第一篇　　极限论

第一部分　　极限初论

第一章　　变量与函数

§1.　　函数的概念

1. 解下列不等式，并画出x的范围：

(1) $-2 < \dfrac{1}{x+2}$

(2) $(x-1)(x+2)(x-3) < 0$

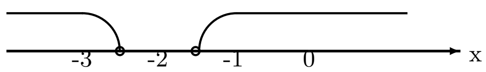
(3) $\dfrac{1}{x-1} < a$

(4) $0 \leqslant \cos x \leqslant \dfrac{1}{2}$

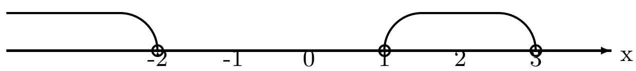
(5) $\begin{cases} x^2 - 16 < & 0 \\ x^2 - 2x \geqslant & 0 \end{cases}$

解：

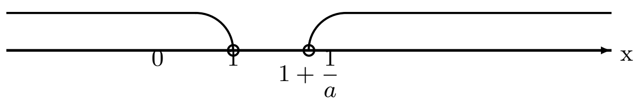
(1) $x < -\dfrac{5}{2}$或$x > -\dfrac{3}{2}$



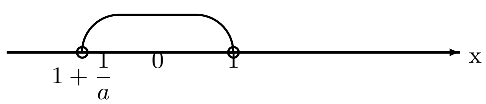
(2) $1 < x < 3$或$x < -2$



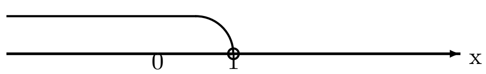
(3) 当$a > 0$时，$x < 1$或$x > 1 + \dfrac{1}{a}$；



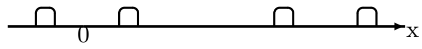
当$a < 0$时，$1 + \dfrac{1}{a} < x < 1$



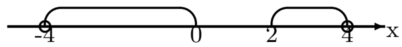
当$a = 0$时，$x < 1$



(4) $2k\pi+\frac{\pi}{3}\leqslant x\leqslant 2k\pi+\frac{\pi}{2}$或$2k\pi-\frac{\pi}{2}\leqslant x\leqslant 2k\pi-\frac{\pi}{3}(k\in Z)$



(5) $-4<x\leqslant 0$或$2\leqslant x<4$



2. 证明下列绝对值不等式：

(1) $|x-y|\geqslant ||x|-|y||$

(2) $|x_1+x_2+x_3+\cdots+x_n|\leqslant |x_1|+|x_2|+\cdots+|x_n|$

(3) $|x+x_1+\cdots+x_n|\geqslant |x|-(|x_1|+\cdots+|x_n|)$

证明：

(1) 因$|x||y|\geqslant xy$，则$(x-y)^2\geqslant (|x|-|y|)^2$，于是$|x-y|\geqslant ||x|-|y||$

(2) 用数学归纳法证明.

(i) 当$n=2$时，由$|x_1+x_2|\leqslant |x_1|+|x_2|$，得结论成立.

(ii) 假设当$n=k$时结论成立，即有$|x_1+x_2+x_3+\cdots+x_k|\leqslant |x_1|+|x_2|+\cdots+|x_k|$.
则当$n=k+1$时，$|x_1+x_2+x_3+\cdots+x_{k+1}|\leqslant |x_1+x_2+x_3+\cdots+x_k|+|x_{k+1}|\leqslant |x_1|+|x_2|+\cdots+|x_k|+|x_{k+1}|$
综上可知，对一切自然数n，$|x_1+x_2+x_3+\cdots+x_n|\leqslant |x_1|+|x_2|+\cdots+|x_n|$均成立.

(3) $|x+x_1+\cdots+x_n|\geqslant |x|-|x_1+x_2+x_3+\cdots+x_n|\geqslant |x|-(|x_1|+\cdots+|x_n|)$

3. 解下列绝对值不等式，并画出x的范围：

(1) $|x|>|x+1|$

(2) $2<\frac{1}{|x|}<4$

(3) $|x|>A$

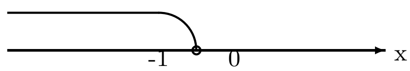
(4) $|x-a|<\eta,\eta$为常数，$\eta>0$

(5) $\left|\frac{x-2}{x+1}\right|>\frac{x-2}{x+1}$

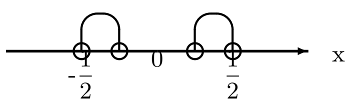
(6) $2<\frac{1}{|x+2|}<3$

解：

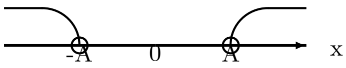
(1) $x<-\frac{1}{2}$



(2) $-\frac{1}{2}<x<-\frac{1}{4}$或$\frac{1}{4}<x<\frac{1}{2}$

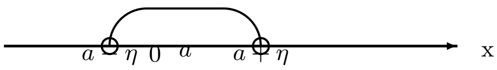


(3) 当$A \geqslant 0$时，$x < -A$或$x > A$

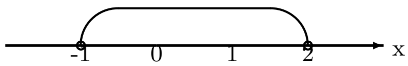


当$A < 0$时，$x \in R$

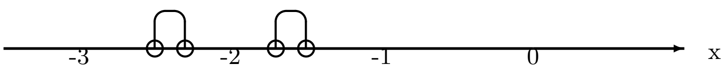
(4) $a - \eta < x < a + \eta$



(5) 原式等价于$\dfrac{x-2}{x+1} < 0$，则$-1 < x < 2$



(6) $-\dfrac{5}{3} < x < -\dfrac{3}{2}$或$-\dfrac{5}{2} < x < -\dfrac{7}{3}$



4. 求下列函数的定义域及它在给定点上的函数值：

(1) $y = f(x) = -x + \dfrac{1}{x}$的定义域及$f(-1), f(1)$和$f(2)$；

(2) $y = f(x) = \sqrt{a^2 - x^2}$的定义域及$f(0), f(a)$和$f\left(-\dfrac{a}{2}\right)$；

(3) $s = s(t) = \dfrac{1}{t}e^{-t}$的定义域及$s(1), s(2)$；

(4) $y = g(\alpha) = \alpha^2 \tan\alpha$的定义域及$g(0), g\left(\dfrac{\pi}{4}\right), g\left(-\dfrac{\pi}{4}\right)$；

(5) $x = x(\theta) = \sin\theta + \cos\theta$的定义域及$x\left(-\dfrac{\pi}{2}\right), x(-\pi)$

(6) $y = f(x) = \dfrac{1}{(x-1)(x+2)}$的定义域及$f(0), f(-1)$

解：

(1) 函数的定义域为$X=(-\infty,0)\bigcup(0,\infty)$，$f(-1)=0,f(1)=0,f(2)=-\dfrac{3}{2}$

(2) 函数的定义域为$X=[-|a|,|a|]$，$f(0)=|a|,f(a)=0,f\left(-\dfrac{a}{2}\right)=\dfrac{\sqrt{3}}{2}|a|$

(3) 函数的定义域为$(-\infty,0)\bigcup(0,\infty)$，$s(1)=\dfrac{1}{e},s(2)=\dfrac{1}{2e^2}$

(4) 函数的定义域为$\left\{x\,\middle|\,x\in R,x\neq k\pi+\dfrac{\pi}{2},k\in Z\right\}$，$g(0)=0,g\left(\dfrac{\pi}{4}\right)=\dfrac{\pi^2}{16},g\left(-\dfrac{\pi}{4}\right)=-\dfrac{\pi^2}{16}$

(5) 函数的定义域为$X=(-\infty,\infty)$，$x\left(-\dfrac{\pi}{2}\right)=-1,x(-\pi)=-1$

(6) 函数的定义域为$X=(-\infty,-2)\bigcup(-2,1)\bigcup(1,+\infty)$，$f(0)=-\dfrac{1}{2},f(-1)=-\dfrac{1}{2}$

5. 求下列函数的定义域及值域：

(1) $y=\sqrt{2+x-x^2}$

(2) $y=\sqrt{\cos x}$

(3) $y=\ln\left(\sin\dfrac{\pi}{x}\right)$

(4) $y=\dfrac{1}{\sin\pi x}$

解：

(1) 函数的定义域为$X=[-1,2]$，值域为$\left[0,\dfrac{3}{2}\right]$

(2) 函数的定义域为$\left[2k\pi-\dfrac{\pi}{2},2k\pi+\dfrac{\pi}{2}\right]$ $(k\in Z)$，值域为$[0,1]$

(3) 函数的定义域为$\left(\dfrac{1}{2k+1},\dfrac{1}{2k}\right)$ $(k\in Z)$，值域为$(-\infty,0]$

(4) 函数的定义域为$(n-1,n)(n=0,\pm1,\pm2,\cdots)$，值域为$(-\infty,-1]\bigcup[1,+\infty)$

6. 设$f(x)=x+1,\varphi(x)=x-2$，试解方程$|f(x)+\varphi(x)|=|f(x)+|\varphi(x)|$
解：由已知，得$f(x)\varphi(x)\geqslant 0$即$(x+1)(x-2)\geqslant 0$，则$x\geqslant 2$或$x\leqslant -1$.

7. 设$f(x)=(|x|+x)(1-x)$，求满足下列各式的x值：

(1) $f(0)=0$

(2) $f(x)<0$

解：

(1) 要$f(x)=0$，则$|x|+x=0$或$1-x=0$，即$x\leqslant 0$或$x=1$

(2) 因$|x|+x\geqslant 0$，则要$f(x)<0$，只要$1-x<0$即可，即$x>1$

8. 图1-5表示电池组V、固定电阻R_0和可变电阻R组成的电路.在一段不长的时间内，A,B两点间的电压V可以看成一个常量.求出电流I和可变电阻R的函数式.
解：由已知及物理学知识，得$V=I(R_0+R)$.

9. 在一个圆柱形容器内倒进某种溶液，该圆柱形容器的底半径是a，高为h，倒进溶液的高度是x（图1-6）. 该溶液的容积V和x之间的函数关系$V=V(x)$，并写出它的定义域和值域.
解：由已知，得$V=\pi a^2x$，它的定义域为$[0,h]$，值域为$[1,\pi a^2h]$

10. 某灌溉渠的截面积是一个梯形，如图1-7，底宽2米，斜边的倾角为45^o，CD表示水面，求截面$ABCD$的面积S与水深h的函数关系.
解：由已知及图，得$S=h(h+2)$.

11. 有一深为H的矿井，如用半径为R的卷扬机以每秒钟ω弧度的角速度从矿井内起吊重物，求重物底面与地面的距离s和时间t的函数关系（图1-8）.
解：由已知及图，得$s=H-\omega Rt\left(t\in\left[0,\dfrac{H}{\omega t}\right]\right)$

12. 设$y=f(x)=\begin{cases}1+x^2, & x<0\\ x-1, & x\geqslant 0\end{cases}$，求$f(-2),f(-1),f(0),f(1)$和$f\left(\dfrac{1}{2}\right)$.
解：由已知，得$f(-2)=5,f(-1)=2,f(0)=-1,f(1)=0,f\left(\dfrac{1}{2}\right)=-\dfrac{1}{2}$.

13. 设$x(t)=\begin{cases}0, & 0\leqslant t<10\\ 1+t^2, & 10\leqslant t\leqslant 20\\ t-10, & 20<t\leqslant 30\end{cases}$，求$x(0),x(5),x(10),x(15),x(20),x(25),x(30)$，并画出这个函数的图形.

解：由已知，得$x(0)=0,x(5)=0,x(10)=101,x(15)=226,x(20)=401,x(25)=15,x(30)=20$

14. 邮资y是信件重量x的函数.按照邮局的规定，对于国内的外埠平信，按信件重量，每重20克应付邮资8分，不足20克者以20克计算.当信件的重量在60克以内时，试写出这个函数的表达式，并画出它的图形.

解：由已知，得$y=f(x)=\begin{cases}8, & 0<x\leqslant 20\\ 16, & 20<x\leqslant 40\\ 24, & 40<x\leqslant 60\end{cases}$

15. 脉冲发生器产生一个三角波，其波形如图1-9，写出函数关系$u=u(t)(0\leqslant t\leqslant 20)$.

解：由已知及图，得$u=u(t)=\begin{cases}1.5t, & 0\leqslant t\leqslant 10\\ 30-1.5t, & 10<t\leqslant 20\end{cases}$

16. 下列函数f和φ是否相等，为什么？

(1) $f(x)=\dfrac{x}{x},\varphi(x)=1$

(2) $f(x)=x,\varphi(x)=\sqrt{x^2}$

(3) $f(x)=1,\varphi(x)=\sin^2 x+\cos^2 x$

解：

(1) 因f的定义域为$(-\infty,0)\bigcup(0,+\infty)$，$\varphi$的定义域为$(-\infty,+\infty)$，故这两个函数不相等.

(2) 因$f(x)=x,\varphi(x)=|x|$，故这两个函数的函数表达式不一样，则这两个函数不相等.

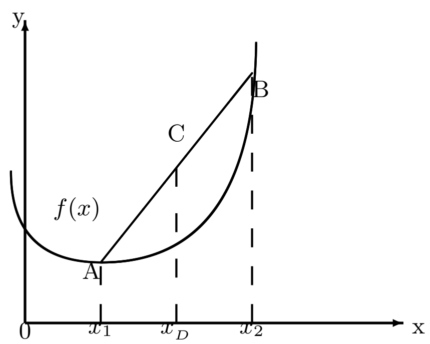
(3) 因$\varphi(x)=\sin^2 x+\cos^2 x=1$恒成立，故这两个函数相等.

17. 证明对于直线函数$f(x)=ax+b$，若自变数值$x=x_n(n=1,2,\cdots)$组成一等差数列，则对应的函数值$y_n=f(x_n)(n=1,2,\cdots)$也组成一等差数列.

证明：设x_{m-1},x_m,x_{m+1}是x_n中任意3个相邻的数$(2\leqslant m\leqslant n)$
据题意，得$2x_m=x_{m-1}+x_{m+1}$
又$y_n=f(x_n)=ax_n+b$，则$y_{m-1}=ax_{m-1}+b,y_m=ax_m+b,y_{m+1}=ax_{m+1}+b$，于是$2y_m=2ax_m+2b,y_{m+1}+y_{m-1}=ax_{m+1}+b+ax_{m-1}+b=2ax_m+2b$，从而$2y_m=y_{m-1}+y_{m+1}$
又x_{m-1},x_m,x_{m+1}是x_n中任意3个相邻的数，则y_{m-1},y_m,y_{m+1}是y_n中任意3个相邻的数，于是$y_n=f(x_n)(n=1,2,\cdots)$也组成一等差数列.

18. 如果曲线$y=f(x)$上的任一条弦都高于它所限的弧（图1-10），证明不等式$\dfrac{f(x_1)+f(x_2)}{2}>f\left(\dfrac{x_1+x_2}{2}\right)$对于所有的$x_1,x_2(x_1\neq x_2)$成立（凡具有上述特性的函数叫做凸函数）.

证明：在曲线上任取两点$A(x_1,f(x_1)),B(x_2,f(x_2))$，连接$AB$，取其中点$C(x_C,y_C)$，则$f(x_1)+f(x_2)=2y_C,x_1+x_2=2x_C$
又曲线上$x_D=\dfrac{x_1+x_2}{2}$所对点的纵坐标为$y_D=f\left(\dfrac{x_1+x_2}{2}\right)$，则$x_C=x_D$
又曲线$y=f(x)$上的任一条弦都高于它所限的弧且x_1,x_2为弦与弧的交点，则$y_C>y_D$即$\dfrac{f(x_1)+f(x_2)}{2}>f\left(\dfrac{x_1+x_2}{2}\right)$对于所有的$x_1,x_2(x_1\neq x_2)$成立.



19. 证明下列各函数在所示区间内是单调增加的函数：

(1) $y=x^2(0\leqslant x<+\infty)$

(2) $y=\sin x\left(-\dfrac{\pi}{2}\leqslant x\leqslant\dfrac{\pi}{2}\right)$

证明：

(1) 设$0 \leqslant x_1 < x_2$
则$y_2 - y_1 = x_2^2 - x_1^2 = (x_2 + x_1)(x_2 - x_1) > 0$，于是函数$y = x^2$当$0 \leqslant x$时严格单调增加.

(2) 设$-\dfrac{\pi}{2} \leqslant x_1 < x_2 \leqslant \dfrac{\pi}{2}$
则$y_2 - y_1 = \sin x_2 - \sin x_1 = 2\cos\dfrac{x_2 + x_1}{2}\sin\dfrac{x_2 - x_1}{2}$
又$-\dfrac{\pi}{2} \leqslant x_1 < x_2 \leqslant \dfrac{\pi}{2}$，则$-\dfrac{\pi}{2} < \dfrac{x_1 + x_2}{2} < \dfrac{\pi}{2}, 0 < \dfrac{x_2}{x_1}2 \leqslant \dfrac{\pi}{2}$，于是$\cos\dfrac{x_1 + x_2}{2} > 0, \sin\dfrac{x_2 - x_1}{2} > 0$，从而$y_2 - y_1 > 0$即函数$y = \sin x$当$-\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$时严格单调增加.

20. 证明下列函数在所示区间内是单调减少的函数：

(1) $y = x^2(-\infty < x \leqslant 0)$

(2) $y = \cos x(0 \leqslant x \leqslant \pi)$

证明:

(1) 设$0 \leqslant x_1 < x_2$
则$y_2 - y_1 = x_2^2 - x_1^2 = (x_2 + x_1)(x_2 - x_1) < 0$，于是函数$y = x^2$当$x \leqslant 0$时严格单调减少.

(2) 设$0 \leqslant x_1 < x_2 \leqslant \pi$
则$y_2 - y_1 = \cos x_2 - \cos x_1 = -2\sin\dfrac{x_2 + x_1}{2}\sin\dfrac{x_2 - x_1}{2}$
又$0 \leqslant x_1 < x_2 \leqslant \pi$，则$0 < \dfrac{x_1 + x_2}{2} < \pi, 0 < \dfrac{x_2}{x_1}2 \leqslant \dfrac{\pi}{2}$，于是$\sin\dfrac{x_1 + x_2}{2} > 0, \sin\dfrac{x_2 - x_1}{2} > 0$，从而$y_2 - y_1 < 0$即函数$y = \cos x$当$0 \leqslant x \leqslant \pi$时严格单调减少.

21. 讨论下列函数的奇偶性：

(1) $y = x + x^2 - x^5$

(2) $y = a + b\cos x$

(3) $y = x + \sin x + e^x$

(4) $y = x\sin\dfrac{1}{x}$

(5) $y = sgnx = \begin{cases} 1, & \text{当}x > 0\text{时} \\ 0, & \text{当}x = 0\text{时} \\ -1 & \text{当}x < 0\text{时} \end{cases}$

(6) $y = \begin{cases} \dfrac{2}{x^2}, & \text{当}\dfrac{1}{2} < x < +\infty\text{时} \\ \sin x^2, & \text{当} -\dfrac{1}{2} \leqslant x \leqslant \dfrac{1}{2}\text{时} \\ \dfrac{1}{2}x^2, & \text{当} -\infty < x < -\dfrac{1}{2}\text{时} \end{cases}$

解:

(1) 因$y = f(x) = x + x^2 - x^5$，则$f(-x) = -x + x^2 + x^5$，故$f(-x) \neq f(x), f(-x) \neq -f(x)$，于是此函数是非奇非偶函数.

(2) 因$y = f(x) = a + b\cos x$，则$f(-x) = a + b\cos(-x) = a + b\cos x = f(x)$，于是此函数是偶函数.

(3) 因$y = f(x) = x + \sin x + e^x$，则$f(-x) = -x - \sin x + e^{-x}$，故$f(-x) \neq f(x), f(-x) \neq -f(x)$，于是此函数是非奇非偶函数.

(4) 因$y = f(x) = x\sin\dfrac{1}{x}$，则$f(-x) = -x\sin\dfrac{1}{-x} = x\sin\dfrac{1}{x} = f(x)$，于是此函数是偶函数.

(5) 因$y = f(x) = \begin{cases} 1, & \text{当}x > 0\text{时} \\ 0, & \text{当}x = 0\text{时} \\ -1 & \text{当}x < 0\text{时} \end{cases}$，
则$f(-x) = \begin{cases} 1, & \text{当} -x > 0\text{时} \\ 0, & \text{当} -x = 0\text{时} \\ -1 & \text{当} -x < 0\text{时} \end{cases} = \begin{cases} -1, & \text{当}x > 0\text{时} \\ 0, & \text{当}x = 0\text{时} \\ 1 & \text{当}x < 0\text{时} \end{cases} = -f(x)$，于是此函数是奇函数.

(6) 因$y=f(x)=\begin{cases}\dfrac{2}{x^2}, & 当\dfrac{1}{2}<x<+\infty时\\ \sin x^2, & 当-\dfrac{1}{2}\leqslant x\leqslant\dfrac{1}{2}时\\ \dfrac{1}{2}x^2, & 当-\infty<x<-\dfrac{1}{2}时\end{cases}$,

则$f(-x)=\begin{cases}\dfrac{2}{(-x)^2}, & 当\dfrac{1}{2}<-x<+\infty时\\ \sin(-x)^2, & 当-\dfrac{1}{2}\leqslant -x\leqslant\dfrac{1}{2}时\\ \dfrac{1}{2}(-x)^2, & 当-\infty<-x<-\dfrac{1}{2}时\end{cases}=\begin{cases}\dfrac{1}{2}x^2, & 当\dfrac{1}{2}<x<+\infty时\\ \sin x^2, & 当-\dfrac{1}{2}\leqslant x\leqslant\dfrac{1}{2}时\\ \dfrac{2}{x^2}, & 当-\infty<x<-\dfrac{1}{2}时\end{cases}$,

故$f(-x)\neq f(x), f(-x)\neq -f(x)$，于是此函数是非奇非偶函数.

22. 试证两个偶函数的乘积是偶函数，两个奇函数的乘积是奇函数，一个奇函数与一个偶函数的乘积是奇函数.

证明: 设$f_1(x), f_2(x)$为定义在$(-a,a)(a>0)$内的偶函数，$g_1(x), g_2(x)$为定义在$(-a,a)(a>0)$内的奇函数，$F_1(x)=f_1(x)f_2(x), F_2(x)=g_1(x)g_2(x), F_3(x)=f_1(x)f_2(x)$

则$f_1(-x)=f_1(x), f_2(-x)=f_2(x), g_1(x)=-g_1(x), g_2(-x)=-g_2(x)$，于是

$$F_1(-x)=f_1(-x)f_2(-x)=f_1(x)f_2(x)=F_1(x)$$
$$F_2(-x)=g_1(-x)g_2(-x)=(-g_1(x))(-g_2(x))=g_1(x)g_2(x)=F_2(x)$$
$$F_3(-x)=f_1(-x)g_1(-x)=f_1(x)(-g_1(x))=-f_1(x)g_1(x)=-F_3(x)$$

从而$F_1(x)$是偶函数；$F_2(x)$是偶函数；$F_3(x)$是奇函数.

23. 设$f(x)$为定义在$(-\infty,+\infty)$内的任何函数，证明$F_1(x)\equiv f(x)+f(-x)$是偶函数，$F_2(x)\equiv f(x)-f(-x)$是奇函数.写出对应于下列函数的$F_1(x), F_2(x)$：

(1) $y=a^x$

(2) $y=(1+x)^n$

证明: 因$F_1(-x)=f(-x)+f(x)=F_1(x)$，则$F_1(x)=f(x)+f(-x)$是偶函数

又$F_2(-x)=f(-x)-f(x)=-F_2(x)$，则$F_2(x)=f(x)-f(-x)$是奇函数.

(1) $F_1(x)=f(x)+f(-x)=a^x+a^{-x}, F_2(x)=f(x)-f(-x)=a^x-a^{-x}$

(2) $F_1(x)=f(x)+f(-x)=(1+x)^n+(1-x)^n, F_2(x)=f(x)-f(-x)=(1+x)^n-(1-x)^n$

24. 说明下列函数哪些是周期函数，并求最小周期：

(1) $y=\sin^2 x$

(2) $y=\sin x^2$

(3) $y=\sin x+\dfrac{1}{2}\sin 2x$

(4) $y=\cos\dfrac{\pi}{4}x$

(5) $y=|\sin x|+|\cos x|$

(6) $y=\sqrt{\tan x}$

(7) $y=x-[x]$

(8) $y=\sin n\pi x$

解:

(1) 因$y=\sin^2 x=\dfrac{1}{2}-\dfrac{1}{2}\cos 2x$，则$T=\dfrac{2\pi}{2}=\pi$

(2) 假设$y=\sin x^2$为一周期函数且$T=\omega>0$

据周期函数的定义，对任何$x\in(-\infty,+\infty)$，有$\sin(x+\omega)^2=\sin x^2$，特别对$x=0$也应该成立，则$\sin\omega^2=0$，于是$\omega^2=k\pi, \omega=\sqrt{k\pi}(k\in Z^+)$

又对$x=\sqrt{2}\omega=\sqrt{2k\pi}$也成立，故$\sin(\sqrt{2}\omega+\omega)^2=\sin\omega^2=0$，则$(\sqrt{2}+1)^2k\pi=n\pi(n\in Z^+)$，于是$(\sqrt{2}+1)^2=\dfrac{k}{n}(k,n\in Z^+)$

又$(\sqrt{2}+1)^2=3+2\sqrt{2}\in Q^-$，而$\dfrac{k}{n}\in Q^+$，则假设不成立，即函数$y=\sin x^2$不是周期函数.

(3) 因$y_1=\sin x$的$T=2\pi$；$y_2=\dfrac{1}{2}\sin 2x$的$T=\pi$，则$y=\sin x+\dfrac{1}{2}\sin 2x$的$T=2\pi$.

(4) $T=\dfrac{2\pi}{\frac{\pi}{4}}=8$

(5) 因$f(x)=|\sin x|+|\cos x|, f\left(x+\frac{\pi}{2}\right)=\left|\sin\left(x+\frac{\pi}{2}\right)\right|+\left|\cos\left(x+\frac{\pi}{2}\right)\right|=|\cos x|+|\sin x|=f(x)$
据经验，知$y=|\sin x|+|\cos x|$的$T=\frac{\pi}{2}$.

(6) 因$f(x)=\tan x$的$T=\pi$，则$y=\sqrt{\tan x}$的$T=\pi$.

(7) 因$y=x-[x]=(x)$，则$y=x-[x]$的$T=1$.

(8) $T=\frac{2\pi}{n\pi}=\frac{2}{n}$

§2. 复合函数和反函数

1. 下列函数能否构成复合函数$y=f(\varphi(x))$，如果能够构成则指出此复合函数的定义域和值域：

(1) $y=f(u)=2^u, u=\varphi(x)=x^2$

(2) $y=f(u)=\ln u, u=\varphi(x)=1-x^2$

(3) $y=f(u)=u^2+u^3, u=\varphi(x)=\begin{cases}1, & \text{当}x\text{为有理数时}\\ -1, & \text{当}x\text{为无理数时}\end{cases}$

(4) $y=f(u)=2$，定义域为U_1，$u=\varphi(x)$，定义域为X，值域为U_2

(5) $y=f(u)=\sqrt{u}, u=\varphi(x)=\cos x$

解:

(1) 因$y=f(u)=2^u$的定义域为$(-\infty,+\infty)$，$u=\varphi(x)=x^2$的值域为$[0,+\infty)$
则此函数能构成复合函数$y=2^{x^2}$，它的定义域为$(-\infty,+\infty)$，值域为$[1,+\infty)$

(2) 因$y=f(u)=\ln u$的定义域为$(0,+\infty)$，$u=\varphi(x)=1-x^2$的值域为$(-\infty,1]$
则此函数能构成复合函数$y=\ln(1-x^2)$，它的定义域为$(-1,1)$，值域为$(-\infty,0]$

(3) 因$y=f(u)=u^2+u^3$的定义域为$(-\infty,+\infty)$，
$u=\varphi(x)=\begin{cases}1, & \text{当}x\text{为有理数时}\\ -1, & \text{当}x\text{为无理数时}\end{cases}$ 的值域为$\{-1,1\}$
则此函数能构成复合函数$y=\begin{cases}2, & \text{当}x\text{为有理数时}\\ 0, & \text{当}x\text{为无理数时}\end{cases}$，它的定义域为$(-\infty,+\infty)$，值域为$\{0,2\}$

(4) 因$y=f(u)=2$的定义域为U_1，$u=\varphi(x)$的值域为U_2
当$U_1\bigcap U_2\neq\phi$时，此函数能构成复合函数$y=2$，它的定义域视具体函数而定，值域为$\{2\}$；
当$U_1\bigcap U_2=\phi$时，此函数不能构成复合函数

(5) 因$y=f(u)=\sqrt{u}$的定义域为$[0,+\infty)$，$u=\varphi(x)=\cos x$的值域为$[-1,1]$
则此函数能构成复合函数$y=\sqrt{\cos x}$，它的定义域为$\left[2k\pi-\dfrac{\pi}{2},2k\pi+\dfrac{\pi}{2}\right]$ $(k=0,\pm1,\pm2,\cdots)$，值域为$[0,1]$

2. 设$f(x)=ax^2+bx+c$，证明$f(x+3)-3f(x+2)+3f(x+1)-f(x)\equiv 0$

证明：由已知，得
$f(x+3)-3f(x+2)+3f(x+1)-f(x)=a(x+3)^2+b(x+3)+c-3[a(x+2)^2+b(x+2)+c]+3[a(x+1)^2+b(x+1)+c]-(ax^2+bx+c)=a[(x+3)^2-x^2]+b(x+3-x)-3a[(x+2)^2-(x+1)^2]-3b[x+2-(x+1)]=6ax+9a+3b-3a(2x+3)-3b\equiv 0$

3. (1) 设$y=f(x)=a+bx+\dfrac{c}{x}$，求$f\left(\dfrac{2}{x}\right)$

(2) 设$y=f(x)=x^2\ln(1+x)$，求$f(e^{-x})$

(3) 设$y=f(x)=\sqrt{1+x+x^2}$，求$f(x^2)$及$f(-x^2)$

(4) 设$y=f(t)=\dfrac{1}{\sqrt{a^2+x^2}}$，求$f(a\tan x)$

解:

(1) 因$y=f(x)=a+bx+\dfrac{c}{x}$，则$f\left(\dfrac{2}{x}\right)=a+\dfrac{2b}{x}+\dfrac{c}{\frac{2}{x}}=a+\dfrac{2b}{x}+\dfrac{cx}{2}=\dfrac{cx^2+2ax+4b}{2x}$

(2) 因$y=f(x)=x^2\ln(1+x)$，则$f(e^{-x})=(e^{-x})^2\ln(1+e^{-x})=\dfrac{\ln(e^x+1)-x}{e^{2x}}$

(3) 因$y=f(x)=\sqrt{1+x+x^2}$，则$f(x^2)=\sqrt{1+x^2+x^4}, f(-x^2)=\sqrt{1-x^2+x^4}$

(4) 因$y=f(t)=\dfrac{1}{\sqrt{a^2+x^2}}$，则$f(a\tan x)=\dfrac{1}{\sqrt{a^2+(a\tan x)^2}}=\dfrac{1}{\sqrt{a^2\sec^2 x}}=\dfrac{1}{|a\sec x|}$

4. 若$f(x)=x^2, \varphi(x)=2^x$，求$f(\varphi(x))$及$\varphi(f(x))$.

解：因$f(x)=x^2, \varphi(x)=2^x$，则$f(\varphi(x))=(2^x)^2=2^{2x}=4^x, \varphi(f(x))=2^{x^2}$

5. 若$\varphi(x)=x^3+1$，求$\varphi(x^2), (\varphi(x))^2$及$\varphi(\varphi(x))$.

解：因$\varphi(x)=x^3+1$，则
$\varphi(x^2)=(x^2)^3+1=x^6+1, (\varphi(x))^2=(x^3+1)^2=x^6+2x^3+1, \varphi(\varphi(x))=(x^3+1)^3+1=x^9+3x^6+3x^3+2$

6. 设$f(x)=\dfrac{1}{1-x}$，求$f(f(x)),f(f(f(x))),f\left(\dfrac{1}{f(x)}\right)$.

解: 因$f(x)=\dfrac{1}{1-x}$，则

$f(f(x))=\dfrac{1}{1-\dfrac{1}{1-x}}=\dfrac{x-1}{x},f(f(f(x)))=\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-x}}}=\dfrac{1}{1-\dfrac{x-1}{x}}=x,f\left(\dfrac{1}{f(x)}\right)=\dfrac{1}{1-(1-x)}=\dfrac{1}{x}$

7. 求下列函数的反函数及反函数的定义域:

(1) $y=x^2(-\infty<x\leqslant 0)$

(2) $y=\sqrt{1-x^2}(-1\leqslant x\leqslant 0)$

(3) $y=\sin x\left(\dfrac{\pi}{2}\leqslant x\leqslant\dfrac{3}{2}\pi\right)$

(4) $y=\begin{cases} x, & 当-\infty<x<1时\\ x^2, & 当1\leqslant x\leqslant 4时\\ 2^x, & 当4<x<+\infty时\end{cases}$

解:

(1) 因$y=x^2(-\infty<x\leqslant 0)$，则$x=-\sqrt{y}(0\leqslant y<+\infty)$，从而此函数的反函数为$y=-\sqrt{x}(0\leqslant y<+\infty)$

(2) 因$y=\sqrt{1-x^2}(-1\leqslant x\leqslant 0)$，则$x=-\sqrt{1-y^2}(0\leqslant y\leqslant 1)$，从而此函数的反函数为$y=-\sqrt{1-x^2}(0\leqslant x\leqslant 1)$

(3) 因$y=\sin x\left(\dfrac{\pi}{2}\leqslant x\leqslant\dfrac{3}{2}\pi\right)$，则$x=\pi-\arcsin y(-1\leqslant y\leqslant 1)$，从而此函数的反函数为$y=\pi-\arcsin x(-1\leqslant x\leqslant 1)$

(4) 因$y=\begin{cases} x, & 当-\infty<x<1时\\ x^2, & 当1\leqslant x\leqslant 4时\\ 2^x, & 当4<x<+\infty时\end{cases}$，则$x=\begin{cases} y, & 当-\infty<y<1时\\ \sqrt{y}, & 当1\leqslant y\leqslant 16时\\ \log_2 y, & 当16<x<+\infty时\end{cases}$，从而此函数的反函数为$y=\begin{cases} x, & 当-\infty<x<1时\\ \sqrt{x}, & 当1\leqslant x\leqslant 16时\\ \log_2 x, & 当16<x<+\infty时\end{cases}$.

