

Vector calculus and applications

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<https://github.com/Crisly/MathcourseBCN>

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Overview

- 1 Background
- 2 Vector addition and scalar multiplication
- 3 Vector inner and outer products
- 4 summary

Take home messages

This session is part of a broader course where you are given a hand-picked selection of topics. These vary quite a bit but there is common theme: Make sense of the mathematics we see (but don't quite grasp) in papers. This session is about vectors and stuff you can do with vectors.

- Know what vectors are and how to use them
- Know how to add and subtract vectors
- Know and use the dot or inner product
- Know and use the cross or outer product

Background

Q

What is actually a vector and what a scalar?

Q

What is actually a vector and what a scalar?

A

A (Euclidian) **vector** describes a geometric entity that has both a *direction* and *magnitude*. In contrast, a **scalar** only represents a magnitude but *not* a direction. The components (a_1, a_2, \dots, a_n) of \mathbf{a} are the lengths of the projection of \mathbf{a} along the n coordinate axes.

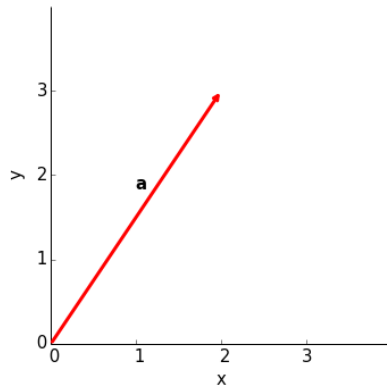


Figure: $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Q

Scalar or vector?

examples

Speed, Weight, Velocity, Force?

Given two vectors

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\mathbf{w} = \begin{pmatrix} c \\ d \end{pmatrix}$$

Vector addition is then simply

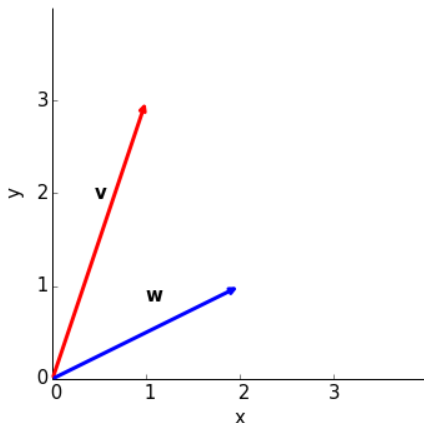
$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$$

similarly:

$$\mathbf{v} - \mathbf{w} = \begin{pmatrix} a - c \\ b - d \end{pmatrix}$$

The geometry of vector addition

If $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, draw $\mathbf{v} + \mathbf{w}$



$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} 1 + 2 \\ 3 + 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

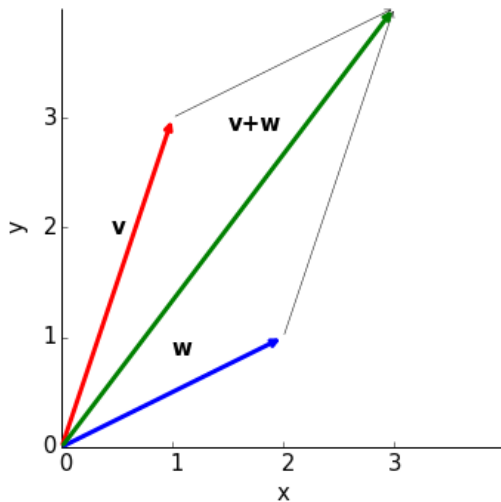


Figure:

Vector addition

Question number 2 of the preparatory questions:

Let $P = (-2, -1)$, $Q = (-3, 3)$, and $R = (-1, -4)$ in the xy -plane.

- Draw these vectors: \mathbf{v} joining P to Q ; \mathbf{w} joining Q to R , and \mathbf{u} joining R to P
- What are the component vectors of \mathbf{v} , \mathbf{w} and \mathbf{u} ?
- What is $\mathbf{v} + \mathbf{w} + \mathbf{u}$?

