Vector calculus and applications, part II

Cris Lanting

University Medical Center Groningen
Dept. of Otorhinolaryngology

c.p.lanting@umcg.nl
https://github.com/Crisly/MathcourseBCN

October 8, 2014

Overview

Take home messages from last session

Last session we've seen the following and we ...

- Know what vectors are and how to use them
- Know how to add and subtract vectors
- Know and use the dot or inner product

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 + \ldots + v_n w_n = \sum_i^n v_i w_i$$

Given a vector

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

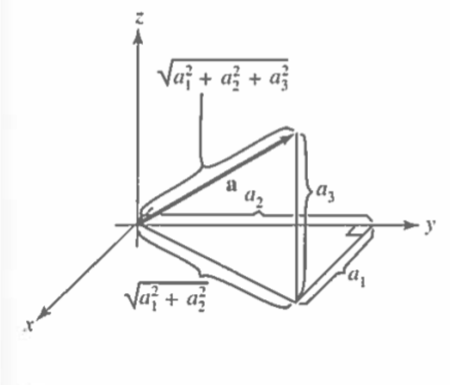
The length –or norm– of a vector a is equal to

$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Since $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2$, it follows that

$$||\mathbf{a}|| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = (\mathbf{a} \cdot \mathbf{a})^{1/2}$$

The norm of a vector and the inner product are thus related.



Orthogonal projection

The orthogonal projection **p** of **w** on **v** is the vector whose tip is obtained by dropping a perpendicular line to the line l (along v) from the top of **w**. We can obtain the projection as:

$$\mathbf{p} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2}\mathbf{v}$$

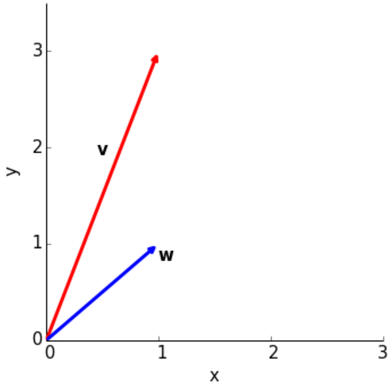


Figure:

So, $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

and $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\mathbf{v} \cdot \mathbf{w} = 1 * 1 + 3 * 1 = 4$$

$$\|\mathbf{v}\|^2 = 1^2 + 3^2 = 10$$

$$\mathbf{p} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{4}{10}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 1.2 \end{pmatrix}$$

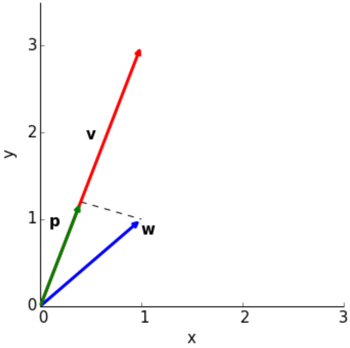


Figure:

Orthogonal vectors

Two vectors **a** and **b** are orthogonal iff their inner product is equal to zero.

$$\mathbf{a} \cdot \mathbf{b} = 0$$

Then, the angle between **a** and **b** is equal to

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = 0$$

from which we can see (remember trigonometry?!) that the angle is equal to 90°.
During this session we're going see how esp. the inner product is used in applications: correlation, covariance, lineair (in)dependence, orthogonality , PCA and SVM.

Correlation and the inner product

Correlation is a measurement of linearity between random variables x and y. Traditionally, the correlation is defined as the covariance of the two variables dived by product of their standard deviations:

$$\rho(x,y) = \frac{cov(x,y)}{\sigma_x \sigma_y}$$

For limited (and zero-mean) samples, this unfolds as the correlation coefficient:

$$r(x,y) = \frac{\frac{1}{n-1}\sum_i^n (x_i y_i)}{\sqrt{\frac{1}{n-1}\sum_i^n x_i^2}\sqrt{\frac{1}{n-1}\sum_i^n y_i^2}}$$

This looks familiar (apart from scaling $\frac{1}{n-1}$):

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \sum_i^n x_i y_i$$

Correlation and the inner product

So, it turns out that (for zero mean variables) the covariance is equal to the inner-product:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \sum_i^n x_i y_i = covar(x, y)$$