§3.　基本初等函数

1. 把下列在$[0,1)$上定义的函数延拓到整个实轴上去，使它成为以1为周期的函数：

   (1) $y=x^2$

   (2) $y=\sin x$

   (3) $y=e^x$

   解:

   (1) 延拓后的函数为$y=(x-n)^2(n\leqslant x<n+1,n\in Z)$

   (2) 延拓后的函数为$y=\sin(x-n)(n\leqslant x<n+1,n\in Z)$

   (3) 延拓后的函数为$y=e^{x-n}(n\leqslant x<n+1,n\in Z)$

2. 把下列在$[0,+\infty)$上定义的函数延拓到整个实轴上去，(a)使它们成为奇函数；(b)使它们成为偶函数：

   (1) $y=x^2$

   (2) $y=\sin x$

   解:

   (1) 延拓后的函数为：

   (a) $f(x)=\begin{cases} x^2, & x\geqslant 0\\ -x^2, & x<0\end{cases}$

   (b) $f(x)=x^2$

   (2) 延拓后的函数为：

   (a) $f(x)=\sin x$

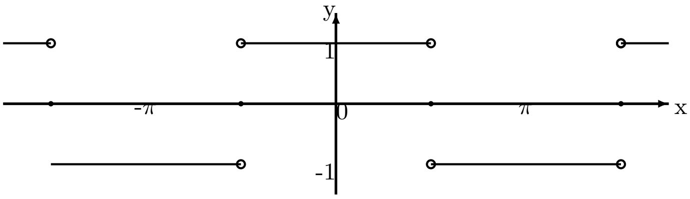
   (b) $f(x)=\sin|x|$

3. 做下列函数的图形：

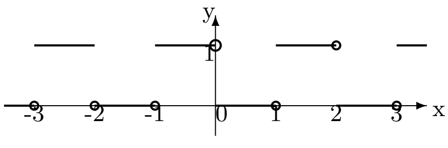
   (1) $y=sgn\cos x$

   (2) $y=[x]-2\left[\dfrac{x}{2}\right]$

   解:



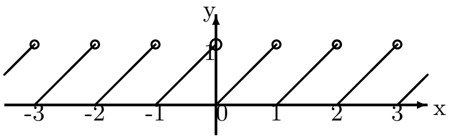
(1)



(2)

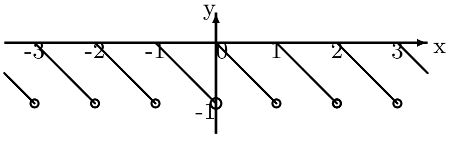
4. 作函数$y=(x)$的图形.

   解:



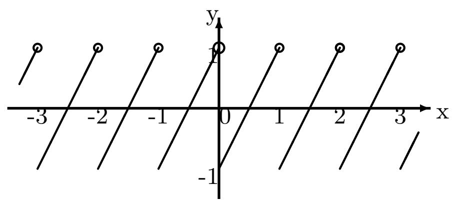
5. 作函数$y=[x]-x$的图形.

   解:



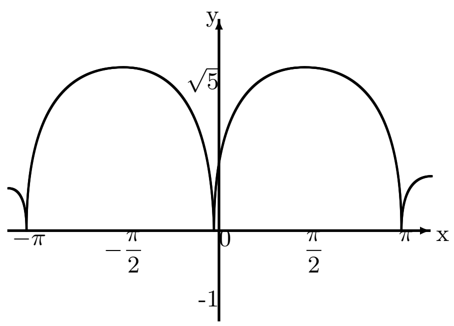
6. 一个函数是用下述方法决定的：在每一个小区间$n \leqslant x < n+1$(其中n为整数)内$f(x)$是线性的且$f(n) = -1, f\left(n+\frac{1}{2}\right) = 0$，试作此函数的图形.

解:



7. 作函数$y = |\sin x + 2\cos x|$的图形.

解:



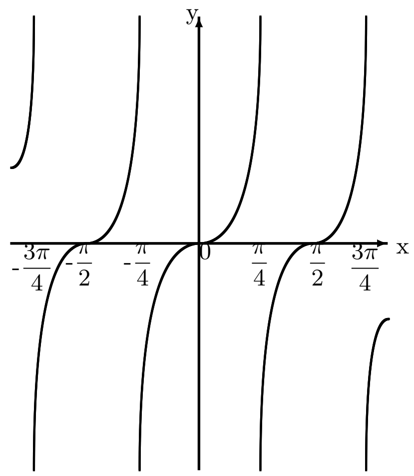
8. 若已知函数$f(x) = \tan x$，作下列函数的图形：

(1) $y = f(2x)$

(2) $y = f(kx+b)(k \neq 0)$

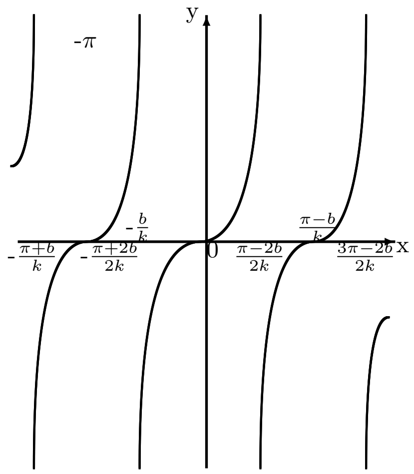
(3) $y = f\left(\frac{x}{2}\right) - 1$

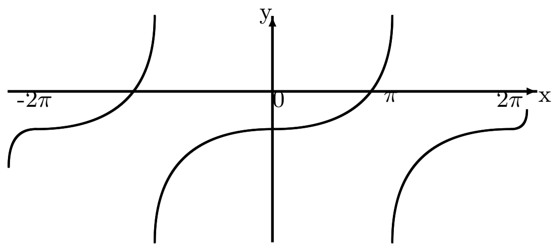
解:



(1)

(2) $(k, b > 0)$

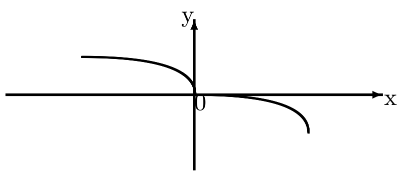




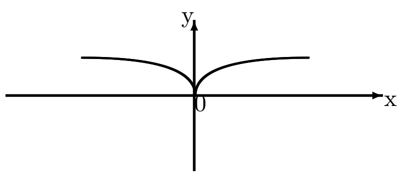
(3)

9. 若已知函数$y=f(x)$的图形，作函数$y_1=|f(x)|, y_2=f(-x), y_3=-f(-x)$的图形，并说明$y_1, y_2, y_3$的图形与$y$的图形的关系.

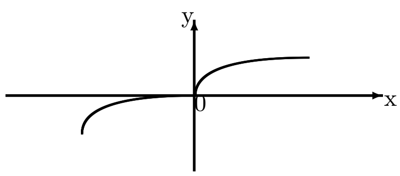
解：设$y=f(x)$的图形如下：



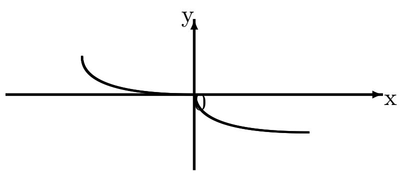
则y_1的图形为：



则y_2的图形为：



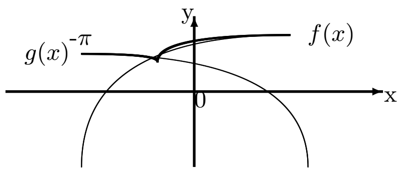
则y_3的图形为：



y_1的图形当$f(x)<0$时与y的图形关于x轴对称，当$f(x)>0$时与y的图形一样；
y_2的图形与y的图形关于y轴对称，
y_3的图形与y的图形关于原点对称，

10. 若已知$f(x), g(x)$的图形，试作函数$y=\dfrac{1}{2}\{f(x)+g(x)+|f(x)-g(x)|\}$的图形，并说明$y$的图形与$f(x), g(x)$图形的关系.

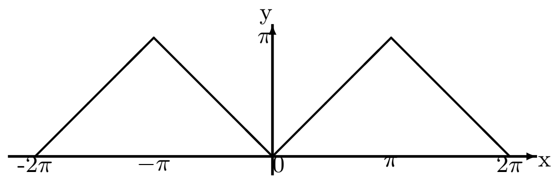
解： $y=\max\{f(x), g(x)\}$



11. 对于定义在$[0,\pi]$上的函数$y=x$，先把它延拓到$[0,2\pi]$使它关于$x=\pi$为对称，然后再把已延拓到$[0,2\pi]$上的函数延拓到整个实轴上使函数为以2π为周期的函数.

解：所求函数为:$f(x)=\begin{cases} x, & x\in[0,\pi] \\ 2\pi-x, & x\in[\pi,2\pi] \\ x-2n\pi, & x\in[2n\pi,(2n+1)\pi](n=\pm1,\pm2,\cdots) \\ 2n\pi-x, & x\in[(2n-1)\pi,2n\pi](n=0,-1,\pm2,\cdots) \end{cases}$

$=\pi\left|\dfrac{x}{\pi}-2\left[\dfrac{x+\pi}{2\pi}\right]\right|$



第二章　极限与连续

§1.　数列的极限和无穷大量

1. 写出下列数列的前四项：

(1) $x_n = \dfrac{1}{3n}\sin n^3$

(2) $x_n = \dfrac{m(m-1)\cdots(m-n+1)}{n!}x^n$

(3) $x_n = \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}}$

(4) $x_1 = a > 0, y_1 = b > 0, x_{n+1} = \sqrt{x_n y_n}, y_{n+1} = \dfrac{x_n + y_n}{2}$

(5) $x_{2n} = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n} \quad (n = 1, 2, 3, \cdots)$

$x_{2n+1} = \dfrac{1}{n} \quad (n = 1, 2, \cdots)$

解:

(1) $x_1 = \dfrac{1}{3}\sin 1, \quad x_2 = \dfrac{1}{6}\sin 8, \quad x_3 = \dfrac{1}{9}\sin 27, \quad x_4 = \dfrac{1}{12}\sin 64$

(2) $x_1 = mx, \quad x_2 = \dfrac{m(m-1)}{2}x^2, \quad x_3 = \dfrac{m(m-1)(m-2)}{6}x^3,$

$x_4 = \dfrac{m(m-1)(m-2)(m-3)}{24}x^4$

(3) $x_1 = \frac{1}{\sqrt{2}}, \quad x_2 = \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{6}}, \quad x_3 = \dfrac{1}{\sqrt{10}} + \dfrac{1}{\sqrt{11}} + \dfrac{1}{\sqrt{12}},$

$x_4 = \dfrac{1}{\sqrt{17}} + \dfrac{1}{\sqrt{18}} + \dfrac{1}{\sqrt{19}} + \dfrac{1}{\sqrt{20}}$

(4) $x_1 = a, \quad x_2 = \sqrt{ab}, \quad x_3 = \sqrt{\sqrt{ab}\dfrac{a+b}{2}},$

$x_4 = \sqrt[8]{ab} \cdot \sqrt[4]{\dfrac{a+b}{2}} \cdot \dfrac{\sqrt{a}+\sqrt{b}}{2}$

$y_1 = b, \quad y_2 = \dfrac{a+b}{2}, \quad y_3 = \dfrac{(\sqrt{a}+\sqrt{b})^2}{4},$

$y_4 = \dfrac{(\sqrt{a}+\sqrt{b})^2}{4} + \dfrac{\sqrt[4]{ab}\sqrt{2(a+b)}}{16}$

(5) $x_2 = 1, \quad x_3 = 1, \quad x_4 = \dfrac{3}{2}, \quad x_5 = \dfrac{1}{2}$

2. 按定义证明以下数列为无穷小量：

(1) $\dfrac{n+1}{n^2+1}$

(2) $\dfrac{\sin n}{n}$

(3) $\dfrac{n+(-1)^n}{n^2-1}$

(4) $\dfrac{1}{n!}$

(5) $\dfrac{1}{n} - \dfrac{1}{2n} + \dfrac{1}{3n} - \cdots + (-1)^{n+1}\dfrac{1}{n^2}$

(6) $(-1)^n(0.999)^n$

(7) $\dfrac{1}{n} + e^{-n}$

(8) $\dfrac{e^{-n}}{n}$

(9) $\sqrt{n+1}-\sqrt{n}$

(10) $\dfrac{1+2+3+\cdots+n}{n^3}$

证明:

(1) 对$\forall\varepsilon>0$，由于$\left|\dfrac{n+1}{n^2+1}-0\right|=\dfrac{n+1}{n^2+1}<\dfrac{2n}{n^2}=\dfrac{2}{n}$，要使$\left|\dfrac{n+1}{n^2+1}-0\right|<\varepsilon$，只要$\dfrac{2}{n}<\varepsilon$即可。取$N=\left[\dfrac{2}{\varepsilon}\right]+1$, 则当$n>N$时，$\left|\dfrac{n+1}{n^2+1}-0\right|<\varepsilon$总成立，所以$\dfrac{n+1}{n^2+1}\to 0(n\to\infty)$

(2) 对$\forall\varepsilon>0$，由于$\left|\dfrac{\sin n}{n}-0\right|=\left|\dfrac{\sin n}{n}\right|\leqslant\dfrac{1}{n}$，要使$\left|\dfrac{\sin n}{n}-0\right|<\varepsilon$，只要$\dfrac{1}{n}<\varepsilon$即可。取$N=\left[\dfrac{1}{\varepsilon}\right]+1$，则当$n>N$时，$\left|\dfrac{\sin n}{n}-0\right|<\varepsilon$总成立，所以$\dfrac{\sin n}{n}\to 0(n\to\infty)$

(3) 对$\forall\varepsilon>0$，由于$\left|\dfrac{n+(-1)^n}{n^2-1}-0\right|=\dfrac{n+(-1)^n}{n^2-1}<\dfrac{n+1}{n^2-1}=\dfrac{1}{n-1}$，要使$\left|\dfrac{n+(-1)^n}{n^2-1}-0\right|<\varepsilon$，只要$\dfrac{1}{n-1}<\varepsilon$即可。取$N=\left[\dfrac{1}{\varepsilon}\right]+1$，则当$n>N$时，$\left|\dfrac{n+(-1)^n}{n^2-1}-0\right|<\varepsilon$总成立，所以$\dfrac{n+(-1)^n}{n^2-1}\to 0(n\to\infty)$

(4) 对$\forall\varepsilon>0$，由于$\left|\dfrac{1}{n!}-0\right|=\dfrac{1}{n!}<\dfrac{1}{n}$，要使$\left|\dfrac{1}{n!}-0\right|<\varepsilon$，只要$\dfrac{1}{n}<\varepsilon$即可。取$N=\left[\dfrac{1}{\varepsilon}\right]+1$，则当$n>N$时，$\left|\dfrac{1}{n!}-0\right|<\varepsilon$总成立，所以$\dfrac{1}{n!}\to 0(n\to\infty)$

(5) 设$S_n=\dfrac{1}{n}-\dfrac{1}{2n}+\dfrac{1}{3n}-\cdots+(-1)^{n+1}\dfrac{1}{n^2}$
对$\forall\varepsilon>0$，由于$S_n=\dfrac{1}{n}(1-\dfrac{1}{2}+\dfrac{1}{3}-\cdots+(-1)^{n+1}\dfrac{1}{n})$
设$\delta_n=1-\dfrac{1}{2}+\dfrac{1}{3}-\cdots+(-1)^{n+1}\dfrac{1}{n}$，则$S_n=\dfrac{\delta_n}{n}$当$n=2k+1$时，有$0<\delta_n=1-(\dfrac{1}{2}-\dfrac{1}{3})-(\dfrac{1}{4}-\dfrac{1}{5})-\cdots-(\dfrac{1}{2k}-\dfrac{1}{2k+1})<1$；当$n=2k$时，有$0<\delta_n=1-(\dfrac{1}{2}-\dfrac{1}{3})-(\dfrac{1}{4}-\dfrac{1}{5})-\cdots-(\dfrac{1}{2k-2}-\dfrac{1}{2k-1})-\dfrac{1}{2k}<1$。
总之，有$0<\delta_n<1$ 从而$|S_n-0|=S_n=\dfrac{\delta_n}{n}<\dfrac{1}{n}$ 要使$|S_n-0|<\varepsilon$，只要$\dfrac{1}{n}<\varepsilon$即可。取$N=\left[\dfrac{1}{\varepsilon}\right]+1$，则当$n>N$时，$|S_n-0|<\varepsilon$总成立，所以$\dfrac{1}{n}-\dfrac{1}{2n}+\dfrac{1}{3n}-\cdots+(-1)6n+1\dfrac{1}{n^2}\to 0(n\to\infty)$

(6) 对$\forall\varepsilon>0$，由于$n>\ln n$，则$e^n>n$，于是$e^{-n}<\dfrac{1}{n}$，从而$\left|\dfrac{1}{n}+e^{-n}-0\right|=\dfrac{1}{n}+e{-}n<\dfrac{2}{n}$，要使$|(-1)^n(0.999)^n-0|<\varepsilon$，只要$(0.999)^n<\varepsilon$即可。取$N=\left[2500\ln\dfrac{1}{\varepsilon}\right]+1$，则当$n>N$时，$|(-1)^n(0.999)^n-0|<\varepsilon$总成立，所以$(-1)^n(0.999)^n\to 0(n\to\infty)$

(7) 对$\forall\varepsilon>0$，由于$\left|\dfrac{1}{n}+e^{-n}-0\right|=\dfrac{1}{n!}<\dfrac{1}{n}$，要使$\left|\dfrac{1}{n}+e^{-n}-0\right|<\varepsilon$，只要$\dfrac{2}{n}<\varepsilon$即可。取$N=\left[\dfrac{2}{\varepsilon}\right]+1$，则当$n>N$时，$\left|\dfrac{1}{n}+e^{-n}-0\right|<\varepsilon$总成立，所以$\dfrac{1}{n}+e^{-n}\to 0(n\to\infty)$

(8) 对$\forall\varepsilon>0$，由于$e^{-n}<e^0=1$，则$\left|\dfrac{e^{-n}}{n}-0\right|=\dfrac{e^{-n}}{n}<\dfrac{1}{n}$，要使$\left|\dfrac{e^{-n}}{n}-0\right|<\varepsilon$，只要$\dfrac{1}{n}<\varepsilon$即可。取$N=\left[\dfrac{1}{\varepsilon}\right]+1$，则当$n>N$时，$\left|\dfrac{e^{-n}}{n}-0\right|<\varepsilon$总成立，所以$\dfrac{e^{-n}}{n}\to 0(n\to\infty)$

(9) 对$\forall\varepsilon>0$，由于$|\sqrt{n+1}-\sqrt{n}-0|=\dfrac{1}{\sqrt{n+1}+\sqrt{n}}<\dfrac{1}{2\sqrt{n}}$，要使$|\sqrt{n+1}-\sqrt{n}-0|<\varepsilon$，只要$\dfrac{1}{2\sqrt{n}}<\varepsilon$即可。取$N=\left[\dfrac{1}{4\varepsilon^2}\right]+1$，则当$n>N$时，$|\sqrt{n+1}-\sqrt{n}-0|<\varepsilon$总成立，所以$\sqrt{n+1}-\sqrt{n}\to 0(n\to\infty)$

(10) 对$\forall\varepsilon>0$，由于$\left|\dfrac{1+2+3+\cdots+n}{n^3}-0\right|=\dfrac{n+1}{2n^2}<\dfrac{2n}{2n^2}=\dfrac{1}{n}$，要使$\left|\dfrac{1}{n}-0\right|<\varepsilon$，只要$\dfrac{1}{n}<\varepsilon$即可。取$N=\left[\dfrac{1}{\varepsilon}\right]+1$，则当$n>N$时，$\left|\dfrac{1+2+3+\cdots+n}{n^3}-0\right|<\varepsilon$总成立，所以$\dfrac{1+2+3+\cdots+n}{n^3}\to 0(n\to\infty)$

3. 举例说明下列关于无穷小量的定义是错误的：

(1) 对任意$\varepsilon > 0$，存在N，当$n > N$时，成立$x_n < \varepsilon$；

(2) 对任意$\varepsilon > 0$，存在无限多个x_n，使$|x_n| < \varepsilon$.

解:

(1) 例如：数列$\{-1+(-1)^{n+1}\}$(或$\{-n\}$)即$\{0,-2,0,-2,\cdots\}$ (或$\{-1,-2,-3,\cdots\}$)满足上述条件，但不是无穷小量；

(2) 例如：数列$\{1,\dfrac{1}{2},1,\dfrac{1}{3},\cdots,1,\dfrac{1}{n},\cdots\}$满足上述条件，但不是无穷小量。

4. 按定义证明：

(1) $\lim\limits_{n\to\infty}\dfrac{3n^2+n}{2n^2-1}=\dfrac{3}{2}$

(2) $\lim\limits_{n\to\infty}(0.\overbrace{99\cdots9}^{n})=1$

(3) $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+n}}{n}=1$

(4) $x_n=\dfrac{1}{1\cdot2}+\dfrac{1}{2\cdot3}+\cdots+\dfrac{1}{(n-1)\cdot n}\to1(n\to\infty)$

(5) $\lim\limits_{n\to\infty}r_n=1$，此处 $r_n=\begin{cases}\dfrac{n-1}{n} & \text{当}n\text{为偶数}\\ \dfrac{n+1}{n} & \text{当}n\text{为奇数}\end{cases}$

(6) $\lim\limits_{n\to\infty}r_n=1$，此处 $r_n=\begin{cases}3 & \text{当}n=3k(k=1,2,3,\cdots)\\ \dfrac{3n+1}{n} & \text{当}n=3k+1\\ 2+\dfrac{1+n}{3-\sqrt{n}+n} & \text{当}n=3k+2\end{cases}$

证明:

(1) 对$\forall\varepsilon > 0$，由于$\left|\dfrac{3n^2+n}{2n^2-1}-\dfrac{3}{2}\right|=\dfrac{2n+3}{4n^2-2}<\dfrac{4(n+1)}{4(n+1)(n-1)}=\dfrac{1}{n-1}(n\geqslant2)$，要使$\left|\dfrac{3n^2+n}{2n^2-1}-\dfrac{3}{2}\right|<\varepsilon$，只要$\dfrac{1}{n-1}<\varepsilon$即可。取$N=\max(\left[\dfrac{1}{\varepsilon}\right]+1,2)$, 则当$n>N$时，$\left|\dfrac{3n^2+n}{2n^2-1}-\dfrac{3}{2}\right|<\varepsilon$总成立，所以$\dfrac{3n^2+n}{2n^2-1}\to\dfrac{3}{2}(n\to\infty)$

(2) 对$\forall\varepsilon > 0$，由于$\left|0.\overbrace{99\cdots9}^{n}-1\right|=(0.1)^n=\dfrac{1}{10^n}$，要使$\left|0.\overbrace{99\cdots9}^{n}-1\right|<\varepsilon$，只要$\dfrac{1}{10^n}<\varepsilon$即可。取$N=\left[\lg\dfrac{1}{\varepsilon}\right]+1$, 则当$n>N$时，$\left|0.\overbrace{99\cdots9}^{n}-1\right|<\varepsilon$总成立，所以$0.\overbrace{99\cdots9}^{n}\to1(n\to\infty)$

(3) 对$\forall\varepsilon > 0$，由于$\left|\dfrac{\sqrt{n^2+n}}{n}-1\right|=\dfrac{\sqrt{n^2+n}-n}{n}=<\dfrac{1}{\sqrt{n^2+n}+n}<\dfrac{1}{2n}$，要使$\left|\dfrac{\sqrt{n^2+n}}{n}-1\right|<\varepsilon$，只要$\dfrac{1}{2n}<\varepsilon$即可。取$N=\left[\dfrac{1}{2\varepsilon}\right]+1$, 则当$n>N$时，$\left|\dfrac{\sqrt{n^2+n}}{n}-1\right|<\varepsilon$总成立，所以$\dfrac{\sqrt{n^2+n}}{n}\to1(n\to\infty)$

(4) 对$\forall\varepsilon>0$，由于$x_n=1-\dfrac{1}{2}+\dfrac{1}{2}-\dfrac{1}{3}+\cdots+\dfrac{1}{n-1}-\dfrac{1}{n}=1-\dfrac{1}{n}$，则$|x_n-1|=\dfrac{1}{n}$，要使$|x_n-1|<\varepsilon$，只要$\dfrac{1}{n}<\varepsilon$即可。取$N=\left[\dfrac{1}{\varepsilon}\right]+1$, 则当$n>N$时，$|x_n-1|<\varepsilon$总成立，所以$x_n\to1(n\to\infty)$

(5) 对$\forall\varepsilon > 0$，由于$|r_n-1|=\left|\dfrac{n\pm1}{n}-1\right|=\dfrac{1}{n}$，要使$|r_n-1|<\varepsilon$，只要$\dfrac{1}{n}<\varepsilon$即可。取$N=\left[\dfrac{1}{\varepsilon}\right]+1$, 则当$n>N$时，$|r_n-1|<\varepsilon$总成立，所以$r_n\to1(n\to\infty)$

(6) 对$\forall\varepsilon > 0$，由于$|r_{3k}-3|=0,|r_{3k+1}-3|=\dfrac{1}{n},|r_{3k+2}-3|=\dfrac{\sqrt{n}-2}{3-\sqrt{n}+n}=\dfrac{n-4}{n\sqrt{n}+n+\sqrt{n}+6}<\dfrac{n}{n\sqrt{n}}=\dfrac{1}{\sqrt{n}}$，要使$|r_n-3|<\varepsilon$，只要$\dfrac{1}{n}<\varepsilon$且$\dfrac{1}{\sqrt{n}}<\varepsilon$即可。取$N=\max\left(\left[\dfrac{1}{\varepsilon}\right]+1,\left[\dfrac{1}{\varepsilon^2}\right]+1\right)$, 则当$n>N$时，$|r_n-3|<\varepsilon$总成立，所以$r_n\to3(n\to\infty)$

5. (1) 按定义证明，若$a_n \to a(n \to \infty)$，则对任意自然数k，$a_{n+k} \to a(n \to \infty)$

(2) 按定义证明，若$a_n \to a(n \to \infty)$，则$|a_n| \to |a|$.又反之是否成立？

(3) 若$|a_n| \to 0$，试问$a_n \to a$是否一定成立？为什么？

证明:

(1) 由于$a_n \to a(n \to \infty)$，故对$\forall \varepsilon > 0, \exists N \in Z^+$，当$n > N$时，$|a_n - a| < \varepsilon$，则对$\forall k \in Z^+, n+k > N$时，$|a_{n+k} - a| < \varepsilon$，于是对$\forall \varepsilon > 0, \exists N \in Z^+$，当$n+k > N$时，$|a_{n+k} - a| < \varepsilon$，从而$a_{n+k} \to a(n \to \infty)$

△ 此结论说明：去掉数列的前面有限项，也不影响其收敛性。

(2) (i) 由于$a_n \to a$，故对$\forall \varepsilon > 0, \exists N \in Z^+$，当$n > N$时，$|a_n - a| < \varepsilon$.又$\big||a_n| - |a|\big| < |a_n - a|$，于是对$\forall \varepsilon > 0, \exists N \in Z^+$，当$n > N$时，$\big||a_n| - |a|\big| < \varepsilon$成立，即$|a_n| \to |a|(n \to \infty$

(ii) 反之不一定成立。

例：

(a) 不成立：$a_n = (-1)^n$，则$|a_n| \to 1$，而a_n无极限；

(b) 成立：$a_n = \dfrac{1}{n}$，则$|a_n| \to 0, a_n \to 0$

(3) 由于$|a_n| \to 0$，故对$\forall \varepsilon > 0, \exists N \in Z^+$，当$n > N$时，$\big||a_n| - 0\big| < \varepsilon$，又$|a_n - 0| = \big||a_n| - 0\big|$，于是对$\forall \varepsilon > 0, \exists N \in Z^+$，当$n > N$时，$|a_n - 0| < \varepsilon$成立，即$a_n \to 0(n \to \infty)$。从而若$|a_n| \to 0$，则$a_n \to 0$一定成立。

6. 按定义证明，若$x_n \to a$，且$a > b$，则存在N，当$n > N$时，成立$x_n > b$.

证明: 由于$x_n \to a$，故对$\forall \varepsilon > 0, \exists N \in Z^+$，当$n > N$时，$|x_n - 0| < \varepsilon$，即$a - \varepsilon < x_n < a + \varepsilon$.又$a > b$，故$a - b > 0$，则取$\varepsilon = a - b > 0$，从而$\exists N \in Z^+$，当$n > N$时，有$x_n > a - \varepsilon = a - (a - b) = b$.即存在$N$，当$n > N$时，成立$x_n > b$.

7. 若$\{x_n y_n\}$收敛，能否断定$\{x_n\}, \{y_n\}$亦收敛.

解: 不能。

例：$x_n = (-1)^n, y_n = (-1)^n (n = 1, 2, \cdots), x_n y_n \equiv 1 (n = 1, 2, \cdots)$，则$\{x_n y_n\}$收敛，但$\{x_n\}, \{y_n\}$均不收敛。故若$\{x_n y_n\}$收敛，不能断定$\{x_n\}, \{y_n\}$亦收敛.