[ex 9. p. 13]

Scalar multiplication

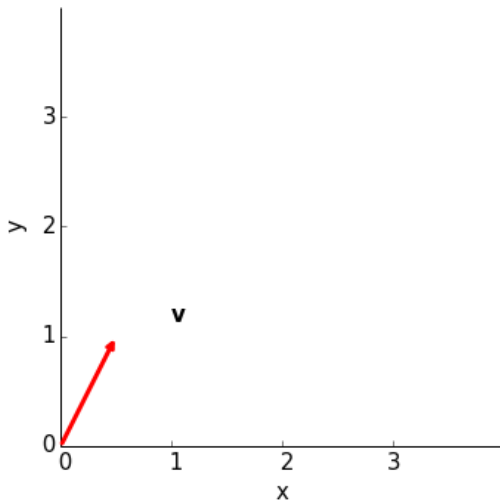
if we have a vector

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$

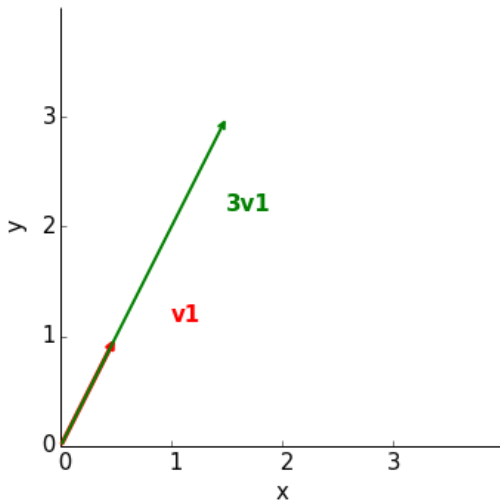
then the scalar multiplication (with scalar s) equals:

$$s\mathbf{v} = \begin{pmatrix} sa \\ sb \end{pmatrix}$$

The geometry of scalar multiplication



scalar multiplication



Vectors: the inner product or dot product

The inner product has some nice features - it is often used to quantify the angle between two vectors. First let's define the inner product of \mathbf{v} and \mathbf{w} .

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

The inner product of two **vectors** is thus a **scalar quantity**

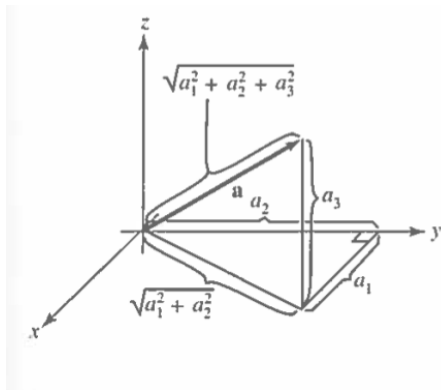
Important concepts in vector calculus are the length of a vector. From the Pythagorean theorem it easily follows that the length of vector

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

is

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

This quality is often called the norm of \mathbf{a} . This can also be used to calculate the distance between the endpoints of e.g., \mathbf{a} and \mathbf{b} as $\|\mathbf{a} - \mathbf{b}\|$.



So, the length –or norm– of a vector \mathbf{a} is equal to

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Since $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2$, it follows that

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = (\mathbf{a} \cdot \mathbf{a})^{1/2}$$

Vectors with norm 1 are called unit vectors. It follows that for any (nonzero) vector, the unit vector can be obtained by dividing \mathbf{a} by $\|\mathbf{a}\|$ and is also called a normalised \mathbf{a} .