Something similar is happening to the standard deviations:

$$||x|| = \sqrt{x_{2_{1+x_2^2 \ldots}} = \sqrt{\sum_i x_i^2}}$$

So

$$\sqrt{\sum_i x_i^2} \sqrt{\sum_i y_i^2} = ||x||||y||$$

Correlation and the inner product

by combining the info so far we get:

$$r(x,y) = \frac{\frac{1}{n-1}\sum_i x_i y_i}{\sqrt{\frac{1}{n-1}\sum_i x_i^2}\sqrt{\sum_i y_i^2}} = \frac{\mathbf{x}\cdot\mathbf{y}}{||x||||y||} = \cos\theta$$

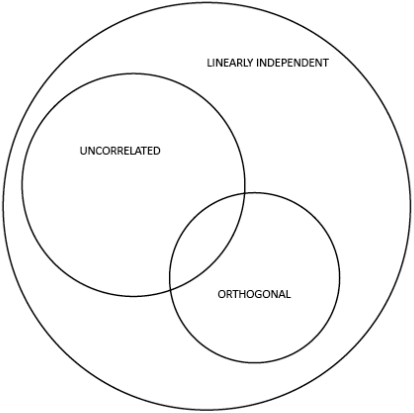
The correlation coefficient can thus be calculated by taking the inner-product of two vectors and by dividing this by their norms. A graphical interpretation is the angle between them: the closer the two (n-dimensional) vectors, the more correlated they are.
Interactive example (iPython notebook)

So far we've seen terms like orthogonality and correlation. They are related, but how?

Linear independence

Uncorrelated

Orthogonal



Linear independence

Two variables are linearly independent iff there is no constant a such that

$$a\mathbf{x} - \mathbf{y} = 0$$

Orthogonal

Two variables are linearly orthogonal iff their inner product equals zero

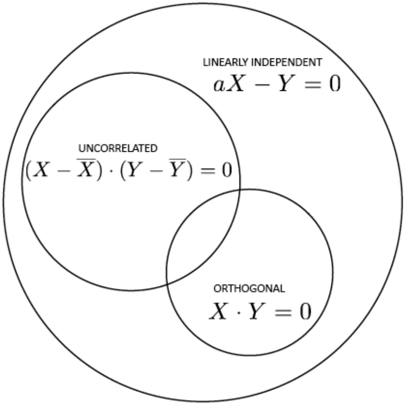
$$\mathbf{x} \cdot \mathbf{y} = 0$$

Uncorrelated

Two variables are uncorrelated iff their demeaned inner product equals zero

$$(\mathbf{x} - \overline{x}) \cdot (\mathbf{y} - \overline{y}) = 0$$

Linear independence, orthogonality and correlation



Linear independence, orthogonality and correlation

Thus, orthogonal denotes that the raw variables are perpendicular whereas uncorrelated denotes that the centred (=demeaned) variables are perpendicular! See e.g. Rodgers (1984) https://www.psych.umn.edu/faculty/waller/classes/FA2010/Readings/rodgers.pdf examples:

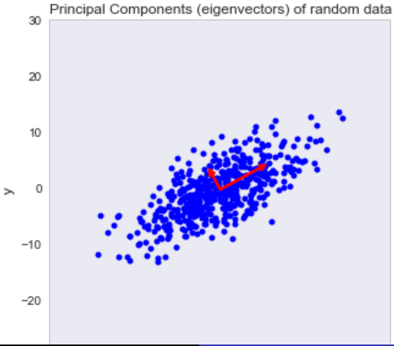
- $x = [1, 1, 2, 3]$ and $y = [2, 3, 4, 5]$?
- $x = [0, 0, 1, 1]$ and $y = [1, 0, 1, 0]$?
- $x = [1, -5, 3, -1]$ and $y = [5, 1, 1, 3]$?
- $x = [-1, -1, 1, 1]$ and $y = [1, -1, 1, -1]$?
- $x = [1, 2, 3, 4]$ and $y = [3, 6, 9, 12]$?

examples:

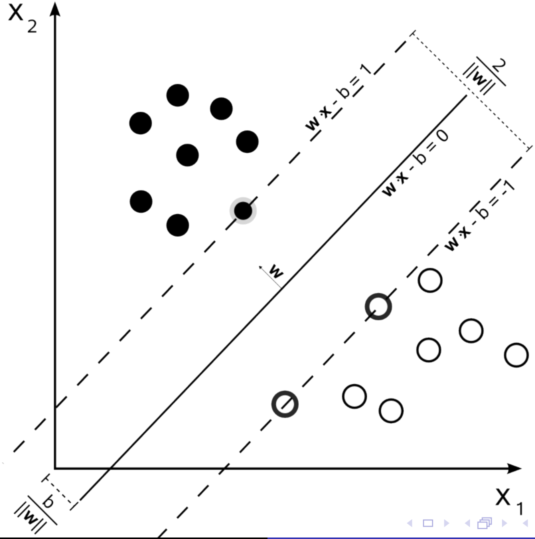
- $x = [1, 1, 2, 3]$ and $y = [2, 3, 4, 5]$: independent, correlated and not orthogonal
- $x = [0, 0, 1, 1]$ and $y = [1, 0, 1, 0]$: independent, uncorrelated and not orthogonal
- $x = [1, -5, 3, -1]$ and $y = [5, 1, 1, 3]$: independent, correlated and orthogonal
- $x = [-1, -1, 1, 1]$ and $y = [1, -1, 1, -1]$: independent, not correlated and orthogonal
- $x = [1, 2, 3, 4]$ and $y = [3, 6, 9, 12]$: not independent, correlated and not orthogonal

PCA, eigenvectors and eigenvalues

PCA often boils down to rotation of data, such that the variance (std) is maximal along the first axis. Eigenvectors (principle components) are linear combinations of the original axes, such that variance is maximised and the corresponding eigenvalues are equal to the variances of the data along the eigenvectors.



Support vector machines



Geometry of the cross product

The cross-product is defined as

$$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||||\mathbf{b}|| \sin \theta$$

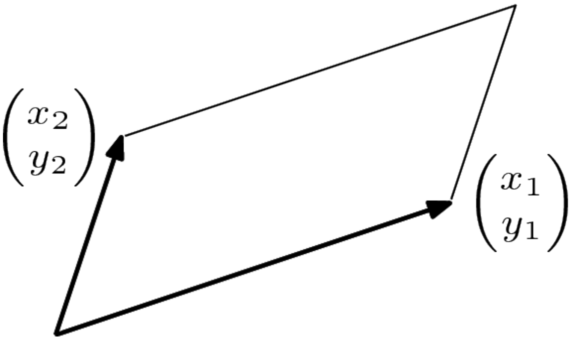


Figure:

Geometry of the cross-product in 3D

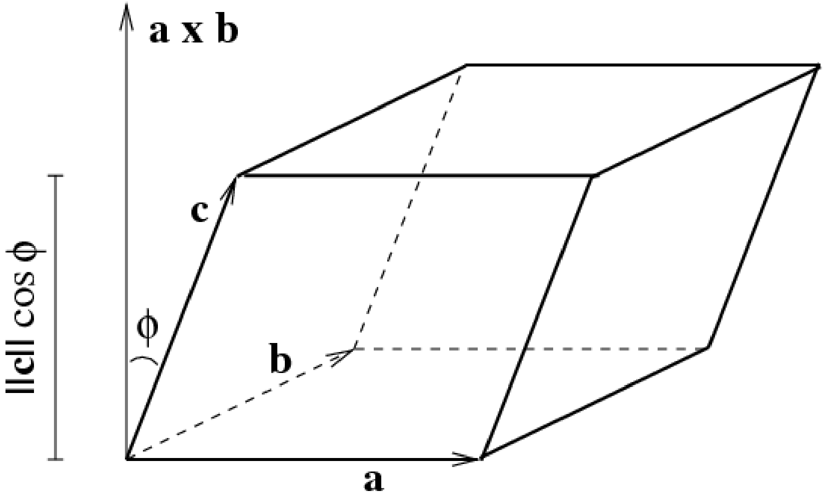


Figure:

summary

Today we've encountered vectors and various related properties:

- scalar multiplication
- vector addition
- inner product $\mathbf{a} \cdot \mathbf{b}$
- outer product $\mathbf{a} \times \mathbf{b}$

bonus infinite sum

Q

What is the sum of all natural numbers $1 + 2 + 3 \ldots \inf$?

$$\sum_{n}^{\inf} n = -\frac{1}{12}$$

Questions