8. 利用极限性质及计算证明：

(1) $\lim\limits_{n \to \infty} \left(\dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} + \cdots + \dfrac{1}{(2n)^2} \right) = 0$

(2) $\lim\limits_{n \to \infty} \left(\dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}} \right) = 1$

(3) 利用$(1+h)^n = \sum\limits_{k=0}^{n} C_n^k h^k = 1 + nh + \dfrac{n(n-1)}{2} h^2 + \cdots + h^n$

证明:

(i) $\lim\limits_{n \to \infty} \dfrac{n}{a^n} = 0 (a > 1)$

(ii) $\lim\limits_{n \to \infty} \dfrac{n^5}{e^n} = 0 (e \approx 2.7)$

证明:

(1) 对$\forall n \in Z^+$，有$0 \leqslant \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} + \cdots + \dfrac{1}{(2n)^2} \leqslant \dfrac{n+1}{n^2}$，且$\lim\limits_{n \to \infty} \dfrac{n+1}{n^2} = 0$，则$\lim\limits_{n \to \infty} \left(\dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} + \cdots + \dfrac{1}{(2n)^2} \right) = 0$

(2) 对$\forall n \in Z^+$，有$\dfrac{n}{n+1} < \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}} < \dfrac{n}{n} = 1$且$\lim\limits_{n \to \infty} \dfrac{n}{n+1} = 1$，则$\lim\limits_{n \to \infty} \left(\dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}} \right) = 1$

(3) (i) 设$a = 1 + h (h > 0)$，由于$0 < \dfrac{n}{a^n} = \dfrac{n}{(1+h)^n} = \dfrac{n}{1 + nh + \frac{n(n_1)}{2} h^2 + \cdots + h^n} < \dfrac{n}{\frac{n(n-1)}{2} h^2} = \dfrac{2}{(n-1)h^2}$，又$\dfrac{2}{h^2}$为定值，$\dfrac{1}{n-1} \to 0 (n \to \infty)$，则$\dfrac{2}{(n-1)h^2} \to 0$.从而$\lim\limits_{n \to \infty} \dfrac{n}{a^n} = 0$

(ii) 设$e = 1+h(h\approx 1.7)$，由于$0 < \dfrac{n^5}{e^n} = \dfrac{n^5}{(1+h)^n} = \dfrac{n^5}{1+nh+C_n^2h^2+\cdots+h^n} < \dfrac{n^5}{C_n^6h^6} < \dfrac{720n^5}{(n-5)^6h^6}$，又$\dfrac{720}{h^6}$为定值，$\dfrac{n^5}{(n-5)^6}\to 0(n\to\infty)$，则$\dfrac{720n^5}{(n-5)^6h^6}\to 0(n\to\infty)$，从而$\lim\limits_{n\to\infty}\dfrac{n^5}{e^n} = 0$

9. 求下列极限：

(1) $\lim\limits_{n\to\infty}\dfrac{3n^3+2n^2-n+1}{2n^3-3n^2+2}$

(2) $\lim\limits_{n\to\infty}\dfrac{6n^2-n+1}{n^3+n^2+2}$

(3) $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{\sqrt[n]{2}}\right)\cos n$

(4) $\lim\limits_{n\to\infty}\dfrac{1+\frac{1}{2}+\cdots+\frac{1}{2^n}}{1+\frac{1}{4}+\cdots+\frac{1}{4^n}}$

(5) $\lim\limits_{n\to\infty}\left[(\sin n!)\left(\dfrac{n-1}{n^2+1}\right)^{10}-\left(\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(n-1)\cdot n}\right)\dfrac{2n^2+1}{n^2-1}\right]$

(6) $\lim\limits_{n\to\infty}\dfrac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}$

解：

(1) $\lim\limits_{n\to\infty}\dfrac{3n^3+2n^2-n+1}{2n^3-3n^2+2} = \dfrac{3}{2}$

(2) $\lim\limits_{n\to\infty}\dfrac{6n^2-n+1}{n^3+n^2+2} = 0$

(3) 由于$\sqrt[n]{2}\to 1(n\to\infty)$，故$1-\sqrt[n]{2}\to 0(n\to\infty)$，又$|\cos n|\leqslant 1$，从而$\lim\limits_{n\to\infty}\left(1-\dfrac{1}{\sqrt[n]{2}}\right)\cos n = 0$

(4) $$\lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{2^n}}{1+\frac{1}{4}+\cdots+\frac{1}{4^n}} = \lim_{n\to\infty}\frac{\dfrac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}}}{\dfrac{1-(\frac{1}{4})^{n+1}}{1-\frac{1}{4}}} = \frac{2}{\frac{4}{3}} = \frac{3}{2}$$

(5) 由于$\{\sin n!\}$为有界数列，$\left(\dfrac{n-1}{n^2+1}\right)^{10}\to 0, 1-\dfrac{1}{n}\to 1, \dfrac{2n^2+1}{n^2+1}\to 2(n\to\infty)$，

故$\lim\limits_{n\to\infty}\left[(\sin n!)\left(\dfrac{n-1}{n^2+1}\right)^{10}-\left(\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(n-1)\cdot n}\right)\dfrac{2n^2+1}{n^2-1}\right] = -2$

(6) $$\lim_{n\to\infty}\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}} = \lim_{n\to\infty}\frac{(\frac{-2}{3})^n+1}{(-2)(\frac{-2}{3})^n+3} = \frac{1}{3}$$

10. 若$x_n\to a>0$，试证：

(1) $\sqrt{x_n}\to\sqrt{a}$

(2) $\sqrt{a_0x_n^m+a_1x_n^{m-1}+\cdots+a_{m-1}x_n+a_m}\to\sqrt{a_0a^m+a_1a^{m-1}+\cdots+a_{m-1}a+a_m}$
(其中$a_0a^m+a_1a^{m-1}+\cdots+a_{m-1}a+a_m>0$)

证明：

(1) 由于$x_n\to a>0$，故对$\forall\varepsilon>0,\exists N\in Z^+$，当$n>N$时，$|x_n-a|<\sqrt{a}\varepsilon$，且$|\sqrt{x_n}-\sqrt{a}| = \left|\dfrac{x_n-a}{\sqrt{x_n}+\sqrt{a}}\right| < \dfrac{|x_n-a|}{\sqrt{a}} < \varepsilon$，即对上述$\varepsilon>0,\exists N\in Z^+$，当$n>N$时，$|\sqrt{x_n}-\sqrt{a}|<\varepsilon$，从而$\sqrt{x_n}\to\sqrt{a}(n\to\infty)$

(2) 由于$x_n\to a(n\to\infty)$，故$a_0x_n^m+a_1x_n^{m-1}+\cdots+a_{m-1}x_n+a_m\to a_0a^m+a_1a^{m-1}+\cdots+a_{m-1}a+a_m>0$，则据(1)得
$\sqrt{a_0x_n^m+a_1x_n^{m-1}+\cdots+a_{m-1}x_n+a_m}\to\sqrt{a_0a^m+a_1a^{m-1}+\cdots+a_{m-1}a+a_m}$

11. 对数列$\{x_n\}$，若$x_{2k}\to a(k\to\infty),x_{2k+1}\to a(k\to\infty)$，证明：$x_n\to a(n\to\infty)$

证明：$\forall\varepsilon>0$，因$x_{2k}\to a(n\to\infty)$，故$\exists K_1\in Z^+$，使当$k>K_1$时，$|x_{2k}-a|<\varepsilon$成立。
又因$x_{2k+1}\to a(n\to\infty)$，故$\exists K_2\in Z^+$，使当$k>K_2$时，$|x_{2k+1}-a|<\varepsilon$成立。
取$N=\max\{2K_1,2K_2+1\}$，则当$n>N$时，若n为偶数，$n=2k>N\geqslant 2K_1,k>K_1,|x_n-a|=|x_{2k}-a|<\varepsilon$，
若n为奇数，$n=2k+1>N\geqslant 2K_2+1,k>K_2,|x_n-a|=|x_{2k+1}-a|<\varepsilon$，
因此$x_n\to a(n\to\infty)$

12. 利用单调有界必有极限，证明$\lim\limits_{n\to\infty}x_n$存在，并求出它：

(1) $x_1=\sqrt{2},\cdots,x_n=\sqrt{2x_{n-1}}$

(2) $x_0=1,x_1=1+\dfrac{x_0}{1+x_0},\cdots,x_{n+1}=1+\dfrac{x_n}{1+x_n}$

证明：

(1) 显然$x_1<x_2$，假设$x_{n-1}<x_n$，则$x_n=\sqrt{2x_{n-1}}<\sqrt{2x_n}$，由归纳法，知$\{x_n\}$是单调增加的，又$x_n=\sqrt{2x_{n-1}}$，故得$x_n^2=2x_{n_1}\leqslant 2x_n$，于是$x_n\leqslant 2$，即$\{x_n\}$由上界。从而$\lim\limits_{n\to\infty}x_n$存在，记$\lim\limits_{n\to\infty}x_n=l$，在$x_n^2=2x_{n-1}$两边令$n\to\infty$，得$l^2=2l$，解之得$l=2$，即$\lim\limits_{n\to\infty}x_n=2$。

(2) 显然$x_n\geqslant 1$，有条件知$x_n=1+\dfrac{x_{n-1}}{1+x_{n-1}}=2-\dfrac{1}{1+x_{n-1}}<2$，故$\{x_n\}$有界。又$x_1=1+\dfrac{x_0}{1+x_0}=1+\dfrac{1}{1+1}=\dfrac{3}{2}>1=x_0$，假设$x_{n_1}<x_n$，则$x_n=2-\dfrac{1}{1+x_{n-1}}<2-\dfrac{1}{1+x_n}=x_{n+1}$，由归纳法，知$\{x_n\}$是单调增加的。从而$\lim\limits_{n\to\infty}x_n$存在，记$\lim\limits_{n\to\infty}x_n=l$，在$x_n=2-\dfrac{1}{1+x_{n-1}}$两边令$n\to\infty$，得$l=2-\dfrac{1}{1+l}$，即$l^2=1+l$，解得$l_1=\dfrac{1+\sqrt{5}}{2},l_2=\dfrac{1-\sqrt{5}}{2}$（不合题意，舍去），即$\lim\limits_{n\to\infty}x_n=\dfrac{1+\sqrt{5}}{2}$。

13. 若$x_1=a>0,y_1=b>0(a<b),x_{n+1}=\sqrt{x_ny_n},y_{n+1}=\dfrac{x_n+y_n}{2}$，证明：$\lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}y_n$.

证明：由于$\sqrt{x_ny_n}\leqslant\dfrac{x_n+y_n}{2}$且此式相等当且仅当$x_n=y_n$，故$x_{n+1}\leqslant y_{n+1}$等号成立当且仅当$x_n=y_n$.又$0<a<b$，故$x_1<y_1$，则由递推公式，得$x_{n+1}<y_{n+1}$且$x_n>0,y_n>0(n\in Z^+)$.而$x_{n+1}=\sqrt{x_ny_n}>\sqrt{x_nx_n}=x_n,y_{n+1}=\dfrac{x_n+y_n}{2}<\dfrac{y_n+y_n}{2}=y_n$，则$x_n<x_{n+1}<y_{n+1}<y_n$.又由$x_1=a>0,y_1=b>0$，得$a<x_n<x_{n+1}<y_{n+1}<y_n<b$，说明$\{x_n\}$与$\{y_n\}$都是单调有界数列，从而$\{x_n\},\{y_n\}$均有极限，设$\lim\limits_{n\to\infty}x_n=\alpha,\lim\limits_{n\to\infty}y_n=\beta$，又由$x_{n+1}=\sqrt{x_ny_n}$，得$x_{n+1}^2=x_ny_n$，在等式两边令$n\to\infty$，得$\alpha^2=\alpha\beta$ 又由$0<a<x_n<x_{n+1}$，得定有$0<a\leqslant\alpha$，从而$\alpha=\beta$即有$\lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}y_n$.

14. 利用单调有界必有极限证明以下数列必有极限：

(1) $x_n=1+\dfrac{1}{2^2}+\cdots+\dfrac{1}{n^2}$

(2) $x_n=\dfrac{1}{3+1}+\dfrac{1}{3^2+1}+\cdots+\dfrac{1}{3^n+1}$

(3) $x_n=\dfrac{n^k}{a^n}(a>1,k$为正整数$)$

(4) $x_n=\sqrt[n]{a}\ (0<a<1)$

证明：

(1) 由于$x_{n+1}-x_n=\dfrac{1}{(n+1)^2}>0$，故$x_{n+1}>x_n$，则$\{x_n\}$为单调增加的. 又$1<x_n<1+\dfrac{1}{1\dot{2}}+\cdots+\dfrac{1}{\dot{n}(n+1)}=1+\left(1-\dfrac{1}{2}+\cdots+\dfrac{1}{n_1}-\dfrac{1}{n}\right)=2-\dfrac{1}{n}<2$，故$\{x_n\}$有界，于是$\{x_n\}$存在极限。

(2) 由于$x_{n+1}-x_n=\dfrac{1}{3^{n+1}+1}>0$，故$x_{n+1}>x_n$，则$\{x_n\}$为单调增加的. 又$\dfrac{1}{4}<x_n<\dfrac{1}{4}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{3^n}<\dfrac{1}{3}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{3^n}=\dfrac{\dfrac{1}{3}}{1-\dfrac{1}{3}}=\dfrac{1}{2}$，故$\{x_n\}$有界，于是$\{x_n\}$存在极限。

(3) 由于$a>1,k$为正整数，故$x_n=\frac{n^k}{a^n}>0$，则$\{x_n\}$有下界。又$\frac{x_{n+1}}{x_n}=\frac{\left(1+\frac{1}{n}\right)^k}{a}=\frac{1}{a}\left(1+\frac{1}{n}\right)^k\to\frac{1}{a}(n\to\infty)<1$，故$\exists N\in Z^+$，当$n>N$时，有$\frac{x_{n+1}}{x_n}<1$，则从$N+1$项开始都有$x_{n+1}<x_n$，于是$\{x_n\}$为单调减少的$(n>N)$，从而$\{x_n\}$存在极限。

(4) 由于$\ln x_n=\frac{1}{n}\ln a=y_n,0<a<1$，故$\{y_n\}$是单调增加的，从而由$x_n=\sqrt[n]{a}=e^{y_n}$得$\{x_n\}$是单调增加的。又$0<x_n=\sqrt[n]{a}<\sqrt[n]{1}=1$，故$\{x_n\}$有界，于是$\{x_n\}$存在极限。

15. 证明：若x_n上升，y_n下降，而x_n-y_n为无穷小量，则x_n和y_n必有同一极限。
证明：由x_n上升，故$x_1\leqslant x_2\leqslant\cdots\leqslant x_n\leqslant\cdots$，又$y_n$下降，故$y_1\geqslant y_2\geqslant\cdots\geqslant y_n\geqslant\cdots$，又$x_n-y_n$为无穷小量，故$\{x_n-y_n\}$有界，设$|x_n-y_n|\leqslant C(n=1,2,\cdots)$（其中$C$为某常数），则$-C\leqslant x_n-y_n\leqslant C$即$x_n\leqslant y_n+C\leqslant y_1+C$，于是$\{x_n\}$有上界，从而$\{x_n\}$存在极限。又$y_n\geqslant x_n-C\geqslant x_1-C$，于是$\{y_n\}$有下界，从而$\{y_n\}$存在极限，则$\lim\limits_{n\to\infty}x_n-\lim\limits_{n\to\infty}y_n=\lim\limits_{n\to\infty}(x_n-y_n)=0$，于是$\lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}y_n$.

16. 设x为任意给定的实数，又设$y_n(x)=\underbrace{\sin\sin\cdots\sin}_{n}x$，证明$\{y_n(x)\}$的极限存在，并求此极限.

证明：先设$0\leqslant x\leqslant\pi$，则$0\leqslant\sin x\leqslant x$，从而有$y_{n+1}(x)=\sin y_n(x)\leqslant y_n(x)$，故$\{y_n(x)\}$是以0为下界的单调下降函数列，必有极限，则得对$\forall x_0\in[0,\pi]$，有$0\leqslant u_0=\lim\limits_{n\to\infty}y_n(x_0)=\sin\left(\lim\limits_{n\to\infty}f_{n-1}(x_0)\right)=\sin u_0$，则$u_0=0$，从而对$\forall x\in[0,\pi]$, $\lim\limits_{n\to\infty}y_n(x)=0$.
同理可证当$x\in[-\pi,0]$时亦有$\lim\limits_{n\to\infty}y_n(x)=0$.
再由周期性可知$\lim\limits_{n\to\infty}y_n(x)=0$

17. 若$\lim\limits_{n\to\infty}x_n=a$，试证：$\lim\limits_{n\to\infty}\frac{x_1+x_2+\cdots+x+n}{n}=a$
证明：由$\lim\limits_{n\to\infty}x_n=a$，得对$\forall\varepsilon>0,\exists N_1\in Z^+$，当$n>N_1$时，有$|x_n-a|<\frac{\varepsilon}{2}$，则有$\left|\frac{x_1+x_2+\cdots+x_n}{n}-a\right|=\left|\frac{(x_1-a)+(x_2-a)+\cdots+(x_n-a)}{n}\right|\leqslant\frac{|x_1-a|+|x_2-a|+\cdots+|x_{N_1}-a|+|x_{N_1+1}-a|+\cdots+|x_n-a|}{n}<$
$\frac{|x_1-a|+|x_2-a|+\cdots+|x_{N_1}-a|}{n}+\frac{n-N_1}{n}\cdot\frac{\varepsilon}{2}<$
$\frac{|x_1-a|+|x_2-a|+\cdots+|x_{N_1}-a|}{n}+\frac{\varepsilon}{2}(\because\frac{n-N_1}{n}<1)$
取$M=\max(|x_1-a|,|x_2-a|,\cdots,|x_{n_1}-a|)$，则$\frac{|x_1-a|+|x_2-a|+\cdots+|x_{N_1}-a|}{n}\leqslant\frac{N_1\cdot M}{n}$，又$N_1\cdot M$为定值，则$\frac{N_1\cdot M}{n}\to0(n\to\infty)$，
于是对上述$\varepsilon>0,\exists N_2=\left[\frac{2N_1\cdot M}{\varepsilon}\right]\in Z^+$，当$n>N_2$时，有
$\frac{|x_1-a|+|x_2-a|+\cdots+|x_{N_1}-a|}{n}<\frac{\varepsilon}{2}$
取$N=\max(N_1,N_2)$，则当$n>N$时，有$\left|\frac{x_1+x_2+\cdots+x_n}{n}-a\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$，
即有$\lim\limits_{n\to\infty}\frac{x_1+x_2+\cdots+x+n}{n}=a$
注：若$\lim\limits_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{n}=a\not\Rightarrow\lim\limits_{n\to\infty}x_n$存在。
例：$x_n=(-1)^{n-1}(n=1,2,\cdots)$，则显然$\lim\limits_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{n}=0$，但$\lim\limits_{n\to\infty}x_n$不存在。

18. 证明：若$\lim\limits_{n\to\infty}a_n=a,\lim\limits_{n\to\infty}b_n=b$，则$\lim\limits_{n\to\infty}\frac{a_1b_n+a_2b_{n-1}+\cdots+a_nb_1}{n}=ab$
证明：

(1) 设$a=0$，去证$\lim\limits_{n\to\infty}\frac{a_1b_n+a_2b_{n-1}+\cdots+a_nb_1}{n}=0$
由$\lim\limits_{n\to\infty}b_n=b$，则据定理$4(P_{38})$，得$\exists M>0$，使$|b_n|\leqslant M(n\in Z^+)$
由$\lim\limits_{n\to\infty}a_n=0$，则对$\forall\varepsilon>0,\exists N_1\in Z^+$，当$n>N_1$时，有$|a_n|<\frac{\varepsilon}{2M}$.取$N=\max\left\{\left[\frac{2(|a_1|+\cdots+|a_n|)M}{\varepsilon}\right]+1,N_1\right\}$，
于是当$n\geqslant N(\geqslant N_1)$时，有$\left|\frac{a_1b_n+a_2b_{n-1}+\cdots+a_nb_1}{n}\right|=\left|\frac{a_1b_n+a_2b_{n-1}+\cdots+a_{N_1}b_{n-N_1+1}+a_{N_1+1}b_{n-N_1}+\cdots+a_nb_1}{n}\right|$
$\leqslant\left|\frac{a_1b_n+a_2b_{n-1}+\cdots+a_{N_1}b_{n-N_1+1}}{n}\right|+\left|\frac{a_{N_1+1}b_{n-N_1}+\cdots+a_nb_1}{n}\right|\leqslant\frac{(|a_1|+\cdots+|a_{N_1}|)M}{n}+\frac{(n-N_1)\cdot\frac{\varepsilon}{2M}\cdot M}{n}<$

$\dfrac{\varepsilon}{2}+\dfrac{\varepsilon}{2}=\varepsilon$，从而$\lim\limits_{n\to\infty}\dfrac{a_1b_n+a_2b_{n-1}+\cdots+a_nb_1}{n}=0$

(2) 当$a\neq 0,b\neq 0$时，由$\lim\limits_{n\to\infty}b_n=b$，得$\lim\limits_{n\to\infty}\dfrac{b_n+b_{n-1}+\cdots+b_1}{n}=b\neq 0$，又$\lim\limits_{n\to\infty}a_n=a$，故$\lim\limits_{n\to\infty}(a_n-a)=0$

由(1)知$\lim\limits_{n\to\infty}\dfrac{(a_1-a)b_n+\cdots+(a_n-a)b_1}{n}=0$，

于是$\lim\limits_{n\to\infty}\left(\dfrac{a_1\cdot\dfrac{b_n}{n}+\cdots+a_n\cdot\dfrac{b_1}{n}}{\dfrac{b_n+\cdots+b_1}{n}}-a\right)=\lim\limits_{n\to\infty}\dfrac{(a_1-a)\cdot\dfrac{b_n}{n}+\cdots+(a_n-a)\cdot\dfrac{b_n}{n}}{\dfrac{b_n+\cdots+b_1}{n}}=\dfrac{0}{b}=0$，

即$\lim\limits_{n\to\infty}\dfrac{a_1\cdot\dfrac{b_n}{n}+\cdots+a_n\cdot\dfrac{b_1}{n}}{\dfrac{b_n+\cdots+b_1}{n}}=a$，

从而$\lim\limits_{n\to\infty}\dfrac{a_1b_n+a_2b_{n-1}+\cdots+a_nb_1}{n}=\lim\limits_{n\to\infty}\left(\dfrac{a_1\cdot\dfrac{b_n}{n}+\cdots+a_n\cdot\dfrac{b_1}{n}}{\dfrac{b_n+\cdots+b_1}{n}}\cdot\dfrac{b_n+\cdots+b_1}{n}\right)=ab$

19. 按定义证明下列数列为无穷大量：

(1) $\sqrt{n}$

(2) $n!$

(3) $\ln n$

(4) $\dfrac{n^2+1}{2n+1}$

(5) $\dfrac{n^2+1}{2n-1}$

(6) $1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}$

证明：

(1) 对$\forall G>0$，要使$|\sqrt{n}|>G$，只要$n>G^2$即可.取$N=[G^2]$，则当$n>N$时，$|\sqrt{n}|>G$总成立，故$\{\sqrt{n}\}$是无穷大量。

(2) 对$\forall G>0$，由于$|n!|>n$，要使$|n!|>G$，只要$n>G$即可.取$N=[G]$，则当$n>N$时，$|n!|>G$总成立，故$\{n!\}$是无穷大量。

(3) 对$\forall G>0$，要使$|\ln n|>G$，只要$n>e^G$即可.取$N=[e^G]$，则当$n>N$时，$|\ln n|>G$总成立，故$\{\ln n\}$是无穷大量。

(4) 对$\forall G>0$，由于$\left|\dfrac{n^2+1}{2n+1}\right|>\dfrac{n^2}{3n}=\dfrac{n}{3}$，要使$\left|\dfrac{n^2+1}{2n+1}\right|>G$，只要$\dfrac{n}{3}>G$即可.取$N=[3G]$，则当$n>N$时，$\left|\dfrac{n^2+1}{2n+1}\right|>G$总成立，故$\{\dfrac{n^2+1}{2n+1}\}$是无穷大量。

(5) 对$\forall G>0$，由于$\left|\dfrac{n^2+1}{2n-1}\right|>\dfrac{n^2}{2n}=\dfrac{n}{2}$，要使$\left|\dfrac{n^2+1}{2n-1}\right|>G$，只要$\dfrac{n}{2}>G$即可.取$N=[2G]$，则当$n>N$时，$\left|\dfrac{n^2+1}{2n-1}\right|>G$总成立，故$\{\dfrac{n^2+1}{2n-1}\}$是无穷大量。

(6) 对$\forall G>0$，由于$\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n=e$且$\left(1+\dfrac{1}{n}\right)^n$单调增加，则$\left(1+\dfrac{1}{n}\right)^n<e$，于是$\ln\left(1+\dfrac{1}{n}\right)<\dfrac{1}{n}$，从而$1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}>\ln 2+\ln\dfrac{3}{2}+\cdots+\ln\left(1+\dfrac{1}{n}\right)=\ln(n+1)>\ln n$，则要使$\left|1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}\right|>G$，只要$\ln n>G$即可.取$N=[e^G]$，则当$n>N$时，$\left|1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}\right|>G$总成立，故$\{1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}\}$是无穷大量。

20. 证明：若$\{x_n\}$是无穷小量，$x_n\neq 0(n=1,2,3,\cdots)$，则$\left\{\dfrac{1}{x_n}\right\}$是无穷大量。

证明：由于$\{x_n\}$是无穷小量，故对$\forall\varepsilon>0,\exists N\in Z^+$，当$n>N$时，有$|x_n|<\varepsilon$

又$x_n \neq 0(n=1,2,3,\cdots)$，故$\dfrac{1}{x_n}$存在且$\left|\dfrac{1}{x_n}\right| > \dfrac{1}{\varepsilon}$

又ε是任意的，故$\dfrac{1}{\varepsilon}$也是任意的，从而$\left\{\dfrac{1}{x_n}\right\}$是无穷大量。

21. 证明：若$\{x_n\}$为无穷大量，$\{y_n\}$为有界变量，则$\{x_n \pm y_n\}$为无穷大量。
并由此计算下列极限：

(1) $\lim\limits_{n\to\infty}\left(\sin n + \dfrac{n^2}{\sqrt{n^2+1}}\right)$

(2) $\lim\limits_{n\to\infty}(n - \arctan n)$

(3) $\lim\limits_{n\to\infty}\left[n + (-1)^n\left(\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)}\right)\right]$

又：两个无穷大量和的极限怎样？试讨论各种可能情形。

i)**证明**：由于$\{y_n\}$为有界变量，故必存在正数M，使$|y_n| \leqslant M$，又$\{x_n\}$为无穷大量，故对$\forall G > M > 0, \exists N \in Z^+$，当$n > N$时，有$|x_n| > G$，则当$n > N$时，有$|x_n \pm y_n| \geqslant |x_n| - |y_n| > G - M$.由$G$的任意性及$G > M > 0$，可知$G - M > 0$且$G - M$是任意的，从而$\{x_n \pm y_n\}$为无穷大量。

ii)**解**：

(1) 由于$\lim\limits_{n\to\infty}\dfrac{n^2}{\sqrt{n^2+1}} = \infty$且$|\sin n| \leqslant 1$，则$\lim\limits_{n\to\infty}\left(\sin n + \dfrac{n^2}{\sqrt{n^2+1}}\right) = \infty$

(2) 由于$\lim\limits_{n\to\infty} n = \infty$且$|\arctan n| \leqslant \dfrac{\pi}{2}$，则$\lim\limits_{n\to\infty}(n - \arctan n) = \infty$

(3) 设$x_n = (-1)^n\left(\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)}\right)$，则$x_n = \dfrac{(-1)^n}{2}\left[1 - \dfrac{1}{3} + \dfrac{1}{3} - \dfrac{1}{5} + \cdots + \dfrac{1}{2n-1} - \dfrac{1}{2n+1}\right] = \dfrac{(-1)^n}{2}\left(1 - \dfrac{1}{2n+1}\right) = \dfrac{(-1)^n}{2}\cdot\dfrac{2n}{2n+1} = \dfrac{(-1)^n}{2+\dfrac{1}{n}}$，故有$\dfrac{1}{3} < |x_n| < \dfrac{1}{2}$.又由$\lim\limits_{n\to\infty} n = \infty$，从而
$\lim\limits_{n\to\infty}\left[n + (-1)^n\left(\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)}\right)\right] = \infty$

iii)**解**：

(1) $x_n = n \to +\infty, y_n = 2n \to +\infty; x_n + y_n = 3n \to +\infty$

(2) $x_n = -n \to -\infty, y_n = -2n \to -\infty; x_n + y_n = -3n \to -\infty$

(3) $x_n = -n \to -\infty, y_n = 2n \to +\infty; x_n + y_n = n \to +\infty$

(4) $x_n = n \to +\infty, y_n = -2n \to -\infty; x_n + y_n = -n \to -\infty$

(5) $x_n = n + a \to +\infty, y_n = -n \to -\infty; x_n + y_n = a$（常量）

(6) $x_n = n + (-1)^n \to +\infty, y_n = -n \to +\infty; x_n + y_n = (-1)^n$无极限

22. 讨论无穷大量和无穷小量的和、差、商的极限的情形。
解：

(1) 和、差：因$y_n \to 0(n\to\infty)$，故$\{y_n\}$有界。又$x_n \to \infty(n\to\infty)$，则由上题结论，有$\{x_n \pm y_n\}$为无穷大量。

(2) 商：当$x_n \neq 0, y_n \neq 0$时，由于$x_n \to \infty, y_n \to 0(n\to\infty)$，则有$y_n \cdot \dfrac{1}{x_n} \to 0$，即$\dfrac{y_n}{x_n} \to 0, \dfrac{x_n}{y_n} \to \infty$

23. 举例说明无穷大量和无穷小量的乘积可能发生的种种情形。
解：

(1) $x_n = n \to +\infty, y_n = \dfrac{1}{n^2} \to 0(n\to\infty); x_n\cdot y_n = \dfrac{1}{n} \to 0(n\to\infty)$

(2) $x_n = n^2 \to +\infty, y_n = \dfrac{1}{n} \to 0(n\to\infty); x_n\cdot y_n = n \to +\infty(n\to\infty)$

(3) $x_n = n \to +\infty, y_n = \dfrac{a}{n} \to 0(n\to\infty); x_n\cdot y_n = a$（常量）

(4) $x_n = n(-1)^n \to \infty, y_n = \dfrac{1}{n} \to 0(n\to\infty); x_n\cdot y_n = (-1)^n$无极限但有界

(5) $x_n = n^2 n^{(-1)^n} \to \infty, y_n = \dfrac{1}{n} \to 0(n\to\infty); x_n\cdot y_n = n\cdot n(-1)^n = n^{1+(-1)^n}$无极限，无界（且不是无穷大量）

24. 若$x_n \to \infty, y_n \to a \neq 0$，证明$x_n y_n \to \infty$

证明：由于$x_n \to \infty(n \to \infty)$，故$\dfrac{1}{x_n} \to 0(n \to \infty)$；又$y_n \to a \neq 0(n \to \infty)$，故$\dfrac{1}{y_n} \to \dfrac{1}{a}(n \to \infty)$，于是$\dfrac{1}{x_n} \cdot \dfrac{1}{y_n} \to 0(n \to \infty)$，从而$x_n y_n \to \infty(n \to \infty)$

25. 若$x_n \to +\infty, y_n \to -\infty$，证明$x_n y_n \to -\infty$.

证明：因$x_n \to +\infty$，则对$\forall G_1 > 0, \exists N_1 \in Z^+$，当$n > N_1$时，有$x_n > G_1$；又$y_n \to -\infty$，则对$\forall G_2 > 0, \exists N_2 \in Z^+$，当$n > N_2$时，有$-y_n > G_2 > 0$. 取$N = \max(N_1, N_2)$，则当$n > N$时，有$-x_n y_n > G_1 G_2$，即$x_n y_n < -G_1 G_2$.由$G_1, G_2$的任意性，得$G_1 G_2$是任意的且$G_1 G_2 > 0$，则得$x_n y_n \to -\infty$.

26. 若$x_n \to +\infty$，证明$\dfrac{x_1 + x_2 + \cdots + x_n}{n} \to +\infty$

证明：因$x_n \to +\infty$，则对$\forall G > 0, \exists N_1 \in Z^+$，当$n > N_1$时，有$x_n > 3G$，于是$\dfrac{x_1 + x_2 + \cdots + x_n}{n} = \dfrac{x_1 + \cdots + x_{N_1}}{n} +$
$\dfrac{x_{N_1+1} + \cdots + x_n}{n} > \dfrac{x_1 + \cdots + x_{N_1}}{n} + \dfrac{n - N_1}{n} \cdot 3G$，
取$M = \max(|x_1|, \cdots, |x_{N_1}|)$，则$\left|\dfrac{x_1 + \cdot + x_{N_1}}{n}\right| \leqslant \dfrac{|x_1| + \cdots + |x_{N_1}|}{n} \leqslant \dfrac{N_1 \cdot M}{n}$，于是对上述$G > 0$，取$N_2 = \left[\dfrac{2N_1 \cdot M}{G}\right]$，则当$n > N_2$时，有$\left|\dfrac{x_1 + \cdots + x_{N_1}}{n}\right| < \dfrac{G}{2}$，从而$\dfrac{x_1 + \cdots + x_{N_1}}{n} > -\dfrac{G}{2}$。又$\lim\limits_{n \to \infty} \dfrac{n - N_1}{n} = 1$，故对于$\varepsilon = \dfrac{1}{2}, \exists N_3 \in Z^+$，当$n > N_3$时，有$\left|\dfrac{n - N_1}{n} - 1\right| < \dfrac{1}{2}$，从而$\dfrac{n - N_1}{n} > \dfrac{1}{2}$，取$N = \max\{N_1, N_2, N_3\}$，则当$n > N$时，有$\dfrac{x_1 + x_2 + \cdots + x_n}{n} > -\dfrac{G}{2} + \dfrac{3G}{2} = G$，由此知$\dfrac{x_1 + x_2 + \cdots + x_n}{n} \to +\infty(n \to \infty)$.

