesting property: it indicates that the area $||\mathbf{a} \times \mathbf{b}||$ equals the absolute value of the determinant.

$$||\mathbf{a} \times \mathbf{b}|| = |a_1 b_2 - a_2 b_1|$$

Geometry of 2x2 determinants

It can also be determined by determining

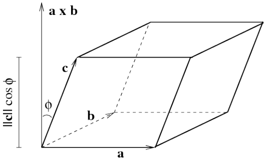
$$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}|| ||\mathbf{b}|| \sin \theta$$



Geometry of 3x3 determinants

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ be three vectors in the (i, j, j) plane. Then the absolute value of the determinant D is the volume of the parallelpiped.

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



Summary

Today we've encountered vectors and various related properties:

- scalar multiplication
- vector addition
- inner product $\mathbf{a} \cdot \mathbf{b}$
- outer product $\mathbf{a} \times \mathbf{b}$

bonus infinite sum

Q

What is the sum of all natural numbers $1 + 2 + 3 \ldots \inf$?

$$\sum_{n}^{\inf} n = -\frac{1}{12}$$

sources

- Vector Calculus 4th edition, Marsden Tromba
- MIT Walter Levine lecture on Vectors, Dot- and cross-products: https://www.youtube.com/watch?v=WarI5ZxW7tA
- Numberphile: https://www.youtube.com/watch?v=w-I6XTVZXww

Questions