One application of the inner product is that it can be used to determine the angle between two vector. How? Let's see.
Remember the cosine rule?

$$c^2 = a^2 + b^2 - 2ab \cos C$$

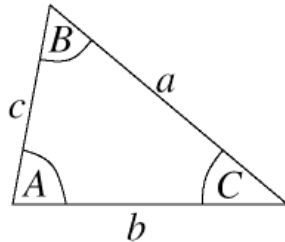


Figure:

So, by knowing the length of the three sides of a triangle we can obtain the angle between each pair of vectors; we know the length of the two vectors \mathbf{a} and \mathbf{b} . We also know the distance between them: $\|\mathbf{b} - \mathbf{a}\|$. Plugging these values into the cosine rule:

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

$$\|\mathbf{b} - \mathbf{a}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\| \cos \theta$$

Since $\|\mathbf{b} - \mathbf{a}\|^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$, $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$ and $\|\mathbf{b}\|^2 = \mathbf{b} \cdot \mathbf{b}$, we can rewrite this as:

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\|\mathbf{a}\|\|\mathbf{b}\| \cos \theta$$

Now

$$\begin{aligned}(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) &= \mathbf{b} \cdot (\mathbf{b} - \mathbf{a}) - \mathbf{a} \cdot (\mathbf{b} - \mathbf{a}) \\&= \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} \\&= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b}\end{aligned}$$

Thus

$$= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

That is,

$$= \mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

Exercise [ex 4, p 29]:

So, the two vector define the angle θ between them by

$$\theta = \cos^{-1} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right)$$

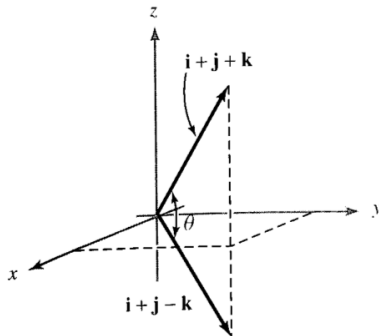


Figure:

Practical application: physics

Q

A bird is flying in a straight line with velocity vector $10 \mathbf{i} + 6 \mathbf{j} + \mathbf{k}$ (in km/h). Suppose that (x, y) are its coordinates on the ground and z its height above the ground. [ex 11, p 35]

- If the bird is at $(1, 2, 3)$ at a certain moment, where is it 1h later?
- How many second does it take the bird to climb 10m?

Orthogonal projection

The orthogonal projection \mathbf{p} of \mathbf{w} on \mathbf{v} is the vector whose tip is obtained by dropping a perpendicular line to the line l (along \mathbf{v}) from the top of \mathbf{w} . We can obtain the projection as:

$$\mathbf{p} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2} \mathbf{v}$$

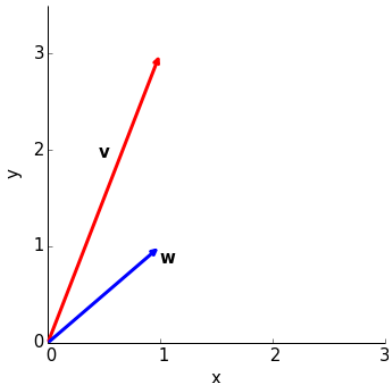


Figure:

$$\text{So, } \mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\text{and } \mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v} \cdot \mathbf{w} = 1 * 1 + 3 * 1 = 4$$

$$\|\mathbf{v}\|^2 = 1^2 + 3^2 = 10$$

$$\mathbf{p} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{4}{10} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 1.2 \end{pmatrix}$$

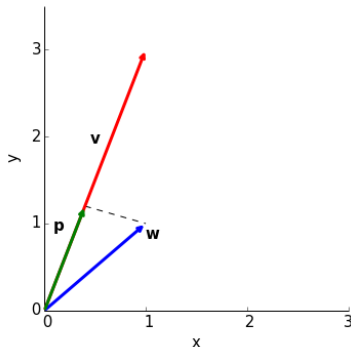


Figure:

Vectors: determinants and the cross-product

So far we've encountered

- vector addition
- scalar-product
- the inner-product closely related to the length of a vector and used to calculate the angle between vectors.

In what follows, we'll discuss another property of vector calculus:
the outer product

Previously, we've encountered the inner product of two vectors that generated a scalar. In what follows we define a product of vectors that is a vector. Given two vectors \mathbf{v} and \mathbf{w} , we can obtain a third vector $\mathbf{v} \times \mathbf{w}$, the cross-product of \mathbf{v} and \mathbf{w} . This vector will have the (pleasing) geometrical property that is it perpendicular to the plane spanned by \mathbf{v} and \mathbf{w} .