§2. 函数的极限

1. 用分析定义证明：

(1) $\lim\limits_{x\to -1}\dfrac{x-3}{x^2-9}=\dfrac{1}{2}$

(2) $\lim\limits_{x\to 3}\dfrac{x-3}{x^2-9}=\dfrac{1}{6}$

(3) $\lim\limits_{x\to 1}\dfrac{x-1}{\sqrt{x}-1}=2$

(4) $\lim\limits_{x\to 1}\dfrac{(x-2)(x-1)}{x-3}=0$

(5) $\lim\limits_{t\to 1}\dfrac{t(t-1)}{t^2-1}=\dfrac{1}{2}$

(6) $\lim\limits_{x\to \infty}\dfrac{x-1}{x+2}=1$

(7) $\lim\limits_{x\to 3}\dfrac{x}{x^2-9}=\infty$

(8) $\lim\limits_{x\to \infty}\dfrac{x^2+x}{x+1}=\infty$

证明:

(1) 对$\forall\varepsilon>0$，由于$\left|\dfrac{x-3}{x^2-9}-\dfrac{1}{2}\right|=\left|\dfrac{1}{x+3}-\dfrac{1}{2}\right|=\left|\dfrac{x+1}{2x+6}\right|$，因$x\to -1$，不妨设$|x+1|<1$，则$-2<x<0$，从而$2<|2x+6|<6$，于是$\left|\dfrac{x+1}{2x+6}\right|<\dfrac{|x+1|}{2}$，要使$\left|\dfrac{x-3}{x^2-9}-\dfrac{1}{2}\right|<\varepsilon$，只要$\dfrac{|x+1|}{2}<\varepsilon$即可。取$\delta=\min\{2\varepsilon,1\}>0$，则当$0<|x-(-1)|<\delta$时，就有$\left|\dfrac{x-3}{x^2-9}-\dfrac{1}{2}\right|<\varepsilon$总成立，故$\lim\limits_{x\to -1}\dfrac{x-3}{x^2-9}=\dfrac{1}{2}$

(2) 对$\forall\varepsilon>0$，由于$\left|\dfrac{x-3}{x^2-9}-\dfrac{1}{6}\right|=\left|\dfrac{1}{x+3}-\dfrac{1}{6}\right|=\left|\dfrac{x-3}{6x+18}\right|$，因$x\to 3$，不妨设$|x-3|<1$，则$2<x<4$，从而$30<|6x+18|<42$，于是$\left|\dfrac{x-3}{6x+18}\right|<\dfrac{|x-3|}{30}$，要使$\left|\dfrac{x-3}{x^2-9}-\dfrac{1}{6}\right|<\varepsilon$，只要$\dfrac{|x-3|}{30}<\varepsilon$即可。取$\delta=\min\{30\varepsilon,1\}>0$，则当$0<|x-3|<\delta$时，就有$\left|\dfrac{x-3}{x^2-9}-\dfrac{1}{6}\right|<\varepsilon$总成立，故$\lim\limits_{x\to 3}\dfrac{x-3}{x^2-9}=\dfrac{1}{6}$

(3) 对$\forall\varepsilon>0$，由于$\left|\dfrac{x-1}{\sqrt{x}-1}-2\right|=|\sqrt{x}+1-2|=|\sqrt{x}-1|=\left|\dfrac{x-1}{\sqrt{x}+1}\right|$，因$x\to 1$，不妨设$|x-1|<1$，则$0<x<2$，从而$1<|\sqrt{x}+1|<\sqrt{2}+1$，于是$\left|\dfrac{x-1}{\sqrt{x}+1}\right|<|x-1|$，要使$\left|\dfrac{x-1}{\sqrt{x}-1}-2\right|<\varepsilon$，只要$|x-1|<\varepsilon$即可。取$\delta=\min\{\varepsilon,1\}>0$，则当$0<|x-1|<\delta$时，就有$\left|\dfrac{x-1}{\sqrt{x}-1}-2\right|<\varepsilon$总成立，故$\lim\limits_{x\to 1}\dfrac{x-1}{\sqrt{x}-1}=2$

(4) 对$\forall\varepsilon>0$，由于$\left|\dfrac{(x-2)(x-1)}{x-3}-0\right|=\left|\left(1+\dfrac{1}{x-3}\right)(x-1)\right|$，因$x\to 1$，不妨设$|x-1|<1$，则$0<x<2$，从而$0<\left|1+\dfrac{1}{x-3}\right|<\dfrac{2}{3}$，于是$\left|1+\dfrac{1}{x-3}\right|<\dfrac{2}{3}|x-1|$，要使$\left|\dfrac{(x-2)(x-1)}{x-3}-0\right|<\varepsilon$，只要$\dfrac{2}{3}|x-1|<\varepsilon$即可。取$\delta=\min\left\{\dfrac{3}{2}\varepsilon,1\right\}>0$，则当$0<|x-1|<\delta$时，就有$\left|\dfrac{(x-2)(x-1)}{x-3}-0\right|<\varepsilon$总成立，故$\lim\limits_{x\to 1}\dfrac{(x-2)(x-1)}{x-3}=0$

(5) 对$\forall\varepsilon>0$，由于$\left|\dfrac{t(t-1)}{t^2-1}-\dfrac{1}{2}\right|=\left|\dfrac{t}{t+1}-\dfrac{1}{2}\right|=\left|\dfrac{t-1}{2t+2}\right|$，因$t\to 1$，不妨设$|t-1|<1$，则$0<t<2$，从而$2<|2t+2|<6$，于是$\left|\dfrac{t-1}{2t+2}\right|<\dfrac{|t-1|}{2}$，要使$\left|\dfrac{t(t-1)}{t^2-1}-\dfrac{1}{2}\right|<\varepsilon$，只要$\dfrac{|t-1|}{2}<\varepsilon$即可。取$\delta=\min\{2\varepsilon,1\}>0$，则当$0<|x-(-1)|<\delta$时，就有$\left|\dfrac{t(t-1)}{t^2-1}-\dfrac{1}{2}\right|<\varepsilon$总成立，故$\lim\limits_{t\to 1}\dfrac{t(t-1)}{t^2-1}=\dfrac{1}{2}$

(6) 对$\forall\varepsilon>0$，由于$\left|\frac{x-1}{x+2}-1\right|=\left|\frac{3}{x+2}\right|$，因$x\to\infty$，不妨设$|x|>2$，则$|x+2|>|x|-2$，于是$\left|\frac{3}{x+2}\right|<\frac{3}{|x|-2}$，要使$\left|\frac{3}{x+2}\right|<\varepsilon$，只要$\frac{3}{|x|-2}<\varepsilon$即可，即$|x|>\frac{3}{\varepsilon}$。取$X=\frac{3}{\varepsilon}+2$，则当$|x|>X$时，就有$\left|\frac{x-1}{x+2}-1\right|<\varepsilon$总成立，故$\lim\limits_{x\to\infty}\frac{x-1}{x+2}=1$

(7) 对$\forall G>0$，由于$\left|\frac{x}{x^2-9}\right|=\left|\frac{x}{x+3}\right|\left|\frac{1}{x-3}\right|$，因$x\to 3$，不妨设$|x-3|<1$，则$2<x<4$，从而$\frac{2}{7}<\left|\frac{x}{x+3}\right|<\frac{4}{5}$，于是$\left|\frac{x}{x+3}\right|\left|\frac{1}{x-3}\right|>\frac{2}{7}\left|\frac{1}{x-3}\right|$，要使$\left|\frac{x}{x^2-9}\right|>G$，只要$\frac{2}{7}\left|\frac{1}{x-3}\right|>G$即可。取$\delta=\min\left\{\frac{2}{7G},1\right\}>0$，则当$0<|x-3|<\delta$时，就有$\left|\frac{x}{x^2-9}\right|>G$总成立，故$\lim\limits_{x\to 3}\frac{x}{x^2-9}=\infty$

(8) 对$\forall G>0$，由于$\left|\frac{x^2+x}{x+1}\right|=|x|$，因$x\to\infty$，取$X=G>0$，则当$|x|>X$时，就有$\left|\frac{x^2+x}{x+1}\right|>G$总成立，故$\lim\limits_{x\to\infty}\frac{x^2+x}{x+1}=\infty$

2. 求极限：

(1) $\lim\limits_{x\to 0}\frac{x^2-1}{2x^2-x-1}$

(2) $\lim\limits_{x\to 1}\frac{x^2-1}{2x^2-x-1}$

(3) $\lim\limits_{x\to\infty}\frac{x^2-1}{2x^2-x-1}$

(4) $\lim\limits_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x}$

(5) $\lim\limits_{t\to 1}\frac{t^2(t-1)}{t^2-1}$

(6) $\lim\limits_{t\to 1}\frac{t^2-\sqrt{t}}{\sqrt{t}-1}$

(7) $\lim\limits_{x\to 3}\frac{\sqrt{1+x}-2}{x-3}$

(8) $\lim\limits_{x\to 0}\frac{(1+x)^5-(1+5x)}{x^2+x^5}$

(9) $\lim\limits_{x\to 0}\frac{(1+mx)^n-(1+nx)^m}{x^2}$（$m,n$为自然数）

(10) $\lim\limits_{x\to 3}\frac{x^2-5+6}{x^2-8x+15}$

(11) $\lim\limits_{x\to\infty}\frac{x^2+3x}{x^2}$

(12) $\lim\limits_{x\to\infty}\frac{5x-7}{2x+\sqrt{x}}$

解：

(1) $\lim\limits_{x\to 0}\frac{x^2-1}{2x^2-x-1}=1$

(2) $\lim\limits_{x\to 1}\frac{x^2-1}{2x^2-x-1}=\lim\limits_{x\to 1}\frac{(x-1)(x+1)}{(2x+1)(x-1)}=\lim\limits_{x\to 1}\frac{x+1}{2x+1}=\frac{2}{3}$

(3) $\lim\limits_{x\to\infty}\frac{x^2-1}{2x^2-x-1}=\frac{1}{2}$

(4) $\lim\limits_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x}=\lim\limits_{x\to 0}(6+11x+6x^2)=6$

(5) $\lim\limits_{t\to 1}\frac{t^2(t-1)}{t^2-1}=\lim\limits_{t\to 1}\frac{t^2}{t+1}=\frac{1}{2}$

(6) $\lim\limits_{t\to 1}\frac{t^2-\sqrt{t}}{\sqrt{t}-1}=\lim\limits_{t\to 1}\frac{\sqrt{t}(\sqrt{t}-1)(t+\sqrt{t}+1}{\sqrt{t}-1}=$
$\lim\limits_{t\to 1}\sqrt{t}(t+\sqrt{t}+1)=3$

(7) $\lim\limits_{x\to 3}\dfrac{\sqrt{1+x}-2}{x-3}=\lim\limits_{x\to 3}\dfrac{1}{\sqrt{1+x}+2}=\dfrac{1}{4}$

(8) $\lim\limits_{x\to 0}\dfrac{(1+x)^5-(1+5x)}{x^2+x^5}=\lim\limits_{x\to 0}\dfrac{10x^2+10x^3+5x^4+x^5}{x^2+x^5}=10$

(9) $\lim\limits_{x\to 0}\dfrac{(1+mx)^n-(1+nx)^m}{x^2}=$
$\lim\limits_{x\to 0}\dfrac{(C_n^2m^2-C_m^2n^2)x^2+(C_n^3m^3-C_m^3n^3)x^3+\cdots+m^nx^n-n^mx^m}{x^2}=C_n^2m^2-C_m^2n^2=\dfrac{n^2m-m^2n}{2}$

(10) $\lim\limits_{x\to 3}\dfrac{x^2-5+6}{x^2-8x+15}=\lim\limits_{x\to 3}\dfrac{(x-2)(x-3)}{(x-3)(x-5)}=\lim\limits_{x\to 3}\dfrac{x-2}{x-5}=-\dfrac{1}{2}$

(11) $\lim\limits_{x\to \infty}\dfrac{x^2+3x}{x^2}=1$

(12) $\lim\limits_{x\to \infty}\dfrac{5x-7}{2x+\sqrt{x}}=\dfrac{5}{2}$

3. 设　$R(x)=\dfrac{P(x)}{Q(x)}$
式中$P(x)$和$Q(x)$为x的多项式，并且$P(a)=Q(a)=0$，问$\lim\limits_{x\to a}$有哪些可能的值？
解：由于$P(x)$和$Q(x)$为x的多项式且$P(a)=Q(a)=0$，
则$P(x)=(x-a)^mP_1(x),Q(x)=(x-a)^nQ_1(x)(P_1(a)\neq 0,Q_1(x)\neq 0)$，于是$\lim\limits_{x\to a}R(x)=\lim\limits_{x\to a}\dfrac{P(x)}{Q(x)}=\lim\limits_{x\to a}\dfrac{(x-a)^mP_1(x)}{(x-a)^nQ_1(x)}$
讨论：

(1) 当$n=m$时，$\lim\limits_{x\to a}R(x)=\dfrac{P_1(a)}{Q_1(a)}$

(2) 当$n>m$时，$\lim\limits_{x\to a}(x-a)^{m-n}=\infty$且$\lim\limits_{x\to a}\dfrac{P_1(x)}{Q_1(x)}=\dfrac{P_1(a)}{Q_1(a)}\neq 0$，故$\lim\limits_{x\to a}R(x)=\infty$

(3) 当$n<m$时，$\lim\limits_{x\to a}(x-a)^{m-n}=0$且$\lim\limits_{x\to a}\dfrac{P_1(x)}{Q_1(x)}=\dfrac{P_1(a)}{Q_1(a)}$，故$\lim\limits_{x\to a}R(x)=0$

4. 求下列极限：

(1) $\lim\limits_{x\to 0}\dfrac{\sin 2x-\sin 3x}{x}$

(2) $\lim\limits_{h\to 0}\dfrac{\cos(x+h)-\cos x}{h}$

(3) $\lim\limits_{x\to +\infty}(\sqrt{x^2+1}-x)$

(4) $\lim\limits_{x\to -\infty}(\sqrt{x^2+1}-x)$

(5) $\lim\limits_{x\to 0}\dfrac{x^2}{1-\cos x}$

(6) $\lim\limits_{x\to +\infty}\dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x+1}$

(7) $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 3x}{x^2}$

(8) $\lim\limits_{x\to 0}\dfrac{\sin 5x-\sin 3x}{\sin 2x}$

(9) $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}$

(10) $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}$

(11) $\lim\limits_{x\to 0}\dfrac{(\sqrt{1+x^2}+x)^n-(\sqrt{1+x^2}-x)^n}{x}$

(12) $\lim\limits_{x\to 0}x\left[\dfrac{1}{x}\right]$

解：

(1) $\lim\limits_{x\to 0}\dfrac{\sin 2x-\sin 3x}{x}=\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}-\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}=2-3=-1$

(2) $\lim\limits_{h\to 0}\dfrac{\cos(x+h)-\cos x}{h}=\lim\limits_{h\to 0}\dfrac{-2\sin\dfrac{2x+h}{2}\sin\dfrac{h}{2}}{h}=\lim\limits_{h\to 0}\dfrac{\sin\dfrac{h}{2}}{h}\sin\dfrac{2x+h}{2}=-\sin x$

(3) $\lim\limits_{x\to +\infty}(\sqrt{x^2+1}-x)=\lim\limits_{x\to +\infty}\dfrac{1}{\sqrt{x^2+1}+x}=0$

(4) $\lim\limits_{x\to -\infty}(\sqrt{x^2+1}-x)=+\infty$

(5) $\lim\limits_{x\to 0}\dfrac{x^2}{1-\cos x}=\lim\limits_{x\to 0}\dfrac{x^2}{\dfrac{x^2}{2}}=2$

(6) $\lim\limits_{x\to +\infty}\dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x+1}=0$

(7) $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 3x}{x^2}=\lim\limits_{x\to 0}\dfrac{2\sin x\sin 2x}{x^2}=4$

(8) $\lim\limits_{x\to 0}\dfrac{\sin 5x-\sin 3x}{\sin 2x}=\lim\limits_{x\to 0}\dfrac{2\sin x\cos 4x}{2x}=1$

(9) 令$y=x-1$，则$\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}=\lim\limits_{y\to 0}-y\tan\left(\dfrac{\pi}{2}(1+y)\right)=\lim\limits_{y\to 0}-y\cot\dfrac{\pi}{2}y=\lim\limits_{y\to 0}\dfrac{y\cos\dfrac{\pi}{2}y}{\sin\dfrac{\pi}{2}y}=$ $\lim\limits_{y\to 0}\dfrac{y}{\dfrac{\pi}{2}y}=\dfrac{2}{\pi}$

(10) $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}=\lim\limits_{x\to a}\dfrac{2\cos\dfrac{x+a}{2}\sin\dfrac{x-a}{2}}{x-a}=\cos a$

(11) $\lim\limits_{x\to 0}\dfrac{(\sqrt{1+x^2}+x)^n-(\sqrt{1+x^2}-x)^n}{x}=$ $\lim\limits_{x\to 0}\dfrac{2C_n^1(1+x^2)^{\frac{n-1}{2}}x+2C_n^3(1+x^2)^{\frac{n-3}{2}}x^2+\cdots}{x}=$ $\lim\limits_{x\to 0}\left[2n(1+x^2)^{\frac{n-1}{2}}+2C_n^3(1+x^2)^{\frac{n-3}{2}}x+\cdots\right]=2n$

(12) 由于$\left[\dfrac{1}{x}\right]=\dfrac{1}{x}-\left(\dfrac{1}{x}\right)$且$0\leqslant\left(\dfrac{1}{x}\right)<1$，
则$\lim\limits_{x\to 0}x\left[\dfrac{1}{x}\right]=\lim\limits_{x\to 0}\left\{1-x\left(\dfrac{1}{x}\right)\right\}=1-\lim\limits_{x\to 0}x\left(\dfrac{1}{x}\right)=1$

5. 若$\lim\limits_{x\to x_0}f(x)=A,\lim\limits_{x\to x_0}g(x)=B$，并且存在$\delta>0$，当$0<|x-x_0|<\delta$时有$f(x)\geqslant g(x)$，证明$A\geqslant B$.
又若当$0<|x-x_0|<\delta$时$f(x)>g(x)$，是否一定成立$A>B$

证明:

(1) 用反证法。假设$A<B$，则由$\lim\limits_{x\to x_0}f(x)=A,\lim\limits_{x\to x_0}g(x)=B$及性质1，得$\exists\delta_0>0$，使当$0<|x-x_0|<\delta_0$时，有$g(x)>f(x)$。这与已知：$\exists\delta>0$，当$0<|x-x_0|<\delta$时，有$f(x)\geqslant g(x)$矛盾，故假设不成立，即$A\geqslant B$成立。

(2) 不一定。例：

(i) 成立。$f(x)=\dfrac{2(x^2+3x^4)}{x^2},g(x)=x^2+3x^4x^2,\exists\delta>0$，当$0<|x|<\delta$时，有$f(x)>g(x)$。又$A=\lim\limits_{x\to x_0}f(x)=2,B=\lim\limits_{x\to x_0}g(x)=1$，故$A>B$成立。

(ii) 不成立。$f(x)=\dfrac{x^2+3x^4}{x^2},g(x)=x^2+x^4x^2,\exists\delta>0$，当$0<|x|<\delta$时，有$f(x)>g(x)$。又$A=\lim\limits_{x\to x_0}f(x)=1,B=\lim\limits_{x\to x_0}g(x)=1$，故有$A=B$。

6. 若在点x_0的邻域内有$g(x)\leqslant f(x)\leqslant h(x)$，并且$g(x)$和$h(x)$在$x_0$的极限存在并且都等于A，证明$\lim\limits_{x\to x_0}f(x)=A$.

证明: 如果对任何$x_n,x_n\to x_0,x_n\neq x_0$，并且可不妨假设$x_n\in O(x_0,\delta)-\{x_0\}$，有$g(x_n)\leqslant f(x_n)\leqslant h(x_n)$以及$g(x_n)\to A,h(x_n)\to A(n\to\infty)$，由数列极限的性质得：$f(x_n)\to A(n\to\infty)$，这就证明了$f(x)\to A(x\to x_0)$.

7. 若 $\lim\limits_{x\to x_0} f(x)=A, \lim\limits_{x\to x_0} g(x)=B\neq 0$，证明 $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{A}{B}$.

证明: 考察 $\left|\dfrac{f(x)}{g(x)}-\dfrac{A}{B}\right|=\left|\dfrac{Bf(x)-Ag(x)}{Bg(x)}\right|=\left|\dfrac{Bf(x)-AB+AB-Ag(x)}{BG(x)}\right|\leqslant\dfrac{|B||f(x)-A|+|A||g(x)-B|}{|B||g(x)|}$,

由于 $\lim\limits_{x\to x_0} f(x)=A, \lim\limits_{x\to x_0} g(x)=B$，故对$\forall\varepsilon>0,\exists\delta_1>0$，当$0<|x-x_0|<\delta_1$时，有$|f(x)-A|<\varepsilon$；对上述$\varepsilon>0,\exists\delta_2>0$，当$0<|x-x_0|<\delta_2$时，有$|g(x)-B|<\varepsilon$

又据乘法运算：$\lim\limits_{x\to x_0} Bg(x)=B^2>\dfrac{B^2}{2}$，则据性质3，得$\exists\delta_3>0$，当$0<|x-x_0|<\delta_3$时，有$Bg(x)>\dfrac{B^2}{2}$

取$\delta=\min\{\delta_1,\delta_2,\delta_3\}$，当$0<|x-x_0|<\delta$时，有$\left|\dfrac{f(x)}{g(x)}-\dfrac{A}{B}\right|<\dfrac{(|A|+|B|)\varepsilon}{\frac{B^2}{2}}=\dfrac{2(|A|+|B|)}{B^2}\varepsilon$

于是，对$\forall\varepsilon>0,\exists\delta>0$，当$0<|x-x_0|<\delta$时，有$\left|\dfrac{f(x)}{g(x)}-\dfrac{A}{B}\right|<\dfrac{2(|A|+|B|)}{B^2}\varepsilon$，从而 $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{A}{B}$.

8. (1) $f(x)=\begin{cases}0 & x>1\\ 1 & x=1\\ x^2+2 & x<1\end{cases}$

求$f(x)$在$x=1$的左右极限。

(2) $f(x)=\begin{cases}x\sin\dfrac{1}{x} & x>0\\ 1+x^2 & x<0\end{cases}$

求$f(x)$在$x=0$的左右极限。

解:

(1) $\lim\limits_{x\to 1-0} f(x)=\lim\limits_{x\to 1-0}(x^2+2)=3, \lim\limits_{x\to 1+0} f(x)=0$

(2) $\lim\limits_{x\to -0} f(x)=\lim\limits_{x\to -0}(1+x^2)=1,$

$\lim\limits_{x\to +0} f(x)=\lim\limits_{x\to +0}(x\sin\frac{1}{x})=0$

9. 说明下列函数在所示点的左右极限情形：

(1) $y=\begin{cases}\dfrac{1}{2x} & 0<x\leqslant 1\\ x^2 & 1<x<2\\ 2x & 2<x<3\end{cases}$ (在$x=1.5,2,1$三点)

(2) $y=x\cdot\sin\dfrac{1}{x}$（在$x=0$点）

(3) $y=\dfrac{2^{\frac{1}{x}}+1}{2^{\frac{1}{x}}-1}$（在$x=0$点）

(4) $y=\dfrac{1}{x}-\left[\dfrac{1}{x}\right]$（在$x=\dfrac{1}{n}$点）

(5) $D(x)=\begin{cases}1 & x\text{为有理数}\\ 0 & x\text{为无理数}\end{cases}$ (在任一点)

(6) $y=\dfrac{(x-1)(-1)^{[x]}}{x^2-1}$(在$x=-1$）

解:

(1) $\lim\limits_{x\to 1.5-0} y=\lim\limits_{x\to 1.5+0} y=2.25,$

$\lim\limits_{x\to 2-0} y=\lim\limits_{x\to 2-0} x^2=4, \lim\limits_{x\to 2+0} y=\lim\limits_{x\to 2+0}(2x)=4$

$\lim\limits_{x\to 1-0} y=\lim\limits_{x\to 1-0}\dfrac{1}{2x}=\dfrac{1}{2}, \lim\limits_{x\to 1+0} y=\lim\limits_{x\to 1+0} x^2=1$

(2) $\lim\limits_{x\to +0} y=\lim\limits_{x\to +0} y=0$

(3) 由于 $\lim\limits_{x\to +0}\dfrac{1}{x}=+\infty, \lim\limits_{x\to -0}\dfrac{1}{x}=-\infty$，

则 $\lim\limits_{x\to +0} 2^{\frac{1}{x}}=+\infty, \lim\limits_{x\to -0} 2^{\frac{1}{x}}=0$，

于是 $\lim\limits_{x\to +0} y=\lim\limits_{x\to =0}\dfrac{2^{\frac{1}{x}}+1}{2^{\frac{1}{x}}-1}=\lim\limits_{x\to +0}\left(1+\dfrac{2}{2^{\frac{1}{x}}-1}\right)=1, \lim\limits_{x\to +-0} y=\lim\limits_{x\to -0}\dfrac{2^{\frac{1}{x}}+1}{2^{\frac{1}{x}}-1}=-1$

(4) $\lim\limits_{x\to\frac{1}{n}+0} y=\lim\limits_{x\to\frac{1}{n}+0}\left(\frac{1}{x}-\left[\frac{1}{x}\right]\right)=n-(n-1)=1$

$\lim\limits_{x\to\frac{1}{n}-0} y=\lim\limits_{x\to\frac{1}{n}-0}\left(\frac{1}{x}-\left[\frac{1}{x}\right]\right)=n-n=0$

(5) 此函数在任一点的左右极限不存在。

设x_0为R上任一点，由有理数和无理数在数轴上的稠密性，可知有理序列$\{x_n^{(1)}\}\to x_0+0$，无理序列$\{x_n^{(2)}\}\to x_0+0$，

故$\lim\limits_{x_n^{(1)}\to x_0+0} D\left(x^{(1)}\right)=1$，$\lim\limits_{x_n^{(2)}\to x_0+0} D\left(x^{(2)}\right)=0$，从而此函数在任一点的右极限不存在

同理，此函数在任一点的左极限也不存在

从而此函数在任一点的左右极限不存在。

(6) $y=\dfrac{(x-1)(-1)^{[x]}}{x^2-1}=\dfrac{(-1)^{[x]}}{x+1}$且$\lim\limits_{x\to-1+0}[x]=-1$, $\lim\limits_{x\to-1-0}[x]=-2$

则$\lim\limits_{x\to-1+0} y=-\infty$, $\lim\limits_{x\to-1+0} y=-\infty$

10. 讨论下列极限：

(1) $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}$

(2) $\lim\limits_{x\to\infty} e^x\sin x$

(3) $\lim\limits_{x\to\infty} x\arctan x$

(4) $\lim\limits_{x\to\infty} x\tan x(x\neq n\pi+\dfrac{\pi}{2})$

解:

(1) 由于$\lim\limits_{x\to\infty}\dfrac{1}{x}=0$且$\sin x$是有界量，故$\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0$

(2) 由于$\lim\limits_{x\to+\infty} e^x=+\infty$，若取$x_n=2n\pi\to+\infty(n\to\infty)$，则$e^{x_n}\sin x_n=e^{2n\pi}\sin 2n\pi=0\to 0(n\to\infty)$；若取$x_n=\dfrac{\pi}{2}+2n\pi\to+\infty(n\to\infty)$，则$e^{x_n}\sin x_n=e^{\frac{\pi}{2}+2n\pi}\sin\left(\dfrac{\pi}{2}+2n\pi\right)=e^{\frac{\pi}{2}+2n\pi}\to+\infty(n\to\infty)$，故$\lim\limits_{x\to+\infty} e^x\sin x$不存在，从而$\lim\limits_{x\to\infty} e^x\sin x$不存在.

(3) 由于$\lim\limits_{x\to-\infty}\arctan x=-\dfrac{\pi}{2}$, $\lim\limits_{x\to+\infty} x\arctan x=\dfrac{\pi}{2}$,

则$\lim\limits_{x\to-\infty} x\arctan x=+\infty$, $\lim\limits_{x\to+\infty} x\arctan x=+\infty$，从而$\lim\limits_{x\to\infty} x\arctan x=+\infty$

(4) 取$x_n=n\pi\to\infty(n\to\infty)$，有$\lim\limits_{n\to\infty} x_n\tan x_n=\lim\limits_{n\to\infty} n\pi\tan n\pi=0$；另取$x_n=\dfrac{\pi}{4}+n\pi\to\infty(n\to\infty)$，有$\lim\limits_{n\to\infty} x_n\tan x_n=\lim\limits_{n\to\infty}\left(\dfrac{\pi}{4}+n\pi\right)\tan\left(\dfrac{\pi}{4}+n\pi\right)=\lim\limits_{n\to\infty}\left(\dfrac{\pi}{4}+n\pi\right)=+\infty$，故$\lim\limits_{x\to\infty} x\tan x(x\neq n\pi+\dfrac{\pi}{2})$不存在.

11. 从条件$\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right)=0$，求常数$a$和$b$.

解: 由于$\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right)=\lim\limits_{x\to\infty}\dfrac{(x^2+1)-ax(x+1)-b(x+1)}{x+1}=\lim\limits_{x\to\infty}\dfrac{(1-a)x^2-(a+b)x-b+1}{x+1}=0$，则有$\begin{cases}1-a=0\\a+b=0\end{cases}$，从而$\begin{cases}a=1\\b=-1\end{cases}$

12. 从条件$\lim\limits_{x\to-\infty}(\sqrt{x^2-x+1}-a_1x-b_1)=0$, $\lim\limits_{x\to-\infty}(\sqrt{x^2-x+1}-a_2x-b_2)=0$，求常数$a_1,b_1,a_2,b_2$.

解: 由于$\lim\limits_{x\to-\infty}(\sqrt{x^2-x+1}-a_1x-b_1)=\lim\limits_{x\to-\infty}\dfrac{(1-a_1^2)x^2-(1+2a_1b_1)x+1-b_1^2}{\sqrt{x^2-x+1}+a_1x+b_1}=0$，则$\begin{cases}1-a_1^2=0\\1+2a_1b_1=0\end{cases}$，于是$\begin{cases}a_1=\pm1\\b_1=\mp\dfrac{1}{2}\end{cases}$.

又据条件可得：若$a_1=1$，则$\lim\limits_{x\to-\infty}(\sqrt{x^2-x+1}-a_1x-b_1)=+\infty$，从而$\begin{cases}a_1=-1\\b_1=\dfrac{1}{2}\end{cases}$，同理$\begin{cases}a_2=1\\b_2=-\dfrac{1}{2}\end{cases}$.

13. 若$\lim\limits_{x\to+\infty}[f(x)-(kx+b)]=0$，则称直线$y=kx+b$是曲线$y=f(x)$当$x\to+\infty$的渐近线.利用这一方程推出渐近线存在的必要且充分的条件.

证明: 若曲线存在渐近线，则有

$$\lim_{x\to+\infty}[f(x)-(kx+b)]=0. \tag{1}$$

因$\dfrac{f(x)}{x}=\dfrac{1}{x}[f(x)-kx-b]+k+\dfrac{b}{x}$，令$x\to+\infty$两端取极限并注意到(1)式，得

$$\lim_{x\to+\infty}\frac{f(x)}{x}=k \tag{2}$$

.

既求出了k，再从(1)式求得

$$b=\lim_{x\to+\infty}[f(x)-kx] \tag{3}$$

.

反之，若(2)、(3)两式成立，立即可看出条件(1)成立.

故曲线$y=f(x)$当$x\to+\infty$时存在渐近线$y=kx+b$的充分必要条件是极限$\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=k$、$\lim\limits_{x\to+\infty}[f(x)-kx]=b$均成立.

14. 若$\lim\limits_{x\to-\infty}f(x)=A>0$，证明存在$X>0$，使得当$x<-X$成立：$\dfrac{A}{2}<f(x)<\dfrac{3}{2}A$.

证明：由于$\lim\limits_{x\to-\infty}f(x)=A>0$，故对给定的$\varepsilon=\dfrac{A}{2}>0,\exists X>0$ ，当$x<-X$时，有$|f(x)-A|<\dfrac{A}{2}$，即$\dfrac{A}{2}<f(x)<\dfrac{3}{2}A$.

15. 若$\lim\limits_{x\to+\infty}f(x)=A,\lim\limits_{x\to+\infty}g(x)=B$，证明$\lim\limits_{x\to+\infty}f(x)g(x)=AB$.

证明：由于$\lim\limits_{x\to+\infty}f(x)=A$，故对$\forall\varepsilon>0,\exists X_1>0$，当$x>X_1$时，有$|f(x)-A|<\varepsilon$且$\exists X_2>0,M>0$，当$x>X_2$时，有$|f(x)|<A$.

又$\lim\limits_{x\to+\infty}g(x)=B$，故对上述$\varepsilon>0,\exists X_3>0$，当$x>X_3$时，有$|g(x)-B|<\varepsilon$.

取$X=\max\{X_1,X_2,X_3\}$，对上述$\varepsilon>0$，当$x>X$时，

有$|f(x)g(x)-AB|=|f(x)g(x)-f(x)B+f(x)B-AB|\leqslant|f(x)||g(x)-B|+|B||f(x)-A|\leqslant M\varepsilon+|B|\varepsilon=(M+|B|)\varepsilon$ ，即$\lim\limits_{x\to+\infty}f(x)g(x)=AB$.

16. 证明$\lim\limits_{x\to+\infty}f(x)=A$的充要条件是：对任何数列$x_n\to+\infty,f(x_n)\to A$.

证明：

$\Rightarrow$ 由于$\lim\limits_{x\to+\infty}f(x)=A$，故对$\forall\varepsilon>0,\exists X>0$，当$x>X$时，有$|f(x)-A|<\varepsilon$.

又$x_n\to+\infty(n\to\infty)$，故对上述$X>0,\exists N\in Z^+$，当$n>N$时，有$x_n>X$，从而$|f(x_n)-A|<\varepsilon$，于是$\lim\limits_{n\to\infty}f(x_n)=A$.

$\Leftarrow$ 用反证法。假设$\lim\limits_{x\to+\infty}f(x)\neq A$，则$\exists\varepsilon_0>0$，对$\forall X>0$，至少有一个$x'$，当$x'>X$时，有$|f(x')-A|\geqslant\varepsilon_0$.

特别地，取X为$1,2,3,\cdots$，可得$x_1',x_2',x_3',\cdots$，使得

$x_1'>1$时，有$|f(x_1')-A|\geqslant\varepsilon_0$；$x_2'>2$时，有$|f(x_2')-A|\geqslant\varepsilon_0$；$x_3'>3$时，有$|f(x_3')-A|\geqslant\varepsilon_0$；$\cdots$

从左边可以看出$x_n'\to+\infty(n\to\infty)$，而从右边看出$\lim\limits_{n\to\infty}f(x_n')\neq A$，与已知矛盾，则假设不成立，故$\lim\limits_{x\to+\infty}f(x)=A$

17. 证明$\lim\limits_{x\to x_0+0}f(x)=+\infty$的充要条件是：对任何数列$x_n:x_n>x_0,x_n\to x_0$，有$f(x_n)\to+\infty$.

证明：

$\Rightarrow$ 由于$\lim\limits_{x\to x_0+0}f(x)=+\infty$，故对$\forall G>0,\exists\delta>0$，当$0<x-x_0<\delta$时，有$f(x)>G$.

又$x_n>x_0,x_n\to x_0(n\to\infty)$，故对上述$\delta>0,\exists N\in Z^+$，当$n>N$时，有$0<x_n-x_0<\delta$，从而$f(x_n)>G$，于是$\lim\limits_{n\to\infty}f(x_n)=+\infty$.

$\Leftarrow$ 用反证法。假设$\lim\limits_{x\to x_0+0}f(x)\neq+\infty$，则$\exists G_0>0$，对$\forall\delta>0$，至少有一个$x'$，当$0<x'-x_0<\delta$时，有$f(x')\leqslant G_0$.

特别地，取δ为$1,\dfrac{1}{2},\dfrac{1}{3},\cdots$，可得$x_1',x_2',x_3',\cdots$，使得

$0<x_1'-x_0<1$时，有$f(x_1')\leqslant G_0$；$0<x_2'-x_0<\dfrac{1}{2}$时，有$f(x_2')\leqslant G_0$；$0<x_3'-x_0<\dfrac{1}{3}$时，有$f(x_3')\leqslant G_0$；$\cdots$

从左边可以看出$x_n'>x_0,x_n'\to x_0$，而从右边看出$\lim\limits_{x\to x_0+0}f(x)\neq+\infty$，与已知矛盾，则假设不成立，故$\lim\limits_{x\to x_0+0}f(x)=+\infty$

18. 举出符合下列要求的$f(x)$

    (1) $f(+0)=0,f(-0)=1$

(2) $f(+0)$不存在，也非$\infty, f(-0)=0$

(3) $f(+\infty)=0, f(-\infty)$不存在

(4) $f(+\infty)=f(-\infty)=A$（常数）

(5) $f(x_0+0)$和$f(x_0-0)$都不存在

(6) $f(x_0+0)=+\infty, f(x_0-0)=-\infty$

(7) $f(x_0+0)=1, f(x_0-0)=+\infty$

(8) $f(+\infty)$不存在，也非$\infty, f(-\infty)=-\infty$

解:

(1) $f(x)=\begin{cases} 0 & x>0 \\ 1 & x\leqslant 0 \end{cases}$

(2) $f(x)=\begin{cases} \sin\dfrac{1}{x} & x>0 \\ 0 & x\leqslant 0 \end{cases}$

(3) $f(x)=e^{-x}$

(4) $f(x)=\dfrac{Ax+1}{x}$

(5) $f(x)=\sin\dfrac{1}{x-x_0}$

(6) $f(x)=\dfrac{1}{x-x_0}$

(7) $f(x)=1+e^{-\frac{1}{x-x_0}}$

(8) $f(x)=\begin{cases} \sin x & x\geqslant 0 \\ x & x<0 \end{cases}$

§3. 连续函数

1. 按定义证明下列函数在定义域内连续：

(1) $y=\sqrt{x}$

(2) $y=\dfrac{1}{x}$

(3) $y=|x|$

(4) $y=\sin\dfrac{1}{x}$

证明：

(1) 设x_0为$(0,+\infty)$内任一点，$|\sqrt{x}-\sqrt{x_0}|<\dfrac{|x-x_0|}{\sqrt{x}+\sqrt{x_0}}\leqslant\dfrac{|x-x_0|}{\sqrt{x_0}}$
对$\forall\varepsilon>0$，取$\delta=\sqrt{x_0}\varepsilon$，当$|x-x_0|<\delta$时，有$|\sqrt{x}-\sqrt{x_0}|<\dfrac{|x-x_0|}{\sqrt{x_0}}<\varepsilon$，故$y=\sqrt{x}$在$x_0$点连续.
又由x_0在$(0,+\infty)$中的任意性，则$y=\sqrt{x}$在$(0,+\infty)$内连续.
当$x_0=0$时，对上述$\varepsilon>0$，取$\delta=\varepsilon^2$，当$0<x-x_0<\delta$时，有$|\sqrt{x}-\sqrt{x_0}|<\sqrt{x}<\varepsilon$，故$f(+0)=0=f(0)$，
从而$y=\sqrt{x}$在$[0,+\infty)$内连续.

(2) 设x_0为$(0,+\infty)$内任一点，不妨设$|x-x_0|<\dfrac{x_0}{2}$，则$x>\dfrac{x_0}{2},xx_0>\dfrac{x_0^2}{2}$，于是$|\dfrac{1}{x}-\dfrac{1}{x_0}|=\dfrac{|x-x_0|}{xx_0}<\dfrac{|x-x_0|}{\frac{x_0^2}{2}}$
若x_0为$(-\infty,0)$内任一点，不妨设$|x-x_0|<-\dfrac{x_0}{2}$，则$x<\dfrac{x_0}{2},xx_0>\dfrac{x_0^2}{2}$，于是$|\dfrac{1}{x}-\dfrac{1}{x_0}|=\dfrac{|x-x_0|}{xx_0}<\dfrac{|x-x_0|}{\frac{x_0^2}{2}}$
设x_0为$(-\infty,0)\bigcup(0,+\infty)$内任一点，
对$\forall\varepsilon>0$，取$\delta=\min\left\{\dfrac{|x_0|}{2},\dfrac{x_0^2}{2}\varepsilon\right\}>0$，当$|x-x_0|<\delta$时，有$\left|\dfrac{1}{x}-\dfrac{1}{x_0}\right|=\dfrac{|x-x_0|}{xx_0}>\varepsilon$，故$y=\dfrac{1}{x}$在$x_0$点连续
又由x_0在$(-\infty,0)\bigcup(0,+\infty)$内的任意性，得$y=\dfrac{1}{x}$在$(-\infty,0)\bigcup(0,+\infty)$内连续.

(3) 设x_0为$(-\infty,+\infty)$内任一点，$||x|-|x_0||\leqslant|x-x_0|$.
对$\forall\varepsilon>0$，取$\delta=\varepsilon>0$，当$|x-x_0|<\delta$时，有$||x|-|x_0||\leqslant|x-x_0|<\varepsilon$，故$y=|x|$在$x_0$点连续
又由x_0在$(-\infty,+\infty)$内的任意性，得$y=|x|$在$(-\infty,+\infty)$内连续.

(4) 设x_0为$(0,+\infty)$内任一点，不妨设$|x-x_0|<\dfrac{x_0}{2}$，则$x>\dfrac{x_0}{2},xx_0>\dfrac{x_0^2}{2}$，于是$\left|\sin\dfrac{1}{x}-\sin\dfrac{1}{x_0}\right|=2\left|\sin\dfrac{x+x_0}{2xx_0}\right|\left|\cos\dfrac{x-x_0}{2xx_0}\right|\leqslant\dfrac{|x-x_0|}{xx_0}<\dfrac{|x-x_0|}{\frac{x_0^2}{2}}$
若x_0为$(-\infty,0)$内任一点，不妨设$|x-x_0|<-\dfrac{x_0}{2}$，则$x<\dfrac{x_0}{2},xx_0>\dfrac{x_0^2}{2}$，于是$\left|\sin\dfrac{1}{x}-\sin\dfrac{1}{x_0}\right|\leqslant\dfrac{|x-x_0|}{xx_0}<\dfrac{|x-x_0|}{\frac{x_0^2}{2}}$
设x_0为$(-\infty,0)\bigcup(0,+\infty)$内任一点，
对$\forall\varepsilon>0$，取$\delta=\min\left\{\dfrac{|x_0|}{2},\dfrac{x_0^2}{2}\varepsilon\right\}>0$，当$|x-x_0|<\delta$时，有$\left|\sin\dfrac{1}{x}-\sin\dfrac{1}{x_0}\right|\leqslant\dfrac{|x-x_0|}{xx_0}<\varepsilon$，故$y=\sin\dfrac{1}{x}$在$x_0$点连续
又由x_0在$(-\infty,0)\bigcup(0,+\infty)$内的任意性，得$y=\sin\dfrac{1}{x}$在$(-\infty,0)\bigcup(0,+\infty)$内连续.

2. 利用连续函数的运算，求下列函数的连续范围：

(1) $y=\tan x$

(2) $y=\dfrac{1}{x^n}$

(3) $y=\sec x+\csc x$

(4) $y=\dfrac{1}{\sqrt{\cos x}}$

(5) $y=\dfrac{\ln(1+x)}{x^2-2x}$

(6) $y=\dfrac{[x]\tan x}{1+\sin x}$

解:

(1) 因$y=\tan x=\dfrac{\sin x}{\cos x}$，则当$\cos x\neq 0$时，$y=\tan x$连续，故$y=\tan x$的连续范围为$\left(-\dfrac{\pi}{2}+k\pi,\dfrac{\pi}{2}+k\pi\right)$ $(k\in Z)$.

(2) 若$n>0$，则$y=\dfrac{1}{x^n}$的连续范围为$(-\infty,0)\bigcup(0,+\infty)$；若$n\leqslant 0$，则$y=\dfrac{1}{x^n}$连续，即它的连续范围为$(-\infty,+\infty)$.

(3) 因$\sec x$的连续范围为$\left(k-\dfrac{1}{2}\right)\pi<x<\left(k+\dfrac{1}{2}\right)\pi(k=0,\pm1,\pm2,\cdots)$，$\csc x$的连续范围为$k\pi<x<(k+1)\pi(k=0,\pm1,\pm2,\cdots)$，
故$y=\sec x+\csc x$的连续范围为$\left(k\pi-\dfrac{\pi}{2}\right)\bigcup\left(k\pi,k\pi+\dfrac{\pi}{2}\right)((k=0,\pm1,\pm2,\cdots)$.

(4) 当$\cos x>0$时，$y=\dfrac{1}{\sqrt{\cos x}}$连续，故$y=\dfrac{1}{\sqrt{\cos x}}$的连续范围为$\left(-\dfrac{\pi}{2}+2k\pi,\dfrac{\pi}{2}+2k\pi\right)$.

(5) 因$\ln(1+x)$当$x>-1$时连续，$\dfrac{1}{x^2-2x}$当$x\neq 0,x\neq 2$时连续，故$y=\dfrac{\ln(1+x)}{x^2-2x}$的连续范围为$(-1,0)\bigcup(0,2)\bigcup(2,+\infty)$.

(6) 因$y=\dfrac{[x]\tan x}{1+\sin x}=\dfrac{[x]\sin x}{(1+\sin x)\cos x}$，则当$\sin x\neq 1,\cos x\neq 0,x\notin Z/\{0\}$时，$y=\dfrac{[x]\tan x}{1+\sin x}$连续，
故$y=\dfrac{[x]\tan x}{1+\sin x}$的连续范围为$x\in\left(k\pi-\dfrac{\pi}{2},k\pi+\dfrac{\pi}{2}\right)$且$x\notin Z/\{0\}(k\in Z)$.

3. 研究下列函数的连续性，并画出其图形.

(1) $y=\begin{cases}\dfrac{x^2-4}{x-2}, & 若x\neq 2\\ 4, & x=2\end{cases}$

(2) $y=\begin{cases}\left|\dfrac{\sin x}{x}\right|, & x\neq 0\\ 1, & x=0\end{cases}$

(3) $y==\begin{cases}\dfrac{\sin x}{|x|}, & x\neq 0\\ 1, & x=0\end{cases}$

(4) y=[x]

解:

(1) 因$\lim\limits_{x\to2}y=\lim\limits_{x\to2}\dfrac{x^2-4}{x-2}=\lim\limits_{x\to2}(x+2)=4$，且当$x=2$时，$y=4$，故函数在$x=2$连续
当$x\neq 2$时，$y=\dfrac{x^2-4}{x-2}=x+2$显然连续，
故$y=\begin{cases}\dfrac{x^2-4}{x-2}, & 若x\neq 2\\ 4, & x=2\end{cases}$ 在$(-\infty,+\infty)$内连续.

(2) 当$x\neq 0$时，$y=\left|\dfrac{\sin x}{x}\right|=\dfrac{\sin x}{x}$或$y=-\dfrac{\sin x}{x}$显然连续。又$\lim\limits_{x\to0}\left|\dfrac{\sin x}{x}\right|=1=f(0)$，故函数在$x=0$连续，
于是$y=\begin{cases}\left|\dfrac{\sin x}{x}\right|, & x\neq 0\\ 1, & x=0\end{cases}$ 在$(-\infty,+\infty)$内连续.

(3) 因$\lim\limits_{x\to+0}y=\lim\limits_{x\to+0}\dfrac{\sin x}{|x|}=1,\lim\limits_{x\to-0}y=\lim\limits_{x\to-0}\dfrac{\sin x}{|x|}=-1$，故$\lim\limits_{x\to0}y$不存在。又当$x>0$时，$y=\dfrac{\sin x}{|x|}=\dfrac{\sin x}{x}$，当$x<0$时，$y=\dfrac{\sin x}{|x|}=-\dfrac{\sin x}{x}$，显然连续，故此函数在除0外连续，即在$(-\infty,0)\bigcup(0,+\infty)$内连续.

(4) 因$\lim\limits_{x\to k+0}y=\lim\limits_{x\to k+0}[x]=k,\lim\limits_{x\to k-0}y=\lim\limits_{x\to k-0}[x]=k-1(k\in Z)$，则$\lim\limits_{x\to k}y$不存在，故$x=k(k\in Z)$为$y=[x]$的间断点，但在间断点处右连续
当$k<x<k+1(k\in Z)$时，$y=[x]$显然连续，故此函数在除$k(k\in Z)$外连续.

4. 若$f(x)$连续，$|f(x)|$和$f^2(x)$是否也连续？又若$|f(x)|$或$f^2(x)$连续，$f(x)$是否连续？

解:

(1) 设$f(x)$在其定义域I上连续，x_0为I上任一点
因$f(x)$在x_0点连续，故对$\forall\varepsilon>0,\exists\delta>0$，当$|x-x_0|<\delta$时，有$|f(x)-f(x_0)|<\varepsilon$
而$||f(x)|-|f(x_0)||\leqslant|f(x)-f(x_0)|<\varepsilon$，即对$\forall\varepsilon>0,\exists\delta>0$，当$|x-x_0|<\delta$时，有$|f(x)-f(x_0)|<\varepsilon$，故$|f(x)|$在$x_0$点连续
又由x_0在I上的任意性，知$|f(x)|$在I上也连续
同样$|f^2(x)-f^2(x_0)|=|f(x)-f(x_0)||f(x)+f(x_0)|=|f(x)-f(x_0)||f(x)-f(x_0)+2f(x_0)|\leqslant|f(x)-f(x_0)|(|f(x)-f(x_0)|+2f(x_0))<\varepsilon(\varepsilon+2f(x_0))$，故$f^2(x)$在$x_0$点连续
又由x_0在I上的任意性，知$f^2(x)$在I上也连续

(2) 反过来，若$|f(x)|$或$f^2(x)$连续，$f(x)$不一定连续.

(i) 不连续。例：$f(x)=\begin{cases}1, & x\geqslant0\\ -1, & x<0\end{cases}$，$|f(x)|=1$和$f^2(x)=1$均在$(-\infty,+\infty)$内连续，但$f(x)$在$x=0$点不连续；

(ii) 连续。例：$f(x)=x$,则$f(x)$、$|f(x)|$、$f^2(x)$在$(-\infty,+\infty)$内均连续。

5. (1) 函数$f(x)$当$x=x_0$时连续，而函数$g(x)$当$x=x_0$时不连续，问此二函数的和在x_0点是否连续？
(2) 当$x=x_0$时函数$f(x)$和$g(x)$二者都不连续，问此二函数的和$f(x)+g(x)$在已知点x_0是否必为不连续？

解:

(1) 用反证法。假设$f(x)+g(x)$在x_0点连续。
因$f(x)$当$x=x_0$时连续，则由连续函数性质，得$g(x)=[f(x)+g(x)]-f(x)$当x_0时连续与已知矛盾。
故假设不成立，即$f(x)+g(x)$在x_0点连续.

(2) 不一定。

(i) 连续：例：$f(x)=\begin{cases}1, & x\geqslant0\\ -1, & x<0\end{cases}, g(x)=\begin{cases}-1, & x\geqslant0\\ 1, & x<0\end{cases}$在$x=0$都不连续，但$f(x)+g(x)=0$在$x=0$连续.

(ii) 不连续：例：$f(x)=g(x)=\dfrac{1}{x}$在$x=0$都不连续，$f(x)+g(x)=\dfrac{2}{x}$在$x=0$不连续.

6. (1) 函数$f(x)$在x_0连续，而函数$g(x)$在x_0不连续；
(2) 当$x=x_0$时函数$f(x)$和$g(x)$二者都不连续，问此二函数的乘积$f(x)g(x)$在已知点x_0是否必不连续？

解:

(1) 不一定。

(i) 连续：例：$f(x)=0$在$x=0$连续，$g(x)=\begin{cases}1, & x\geqslant0\\ 0, & x<0\end{cases}$在$x=0$不连续，但$f(x)g(x)=0$在$x=0$连续.

(ii) 不连续：例：$f(x)=x$在$x=0$连续，$g(x)=\dfrac{1}{x^2}$在$x=0$不连续，$f(x)g(x)=\dfrac{1}{x}$在$x=0$不连续.

(2) 不一定。

(i) 连续：例：$f(x)=\begin{cases}1, & x\geqslant0\\ -1, & x<0\end{cases}, g(x)=\begin{cases}-1, & x\geqslant0\\ 1, & x<0\end{cases}$在$x=0$都不连续，但$f(x)g(x)=-1$在$x=0$连续.

(ii) 不连续：例：$f(x)=g(x)=\dfrac{1}{x}$在$x=0$都不连续，$f(x)g(x)=\dfrac{1}{x^2}$在$x=0$不连续.

7. 若$f(x)$在$[a,\infty)$连续，并且$\lim\limits_{x\to\infty}f(x)$存在，证明$f(x)$在$[a,\infty)$有界.

证明：由于$\lim\limits_{x\to\infty}f(x)$存在，不妨设$\lim\limits_{x\to\infty}f(x)=A$
则对$\varepsilon=1,\exists X>0$，当$x>X$时，有$|f(x)-A|<\varepsilon=1$成立，从而得$|f(x)|=|f(x)-A+A|\leqslant|f(x)-A|+$

$|A| < 1 + |A|$
取$X_1 = \max\{X, a+1\}$，则$f(x)$在(X_1, ∞)内有界，且$|f(x)| < |A| + 1, x \in (X_1, \infty)$
又由于$f(x)$在$[a, X_1]$上连续，故$f(x)$在$[a, X_1]$上有界，设其界为$M > 0$，即$\forall x \in [a, X_1]$，有$|f(x)| \leqslant M$
取$G = \max\{|A| + 1, M\}$，则$\forall x \in [a, \infty), f(x) \leqslant G$，
即$f(x)$在$[a, \infty)$有界.

8. 若对任一$\varepsilon > 0$，$f(x)$在$[a+\varepsilon, b-\varepsilon]$连续，问：

   (1) $f(x)$是否(a, b)在连续？

   (2) $f(x)$是否在$[a, b]$连续？

   解:

   (1) 任取$x_0 \in (a, b)$，取$\varepsilon = \min\left\{\dfrac{x_0 - a}{2}, \dfrac{b - x_0}{2}\right\}$，则$x_0 \in [a+\varepsilon, b-\varepsilon]$
   因对任一$\varepsilon > 0$，$f(x)$在$[a+\varepsilon, b-\varepsilon]$连续，故$f(x)$在$x_0$点连续
   由$x_0 \in (a, b)$的任意性，得$f(x)$在(a, b)内连续.

   (2) 不一定连续。

   (i) 不连续。例：$f(x)$在$[0+\varepsilon, 1-\varepsilon](\varepsilon > 0)$内连续，但$f(x)$在$[0, 1]$上不连续，在$x = 0$点断开.

   (ii) 连续。例：$f(x)$在$[1+\varepsilon, 2-\varepsilon](\varepsilon > 0)$内连续，且$f(x)$在$[1, 2]$上连续.

9. 若$f(x)$在x_0点连续，并且$f(x_0) > 0$，证明存在x_0的δ邻域$O(x_0, \delta)$，当$x \in O(x_0, \delta)$时，$f(x) \geqslant c > 0$，c为某个常数.
   证明: 由于$f(x)$在x_0点连续，且$f(x_0) > 0$，则设$f(x_0) > c > 0$
   对给定的$\varepsilon = f(x_0) - c > 0, \exists \delta > 0$，当$|x - x_0| < \delta$时，有$|f(x) - f(x_0)| < \varepsilon = f(x_0) - c$，则$f(x_0) - [f(x_0) - c] \leqslant f(x)$，即$f(x) \geqslant c > 0$.

10. 证明若连续函数在有理点的函数值为0，则此函数恒为0.
   证明: 设$f(x)$为实轴上的连续函数，x_0 为任意一个无理点.
   由有理点在数轴上的稠密性，可以取无理数列$\{x_n\}$，使得$x_n \to x_0 (n \to \infty)$.
   因$f(x)$在x_0连续，则$f(x_0) = \lim\limits_{n\to\infty} f(x_n) = 0$，
   由x_0点的任意性，得$f(x)$在所有无理点的函数值都为0.
   又$f(x)$在有理点的函数值为0，则此函数恒为0.

11. 若$f(x)$在$[a, b]$连续，恒正，按定义证明$\dfrac{1}{f(x)}$在$[a, b]$连续.

   证明: 由于$f(x)$在$[a, b]$连续，恒正，则$f(x)$在(a, b)连续，$f(x) > 0$，$\dfrac{1}{f(x)}$存在，$x \in [a, b]$
   设x_0为(a, b)内任一点，则对$\forall \varepsilon > 0, \exists \delta > 0$，当$|x - x_0| < \delta$时，有$|f(x) - f(x_0)| < \varepsilon$.
   又$f(x)$在$[a, b]$连续，则由闭区间连续函数性质2，可设$f(x)$在$[a, b]$上的最小值为$m > 0$，即$f(x) \geqslant m, x \in [a, b]$，于是$\left|\dfrac{1}{f(x)} - \dfrac{1}{f(x_0)}\right| = \dfrac{|f(x) - f(x_0)|}{f(x) f(x_0)} < \dfrac{\varepsilon}{m^2}$，故$\lim\limits_{x \to x_0} \dfrac{1}{f(x)} = \dfrac{1}{f(x_0)}$，从而$\dfrac{1}{f(x)}$在$x_0$连续.
   由x_0在(a, b)内的任意性，得$f(x)$在(a, b)内连续.
   又$f(a+0) = f(a) > 0$，则$\dfrac{1}{f(a+0)} = \dfrac{1}{f(a)}$，故$f(x)$在$[a, b)$连续；
   又$f(b-0) = f(b) > 0$，则$\dfrac{1}{f(b-0)} = \dfrac{1}{f(b)}$，故$f(x)$在$[a, b]$连续.

12. 若$f(x)$和$g(x)$都在$[a, b]$连续，试证明$\max(f(x), g(x))$以及$\min(f(x), g(x))$都在$[a, b]$连续.
   证明: 由于$f(x)$和$g(x)$都在$[a, b]$连续，故$f(x) - g(x)$和$f(x) + g(x)$都在$[a, b]$连续.
   由第4题结论，有$|f(x) - g(x)|$在$[a, b]$连续.
   令$\varphi(x) = \max(f(x), g(x)) = \dfrac{1}{2}(f(x) + g(x) + |f(x) - g(x)|)$，
   $\psi(x) = \min(f(x), g(x)) = \dfrac{1}{2}(f(x) + g(x) - |f(x) - g(x)|)$，
   故$\varphi(x), \psi(x)$都在$[a, b]$连续.

13. 若$f(x)$是连续的，证明对任何$c > 0$，函数$g(x) = \begin{cases} -c, & 若f(x) < -c \\ f(x), & 若|f(x)| \leqslant c \\ c, & 若f(x) > c \end{cases}$ 是连续的.

   证明: 由于$g(x) = \max(-c, \min(f(x), c))$
   又由于$f(x)$连续，且对任何$c > 0$，$\varphi(x) = c$连续，$\psi(x) = -c$连续，
   则由上题结论，得$\min(f(x), c)$连续，从而再由上题结论，得$g(x)$连续.

14. 研究下列函数各个不连续点的性质（即为何种不连续点）：

   (1) $y = \dfrac{x}{(1+x)^2}$

(2) $y=\dfrac{1+x}{1+x^3}$

(3) $y=\dfrac{x^2-1}{x^3-3x+2}$

(4) $y=\dfrac{x}{\sin x}$

(5) $y=\cos^2\dfrac{1}{x}$

(6) $y=[x]+[-x]$

(7) $y=\dfrac{1}{\ln x}$

(8) $y=\dfrac{x^2-x}{|x|(x^2-1)}$

(9) $y=\begin{cases}\dfrac{1}{q}, & x=\dfrac{p}{q}(q>0,q,p\text{为互质的整数})\\ 0, & x\text{为无理数}\end{cases}$

(10) $y=\begin{cases}x, & \text{当}|x|\leqslant 1\\ 1, & \text{当}|x|>1\end{cases}$

(11) $y=\begin{cases}\cos\dfrac{\pi x}{2}, & \text{当}|x|\leqslant 1\\ |x-1|, & \text{当}|x|>1\end{cases}$

(12) $y=\begin{cases}\sin\pi x, & \text{当}x\text{为有理数}\\ 0, & \text{当}x\text{为无理数}\end{cases}$

解:

(1) 因$\lim\limits_{x\to-1-0}\dfrac{x}{(1+x)^2}=-\infty$，故$x=-1$为第二类不连续点（无穷间断点）.

(2) 因$\lim\limits_{x\to-1}\dfrac{1+x}{1+x^3}=\dfrac{1}{3}$，但$y$在$x=-1$点没有定义，故$x=-1$为可移不连续点.

(3) 因$y=\dfrac{x^2-1}{x^3-3x+2}=\dfrac{(x-1)(x+1)}{(x-1)(x^2+x+1)-3(x-1)}=\dfrac{(x-1)(x+1)}{(x-1)(x^2+x-2)}=\dfrac{(x-1)(x+1)}{(x-1)^2(x+2)}$，
又$\lim\limits_{x\to1-0}y=-\infty$, $\lim\limits_{x\to-2-0}y=-\infty$，故$x=-2,x=1$为第二类不连续点.

(4) 因$\lim\limits_{x\to0}\dfrac{x}{\sin x}=1$但$y$在$x=0$无定义，故$x=0$为可移不连续点；
又$\lim\limits_{\substack{x\to k\pi\\ k\in Z,k\neq0}}\dfrac{x}{\sin x}=\infty$，故$x=k\pi(k\in Z,k\neq0)$为第二类不连续点.

(5) 因$\lim\limits_{x\to0}\cos^2\dfrac{1}{x}$在$[0,1]$间振荡，为振荡型极限，故此极限不存在，于是$x=0$为第二类不连续点.

(6) 因$x\to k+0$时，$-x\to-k-0$，故$\lim\limits_{x\to k+0}y=\lim\limits_{x\to k+0}([x]+[-x])=k+(-k-1)=-1$；
又因$x\to k-0$时，$-x\to-k+0$，故$\lim\limits_{x\to k-0}y=\lim\limits_{x\to k-0}([x]+[-x])=k-1+(-k)=-1(k\in Z)$
又当$x=k$时，$y=[x]+[-x]=[k]+[-k]=0(k\in Z)$，故整数点均为可移不连续点.

(7) 因$\lim\limits_{x\to1+0}\dfrac{1}{\ln x}=+\infty$，故$x=-1$为第二类不连续点；
因$\lim\limits_{x\to-0}\dfrac{1}{\ln x}$不存在，故$x=0$为第二类不连续点.

(8) $y=\dfrac{x(x-1)}{|x|(x-1)(x+1)}$
因$\lim\limits_{x\to1}y=\dfrac{1}{2}$但$y$在$x=1$无定义，故$x=1$为可移不连续点；
因$\lim\limits_{x\to+0}y=1$, $\lim\limits_{x\to-0}y=-1$，故$x=0$为第一类不连续点（跳跃间断点）；
因$\lim\limits_{x\to-1+0}y=-\infty$，故$x=-1$为第二类间断点.

(9) 因此函数是以1为周期的函数，故可在区间$[0,1]$讨论，其它区间的情形与此类似.
在$[0,1]$上，分母为1的有理数有两个：$\dfrac{0}{1},\dfrac{1}{1}$；分母为2的有理数有一个：$\dfrac{1}{2}$；
分母为3的有理数有两个：$\dfrac{1}{3},\dfrac{2}{3}$；分母为4的有理数有两个：$\dfrac{1}{4},\dfrac{3}{4}$；
分母为5的有理数有四个：$\dfrac{1}{5},\dfrac{2}{5},\dfrac{3}{5},\dfrac{4}{5}$；分母为6的有理数有两个：$\dfrac{1}{6},\dfrac{5}{6}$；$\cdots$

总之，分母不超过k的有理数个数$l \leqslant 2+1+2+3+\cdots+(k-1)=\dfrac{k(k-1)}{2}+2$，即分母不超过$k$的有理数只有有限个。

下面来证，在任一点$x_0 \in [0,1]$，当$x \to x_0$时，$y \to 0$.

对$\forall \varepsilon > 0$，取$k=\left[\dfrac{1}{\varepsilon}\right]$，设在$[0,1]$上，分母不超过$k$的有理数为$r_1, r_2, \cdots, r_l$.

取$\delta = \min$
$limits_{1\leqslant i\leqslant l}|r_i - x_0|$，则当$0<|x-x_0|<\delta$，即$x \notin \{r_1, r_2, \cdots, r_n\}$，也就是$x$或者为无理数，或者为有理数$\dfrac{p}{q}$，且$q \geqslant k+1 > k$时，就有$|y-0| = \begin{cases} \dfrac{1}{q} \leqslant \dfrac{1}{k+1}, & x\text{为有理数}x=\dfrac{p}{q}, q>k \\ 0<\varepsilon, & x\text{为无理数} \end{cases}$.

故$\lim\limits_{x\to x_0} y = 0$，于是得：任何无理点都是此函数的连续点，任何有理点都是此函数的可移不连续点.

(10) 因$\lim\limits_{x\to -1+0} y = -1$, $\lim\limits_{x\to -1-0} y = 1$，故$x=-1$为第一类不连续点.

(11) 因$\lim\limits_{x\to -1+0} y = 0$, $\lim\limits_{x\to -1-0} y = 2$，故$x=-1$为第一类不连续点.

(12) (i) $x_0 \neq n, n \in Z$，
取有理点列$r_n \to x_0$且$r_n > x_0$，则$\lim\limits_{r_n \to x_0+0} f(r_n) = \sin \pi x_0 \neq 0$；
取无理点列$x_n \to x_0$且$x_n > x_0$，则$\lim\limits_{x_n \to x_0+0} f(x_n) = 0$。
故$f(x_0+0)$不存在，从而$x \neq n (n \in Z)$为函数的第二类不连续点.

(ii) $x_0 = n, n \in Z$，
当x为无理数时，$|f(x)-f(n)| = 0$；
当x为有理数时，$|f(x)-f(n)| \leqslant \pi |x-n|$，对$\forall \varepsilon > 0, \exists \delta = \dfrac{\varepsilon}{\pi} > 0$，使$|x-n|<\delta$时，有$|f(x)-f(n)| < \varepsilon$，故$f(x)$在$x=n (n\in Z)$连续.

15. 当$x=0$时下列函数$f(x)$无定义，试定义$f(0)$的数值，使$f(x)$在$x=0$连续：

(1) $f(x) = \dfrac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$

(2) $f(x) = \dfrac{\tan 2x}{x}$

(3) $f(x) = \sin x \cdot \sin \dfrac{1}{x}$

(4) $f(x) = (1+x)^{\frac{1}{x}}$

解：

(1) 因$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1} = \lim\limits_{x\to 0} \dfrac{\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1}{\sqrt{1+x}+1} = \dfrac{3}{2}$，
故$f(0) = \dfrac{3}{2}$.

(2) 因$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\tan 2x}{x} = 2$，
故$f(0) = 2$.

(3) 因$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \sin x \cdot \sin \dfrac{1}{x} = 0$，
故$f(0) = 0$.

(4) 因$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$，
故$f(0) = e$.

16. 若$f(x)$在$[a,b]$连续，$a < x_1 < x_2 < \cdots < x_n < b$，则在$[x_1, x_n]$中必有$\xi$，使$f(\xi) = \dfrac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$.

证明：设$M = \max\limits_{1\leqslant i\leqslant n} f(x_i), m = \min\limits_{1\leqslant i\leqslant n} f(x_i)$
则$\dfrac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} \leqslant M$；
同理得$\dfrac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} \geqslant m$.
由于$f(x)$在$[x_1, x_n] \subset [a,b]$上连续，故由介值定理知，必$\exists \xi \in [x_1, x_n] \subset [a,b]$，使$f(\xi) = \dfrac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$.

17. 用一致连续定义验证：

(1) $f(x)=\sqrt[3]{x}$在$[0,1]$上是一致连续的；

(2) $f(x)=\sin x$在$(-\infty,+\infty)$上是一致连续的；

(3) $f(x)=\sin x^2$在$(-\infty,+\infty)$上不一致连续.

证明：

(1) 对任何$x_1,x_2\in[0,1]$，$|\sqrt[3]{x_1}-\sqrt[3]{x_2}|=\dfrac{|x_1-x_2|}{\sqrt[3]{x_1^2}+\sqrt[3]{x_1x_2}+\sqrt[3]{x_2^2}}=\dfrac{|x_1-x_2|}{\frac{3}{4}(\sqrt[3]{x_1}+\sqrt[3]{x_2})^2+\frac{1}{4}(\sqrt[3]{x_1}-\sqrt[3]{x_2})^2}\leqslant\dfrac{|x_1-x_2|}{\frac{1}{4}(\sqrt[3]{x_1}-\sqrt[3]{x_2})^2}$，
即$\dfrac{1}{4}|\sqrt[3]{x_1}-\sqrt[3]{x_2}|^3\leqslant|x_1-x_2|$，亦即$|\sqrt[3]{x_1}-\sqrt[3]{x_2}|\leqslant\sqrt[3]{4|x_1-x_2|}$
对$\forall\varepsilon>0,\exists\delta=\dfrac{\varepsilon^3}{4}>0$，使得对$\forall x_1,x_2\in[0,1]$，当$|x_1-x_2|<\delta$时，总有$|\sqrt[3]{x_1}-\sqrt[3]{x_2}|\leqslant\sqrt[3]{4|x_1-x_2|}<\varepsilon$
从而$f(x)=\sqrt[3]{x}$在$[0,1]$上是一致连续的.

(2) 对任何$x_1,x_2\in(-\infty,+\infty)$，$|\sin x_1-\sin x_2|=2\left|\cos\dfrac{x_1+x_2}{2}\sin\dfrac{x_1-x_2}{2}\right|\leqslant2\left|\dfrac{x_1-x_2}{2}\right|=|x_1-x_2|$，
对$\forall\varepsilon>0,\exists\delta=\varepsilon>0$，使得对$\forall x_1,x_2\in(-\infty,+\infty)$，当$|x_1-x_2|<\delta$时，总有$|\sin x_1-\sin x_2|\leqslant|x_1-x_2|<\varepsilon$
从而$f(x)=\sin x$在$(-\infty,+\infty)$上是一致连续的.

(3) 取$\varepsilon_0=1$，对任何$\delta>0$，取$x_n'=\sqrt{2n\pi+\frac{\pi}{2}},x_n''=\sqrt{2n\pi-\frac{\pi}{2}}$，$|x_n'-x_n''|=|\sqrt{2n\pi+\frac{\pi}{2}}-\sqrt{2n\pi-\frac{\pi}{2}}|=\left|\dfrac{\pi}{\sqrt{2n\pi+\frac{\pi}{2}}+\sqrt{2n\pi-\frac{\pi}{2}}}\right|\to0(n\to\infty)$
故当n充分大时，一定有$|x_n'-x_n''|<\delta$，
但$|\sin(x_n')^2-\sin^2(x_n'')^2|=|1-(-1)|=2>1=\varepsilon_0$
从而$f(x)=\sin x^2$在$(-\infty,+\infty)$上不一致连续.

§4. 无穷小量和无穷大量的阶

1. 求下列无穷小量当$x \to 0$时的阶和主要部分：

(1) $x^3 + x^6$

(2) $4x^2 + 6x^3 - x^5$

(3) $\sqrt{x \cdot \sin x}$

(4) $\sqrt{x^2 + \sqrt[3]{x}}$

(5) $\sqrt{1+x} - \sqrt{1-x}$

(6) $\tan x - \sin x$

(7) $\ln(1+x)$

解:

(1) 由于$\lim\limits_{x\to 0} \dfrac{x^3+x^6}{x^3} = \lim\limits_{x\to 0}(1+x^3) = 1$，故它是一个3阶无穷小量，它的主要部分为$x^3$.