First, let's define a 2×2 matrix \mathbf{M} as

$$\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the determinant is then defined as:

$$|\mathbf{M}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

or, with a 3×3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or, with a 3×3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

the determinant is then defined as:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

or, with a 3×3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

This looks like a lot of work to get another vector, but there is some history and application to it:

- Linear equations and Cramer's rule (remember)
- 2D: The area of the parallelogram defined by two vectors
- 3D: The volume of a parallelepiped

Cramer's law to linear equations (1750!)

Say we have a system of equations:

$$\begin{aligned}2x + y + z &= 3 \\ x - y - z &= 0 \\ x + 2y + z &= 0\end{aligned}$$

Or rewritten like $Ax = b$,

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

This system can be solved (i.e., finding the values of x, y, z for which all three equations make sense) by using *determinants*.

Cramer's law : step 1

First find the determinant of the equations:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix} =$$
$$2 \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = \dots$$

Cramer's law : step 2

The next step is to plug the answer vector (the vector describing the results) into the three separate determinants – these are going to be used to determine the solution to the system of equations.
So,

$$\Delta_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 3$$

$$\Delta_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -6$$

$$\Delta_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 9$$

Cramer's law : step 3

The final step to the solution is Cramer's rule:

$$x = \frac{\Delta_x}{\Delta} = \frac{3}{3} = 1$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-6}{3} = -2$$

$$z = \frac{\Delta_z}{\Delta} = \frac{9}{3} = 3$$

Exercise 1: check whether the above solution is a valid one.

Exercise 2: check whether this method works for the following

$$\begin{array}{rcl} 2x + y + z & = & 3 \\ x + y/2 + z/2 & = & 3/2 \\ x + 2y + z & = & 0 \end{array}$$

Geometry of 2x2 determinants

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ be two vectors in the (i,j) plane. Then the cross-product can be described as a determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

This is a very interesting property: it indicates that the area $||\mathbf{a} \times \mathbf{b}||$ equals the absolute value of the determinant.

$$||\mathbf{a} \times \mathbf{b}|| = |a_1b_2 - a_2b_1|$$

Geometry of 2x2 determinants

It can also be determined by determining

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$$

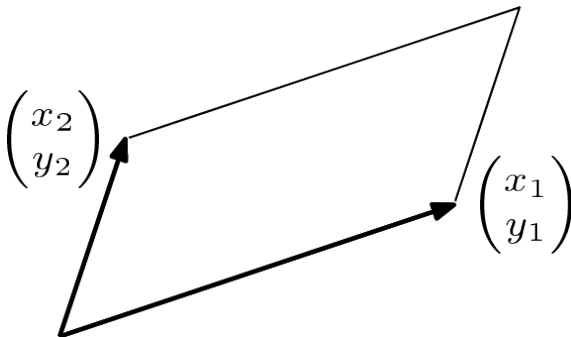


Figure:

Geometry of 3x3 determinants

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ be three vectors in the (i, j, k) plane. Then the absolute value of the determinant D is the volume of the parallelepiped.

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

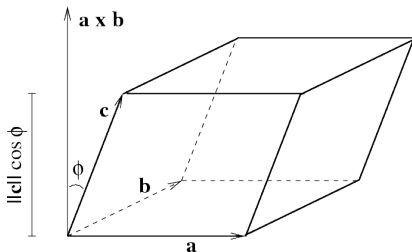


Figure:

summary

Today we've encountered vectors and various related properties:

scalar multiplication

vector addition

inner product $\mathbf{a} \cdot \mathbf{b}$

outer product $\mathbf{a} \times \mathbf{b}$

Next time, we'll cover some more applications

bonus infinite sum

Q

What is the sum of all natural numbers $1 + 2 + 3 \dots \text{inf}$?

$$\sum_n^{\text{inf}} n = -\frac{1}{12}$$

Questions