(2) 由于$\lim\limits_{x\to 0} \dfrac{4x^2+6x^3-x^5}{4x^2} = \lim\limits_{x\to 0}(1+\dfrac{3}{2}x - \dfrac{x^3}{4}) = 1$，故它是一个2阶无穷小量，它的主要部分为$4x^2$.

(3) 由于$\lim\limits_{x\to 0} \dfrac{\sqrt{x\cdot \sin x}}{|x|} = \lim\limits_{x\to 0} \sqrt{\dfrac{\sin x}{x}} = 1$，故它是一个1阶无穷小量，它的主要部分为$|x|$.

(4) 由于$\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+\sqrt[3]{x}}}{\sqrt[6]{x}} = \lim\limits_{x\to 0} \sqrt{x^{\frac{5}{3}}+1} = 1$，故它是一个$\dfrac{1}{6}$阶无穷小量，它的主要部分为$\sqrt[6]{x}$.

(5) 由于$\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim\limits_{x\to 0} \dfrac{2x}{x(\sqrt{1+x}+\sqrt{1-x})} = 1$，故它是一个1阶无穷小量，它的主要部分为$x$.

(6) 由于$\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{\dfrac{x^3}{2}} = \lim\limits_{x\to 0} 2\dfrac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0} \dfrac{2(1-\cos x)}{\cos x \cdot x^2} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$，故它是一个3阶无穷小量，它的主要部分为$\dfrac{x^3}{2}$.

(7) 由于$\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$，故它是一个1阶无穷小量，它的主要部分为$x$.

2. 当$x \to \infty$时，求下列变量的阶和主要部分：

(1) $x^2 + x^6$

(2) $4x^2 + 6x^4 - x^5$

(3) $\sqrt[3]{x^2 \sin \dfrac{1}{x}}$

(4) $\sqrt{1+\sqrt{1+\sqrt{x}}}$

(5) $\dfrac{2x^5}{x^3 - 3x + 1}$

解:

(1) 由于$\lim\limits_{x\to \infty} \dfrac{x^2+x^6}{x^6} = 1$，故它是一个6阶无穷大量，它的主要部分为$x^6$.

(2) 由于$\lim\limits_{x\to \infty} \dfrac{4x^2+6x^4-x^5}{-x^5} = 1$，故它是一个5阶无穷大量，它的主要部分为$-x^5$.

(3) 由于$\lim\limits_{x\to \infty} \dfrac{\sqrt[3]{x^2 \sin \dfrac{1}{x}}}{\sqrt[3]{x}} = \lim\limits_{x\to \infty} \dfrac{\sqrt[3]{x}}{\sqrt[3]{x}} = 1$，故它是一个$\dfrac{1}{3}$阶无穷大量，它的主要部分为$\sqrt[3]{x}$.

(4) 由于$\lim\limits_{x\to \infty} \dfrac{\sqrt{1+\sqrt{1+\sqrt{x}}}}{\sqrt[8]{x}} = \lim\limits_{x\to \infty} \sqrt{\left(\dfrac{1}{x}\right)^{\frac{1}{4}} + \sqrt{\left(\dfrac{1}{x}\right)^{\frac{1}{2}}+1}} = 1$，故它是一个$\dfrac{1}{8}$阶无穷大量，它的主要部分为$\sqrt[8]{x}$.

(5) 由于$\lim\limits_{x\to\infty}\dfrac{\dfrac{2x^5}{x^3-3x+1}}{2x^2}=\lim\limits_{x\to\infty}\dfrac{x^3}{x^3-3x+1}=1$，故它是一个2阶无穷大量，它的主要部分为$2x^2$.

3. 试证：当$\Delta x\to 0$时

(1) $o(\Delta x^m)+o(\Delta x^n)=o(\Delta x^n)(m>n>0)$

(2) $o(\Delta x^m)o(\Delta x^n)=o(\Delta x^{m+n})(m,n>0)$

(3) $|f(x)|\leqslant M$，则$f(x)o(\Delta x)=o(\Delta x)$

(4) $\Delta x^m\cdot o(1)=o(\Delta x^m)$

证明：

(1) 由于$\Delta x\to 0$，故$\Delta x^m\to 0,\Delta x^n\to 0$，于是$\dfrac{o(\Delta x^m)}{\Delta x^m}\to 0,\dfrac{o(\Delta x^n)}{\Delta x^n}\to 0$

又$m>n>0$，故$\dfrac{\Delta x^m}{\Delta x^n}=\Delta x^{m-n}\to 0$，于是$\dfrac{o(\Delta x^m)+o(\Delta x^n)}{\Delta x^n}=\dfrac{o(\Delta x^m)}{\Delta x^m}\cdot\dfrac{\Delta x^m}{\Delta x^n}+\dfrac{o(\Delta x^n)}{\Delta x^n}\to 0$，

从而$o(\Delta x^m)+o(\Delta x^n)=o(\Delta x^n)$

(2) 由于$\Delta x\to 0$，故$\Delta x^m\to 0,\Delta x^n\to 0$，于是$\dfrac{o(\Delta x^m)}{\Delta x^m}\to 0,\dfrac{o(\Delta x^n)}{\Delta x^n}\to 0$

于是$\dfrac{o(\Delta x^m)o(\Delta x^n)}{\Delta x^{m+n}}=\dfrac{o(\Delta x^m)}{\Delta x^m}\cdot\dfrac{o(\Delta x^n)}{\Delta x^n}\to 0$，

从而$o(\Delta x^m)o(\Delta x^n)=o(\Delta x^{m+n})$

(3) $\Delta x\to 0$，故$\dfrac{o(\Delta x)}{\Delta x}\to 0$，又$|f(x)|\leqslant M$，故$f(x)$有界，于是$\dfrac{f(x)o(\Delta x)}{\Delta x}=f(x)\dfrac{o(\Delta x)}{\Delta x}\to 0$，从而$f(x)o(\Delta x)=o(\Delta x)$.

(4) 由$o(1)$于是无穷小量，则$o(1)\to 0$，于是$\dfrac{\Delta x^m\cdot o(1)}{\Delta x^m}=\dfrac{\Delta x^m}{\Delta x^m}o(1)=o(1)\to 0$，从而$\Delta x^m\cdot o(1)=o(\Delta x^m)$.

第二部分　　极限续论

第三章　　关于实数的基本定理及闭区间上连续函数性质的证明

§1.　　关于实数的基本定理

1. 从定义出发证明下确界的唯一性.
证明: 设α,α'都是数集E的下确界，于是$\forall x\in E$，都有$x\geqslant\alpha$，即α是E的下界；$x\geqslant\alpha'$，即α'是E的下界. 由于α是E的下确界，故是下界中的最大者，从而有$\alpha\geqslant\alpha'$；同样由α'是E的下确界，有$\alpha'\geqslant\alpha$.由此知$\alpha=\alpha'$.

2. 设$\beta=\sup E,\beta\notin E$，试证自$E$中可选取数列$\{x_n\}$，其极限为$\beta$；又若$\beta\in E$，则情形如何？
证明:

(1) 由于$\beta=\sup E,\beta\notin E$，则由上确界的定义，得

(i) 对$\forall x\in E$，都有$x<\beta$；

(ii) 对$\forall\varepsilon>0$，至少存在一个数$x_0\in E$，使得$x_0>\beta-\varepsilon$.

取$\varepsilon_n=\dfrac{1}{n}$，对每个$\varepsilon_n$都有$x_n\in E$，使得$\beta>x_n>\beta-\varepsilon_n$，即$0<\beta-x_n<\varepsilon_n$，于是可选一个数列$\{x_n\}\subset E$.
又$\lim\limits_{n\to\infty}(\beta-\varepsilon_n)=\beta-\lim\limits_{n\to\infty}\varepsilon_n=\beta$且$\beta\geqslant\lim\limits_{n\to\infty}x_n\geqslant\lim\limits_{n\to\infty}(\beta-\varepsilon_n)=\beta$，故$\lim\limits_{n\to\infty}x_n=\beta$.

(2) 当$\beta\in E$时，命题不一定成立。例：不成立。$E=(1,\dfrac{1}{2},\dfrac{1}{3},\cdots,\dfrac{1}{n},\cdots),\beta=\sup E=1,1\in E$. 又$\dfrac{1}{n}\to0(n\to\infty)$，则$E$中任一子列的极限均为0，故当$\beta\in E$时，命题不成立。
成立。$E=\left\{\sin\dfrac{\pi}{8},\sin\dfrac{2\pi}{8},\cdots,\sin\dfrac{n\pi}{8},\cdots\right\},\beta=\sup E=1,1\in E$，取$x_n=\sin\dfrac{16n+4}{8}\pi$，则$\lim\limits_{n\to\infty}x_n=1$，故当$\beta\in E$时，命题成立。

3. 举例：

(1) 有上确界无下确界的数列；

(2) 含有上确界但不含有下确界的数列；

(3) 既含有上确界又含有下确界的数列；

(4) 既不含有上确界，又不含有下确界的数列，其中上、下确界都有限.

解:

(1) $\{x_n\}=\{-n\},\sup\{x_n\}=-1$

(2) $\{x_n\}=\{\dfrac{1}{n}\},\sup\{x_n\}=1\in\{x_n\},inf\{x_n\}=0\notin\{x_n\}$

(3) $\{x_n\}=\{1+(-1)^n\},\sup\{x_n\}=2\in\{x_n\},inf\{x_n\}=0\in\{x_n\}$

(4) $E=\left(1,\dfrac{1}{2},1+\dfrac{1}{2},\dfrac{1}{3},1+\dfrac{2}{3},\cdots,\dfrac{1}{n},1+\dfrac{n-1}{n}\right),\sup E=2\notin E,\inf E=0\notin E$

4. 试证收敛数列必有上确界和下确界，趋于$+\infty$的数列必有下确界，趋于$-\infty$的数列必有上确界.
证明:

(1) 对于各项恒为常数的数列，显然上、下确界均可达到.
对于不恒为常数的数列，因$\{x_n\}$收敛，即$\{x_n\}$有极限，则由第二章§1定理4，得数列$\{x_n\}$是有界数列. 从而由本章定理三，得数列$\{x_n\}$有上、下确界，即收敛数列必有上、下确界.
注：还可证明：上、下确界β,α中至少有一个属于$\{x_n\}$.
事实上，若$\alpha=\beta$，则$\alpha=\beta=x_n,n=1,2,\cdots$
若$\alpha\neq\beta$，且$\alpha\notin\{x_n\}$，则由习题2知，存在子列$\left\{x_{n_k}^{(1)}\right\}$收敛于$\alpha$，也存在子列$\left\{x_{n_k}^{(2)}\right\}$收敛于$\beta$，故$\{x_n\}$不收敛，这与已知$\{x_n\}$收敛矛盾，故$\alpha,\beta$中至少有一个属于$\{x_n\}$.

(2) 因$\{x_n\}$是趋于$+\infty$的数列，则$\exists N\in Z^+$，当$n>N$时，恒有$x_n>x_1$，于是$x_1,x_2,\cdots,x_N$中最小者，即为$\{x_n\}$的下确界。

(3) 因$\{x_n\}$是趋于$-\infty$的数列，则$\exists N\in Z^+$，当$n>N$时，恒有$x_n<x_1$，于是$x_1,x_2,\cdots,x_N$中最大者，即为$\{x_n\}$的上确界。

5. 求数列$\{x_n\}$的上、下确界:

(1) $x_n=1-\dfrac{1}{n}$

(2) $x_n=-n[2+(-2)^n]$

(3) $x_{2k}=k,x_{2k+1}=1+\dfrac{1}{k}(k=1,2,3,\cdots)$

解:

(1) $\alpha=0$（可达），$\beta=1$（不可达）

(2) 因$\lim\limits_{k\to\infty}\left[-2k\left(2+(-2)^{2k}\right)\right]=-\infty,\lim\limits_{k\to\infty}\left[-(2k+1)\left(2+(-2)^{2k+1}\right)\right]=+\infty$，故$\{x_n\}$无上、下确界.

(3) 因$\lim\limits_{k\to\infty}x_{2k}=\lim\limits_{x\to\infty}k=+\infty$，故$\{x_n\}$无上确界；
又因$x_{2k}\geqslant 1,k=1,2,3,\cdots;x_{2k+1}>1$且$\min\{x_{2k}\}=1$，故$\inf\{x_n\}=1$（可达）.

6. 证明：单调减少有下界的数列必有极限.
证明: 由于$\{y_n\}$有下界，故$\{y_n\}$必有下确界.
由下确界的定义有：(i)$y_n\geqslant\alpha(n=1,2,3,\cdots)$；(ii)对$\forall\varepsilon>0$，至少有一个$y_{_N}\in\{y_n\}$，使$y_{_N}<\alpha+\varepsilon$.
由于$\{y_n\}$是单调减少数列，故当$n>N$时，有$y_n<\alpha+\varepsilon$，即当$n>N$时，有$0\leqslant y_n-\alpha<\varepsilon$，于是$y_n\to\alpha(n\to\infty)$.
从而单调减少有下界的数列必有极限.

7. 试分析区间套定理的条件：若将闭区间改为开区间，结果如何？若将条件$[a_1,b_1]\supset[a_2,b_2]\supset\cdots$去掉或将条件$b_n-a_n\to 0$去掉，结果怎样？试举例说明.
解:

(1) 在区间套定理中，若将闭区间列改为开区间列，即

(i) $(a_{n+1},b_{n+1})\subset(a_n,b_n)$；

(ii) $\lim\limits_{n\to\infty}(b_n-a_n)=0$

则可以证明$\{a_n\},\{b_n\}$仍收敛于同一极限ξ，即$\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=\xi$，但此时$\xi$可能根本不属于这些开区间，即$\xi\notin(a_n,b_n)(n\in Z^+)$，亦即$\xi$可能不为$(a_n,b_n)$的公共点.

例：开区间列$\left\{(0,\dfrac{1}{n})\right\}$,

(i) $\left(0,\dfrac{1}{n+1}\right)\subset\left(0,\dfrac{1}{n}\right)$；

(ii) $\lim\limits_{n\to\infty}\left(\dfrac{1}{n}-0\right)=\lim\limits_{n\to\infty}\dfrac{1}{n}=0$;

$a_n=0\to 0(n\to\infty);b_n=\dfrac{1}{n}\to 0(n\to\infty)$，则$\xi=0\notin\left(0,\dfrac{1}{n}\right)$，即结论不成立.

(2) 若将条件$[a_{n+1},b_{n+1}]\subset[a_n,b_n]$去掉，即只有条件$b_n-a_n\to 0$成立，则不能保证$\{a_n\}$与$\{b_n\}$收敛.
例：闭区间列$\left[n-\dfrac{1}{n},n+\dfrac{1}{n}\right]$不是一个套一个.$\lim\limits_{n\to\infty}\left[n+\dfrac{1}{n}-\left(n-\dfrac{1}{n}\right)\right]=\lim\limits_{n\to\infty}\dfrac{2}{n}=0$，而$\lim\limits_{n\to\infty}\left(n+\dfrac{1}{n}\right)$与$\lim\limits_{n\to\infty}\left(n-\dfrac{1}{n}\right)$皆不收敛.
故不存在ξ为$\{a_n\},\{b_n\}$的公共极限，即结论不成立.

(3) 若将条件$b_n-a_n\to 0$去掉，即只有条件$[a_{n+1},b_{n+1}]\subset[a_n,b_n]$成立.则可以证明$\{a_n\},\{b_n\}$收敛（与区间套定理证明一样），但不能保证$\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}a_n$成立，从而$[a_n,b_n]$的公共点不唯一，甚至出现一个公共区间.
例：闭区间列$\left[1-\dfrac{1}{n+1},2+\dfrac{1}{n+1}\right]\subset\left[1-\dfrac{1}{n},2+\dfrac{1}{n}\right],n\in Z^+$，且$\lim\limits_{n\to\infty}\left[2+\dfrac{1}{n}-\left(1-\dfrac{1}{n}\right)\right]=1$.
但由$\lim\limits_{n\to\infty}a_n=1,\lim\limits_{n\to\infty}b_n=2$，得$[1,2]\subset\left[1-\dfrac{1}{n},2+\dfrac{1}{n}\right],n\in Z^+$，即结论不成立.

8. 若$\{x_n\}$无界，且非无穷大量，则必存在两个子列$x_{n_k}^{(1)}\to\infty,x_{n_k}^{(2)}\to a$($a$为某有限数).
证明: 先证$\left\{x_{n_k}^{(1)}\right\}$是一个无穷大量.
由于$\{x_n\}$无界，故对任何实数$M>0$，至少有一个$n'\in Z^+$，使得$|x_{n'}|>M$.
取$M=1$，则必存在n_1，使得$\left|x_{n_1}^{(1)}\right|>1$；$M=2$，则必存在$n_2$，使得$\left|x_{n_2}^{(1)}\right|>2$；$\cdots$；$M=K$，则必存

在$n_K > n_{K-1}$，使得$\left|x_{n_K}^{(1)}\right| > K$，$\cdots$.
则可得一子列$\left\{x_{n_k}^{(1)}\right\}$，对$\forall M \in Z^+$，取$K = M$，则当$k > K$时，就有$\left|x_{n_k}^{(1)}\right| > M$，故有$\lim\limits_{k\to\infty} x_{n_k}^{(1)} = \infty$.
由已知$\{x_n\}$不是无穷大量，则由定义得，$\exists M_0 > 0$，对$\forall N \in Z^+$，至少有一个$m \in Z^+$，当$m > N$时，有$|x_m| < M_0$.
现取定一个$N = m_0\ (m_0 \in Z^+)$，则至少有一个$m_1 > m_0$，使得$|x_{m_1}| \leqslant M_0$
再取$N = m_1$，则至少有一个$m_2 > m_1$，使得$|x_{m_2}| \leqslant M_0$，$\cdots$
如此进行下去，则可得一列m_t：$m_1 < m_2 < \cdots < m_t < \cdots$，使得$|x_{m_t}| \leqslant M_0$，即得子列$\{x_{m_t}\}$且$|x_{m_t}| \leqslant M_0 (m_t \in Z^+)$，这说明子列$\{x_{m_t}\}$有界，由致密性定理，知有界子列$\{x_{m_t}\}$必有收敛的子列.
不妨记这个收敛子列为$\{x_{n_k}^{(2)}\}$，它也是$\{x_n\}$的子列且设它收敛于a.即$\lim\limits_{k\to\infty} x_{n_k}^{(2)} = a$（$a$为某有限数）.

9. 有界数列$\{x_n\}$若不收敛，则必存在两个子列$x_{n_k}^{(1)} \to a, x_{n_k}^{(2)} \to b (a \neq b)$.
证明：由于$\{x_n\}$有界，则由致密性定理知它必有收敛的子列$x_{n_k}^{(1)} \to a$.
由于$\{x_n\}$不收敛，故存在$\varepsilon_0 > 0$，在$(a - \varepsilon_0, a + \varepsilon_0)$外有$\{x_n\}$无穷多项，构成$\{x_n\}$的子列，记为$\left\{x_n^{(2)}\right\}$.
由于$\left\{x_n^{(2)}\right\}$有界，故存在子列$x_{n_k}^{(2)} \to b$，显然$a \neq b$.

10. 若在区间$[a,b]$中的两个数列$\left\{x_n^{(1)}\right\}$及$\left\{x_n^{(2)}\right\}$满足$x_n^{(1)} - x_n^{(2)} \to 0 (n \to \infty)$，则在此两数列中能找到具有相同足标$n_k$的子列，使$x_{n_k}^{(1)} \to x_0, x_{n_k}^{(1)} \to x_0 (k \to \infty)$.
证明：因$\left\{x_n^{(1)}\right\} \subset [a,b]$，则$\left\{x_n^{(1)}\right\}$为一有界数列，则由致密性定理，得$\left\{x_n^{(1)}\right\}$必有收敛子列，记为$\left\{x_{n_k}^{(1)}\right\}$，且设$\lim\limits_{k\to\infty} x_n^{(1)} = x_0$.
在$\left\{x_n^{(2)}\right\}$中取出与$\left\{x_{n_k}^{(1)}\right\}$有相同足标的子列$\left\{x_{n_k}^{(2)}\right\}$.
因$x_n^{(1)} - x_n^{(2)} \to 0 (n \to \infty)$，则$\lim\limits_{k\to\infty}\left(x_{n_k}^{(1)} - x_{n_k}^{(2)}\right) = 0$，
于是$\lim\limits_{k\to\infty} x_{n_k}^{(2)} = \lim\limits_{k\to\infty}\left[x_{n_k}^{(1)} - \left(x_{n_k}^{(1)} - x_{n_k}^{(2)}\right)\right] = \lim\limits_{k\to\infty} x_{n_k}^{(1)} - \lim\limits_{k\to\infty}\left(x_{n_k}^{(1)} - x_{n_k}^{(2)}\right) = x_0 - 0 = x_0$.

11. 利用柯西收敛原理讨论下列数列的收敛性：
(1) $x_n = a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n (|q| < 1, |a_k| \leqslant M)$
(2) $x_n = 1 + \dfrac{\sin 1}{2} + \dfrac{\sin 2}{2^2} + \cdots + \dfrac{\sin n}{2^n}$
(3) $x_n = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \cdots + (-1)^{n+1}\dfrac{1}{n}$

证明：

(1) 设$n > m$，则$|x_n - x_m| = \left|a_{m+1}q^{m+1} + a_{m+1}q^{m+1} + \cdots + a_n q^n\right| \leqslant M\left(|q|^{m+1} + |q|^{m+2} + \cdots + |q|^n\right) = M|q|^{m+1}\dfrac{1 - |q|^{n-m}}{1 - |q|} < M|q|^{m+1}\dfrac{1}{1 - |q|} \to 0 (m \to \infty)$
故而对$\forall \varepsilon > 0, \exists N \in Z^+$，当$n > m > N$时，有$M|q|^{m+1}\dfrac{1}{1-|q|} < \varepsilon$，从而有$|x_n - x_m| < \varepsilon$.
由柯西收敛原理，得$\{x_n\}$必收敛.

(2) 设$m > n$，对$\forall \varepsilon > 0$（不妨设$\varepsilon < \dfrac{1}{2}$），由于$|x_m - x_n| = \left|\dfrac{\sin(n+1)}{2^{n+1}} + \dfrac{\sin(n+2)}{2^{n+2}} + \cdots + \dfrac{\sin m}{2^m}\right| \leqslant \dfrac{1}{2^{n+1}} + \dfrac{1}{2^{n+2}} + \cdots + \dfrac{1}{2^m} = \dfrac{1}{2^{n+1}}\left(1 + \dfrac{1}{2} + \cdots + \dfrac{1}{2^{m-n-1}}\right) = \dfrac{1}{2^{n+1}}\dfrac{1 - \left(\dfrac{1}{2}\right)^{m-n}}{1 - \dfrac{1}{2}} < \dfrac{1}{2^n}$，若要$|x_m - x_n| < \varepsilon$，只要$\dfrac{1}{2^n} < \varepsilon$即可.
取$N = \left[\dfrac{\ln\varepsilon}{\ln\dfrac{1}{2}}\right] \in Z^+$，当$m > n > N$时，有$|x_m - x_n| < \varepsilon$.
由柯西收敛原理，得$\{x_n\}$必收敛.
（或：在（1）中令$a_0 = 1, a_k = \sin k, q = \dfrac{1}{2}$，则由（1）即得（2））.

(3) 对$\forall \varepsilon > 0$，对$\forall k \in Z^+$，由于$|x_{n+k} - x_n| = \left|\dfrac{(-1)^{n+2}}{n+1} + \dfrac{(-1)^{n+3}}{n+2} + \cdots + \dfrac{(-1)^{n+k-1}}{n+k}\right| = \left|\dfrac{1}{n+1} - \dfrac{1}{n+2} + \cdots + \dfrac{(-1)^{k-1}}{n+k}\right| = \dfrac{1}{n+1} - \left(\dfrac{1}{n+2} - \dfrac{1}{n+3} + \cdots + \dfrac{(-1)^k}{n+k}\right) < \dfrac{1}{n+1} < \dfrac{1}{n}$，若要$|x_{n+k} - x_n| < \varepsilon$，只要$\dfrac{1}{n} < \varepsilon$即可.

取$N=\left[\dfrac{1}{\varepsilon}\right]$，则当$n+k>n>N$时，有$|x_{n+k}-x_n|<\varepsilon$.
由柯西收敛原理，得$\{x_n\}$必收敛.

12. 利用有限覆盖定理证明魏尔斯特拉斯定理.
证明：设$\{x_n\}$为有界数列，则必存在a,b，使得$a\leqslant x_n\leqslant b$.
用反证法。假设$\{x_n\}$的任一子列都不收敛，则对任何$x_0\in[a,b]$，都有$\varepsilon_0>0$，使得在$O(x_0,\varepsilon_0)$中只含有$\{x_n\}$的有限项.
否则对$\forall\varepsilon>0$，在$O(x_0,\varepsilon)$中含有$\{x_n\}$的无限项.
取$\varepsilon_n=\dfrac{1}{n}$，显然在$O(x_0,\varepsilon_n)$中都含有$\{x_n\}$的无限多项，则在$\{x_n\}$中可取出：$x_{n_1}\in O(x_0,1)$，又可取出$x_{n_2}\in O\left(x_0,\dfrac{1}{2}\right)(n_2>n_1)$，如此进行下去，可得$\{x_n\}$的一个子列$\{x_{n_k}\},|x_{n_k}-x_0|<\dfrac{1}{k}$，对$\forall M\in Z^+$，取$K=M$，则当$k>K$时，就有$|x_{n_k}-x_0|<\dfrac{1}{k}<\dfrac{1}{K}<\dfrac{1}{M}$，则$x_{n_k}\to x_0(k\to\infty)$这与假设矛盾.
由$x_0\in[a,b]$的任意性，得对$[a,b]$中的每个点都有这样一个邻域，使此邻域只含$\{x_n\}$的有限项，所有这些邻域构成$[a,b]$的一个开覆盖.
由有限覆盖定理，则得存在有限个邻域也覆盖$[a,b]$，因而$[a,b]$也只含有$\{x_n\}$的有限项，这与已知$x_n\in[a,b]$矛盾，故假设不成立，则$\{x_n\}$必有收敛子列.

13. 利用魏尔斯特拉斯定理证明单调有界数列必有极限.
证明：设$\{x_n\}$为单调增加有界数列，$x_1\leqslant x_2\leqslant\cdots\leqslant x_n\leqslant\cdots\leqslant M$
据魏尔斯特拉斯定理，存在子列$\{x_{n_k}\}$, $\lim\limits_{k\to\infty}x_{n_k}=a$.
下证：$\lim\limits_{n\to\infty}x_n=a$.
先证$x_n\leqslant a,n=1,2,\cdots$.若不然，$\exists N\in Z^+$，使得$x_N>a$.
由于$n_k\to\infty(k\to\infty)$，故k充分大时，必有$n_k>N$，从而$x_{n_k}\geqslant x_N>a$，于是$a=\lim\limits_{k\to\infty}x_{n_k}\geqslant x_N>a$矛盾.
再证$\lim\limits_{n\to\infty}x_n=a$.
对$\forall\varepsilon>0,\exists k_0$，使$\left|x_{n_{k_0}}-a\right|=a-x_{n_{k_0}}<\varepsilon$.
取$N=n_{k_0}$，则当$n>N$时，有$x_n\geqslant x_{n_{k_0}}=x_N$，从而有$|a-x_n|=a-x_n\leqslant a-x_{n_{k_0}}<\varepsilon$，故$\lim\limits_{n\to\infty}x_n=a$.
即单调增加有界数列必有极限.
同理可得，单调减少有界数列必有极限，从而单调有界数列必有极限.

14. (1) 证明单调有界函数存在左、右极限；
(2) 证明单调有界函数的一切不连续点都为第一类不连续点.
证明：
(1) 由已知可设$f(x)$在(a,b)上单调增加有界，任取$x_0\in(a,b)$，设$\beta(x_0)=\sup\limits_{a<x<x_0}f(x)$，
由上确界定义，对$\forall\varepsilon>0$，至少有一个$x'\in(a,x_0)$，使得$f(x')>\beta(x_0)-\varepsilon$即$f(x')+\varepsilon>\beta(x_0)$
取$\delta=x_0-x'>0$，因$f(x)$在(a,b)上单调增加，故当$\delta>x_0-x>0$即$x'<x$时，有$f(x')<f(x)$，于是有$f(x)+\varepsilon>\beta(x_0)$即$0\leqslant\beta(x_0)-f(x)<\varepsilon$，从而$|\beta(x_0)-f(x)|<\varepsilon$
说明$\lim\limits_{x\to x_0-0}f(x)=\beta(x_0)$.即$f(x)$在$x_0$存在左极限.
同理可得，当$f(x)$在(a,b)上单调减少有界时，$f(x)$在x_0存在左极限，从而单调有界函数存在左极限.
同理可得，单调有界函数存在右极限.
(2) 设x_0为$f(x)$的不连续点，则由（1）的结论知$f(x_0-0)$和$f(x_0+0)$存在，此时$f(x_0-0)\neq f(x_0+0)$。
否则，$f(x_0-0)=f(x_0+0)$，由$f(x)$的单调性，必有$f(x_0)=f(x_0-0)=f(x_0+0)$.
这说明x_0是连续点，与已知矛盾，故$f(x_0-0)\neq f(x_0+0)$，从而x_0是$f(x)$的第一类不连续点.

15. 证明$\lim\limits_{x\to+\infty}f(x)$存在的充分必要条件是：对任意给定$\varepsilon>0$，存在$X>0$，当$x',x''>X$时恒有$|f(x')-f(x'')|<\varepsilon$.
证明：$\Rightarrow$ 已知$\lim\limits_{x\to+\infty}f(x)$存在，不妨设$\lim\limits_{x\to+\infty}f(x)=A$.
对$\forall\varepsilon>0,\exists X>0$，当$x>X$时，有$|f(x)-A|<\dfrac{\varepsilon}{2}$
当$x',x''>X$时，有$|f(x')-A|<\dfrac{\varepsilon}{2},|f(x'')-A|<\dfrac{\varepsilon}{2}$，则$|f(x')-f(x'')|=|f(x')-A-(f(x'')-A)|\leqslant|f(x')-A|+|f(x'')-A|<\varepsilon$，从而对任意给定$\varepsilon>0$，存在$X>0$，当$x',x''>X$时恒有$|f(x')-f(x'')|<\varepsilon$.
$\Leftarrow$ 在$f(x)$的定义域内，任意选取数列$\{x_n\}$，使得$x_n\to+\infty(n\to\infty)$
由已知，对$\forall\varepsilon>0,\exists X>0$，当$x',x''>X$时，恒有$|f(x')-f(x'')|<\varepsilon$.
又因$x_n\to+\infty$，于是对上述$X>0$，定$\exists N\in Z^+$，当$n>N$时，有$x_n>X$，从而当$n,m>N$时，就有$x_n>X,x_m>X$，进而有$|f(x_n)-f(x_m)|<\varepsilon$.

由柯西收敛原理，得$\lim\limits_{n\to\infty}f(x_n)$存在，不妨设$\lim\limits_{n\to\infty}f(x_n)=A$
由x_n的任意性及函数极限与数列极限的关系知，$\lim\limits_{x\to+\infty}f(x)=A$即$\lim\limits_{x\to+\infty}f(x)$存在.

16. 证明$\lim\limits_{x\to x_0}f(x)$存在的充分必要条件是：对任意给定$\varepsilon>0$，存在$\delta>0$，当$0<|x'-x_0|<\delta, 0<|x''-x_0|<\delta$时，恒有$|f(x')-f(x'')|<\varepsilon$.
证明：$\Rightarrow$ 已知$\lim\limits_{x\to x_0}f(x)$存在，不妨设$\lim\limits_{x\to x_0}f(x)=A$.
对$\forall\varepsilon>0,\exists\delta>0$，当$0<|x-x_0|<\delta$时，有$|f(x)-A|<\dfrac{\varepsilon}{2}$
当$0<|x'-x_0|<\delta, 0<|x''-x_0|<\delta$时，有$|f(x')-A|<\dfrac{\varepsilon}{2}, |f(x'')-A|<\dfrac{\varepsilon}{2}$，则$|f(x')-f(x'')|=|f(x')-A-(f(x'')-A)|\leqslant|f(x')-A|+|f(x'')-A|<\varepsilon$，从而对任意给定$\varepsilon>0$，存在$\delta>0$，当$0<|x'-x_0|<\delta, 0<|x''-x_0|<\delta$时，恒有$|f(x')-f(x'')|<\varepsilon$.
$\Leftarrow$ 在$f(x)$的定义域内，任意选取数列$\{x_n\}$，使得$x_n\to x_0$且$x_n\neq x_0(n\to\infty)$
由已知，对$\forall\varepsilon>0,\exists\delta>0$，当$x',x''\in D(f)$，且当$0<|x'-x_0|<\delta, 0<|x''-x_0|<\delta$时，就有$|f(x')-f(x'')|<\varepsilon$.
又因$x_n\to x_0, x_n\neq x_0(n\to\infty)$，于是对上述$\delta>0$，定$\exists N\in Z^+$，当$n>N$时，有$0<|x_n-x_0|<\delta$，从而当$n,m>N$时，就有$0<|x_n-x_0|<\delta, 0<|x_m-x_0|<\delta$，进而有$|f(x_n)-f(x_m)|<\varepsilon$.
由数列的柯西收敛原理，得$\lim\limits_{n\to\infty}f(x_n)$存在，不妨设$\lim\limits_{n\to\infty}f(x_n)=A$
由$\{x_n\}$是任意以x_0为极限的数列且$x_n\neq x_0$ 及函数极限与数列极限的关系知，$\lim\limits_{x\to x_0}f(x)=A$即$\lim\limits_{x\to x_0}f(x)$存在.

17. 证明$f(x)$在x_0点连续的充分必要条件是：对任意给定$\varepsilon>0$，存在$\delta>0$，当$|x'-x_0|<\delta, |x''-x_0|<\delta$时，恒有$|f(x')-f(x'')|<\varepsilon$.
证明：$\Rightarrow$ 已知$f(x)$在x_0点连续，则对$\forall\varepsilon>0,\exists\delta>0$，当$|x-x_0|<\delta$时，有$|f(x)-f(x_0)|<\dfrac{\varepsilon}{2}$
当$|x'-x_0|<\delta, |x''-x_0|<\delta$时，有$|f(x')-f(x_0)|<\dfrac{\varepsilon}{2}, |f(x'')-f(x_0)|<\dfrac{\varepsilon}{2}$，则$|f(x')-f(x'')|=|f(x')-f(x_0)-(f(x'')-f(x_0))|\leqslant|f(x')-f(x_0)|+|f(x'')-f(x_0)|<\varepsilon$，从而对任意给定$\varepsilon>0$，存在$\delta>0$，当$|x'-x_0|<\delta, |x''-x_0|<\delta$时，恒有$|f(x')-f(x'')|<\varepsilon$.
$\Leftarrow$ 取$x'=x_0, x''=x$，则由已知，得对$\forall\varepsilon>0,\exists\delta>0$，当$|x-x_0|<\delta$时，就有$|f(x)-f(x_0)|<\varepsilon$.
从而$f(x)$在x_0点连续.

§2. 闭区间上连续函数性质的证明

1. 证明：若单调有界函数$f(x)$可取到$f(a),f(b)$之间的一切值，则$f(x)$在$[a,b]$连续.

证明： 不妨设$f(x)$为单调增加有界函数.

由本章§1, 14题（1）知，$f(x)$在$[a,b]$的端点$a(b)$处的右（左）极限存在，此时$f(a)=f(a+0)(f(b)=f(b-0))$，

若不然，必有$f(a)<f(a+0)=\inf\limits_{a<x<b}f(x)(f(b)>f(b-0)=\sup\limits_{a<x<b}f(x))$，于是由$f(x)$可取到$f(a)$与$f(b)$之间的一切值，得对任何$f(a)<y<f(a+0)(f(b-0)<y<f(b))$，必有$x\in(a,b)$，使得$f(x)=y$，此与$f(a+0)=\inf\limits_{a<x<b}f(x)(f(b-0)=\sup\limits_{a<x<b}f(x))$矛盾.

由此可知$f(x)$在$a(b)$右（左）连续.

若有$x_0\in(a,b)$，使$f(x)$在x_0点不连续。由§1, 14(2)的结论，知x_0必为第一类间断点，即$f(x_0+0)$和$f(x_0-0)$存在，但$f(x_0+0)\neq f(x_0-0)$.

又因$f(x)$为单调增函数，故$f(x_0-0)\leqslant f(x_0)<f(x_0+0)$或$f(x_0-0)<f(x_0)\leqslant f(x_0+0)$，这时$f(x)$取不到$(f(x_0-0),f(x_0+0))$之间异于$f(x_0)$的值，这与已知矛盾，故假设不成立.

于是$f(x)$在$[a,b]$连续.

同理，当$f(x)$为单调减少有界函数时，$f(x)$在$[a,b]$连续.

从而$f(x)$在$[a,b]$连续.

2. 证明：函数$f(x)$在(a,b)连续，并且$f(a+0),f(b-0)$存在，则$f(x)$可取到$f(a+0)$和$f(b-0)$之间的(但可能不等于$f(a+0),f(b-0)$)一切值.

证明：由于$f(a+0),f(b-0)$存在，则补充定义$f(a)=f(a+0),f(b)=f(b-0)$.

又$f(x)$在(a,b)连续，则$f(x)$在$[a,b]$连续，因而$f(x)$在$[a,b]$上必有最大值M和最小值m.

再由介值定理，知$f(x)$可以取到M和m间的一切值.

若$M=f(a+0)$(或$f(b-0)$)，$m=f(b-0)$(或$f(b-0)$)，这时$f(x)$可取到$(f(a+0),f(b-0))$中的一切值(但可能不等于$f(a+0),f(b-0)$).

若$M>f(a+0)$(或$f(b-0)$)，$m<f(b-0)$(或$f(b-0)$)，这时$f(x)$可取到$(f(a+0),f(b-0))$中的一切值(可能等于$f(a+0),f(b-0)$). 故$f(x)$可取到$f(a+0)$和$f(b-0)$之间的(但可能不等于$f(a+0),f(b-0)$)一切值.

3. 证明(a,b)上的连续函数为一致连续的充分必要条件是：$f(a+0),f(b-0)$存在.

证明：$\Leftarrow$设$f(x)$为(a,b)上的连续函数

因$f(a+0),f(b-0)$存在，则补充定义$f(a)=f(a+0),f(b)=f(b-0)$，于是$f(x)$在$[a,b]$连续，则由康托定理，得$f(x)$在$[a,b]$上一致连续，从而$f(x)$在(a,b)上一致连续.

$\Rightarrow$因$f(x)$在(a,b)上一致连续，则由定义，得对$\forall\varepsilon>0,\exists\delta(\varepsilon)>0$，当$x_1,x_2\in(a,b)$且$|x_1-x_2|<\delta(\varepsilon)$时，有$|f(x_1)-f(x_2)|<\varepsilon$.

对a，当$0<x_1-a<\dfrac{\delta(\varepsilon)}{2},0<x_2-a<\dfrac{\delta(\varepsilon)}{2}$时，$|x_1-x_2|=|(x_1-a)-(x_2-a)|\leqslant|x_1-a|+|x_2-a|<\delta(\varepsilon)$，则有$|f(x_1)-f(x_2)|<\varepsilon$.

由柯西收敛原理，得$\lim\limits_{x\to a+0}f(x)$存在，即$f(a+0)$存在且有限.

同理，得$f(b-0)$存在且有限.

4. 若函数$f(x)$在$(-\infty,+\infty)$上的任一有限闭区间上连续，则它在$(-\infty,+\infty)$上的任一有限开区间上也一致连续.

证明：设(a,b)为$(-\infty,+\infty)$上的任一有限开区间，则$[a,b]$为$(-\infty,+\infty)$上的任一有限闭区间.

因$f(x)$在$[a,b]$上连续，则由康托定理，得$f(x)$在$[a,b]$上一致连续，因而$f(x)$在(a,b)上一致连续.

由(a,b)的任意性，得$f(x)$在$(-\infty,+\infty)$上的任一有限开区间上也一致连续.

5. 函数$f(x)=x^2$在$(-\infty,+\infty)$及$(-l,l)$上$(l>0)$是否一致连续？

解：

(1) $f(x)=x^2$在$(-\infty,+\infty)$上不一致连续.

设$x_1>x_2>0$，且$x_1,x_2\in(-\infty,+\infty),|f(x_1)-f(x_2)|=|x_1^2-x_2^2|=|x_1+x_2||x_1-x_2|=(x_1+x_2)(x_1-x_2)>2x_2(x_1-x_2)$，存在$\varepsilon_0>0$，对$\forall\eta>0$，取$x_2=\dfrac{2\varepsilon_0}{\eta},x_1=x_2+\dfrac{\eta}{2}$，

显然有$x_1>x_2>0$且$|x_1-x_2|=\dfrac{\eta}{2}<\eta$，但$|f(x_1)-f(x_2)|>2x_2(x_1-x_2)=2\cdot\dfrac{2\varepsilon_0}{\eta}\cdot\dfrac{\eta}{2}=2\varepsilon_0>\varepsilon_0$，

从而$f(x)=x^2$在$(-\infty,+\infty)$上不一致连续.

(2) $f(x)=x^2$在$(-l,l)(l>0)$上一致连续.

因$f(x)$在$[-l,l](l>0)$上是连续的，则由康托定理，得$f(x)$在$[-l,l]$上一致连续，从而$f(x)=x^2$在$(-l,l)$上一致连续.

6. 若$f(x)$在(a,b)内有定义，并且对(a,b)内任何x，存在x的某个邻域O_x，使得$f(x)$在O_x内有界.问：$f(x)$在(a,b)内是否有界？又若将(a,b)改为$[a,b]$，如何？

证明：

(1) $f(x)$在(a,b)不一定有界.

例：无界：$f(x)=\dfrac{1}{x}$在$(0,1)$内有定义，且对$\forall x\in(a,b)$连续，故必局部有界，即存在x的邻域$O_x(O(x,\delta_x))$，使得它在$O_x(O(x,\delta_x))$内有界，但它在$(0,1)$内无界.

有界：$f(x)=\sin x$在$\left(0,\dfrac{\pi}{2}\right)$有定义，对$\left(0,\dfrac{\pi}{2}\right)$内的任何$x$，存在$x$的某个邻域$O_x$，使得$f(x)$在$O_x$内有界；$f(x)$在$\left(0,\dfrac{\pi}{2}\right)$上有界，且$0<f(x)<1$.

(2) $f(x)$在$[a,b]$一定有界.

因$f(x)$在$[a,b]$内有定义，则补充定义：$f(x)$在$(a-\delta,a)$的值为$f(a)$，$f(x)$在$(b,b+\delta)$的值为$f(b)$.

由已知对$[a,b]$内任何x，存在x的某个邻域O_x，使得$f(x)$在O_x内有界，即$\exists M>0$，使$|f(x)|\leqslant M$，因此在$[a,b]$上每一点都得到这样一个邻域（亦即开区间），这些开区间的全体构成一个开区间集，它覆盖了$[a,b]$.

由有限覆盖定理，得在这些开区间集中必有有限个开区间覆盖了$[a,b]$，记这有限个开区间为$(x_1-\delta_1,x_1+\delta_1),(x_2-\delta_2,x_2+\delta_2),\cdots,(x_k-\delta_k,x_k+\delta_k)$，相应的$M$分别记为$M_1,M_2,\cdots,M_k$，如今只要取$M^*=\max\{M_1,M-2,\cdots,M_k\}$.

对$[a,b]$上任意一点x，由区间覆盖概念，在这k个开区间$O(x_i,\delta_i)(i=1,2,\cdots,k)$中至少有一个包含$x$，记它为$O(x_i,\delta_i)$，且在这个开区间上，有$|f(x)|\leqslant M_i$，故$|f(x)|\leqslant M_i\leqslant M^*$.

由于x为$[a,b]$上的任意一点，则在$[a,b]$上总成立$|f(x)|\leqslant M^*$，从而证明了$f(x)$在$[a,b]$上有界.

7. 证明(a,b)上的一致连续函数必有界.

证明:因$f(x)$为(a,b)上的一致连续函数，则由习题3，得$f(x)$在(a,b)上连续且$f(a+0),f(b-0)$存在，于是补充定义：$f(a)=f(a+0),f(b)=f(b-0)$，则$f(x)$在$[a,b]$上连续，于是$f(x)$在$[a,b]$上有界，从而$f(x)$在(a,b)上有界.

8. 按定义证明，两个一致连续函数的和仍一致连续.有问：两个一致连续函数的积如何？

证明:

(1) 设$f(x)$与$g(x)$在任一区间X上一致连续.

因$f(x)$在区间X上一致连续，则由定义对$\forall\varepsilon>0,\exists\delta_1>0$，对区间$X$内任何两点$x',x''$，只要$|x'-x''|<\delta_1$，就有$|f(x')-f(x'')|<\dfrac{\varepsilon}{2}$.

又因$g(x)$在区间X上一致连续，则由定义对上述$\varepsilon>0,\exists\delta_2>0$，对区间$X$内任何两点$x',x''$，只要$|x'-x''|<\delta_2$，就有$|g(x')-g(x'')|<\dfrac{\varepsilon}{2}$.

取$\delta=\min\{\delta_1,\delta_2\}$，则当$|x'-x''|<\delta$时，有$|f(x')+g(x')-(f(x'')+g(x''))|=|f(x')-f(x'')+(g(x')-g(x''))|\leqslant|f(x')-f(x'')|+|g(x')-g(x'')|<\dfrac{\varepsilon}{2}+\dfrac{\varepsilon}{2}=\varepsilon$.

从而$f(x)$在区间X上一致连续.

(2) (i) 若区间X为有限区间，则结论成立.

设$f(x),g(x)$在区间X上一致连续，则由上题结论，知存在常数$L>0,M>0$，使$|f(x)|<L,g(x)<M(x\in X)$.

又由一致收敛定义，得$\forall\varepsilon>0,\exists\delta_1>0$，对区间$X$内任何两点$x',x''$，只要$|x'-x''|<\delta_1$，就有$|f(x')-f(x'')|<\dfrac{\varepsilon}{2M}$.

同样，对上述$\varepsilon>0,\exists\delta_2>0$，对区间$X$内任何两点$x',x''$，只要$|x'-x''|<\delta_2$，就有$|g(x')-g(x'')|<\dfrac{\varepsilon}{2L}$.

取$\delta=\min\{\delta_1,\delta_2\}$，则当$|x'-x''|<\delta$时，就有$|f(x')-f(x'')|<\dfrac{\varepsilon}{2M},|g(x')-g(x'')|<\dfrac{\varepsilon}{2L}$同时成立.

由此可知，$|f(x')g(x')-f(x'')g(x'')|=$
$|[f(x')-f(x'')]g(x')+f(x'')[g(x')-g(x'')]|\leqslant|f(x')-f(x'')||g(x')|+|f(x'')||g(x')-g(x'')|<\dfrac{\varepsilon}{2M}\cdot M+L\cdot\dfrac{\varepsilon}{2L}=\dfrac{\varepsilon}{2}+\dfrac{\varepsilon}{2}=\varepsilon$.

从而$f(x)g(x)$在区间X上一致连续.

(ii) 当$f(x),g(x)$在$(-\infty,+\infty)$都一致连续时，$f(x)g(x)$在$(-\infty,+\infty)$上不一定一致连续.

例：

(a) 不一致连续.

$f(x)=g(x)=x$，因对$\forall\varepsilon>0$，及$x_1,x_2\in(-\infty,+\infty)$，取$\delta=\varepsilon$，当$|x_1-x_2|<\delta$时，有$|x_1-x_2|<\varepsilon$，故$f(x)=g(x)=x$在$(-\infty,+\infty)$上一致连续.

但$f(x)g(x)=x^2$，由第5题可知$f(x)g(x)$在$(-\infty,+\infty)$上不一致连续.

(b) 一致连续.

$f(x)=1$，因对$\forall\varepsilon>0$，对任何$x_1,x_2\in(-\infty,+\infty)$，取$\delta=\varepsilon$，当$|x_1-x_2|<\delta$时，有$|f(x_1)-f(x_2)|<\varepsilon$，故$f(x)=1$在$(-\infty,+\infty)$上一致连续.

$g(x)=x$，则由可知$g(x)=x$在$(-\infty,+\infty)$上一致连续，且$f(x)g(x)=x$在$(-\infty,+\infty)$上一致连续.

第二篇　　单变量微积分学

第一部分　　单变量微分学

第四章　　导数与微分

§1.　　导数的引进与定义

1. 过曲线$y=x^2$上两点$A(2,4)$和$B(2+\Delta x,2+\Delta y)$作割线，分别求出当$\Delta x=1$及$\Delta x=0.1$时割线的斜率，并求出曲线在A点的切线斜率.

   解：$k_{AB}=\dfrac{(2+\Delta x)^2-2^2}{\Delta x}=4+\Delta x$
   当$\Delta x=1$时，$k_{AB}=5$；当$\Delta x=0.1$时，$k_{AB}=4.1$
   曲线在A点的切线斜率为$k=\lim\limits_{\Delta x\to 0}k_{AB}=\lim\limits_{\Delta x\to 0}(4+\Delta x)=4$.

2. 求抛物线$y=x^2$在$A(1,1)$点和在$B(-2,4)$点的切线方程和法线方程.

   解：因$y'=2x$，故在点$A(1,1)$：$k_1=2$，切线方程为：$y-1=2(x-1)$即$2x-y-1=0$；法线方程为$y-1=-\dfrac{1}{2}(x-1)$即$x+2y-3=0$
   在点$B(-2,4)$：$k_2=-4$，切线方程为：$y-4=-4(x+2)$即$4x+y+4=0$；法线方程为$y-4=\dfrac{1}{4}(x+2)$即$x-4y+18=0$

3. 若$y=f(x)=x^3$，求

   (1) 过曲线上二点$x_0,x_0+\Delta x$之割线的斜率（设$x_0=2,\Delta x$分别为0.1，0.01，0.001）；

   (2) 在$x=x_0$时曲线切线的斜率.

   解：

   (1) 因$k=\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\dfrac{(x_0+\Delta x)^3-x^3}{\Delta x}=3x_0^2+3x_0\Delta x+(\Delta x)^2$，
   故：当$\Delta x=0.1$时，$k=12.61$；当$\Delta x=0.01$时，$k=12.0601$；当$\Delta x=0.001$时，$k=12.006001$.

   (2) $f'(x)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=3x^2$，
   于是$f'(x_0)=3x_0^2$

4. 若$s=vt-\dfrac{1}{2}gt^2$，求

   (1) 在$t=1,t=1+\Delta t$之间的平均速度（设$\Delta t=1,0.1,0.01$）；

   (2) 在$t=1$的瞬时速度.

   解：

   (1) 因$\bar{v}=\dfrac{v(1+\Delta t)-\dfrac{1}{2}g(1+\Delta t)^2-\left(vt-\dfrac{1}{2}gt^2\right)}{\Delta t}=v-g-\dfrac{1}{2}g\Delta t^2$，
   故：当$\Delta t=1$时，$\bar{v}=v-\dfrac{3}{2}g$；当$\Delta t=0.1$时，$\bar{v}=v-\dfrac{21}{20}g$；当$\Delta t=0.01$时，$\bar{v}=v-\dfrac{201}{200}g$.

   (2) 在$t=1$的瞬时速度$v=\lim\limits_{\Delta t\to 0}\bar{v}=v-g$.

5. 抛物线$y=x^2$在哪一点的切线平行于直线$y=4x-5$？在哪一点的切线垂直于直线$2x-6y+5=0$？

   解：因直线$y=4x-5$的斜率为$k=4$，则由$f'(x)=2x=k$，得$x=2$，即$(2,4)$点的切线平行于直线$y=4x-5$；
   因直线$2x-6y+5=0$的斜率为$k=\dfrac{1}{3}$，则由$f'(x)=2x=-\dfrac{1}{k}=-3$，得$x=-\dfrac{3}{2}$，即$(-\dfrac{3}{2},\dfrac{9}{4})$点的切线垂直于直线$2x-6y+5=0$.

6. 求下列函数在所示点的$\dfrac{\Delta y}{\Delta x}$：

   (1) $y=\sqrt{x}$(设$x=2,\Delta x=0.01$)

   (2) $y=\dfrac{1}{x}$(设$x=4,\Delta x=0.04$)

解:

(1) $\dfrac{\Delta y}{\Delta x}=\dfrac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}=\dfrac{\sqrt{2.01}-\sqrt{2}}{0.01}=100\left(\sqrt{2.01}-\sqrt{2}\right)=\dfrac{1}{\sqrt{2.01}+\sqrt{2}}$

(2) $\dfrac{\Delta y}{\Delta x}=\dfrac{\dfrac{1}{x+\Delta x}-\dfrac{1}{x}}{\Delta x}=-\dfrac{1}{x(x+\Delta x)}=-\dfrac{1}{4(4+0.04)}=-\dfrac{25}{404}$

7. 证明：

(1) $\Delta(f(x)\pm g(x))=\Delta f(x)\pm\Delta g(x)$

(2) $\Delta[f(x)\cdot g(x)]=g(x+\Delta x)\cdot\Delta f(x)+f(x)\cdot\Delta g(x)$

证明:

(1) $\Delta(f(x)\pm g(x))=[f(x+\Delta x)\pm g(x+\Delta)]-[f(x)\pm g(x)]=[f(x+\Delta x)-f(x)]\pm[g(x+\Delta x)-g(x)]=\Delta f(x)\pm\Delta g(x)$

(2) $\Delta[f(x)\cdot g(x)]=f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x)=f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x+\Delta x)+f(x)\cdot g(x+\Delta x)-f(x)\cdot g(x)=[f(x+\Delta x)-f(x)]\cdot g(x+\Delta x)+f(x)[g(x+\Delta x)-g(x)]=g(x+\Delta x)\cdot\Delta f(x)+f(x)\cdot\Delta g(x)$

§2. 简单函数的导数

1. 由导数定义求$y=\cos x$的导数.

解: $y'=\lim\limits_{\Delta x\to 0}\dfrac{\cos(x+\Delta x)-\cos x}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{-2\sin\dfrac{2x+\Delta x}{2}\sin\dfrac{\Delta x}{2}}{\Delta x}=-\lim\limits_{\Delta x\to 0}\sin\left(x+\dfrac{\Delta x}{2}\right)\dfrac{\sin\dfrac{\Delta x}{2}}{\dfrac{\Delta x}{2}}=$ $-\sin x$，即$(\cos x)'=-\sin x$.

2. 由导数定义求$y=\sqrt[3]{x}$的导数.

解: $y'=\lim\limits_{\Delta x\to 0}\dfrac{\sqrt[3]{x+\Delta x}-\sqrt[3]{x}}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{x^{\frac{1}{3}}\left[\left(1+\dfrac{\Delta x}{x}\right)^{\frac{1}{3}}-1\right]}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{x^{-\frac{2}{3}}\left[\left(1+\dfrac{\Delta x}{x}\right)^{\frac{1}{3}}-1\right]}{\dfrac{\Delta x}{x}}=\dfrac{x^{-\frac{2}{3}}}{3}=$ $\dfrac{1}{3\sqrt[3]{x^2}}$，即$(\sqrt[3]{x})'=\dfrac{1}{3\sqrt[3]{x^2}}$

3. 按定义证明：可导的偶函数其导函数是奇函数，可导的奇函数其导函数是偶函数.

证明: 设$f(x)$为可导的偶函数，则$f(-x)=f(x)$；$g(x)$为可导的奇函数，则$g(-x)=-g(x)$

于是$f'(-x)=\lim\limits_{\Delta x\to 0}\dfrac{f(-x+\Delta x)-f(-x)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{f(x-\Delta x)-f(x)}{\Delta x}=\lim\limits_{-\Delta x\to 0}\dfrac{-[f(x-\Delta x)-f(x)]}{-\Delta x}=$ $-f'(x)$即可导的偶函数其导函数是奇函数；

$g'(-x)=\lim\limits_{\Delta x\to 0}\dfrac{g(-x+\Delta x)-g(-x)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{-g(x-\Delta x)+g(x)}{\Delta x}=\lim\limits_{-\Delta x\to 0}\dfrac{g(x-\Delta x)-g(x)]}{-\Delta x}=g'(x)$即可导的奇函数其导函数是偶函数.

4. 按定义证明：可导的周期函数，其导函数仍为周其函数.

证明: 设$f(x)$为可导的周期为T的函数，则$f(x+T)=f(x)$，

于是$f'(x+T)=\lim\limits_{\Delta x\to 0}\dfrac{f(x+T+\Delta x)-f(x+T)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x)$即可导的周期函数，其导函数仍为周其函数.

§3. 求导法则

1. 利用已经给出的导数公式，求下列函数的导数：

(1) $y = x^5$

(2) $y = x^{11}$

(3) $y = x^6$

(4) $y = 2^x$

(5) $y = \log_{10} x$

(6) $y = 10^x$

解：

(1) $y' = (x^5)' = 5x^4$

(2) $y' = (x^{11})' = 11x^{10}$

(3) $y' = (x^6)' = 6x^5$

(4) $y' = (2^x)' = 2^x \ln 2$

(5) $y' = (\log_{10} x)' = \dfrac{1}{x \ln 10}$

(6) $y' = (10^x)' = 10^x \ln 10$

2. 求下列函数的导数：

(1) $f(x) = 2x^2 - 3x + 1$，并求$f'(0), f'(1)$

(2) $f(x) = x^5 + 3\sin x$，并求$f'(0), f'\left(\dfrac{\pi}{2}\right)$

(3) $f(x) = e^x + 2\cos x - 7x$，并求$f'(0), f'(\pi)$

(4) $f(x) = 4\sin x - \ln x + x^2$

(5) $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$，并求$f'(0), f'(1)$

解：

(1) $f'(x) = 4x - 3$，$f'(0) = -3, f'(1) = 1$

(2) $f'(x) = 5x^4 + 3\cos x$，并求$f'(0) = 3, f'\left(\dfrac{\pi}{2}\right) = \dfrac{5\pi^4}{16}$

(3) $f'(x) = e^x - 2\sin x - 7$，并求$f'(0) = -6, f'(\pi) = e^\pi - 7$

(4) $f'(x) = 4\cos x - \dfrac{1}{x} + 2x$

(5) $f(x) = na_n x^{n-1} + (n_1)a_{n-1}x^{n-2} + \cdots + a_1$，并求$f'(0) = a_1, f'(1) = \sum\limits_{i=1}^{n} ia_i$

3. 求下列函数的导数：

(1) $y = x^2 \sin x$，并求$f'(0), f'\left(\dfrac{\pi}{2}\right)$

(2) $y = x\cos x + 3x^2$，并求$f'(-\pi)$和$f'(\pi)$

(3) $y = x\tan x + 7x - 6$

(4) $y = e^x \sin x - 7\cos x + 5x^2$

(5) $y = 4\sqrt{x} + \dfrac{1}{x} - 2x^3$

(6) $y = (3x^2 + 2x - 1)\sin x$

解：

(1) $y' = 2x\sin x + x^2\cos x$，$f'(0) = 0, f'\left(\dfrac{\pi}{2}\right) = \pi$

(2) $y' = \cos x - x\sin x + 6x$，$f'(-\pi) = -1 - 6\pi, f'(\pi) = -1 + 6\pi$

(3) $y' = \tan x + x\sec^2 x + 7$

(4) $y' = e^x \sin x + e^x \cos x + 7\sin x + 10x = e^x(\sin x + \cos x) + 7\sin x + 10x$

(5) $y' = \dfrac{2}{\sqrt{x}} - \dfrac{1}{x^2} - 6x^2$

(6) $y' = (3x^2 + 2x - 1)\cos x + (6x + 2)\sin x$

4. 求下列函数的导数：

(1) $y = \dfrac{2 + \sin x}{x}$

(2) $y = \cot x$

(3) $y = \dfrac{3x^2 + 7x - 1}{\sqrt{x}}$

(4) $y = \dfrac{(1 + x^2)\sin x}{2x}$

(5) $y = \dfrac{x \ln x}{1 + x}$

(6) $y = \dfrac{xe^x - 1}{\sin x}$

解：

(1) $y' = \dfrac{x(2 + \sin x)' - (x + \sin x)}{x^2} = \dfrac{x\cos x - \sin x - 2}{x^2}$

(2) $y' = \left(\dfrac{\cos x}{\sin x}\right)' = \dfrac{\sin x(\cos x)' - \cos x(\sin x)'}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

(3) $y' = \dfrac{\sqrt{x}(3x^2 + 7x - 1)' - (\sqrt{x})'(3x^2 + 7x - 1)}{x} = \dfrac{\sqrt{x}(6x + 7) - \dfrac{3x^2 + 7x - 1}{2\sqrt{x}}}{x} = \dfrac{9x^2 + 7x + 1}{2x\sqrt{x}} =$
$\dfrac{9}{2}x^{\frac{1}{2}} + \dfrac{7}{2}x^{-\frac{1}{2}} + \dfrac{1}{2}x^{-\frac{3}{2}}$

(4) $y' = \dfrac{2x[(1 + x^2)\sin x]' - 2(1 + x^2)\sin x}{4x^2} =$
$\dfrac{2x[2x\sin x + (1 + x^2)\cos x] - 2(1 + x^2)\sin x}{4x^2} = \dfrac{(x^2 - 1)\sin x + x(1 + x^2)\cos x}{2x^2}$

(5) $y' = \dfrac{(1 + x)(x\ln x)' - x\ln x}{(1 + x)^2} = \dfrac{(1 + x)(\ln x + 1) - x\ln x}{(1 + x)^2} = \dfrac{x + \ln x + 1}{(1 + x)^2}$

(6) $y' = \dfrac{\sin x(xe^x - 1)' - (\sin x)'(xe^x - 1)}{\sin^2 x} = \dfrac{e^x \sin x(x + 1) - \cos x(xe^x - 1)}{\sin^2 x}$

5. 求下列函数的导数：

(1) $y = \dfrac{\sqrt{x} + \cos x}{x - 1} - 7x^2$

(2) $y = \dfrac{x\sin x + \cos x}{x\sin x - \cos x}$

(3) $y = x^2 e^x \sin x + \dfrac{3 + x^2}{\sqrt{x}} - x\ln x + 8x^2$

(4) $y = \dfrac{\sin x}{1 + \tan x}$

(5) $y = \dfrac{x\cos x - \ln x}{x + 1}$

(6) $y = \dfrac{1}{x + \cos x}$

解：

(1) $y' = \dfrac{(x - 1)(\dfrac{1}{2\sqrt{x}} - \sin x) - (\sqrt{x} + \cos x)}{(x - 1)^2} - 14x =$
$\dfrac{(x - 1)(1 - 2\sqrt{x}\sin x) - (2x + 2\sqrt{x}\cos x)}{2\sqrt{x}(x - 1)^2} - 14x$

(2) $y' = \dfrac{(x\sin x - \cos x)(\sin x + x\cos x - \sin x) - (x\sin x + \cos x)(\sin x + x\cos x + \sin x)}{(x\sin x - \cos x)^2} = -\dfrac{2(\sin x\cos x + x)}{(x\sin x - \cos x)^2} =$
$-\dfrac{2x + \sin 2x}{(x\sin x - \cos x)^2}$

(3) $y' = 2xe^x\sin x + x^2e^x\sin x + x^2e^x\cos x + \dfrac{2x\sqrt{x} - \dfrac{3+x^2}{2\sqrt{x}}}{x} - \ln x - 1 + 16x = xe^x(2\sin x + x\sin x + x\cos x) + \dfrac{3x^2-1}{2x\sqrt{x}} - \ln x - 1 + 16x$

(4) $y' = \dfrac{\cos x(1+\tan x) - \sin x\cdot\sec^2 x}{(1+\tan x)^2}$

(5) $y' = \dfrac{(x+1)(\cos x - x\sin x - \frac{1}{x}) - (x\cos x - \ln x)}{(x+1)^2} = \dfrac{x\cos x - (x^2\sin x+1)(x+1) + x\ln x}{x(x+1)^2}$

(6) $y' = -\dfrac{1-\sin x}{(x+\cos x)^2} = \dfrac{\sin x - 1}{(x+\cos x)^2}$

6. 求曲线$y+1=(x-2)^3$在点$A(3,0)$处的切线方程及法线方程.
解：因$y+1=(x-2)^3$，则$y=(x-2)^3-1$，于是$y'=3(x-2)^2$，则所求切线的斜率为$k=y\,'|_{x=3}=3$，从而所求切线方程为：$y=3(x-3)$即$3x-y-9=0$；所求法线方程为：$y=-\dfrac{1}{3}(x-3)$即$x+3y-3=0$.

7. 求曲线$y=\ln x$在点$(1,0)$处的切线方程和法线方程.
解：因$y=\ln x$，则$y\,'=\dfrac{1}{x}$，于是所求切线的斜率为$k=y\,'|_{x=1}=1$，
从而所求切线方程为：$y=x-1$即$x-y-1=0$；所求法线方程为：$y=-(x-1)$即$x+y-1=0$.

8. 抛物线$y=x^2-2x+4$在哪一点的切线平行于x轴？在哪一点的切线与x轴的交角为45^o？
解：因$y=x^2-2x+4$，故$y\,'=2x-2$.
又平行于x轴的切线斜率为$k=0$，则$2x-2=0$，于是$x=1$，即所求点为$(1,3)$；
又与x轴的交角为45^o的切线斜率为$k=1$，则$2x-2=1$，于是$x=\dfrac{3}{2}$，即所求点为$\left(\dfrac{3}{2},\dfrac{13}{4}\right)$.

9. 沿直线运动的物体，其运动方程为$s=3t^4-20t^3+36t^2$，求其速度，并问物体何时向前运动？何时向后运动？
解：因$s=3t^4-20t^3+36t^2$，故$v=s'=12t^3-60t^2+72t$.
当$v>0$即$0<t<2$或$t>3$时，物体向前运动；当$v<0$即$2<t<3$时，物体向后运动.

10. 由于外力作用，一球沿着斜面向上滚，初速度为5，运动方程为$s=5t-t^2$，试问此球何时开始向下滚？
解：因$s=5t-t^2$，故$v=s'=5-2t$，当$v=0$即$t=\dfrac{5}{2}$时，球开始向下滚.

11. 在$x=2$处，作曲线$y=0.1x^3$的切线，试问除切点外，此切线与曲线还在何处相交？
解：因$y=0.1x^3$，故$y\,'=0.3x^2$，于是在$x=2$处，切线的斜率为$k=y\ |x=2=1.2$，从而此曲线在切点$(2,0.8)$处的切线方程为$y-0.8=1.2(x-2)$，即$6x-5y-8=0$；由$\begin{cases} y=0.1x^3 \\ 6x-5y-8=0 \end{cases}$，得$x^3-12x+16=0$，则$(x-2)^2(x+4)=0$，解得$x_1=x_2=2,x_3=-4$，则此切线与曲线还在点$(-4,-6.4)$处相交.

12. 曲线$y=x^n$（n为正整数）上点$(1,1)$处的切线交x轴于点$(\xi_n,0)$，求$\lim\limits_{n\to\infty}y(\xi_n)$.
解：因$y=x^n$，则$y\,'=nx^{n-1}$，则此曲线在$x=1$处的切线斜率为$k=y\,'|_{x=1}=n$，于是此曲线在点$(1,1)$处的切线方程为$y-1=n(x-1)$即$y=nx-n+1$.
当$y=0$时，$x=\dfrac{n-1}{n}$即$\xi_n=\dfrac{n-1}{n}$，则$\lim\limits_{n\to\infty}y(\xi_n)=\lim\limits_{n\to\infty}\left(\dfrac{n-1}{n}\right)^n=\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^n=\dfrac{1}{e}$.

13. 设抛物线方程为$y=x^2+ax+b$，试问点(x_0,y_0)位于何处时，可以从点(x_0,y_0)对此抛物线作出两条切线或一条切线，或作不出切线？
解：设(x_0,y_0)为平面上任一点，(x,y)为过(x_0,y_0)的切线与抛物线的交点.
由已知，得与抛物线相交的切线的斜率为$k=y\,'=2x+a$，则所求切线为$y-y_0=(2x+a)(x-x_0)$即$y_0-y=(2x-a)(x_0-x)$，
又$y=x^2+ax+b$，则$y_0-(x^2+ax+b)=(2x+a)(x_0-x)$，故$x^2-2x_0x+y_0-ax_0$，则$\Delta=4x_0^2-4(y_0-b-ax_0)$
当$\Delta>0$即$y_0<x_0^2+ax_0+b$时，可作两条切线；当$\Delta=0$即$y_0=x_0^2+ax_0+b$时，可作一条切线；当$\Delta<0$即$y_0>x_0^2+ax_0+b$时，作不出切线.

14. 问底数a为什么值时，直线$y=x$才能与对数曲线$y=\log_a x$相切？在何处相切？
解：由题意，得$x'=(\log_a x)'$，即$1=\dfrac{1}{x\ln a}$，则$x=\dfrac{1}{\ln a}$，于是$y=\dfrac{1}{\ln a}$.
又由于在切点相切，其纵坐标必须相等，则$\log_a x=\dfrac{1}{\ln a}$，于是$x=e$，则可得$\ln a=\dfrac{1}{e}$，即$a=e^{\frac{1}{e}}$即当底数$a=e^{\frac{1}{e}}$时，直线$y=x$才能与对数曲线$y=\log_a x$相切，在点$(e,e)$处相切.

§4. 复合函数求导法

1. 求下列函数的导数：

(1) $y = 2\sin 3x$

(2) $y = 4\cos(3t-1)$

(3) $y = 3e^{2x} + 5\cos 2x$

(4) $y = (x+1)^2$

(5) $y = (1-x+x^2)^3$

(6) $y = 3e^{-2t} + 1$

(7) $y = \ln(x+1)$

(8) $y = (3x+1)^4$

(9) $y = \sqrt{1+x^2}$

(10) $y = \left(1 - \dfrac{1}{x}\right)^2$

(11) $y = \tan\dfrac{x}{2} + \sin 3x$

(12) $y = \ln\sin x$

(13) $y = \dfrac{x}{\sqrt{1+x^2}}$

(14) $y = \dfrac{1}{\sqrt{2\pi}} e^{-3t^2}$

解：

(1) $y' = 6\cos 3x$

(2) $y' = -12\sin(3t-1)$

(3) $y' = 6e^{2x} - 10\sin 2x$

(4) $y' = 2(x+1)$

(5) $y' = 3(1-x+x^2)^2(2x-1)$

(6) $y' = -6e^{-2t}$

(7) $y' = \dfrac{1}{x+1}$

(8) $y' = 12(3x+1)^3$

(9) $y' = \dfrac{x}{\sqrt{1+x^2}}$

(10) $y' = 2\left(1-\dfrac{1}{x}\right)\cdot\left(-\dfrac{1}{x^2}\right) = \dfrac{2(x-1)}{x^3}$

(11) $y' = \dfrac{1}{2}\sec^2\dfrac{x}{2} + 3\cos 3x$

(12) $y' = \dfrac{\cos x}{\sin x} = cotx$

(13) $y' = \dfrac{\sqrt{1+x^2} - \dfrac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \dfrac{1}{(1+x^2)^{\frac{3}{2}}}$

(14) $y' = \dfrac{-3\sqrt{2}t}{\sqrt{\pi}} e^{-3t^2}$

2. 求下列函数的导数：

(1) $y = \sin^3 2x$

(2) $y = (at+b)e^{-2t}(a, b\text{为常数})$

(3) $y = e^{2t}\sin 3t + \dfrac{t^2}{2}$

(4) $y = \ln\dfrac{1-x^2}{1+x^2}$

(5) $y = \dfrac{e^{-kt}\sin\omega t}{1+t}(k,\omega\text{为常数})$

(6) $y = \dfrac{4}{(x+\cos 2x)^2}$

(7) $y = e^{-t}(\cos t + \sin t)$

(8) $y = \dfrac{x}{\sqrt{1+\cos^2 x}}$

(9) $y = (x-1)\sqrt{x^2+1}$

(10) $y = (2+3t)\sin 2t + 7t^2 - 7$

解:

(1) $y' = 6\sin^2 2x\cos x = 3\sin 4x\sin 2x$

(2) $y' = ae^{-2t} - 2(at+b)e^{-2t} = -(2at+2b-a)e^{-2t}$

(3) $y' = 2e^{2t}\sin 3t + 3e^{2t}\cos 3t + t = e^{2t}(2\sin 3t + 3\cos 3t) + t$

(4) $y' = \dfrac{1+x^2}{1-x^2}\cdot\dfrac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2} = \dfrac{4x}{x^4-1}$

(5) $y' = \dfrac{(1+t)e^{-kt}(-k\sin\omega t + \omega\cos\omega t) - (e^{-kt}\sin\omega t}{(1+t)^2} = \dfrac{-(kt+k+1)e^{-kt}\sin\omega t + \omega(1+t)e^{-kt}\cos\omega t}{(1+t)^2}$

(6) $y' = -\dfrac{4[(x+\cos 2x)^2]'}{(x+\cos 2x)^4} = -\dfrac{8(1-2\sin 2x)}{(x+\cos 2x)^2}$

(7) $y' = -e^{-t}(\cos t + \sin t) + e^{-t}(-\sin t + \cos t) = -2e^{-t}\sin t$

(8) $y' = \dfrac{\sqrt{1+\cos^2 x} - x\dfrac{-2\sin x\cos x}{2\sqrt{1+\cos^2 x}}}{1+\cos^2 x} = \dfrac{1+\cos^2 x + x\sin x\cos x}{(1+\cos^2 x)^{\frac{3}{2}}}$

(9) $y' = \sqrt{x^2+1} + (x-1)\dfrac{2x}{2\sqrt{x^2+1}} = \dfrac{2x^2-x+1}{\sqrt{x^2+1}}$

(10) $y' = 3\sin 2t + 2(2+3t)\cos 2t + 14t$

3. 求下列函数的导数:

(1) $y = e^{-kt}(3\cos\omega t + 4\sin\omega t)(k,\omega\text{为常数})$

(2) $y = x\arctan x$

(3) $y = (2x^2+1)^2 e^{-x}\sin 3x$

(4) $y = \dfrac{e^{-t}\sin 3t}{\sqrt{1+t^2}}$

(5) $y = (3t+1)e^t(\cos 3t - 7\sin 3t)$

(6) $y = t\arcsin 3t + 7e^{-2t}\ln t + 8t$

(7) $y = x\sqrt{a^2-x^2} + \dfrac{x}{\sqrt{a^2-x^2}}(a\text{为常数})$

解:

(1) $y' = -ke^{-kt}(3\cos\omega t + 4\sin\omega t) + e^{-kt}(-3\omega\sin\omega t + 4\omega\cos\omega t) = e^{-kt}[(4\omega-3k)\cos\omega t - (3\omega+4k)\sin\omega t]$

(2) $y' = \arctan x + \dfrac{x}{1+x^2}$

(3) $y' = 4x(2x^2+1)e^{-x}\sin 3x - (2x^2+1)^2 e{-}x\sin 3x + 3(2x^2+1)^2 e{-}x\cos 3x = e^{-x}(2x^2+1)[(-2x^2+8x-1)\sin 3x + 3(2x^2+1)\cos 3x]$

(4) $y' = \dfrac{e^{-t}(-\sin 3t + 3\cos 3t)\sqrt{1+t^2} - e^{-t}\sin 3t\dfrac{t}{\sqrt{1+t^2}}}{1+t^2} = \dfrac{e^{-t}[3(1+t^2)\cos 3t - (t^2+t+1)\sin 3t]}{(1+t^2)^{\frac{3}{2}}}$

(5) $y' = 3e^t(\cos 3t - 7\sin 3t) + (3t+1)e^t(\cos 3t - 7\sin 3t) + (3t+1)e^t(-3\sin 3t - 21\cos 3t) = -e^t[(60t + 17)\cos 3t + (30t+31)\sin 3t]$

(6) $y' = \arcsin 3t + \dfrac{3t}{\sqrt{1-9t^2}} - 14e^{-2t}\ln t + \dfrac{7e^{-2t}}{t} + 8$

(7) $y' = \sqrt{a^2-x^2} - \dfrac{x^2}{\sqrt{a^2-x^2}} + \dfrac{\sqrt{a^2-x^2} + \dfrac{x^2}{\sqrt{a^2-x^2}}}{a^2-x^2} = \dfrac{(a^2-2x^2)(a^2-x^2)+a^2}{(a^2-x^2)^{\frac{3}{2}}}$

4. 求下列函数的导数：

(1) $y = \sin^n x \cos nx$

(2) $y = \sinh^n x \cosh nx$

(3) $y = e^{-x^2+2x}$

(4) $y = (\sin x + \cos x)^n$

(5) $y = \arcsin(\sin x \cdot \cos x)$

(6) $y = \ln\sqrt{\dfrac{(x+2)(x+3)}{x+1}}$

(7) $y = \arctan\dfrac{2x}{1-x^2}$

(8) $y = \dfrac{x}{a^2\sqrt{a^2+x^2}}$

解:

(1) $y' = n\sin^{n-1} x\cos x\cos nx - n\sin^n x\sin nx =$
$n\sin^{n-1} x\cos(n+1)x$

(2) $y' = n\sinh^{n-1} x\cosh x\cosh nx + n\sinh^n x\sinh nx =$
$n\sinh^n x\cosh(n+1)x$

(3) $y' = -2(x-1)e^{-x^2+2x}$

(4) $y' = n(\sin x+\cos x)^{n-1}(\cos x - \sin x) = n(\sin x + \cos x)^{n-2}\cos 2x$

(5) $y' = \dfrac{\cos 2x}{\sqrt{1-(\sin x\cdot\cos x)^2}} = \dfrac{2\cos 2x}{\sqrt{4-sin^2 2x}}$

(6) 因$y = \ln\sqrt{\dfrac{(x+2)(x+3)}{x+1}} = \dfrac{1}{2}[\ln(x+2)+\ln(x+3)-\ln(x+1)]$，故$y' = \dfrac{1}{2}\left[\dfrac{1}{x+2}+\dfrac{1}{x+3}-\dfrac{1}{x+1}\right] = \dfrac{x^2+2x-1}{2(x+1)(x+2)(x+3)}$.

(7) $y' = \dfrac{1}{1+\left(\dfrac{2x}{1-x^2}\right)^2}\cdot\dfrac{2(1-x^2)+4x^2}{(1-x^2)^2} = \dfrac{2}{1+x^2}$

(8) $y' = \dfrac{\sqrt{a^2+x^2} - \dfrac{x^2}{\sqrt{a^2+x^2}}}{a^2(a^2+x^2)} = \dfrac{1}{(a^2+x^2)^{\frac{3}{2}}}$

5. 利用取对数再求导的方法，求下列函数的导数：

(1) $y = x\sqrt{\dfrac{1-x}{1+x}}$

(2) $y = \dfrac{x^2}{1-x}\sqrt{\dfrac{x+1}{1+x+x^2}}$

(3) $y = (x-\alpha_1)^{\alpha_1}(x-\alpha_2)^{\alpha_2}\cdots(x-\alpha_n)^{\alpha_n}$

(4) $y = (x+\sqrt{1+x^2})^n$

(5) $y = x^m m^x$

解:

(1) 因$y=x\sqrt{\frac{1-x}{1+x}}$，则 $\ln y=\ln x+\frac{1}{2}\ln(1-x)-\frac{1}{2}\ln(1+x)$，两边对$x$求导，得$\frac{1}{y}y'=\frac{1}{x}+\frac{-1}{2(1-x)}-\frac{1}{2(1+x)}$，

则$y'=\frac{1-x-x^2}{(1+x)\sqrt{1-x^2}}(0<|x|<1)$

(2) 因$y=\frac{x^2}{1-x}\sqrt{\frac{x+1}{1+x+x^2}}$，则 $\ln y=2\ln x-\ln(1-x)+\frac{1}{2}\ln(x+1)-\frac{1}{2}\ln(1+x+x^2)$，两边对$x$求导，得$\frac{1}{y}y'=\frac{2}{x}++\frac{1}{1-x}+\frac{1}{2(1+x)}-\frac{1+2x}{2(1+x+x^2)}$，

则$y'=\frac{x^2}{1-x}\sqrt{\frac{x+1}{1+x+x^2}}\left(\frac{2}{x}+11-x+12(x+1)-\frac{2x+1}{2(1+x+x^2)}\right)$

(3) 因$y=(x-\alpha_1)^{\alpha_1}(x-\alpha_2)^{\alpha_2}\cdots(x-\alpha_n)^{\alpha_n}=\prod\limits_{i=1}^{n}(x-\alpha_i)^{\alpha_i}$及$y$在对数符号内，故应设$\prod\lim\limits_{i=1}^{n}(x-\alpha_i)^{\alpha_i}>0$，则 $\ln y=\sum\limits_{i=1}^{n}\alpha_i\ln|x-\alpha_i|$，两边对$x$求导数，得$\frac{1}{y}y'=\sum\limits_{i=1}^{n}\frac{\alpha_i}{x-\alpha_i}$，

则$y'=\sum\limits_{i=1}^{n}\frac{\alpha_i}{x-\alpha_i}\prod\limits_{i=1}^{n}(x-\alpha_i)^{\alpha_i}(x\in D)$其中$D=\left\{\prod\limits_{i=1}^{n}(x-\alpha_i)^{\alpha_i}>0\right\}$

(4) 因$y=(x+\sqrt{1+x^2})^n$，则 $\ln y=n\ln(x+\sqrt{1+x^2})$，两边对$x$求导，得$\frac{1}{y}y'=n\frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}=\frac{n}{\sqrt{1+x^2}}$，则$y'=\frac{n}{\sqrt{1+x^2}}(x+\sqrt{1+x^2})^n$

(5) 因$y=x^m m^x$，则$\ln y=m\ln|x|+x\ln m$，两边对x求导，得$\frac{1}{y}y'=\frac{m}{x}+\ln m$，则$y'=x^{m-1}m^{x+1}+x^m m^x\ln m$

6. 设$f(x)$是对x可求导的函数，求$\frac{dy}{dx}$.

(1) $y=f(x^2)$

(2) $y=f(e^x)\cdot e^{f(x)}$

(3) $y=f(f(f(x)))$

解:

(1) $\frac{dy}{dx}=2xf'(x^2)$

(2) $\frac{dy}{dx}=e^x f'(e^x)\cdot e^{f(x)}+f'(x)f(e^x)e^{f(x)}=e^{f(x)}(e^x f'(e^x)+f(e^x)f'(x))$

(3) $\frac{dy}{dx}=f'(f(f(x)))f'(f(x))f'(x)$

7. 设$\varphi(x),\psi(x)$为对x可求导的函数，求$\frac{dy}{dx}$.

(1) $y=\sqrt{\varphi^2(x)+\psi^2(x)}$

(2) $y=\arctan\frac{\varphi(x)}{\psi(x)}(\psi(x)\neq 0)$

(3) $y=\sqrt[\varphi(x)]{\psi(x)}(\varphi(x)\neq 0,\psi(x)>0)$

(4) $y=\log_{\varphi(x)}\psi(x)(\varphi(x)>0,\psi(x)\neq 0)$

解:

(1) $\frac{dy}{dx}=\frac{\varphi(x)\varphi'(x)+\psi(x)\psi'(x)}{\sqrt{\varphi^2(x)+\psi^2(x)}}$

(2) $\frac{dy}{dx}=\frac{\varphi'(x)\psi(x)-\psi'(x)\varphi(x)}{\varphi^2(x)+\psi^2(x)}$

(3) $\frac{dy}{dx}=\sqrt[\varphi(x)]{\psi(x)}\left(\frac{\psi'(x)}{\varphi(x)\psi(x)}-\frac{\varphi'(x)\ln\psi(x)}{\varphi^2(x)}\right)$

(4) $\dfrac{dy}{dx}=\dfrac{\dfrac{\psi'(x)}{\psi(x)}\ln\varphi(x)-\dfrac{\varphi'(x)}{\varphi(x)}\ln\psi(x)}{(\ln\varphi(x))^2}=\dfrac{\psi'(x)}{\psi(x)\ln\varphi(x)}-\dfrac{\varphi'(x)\ln\psi(x)}{\varphi(x)(\ln\varphi(x))^2}=$
$\log_{\varphi(x)}\psi(x)\left[\dfrac{\psi'(x)}{\psi(x)\ln\psi(x)}-\dfrac{\varphi'(x)}{\varphi(x)\ln\varphi(x)}\right]$

8. 求图4-7所示曲柄连杆机构滑块运动的速度.

解：因$s=\sqrt{l^2-r^2\sin^2\omega t}-r\cos\omega t$，故$v=s'=r\omega\sin\omega t-\dfrac{r^2\omega\sin 2\omega t}{2\sqrt{l^2-r^2\sin^2\omega t}}$.

9. 求曲线$y=\sqrt{1-x^2}$在$x=\dfrac{1}{2}$处的切线方程和法线方程.

解：因$y'=-\dfrac{x}{\sqrt{1-x^2}}$，则在$x=\dfrac{1}{2}$处的切线斜率为$k=-\dfrac{\sqrt{3}}{3}$，

于是所求切线方程为：$y-\dfrac{\sqrt{3}}{2}=-\dfrac{\sqrt{3}}{3}\left(x-\dfrac{1}{2}\right)$即$x+\sqrt{3}y-2=0$；

所求法线方程为：$y-\dfrac{\sqrt{3}}{2}=\sqrt{3}\left(x-\dfrac{1}{2}\right)$即$\sqrt{3}x-y=0$.

10. 求曲线$y=e^{-x}$上的一点，使过该点的切线与直线$y=-ex$平行，并写出该点的法线方程.

解：因$k=y'=-e^{-x}=-e$，则$x=-1$，则过$(-1,e)$点的切线与直线$y=-ex$平行，过该点的法线方程为$y-e=\dfrac{1}{e}(x+1)$即$x-ey+e^2+1=0$.

11. 求曲线$y=\sqrt{1-x^2}$上的水平切线.

解：因$k=y'=-\dfrac{x}{\sqrt{1-x^2}}=0$，则$x=0$，于是此曲线在$(0,1)$处的切线为水平切线，切线方程为$y=1$.

12. 求曲线$y=\dfrac{1}{2}(1+2x^2\pm\sqrt{1+4x^2})$上横坐标$x=U$的点处的切线方程.这切线还与曲线交于何处？

解：因$y'=2x\pm\dfrac{2x}{\sqrt{1+4x^2}}$，则曲线在$x=U$处的切线斜率为$k=2U\pm\dfrac{2U}{\sqrt{1+4U^2}}$，于是此曲线在切点$(U,\dfrac{1}{2}(1+2U^2\pm\sqrt{1+4U^2}))$处的切线方程为$y-\dfrac{1}{2}(1+2U^2\pm\sqrt{1+4U^2}))=(2U\pm\dfrac{2U}{\sqrt{1+4U^2}})(x-U)$，

即$2U(\sqrt{1+4U^2}\pm1)x-\sqrt{1+4U^2}y\pm\dfrac{1}{2}+\dfrac{1}{2}(1-2U^2)\sqrt{1+4U^2}=0$，此切线还与曲线交于

$\left(\dfrac{U(\sqrt{1+4U^2}\pm1)}{\sqrt{1+4U^2}},\dfrac{1}{2}\left(1+\dfrac{2U^2(\sqrt{1+4U^2}\pm1)^2}{1+4U^2}\pm\sqrt{1+\dfrac{4U^2(\sqrt{1+4U^2}\pm1)^2}{1+4U^2}}\right)\right)$.

13. 设$y=f(x)$在x_0可导，记$\varphi(t)=f(x_0+at)$，a为常数，求$\varphi'(0)$.

解：若$a=0$，则$\varphi(t)=f(x_0)$，则$\varphi'(0)=0$

若$a\neq0$，则$\varphi'(x)=\lim\limits_{t\to0}\dfrac{\varphi(x)-\varphi(0)}{t}=\lim\limits_{t\to0}\dfrac{f(x_0+at)-f(x_0)}{t}=a\lim\limits_{t\to0}\dfrac{f(x_0+at)-f(x_0)}{at}=af'(x_0)$.

§5. 微分及其运算

1. 求下列函数在指定点的微分：

(1) $y=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$，求$dy(0),dy(1)$

(2) $y=\sec x+\tan x$，求$dy(0),dy\left(\frac{\pi}{4}\right),dy(\pi)$

(3) $y=\frac{1}{a}\arctan\frac{x}{a}$，求$dy(0),dy(a)$

(4) $y=\frac{1}{x}+\frac{1}{x^2}$，求$dy(0.1),dy(0.01)$

解：

(1) 因$dy=[na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\cdots+a_1]dx$，则$dy(0)=a_1dx,dy(1)=\sum\limits_{i=1}^{n}ia_idx$

(2) 因$dy=(\tan x\sec x+\sec^2 x)dx$，则$dy(0)=dx,dy\left(\frac{\pi}{4}\right)=(\sqrt{2}+2)dx,dy(\pi)=dx$

(3) 因$dy=\frac{dx}{a^2+x^2}dx$，则$dy(0)=\frac{dx}{a^2}dx,dy(a)=\frac{dx}{2a^2}dx$

(4) 因$y=-\frac{x+2}{x^3}dx$，则$dy(0.1)=-2100dx,dy(0.01)=-2010000dx$

2. 求下列函数$y=y(x)$的微分：

(1) $y=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$

(2) $y=x^2\sin x$

(3) $y=\frac{x}{1-x^2}$

(4) $y=x\ln x-x$

(5) $y=(1-x^2)^n$

(6) $y=\sqrt{x}+\ln x-\frac{1}{\sqrt{x}}$

(7) $y=\ln\tan x$

(8) $y=\sin ax\cos bx$

(9) $y=e^{ax}\cos bx$

(10) $y=\arcsin\sqrt{1-x^2}$

解：

(1) $dy=(1-x+x^2-x^3)dx$

(2) $dy=(2x\sin x+x^2\cos x)dx$

(3) $dy=\frac{1+x^2}{(1-x^2)^2}dx$

(4) $dy=\ln xdx$

(5) $dy=-2nx(1-x^2)^{n-1}dx$

(6) $dy=\frac{x+2\sqrt{x}+1}{x^{\frac{3}{2}}}dx$

(7) $dy=\frac{2}{\sin 2x}dx$

(8) $dy=(a\cos ax\cos bx-b\sin ax\sin bx)dx$

(9) $dy=e^{ax}(a\cos bx-b\sin bx)dx$

(10) $dy=-\frac{x}{|x|\sqrt{1-x^2}}dx$

3. 求下列函数y的微分：

(1) $y=\sin^2 t,t=\ln(3x+1)$

(2) $y=\ln(3t+1), t=\sin^2 x$

(3) $y=e^{3u}, u=\dfrac{1}{2}\ln t, t=x^2-2x+5$

(4) $y=\arctan u, u=(\ln t)^2, t=1+x^2-\cot x$

解:

(1) $dy=\dfrac{3\sin(2\ln(3x+1))}{3x+1}dx$

(2) $y=\dfrac{3\sin 2x}{3\sin^2 x+1}dx$

(3) $y=\dfrac{3(3x^2-2)}{2(x^3-2x+5)}e^{\frac{3}{2}\ln(x^2-2x+5)}dx$

(4) $y=\dfrac{2\ln(1+x^2-\cot x)(2x+\csc^2 x)}{[1+(\ln(1+x^2-\cot x))^4](1+x^2-\cot x)}dx$

4. 若u,v,w为x的可微分函数，求函数y的微分dy：

(1) $y=u\cdot v\cdot w$

(2) $y=\dfrac{u\cdot w}{v^2}$

(3) $y=\dfrac{1}{\sqrt{u^2+v^2}}$

(4) $y=\ln\sqrt{u^2+v^2}$

(5) $y=\arctan\dfrac{u}{v}$

解:

(1) $dy=(u'\cdot v\cdot w+u\cdot v'\cdot w+u\cdot v\cdot w')dx$

(2) $dy=\dfrac{v^2(u'w+uw')-2uvv'w}{v^4}dx$

(3) $dy=-\dfrac{uu'+vv'}{(u^2+v^2)^{\frac{3}{2}}}dx(u^2+v^2>0)$

(4) $dy=\dfrac{uu'+vv'}{u^2+v^2}dx$

(5) $dy=\dfrac{u'v-uv'}{u^2+v^2}dx(v\neq 0)$

§6. 隐函数及参数方程所表示函数的求导法

1. 求下列隐函数的导数$\dfrac{dy}{dx}$：

(1) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$，其中$a,b$为常数

(2) $y^2=2px$，其中p为常数

(3) $x^2+xy+y^2=a^2$，其中a为常数

(4) $x^3+y^3-xy=0$

(5) $y=x+\dfrac{1}{2}\sin y$

(6) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$，其中$a$为常数

(7) $y-\cos(x+y)=0$

(8) $y=x+\arctan y$

(9) $y=1-\ln(x+y)+e^y$

(10) $\arctan\dfrac{y}{x}=\ln\sqrt{x^2+y^2}$

解:

(1) 在方程两端对x求导数，并注意到y是x的函数，就有$\dfrac{2x}{a^2}+\dfrac{2yy'}{b^2}=0$，则$y'=-\dfrac{b^2x}{a^2y}(y\neq 0)$.

(2) 在方程两端对x求导数，并注意到y是x的函数，就有$2yy'=2p$，则$y'=\dfrac{p}{y}(y\neq 0)$.

(3) 在方程两端对x求导数，并注意到y是x的函数，就有$2x+xy'+y+2yy'=0$，则$y'=-\dfrac{2x+y}{x+2y}$.

(4) 在方程两端对x求导数，并注意到y是x的函数，就有$3x^2+3y^2y'-xy'-y=0$，则$y'=\dfrac{3x^2-y}{x-3y^2}$.

(5) 在方程两端对x求导数，并注意到y是x的函数，就有$y'=1+\dfrac{y'}{2}\cos y$，则$y'=\dfrac{2}{2-\cos y}$.

(6) 在方程两端对x求导数，并注意到y是x的函数，就有$\dfrac{2}{3}x^{-\frac{1}{3}}+\dfrac{2}{3}y^{-\frac{1}{3}}y'=0$，则$y'=-\sqrt[3]{\dfrac{x}{y}}$.

(7) 在方程两端对x求导数，并注意到y是x的函数，就有$y'+(1+y')\sin(x+y)=0$，则$y'=-\dfrac{\sin(x+y)}{1+\sin(x+y)}$.

(8) 在方程两端对x求导数，并注意到y是x的函数，就有$y'=1+\dfrac{y'}{1+y^2}$，则$y'=\dfrac{1+y^2}{y^2}$.

(9) 在方程两端对x求导数，并注意到y是x的函数，就有$y'=-\dfrac{1+y'}{x+y}+y'e^y$，则$y'=\dfrac{1}{(x+y)e^y-x-y-1}$.

(10) 在方程两端对x求导数，并注意到y是x的函数，就有$\dfrac{xy'-y}{x^2+y^2}=\dfrac{x+yy'}{x^2+y^2}$，则$y'=\dfrac{x+y}{x-y}$.

2. 求下列隐函数在指定点的导数$\dfrac{dy}{dx}$：

(1) $y=\cos x+\dfrac{1}{2}\sin y$，点$\left(\dfrac{\pi}{2},0\right)$

(2) $ye^x+\ln y=1$，点$(0,1)$

解:

(1) 在方程两端对x求导数，并注意到y是x的函数，就有$y'=-\sin x+\dfrac{y'}{2}\cos y$，则$y'=\dfrac{2\sin x}{\cos y-2}$，于是在点$\left(\dfrac{\pi}{2},0\right)$处，$y'=-2$.

(2) 在方程两端对x求导数，并注意到y是x的函数，就有$e^x(y+y')+\dfrac{y'}{y}=0$，则$y'=-\dfrac{y^2e^x}{ye^x+1}$，于是在点$(0,1)$处，$y'=-\dfrac{1}{2}$.

3. 求曲线$x^{\frac{3}{2}}+y^{\frac{3}{2}}=16$在点$(4,4)$的切线方程和法线方程.

解：在方程两端对x求导数，并注意到y是x的函数，就有$\frac{3}{2}x^{\frac{1}{2}}+\frac{3}{2}y^{\frac{1}{2}}y'=0$，则$y'=-\sqrt{\frac{x}{y}}$，于是$y'|_{\substack{x=4\\y=4}}=-1$，从而切线方程为$y-4=-(x-4)$，即$x+y-8=0$
法线方程为$y-4=x-4$，即$x=y$.

4. 求下列参数方程在所示点的导数：

(1) $\begin{cases} x= & a\cos t \\ y= & b\sin t \end{cases}$ 在$t=\frac{\pi}{3}$和$\frac{\pi}{4}$处

(2) $\begin{cases} x= & t-\sin t \\ y= & 1-\cos t \end{cases}$，在$t=\frac{\pi}{2},\pi$处

(3) $\begin{cases} x= & 1-t^2 \\ y= & t-t^3 \end{cases}$，在$t=\frac{\sqrt{2}}{2},\frac{\sqrt{3}}{3}$处

(4) $\begin{cases} x= & a(t-\sin t) \\ y= & a(1-\cos t) \end{cases}$ (a是常数)，在$t=0,\frac{\pi}{2}$处

解：

(1) 因$x'(t)=-a\sin t, y'(t)=b\cos t$，则$\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=-\frac{b}{a}\cot t$，于是，当$t=\frac{\pi}{3}$时，$y'=-\frac{\sqrt{3}b}{3a}$；当$t=\frac{\pi}{4}$时，$y'=-\frac{b}{a}$

(2) 因$x'(t)=1-\cos t, y'(t)=\sin t$，则$\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\frac{\sin t}{1-\cos t}$，于是，当$t=\frac{\pi}{2}$时，$y'=1$；当$t=\pi$时，$y'=0$

(3) 因$x'(t)=-2t, y'(t)=1-3t^2$，则$\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\frac{3t^2-1}{2t}$，于是，当$t=\frac{\sqrt{2}}{2}$时，$y'=\frac{\sqrt{2}}{4}$；当$t=\frac{\sqrt{3}}{3}$时，$y'=0$

(4) 因$x'(t)=a(1-\cos t), y'(t)=a\sin t$，则$\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\cot\frac{t}{2}$，于是，当$t=0$时，$y'$无意义；当$t=\frac{\pi}{2}$时，$y=1$

5. 求下列参数方程的导数：

(1) $\begin{cases} x= & a\cosh t \\ y= & b\sinh t \end{cases}$

(2) $\begin{cases} x= & \sin^2 t \\ y= & \cos^2 t \end{cases}$

(3) $\begin{cases} x= & a\cos^3 t \\ y= & a\sin^3 t \end{cases}$

(4) $\begin{cases} x= & e^{2t}\cos^2 t \\ y= & e^{2t}\sin^2 t \end{cases}$

解：

(1) $\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\frac{a\sinh t}{b\cosh t}=\frac{a}{b}\coth t$

(2) $\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\frac{-2\cos t\sin t}{2\sin t\cos t}=-1$

(3) $\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\frac{3\sin^2 t\cos t}{-3\cos^2 t\sin t}=-!tant$

(4) $\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\frac{e^{2t}(2\sin^2 t+2\sin t\cos t)}{e^{2t}(2\cos^2 t-2\cos t\sin t)}=\tan t\cdot\frac{\sin t+\cos t}{\cos t-\sin t}$

6. 一圆锥形容器，深10尺，上顶圆半径为4尺（图4-11）：

(1) 灌入水时，求水的体积V对水面高度h的变化率；

(2) 求体积V对容器截面圆半径R的变化率.

解：因体积V与容器截面圆半径R，水面高度h的关系为$V=\dfrac{1}{3}\pi R^2h$，且由已知，得$\dfrac{R}{4}=\dfrac{h}{10}$即$h=\dfrac{5}{2}R$，于是

(1) $V=\dfrac{1}{3}\pi\left(\dfrac{2}{5}h\right)^2h=\dfrac{4}{75}\pi h^3$，从而$\dfrac{dV}{dh}=\dfrac{4}{25}\pi h^2$；

(2) $V=\dfrac{1}{3}\pi R^2\cdot\dfrac{5}{2}R=\dfrac{5}{6}\pi R^3$，从而$\dfrac{dV}{dR}=\dfrac{5}{2}\pi R^2$.

7. 一圆锥形容器底面朝上放着，它的顶角为$2\arctan\dfrac{3}{4}$，今向里面倒进某种液体，

(1) 当液体半径r为3，半径增加的速度$\dfrac{dr}{dt}$为$\dfrac{1}{4}$时，体积增加的速度$\dfrac{dV}{dt}$是多少？

(2) 当液体半径为6，体积增加的速度为24时，半径增加的速度是多少？

解：因体积V与液体半径r的关系为$V=\dfrac{4}{9}\pi r^3$，V,r都是时间t的函数，两边对t求导，得$\dfrac{dV}{dt}=\dfrac{4}{9}\pi(3r^2)\dfrac{dr}{dt}$即$\dfrac{dV}{dt}=\dfrac{4}{3}\pi r^2\dfrac{dr}{dt}$，则

(1) 当$r=3,\dfrac{dr}{dt}=\dfrac{1}{4}$时，$\dfrac{dV}{dt}=3\pi$；

(2) 由$\dfrac{dr}{dt}=\dfrac{3}{4\pi r^2}\dfrac{dV}{dt}$，得当$r=6,\dfrac{dV}{dt}=24$时，$\dfrac{dr}{dt}=\dfrac{1}{2\pi}$.

8. 水从高为18厘米、底半径为6厘米的圆锥形漏斗流入半径为5厘米的圆柱形筒内.已知漏斗中水深为12厘米时，漏斗中水面的下降速度为1厘米/分，求此时圆筒中水面的上升速度.

解：设从开始漏水起经t分钟后，圆锥形漏斗中溶液的深度为x厘米，圆柱形筒中的水面升高了y厘米。此时，漏斗中漏出的溶液的体积为$\dfrac{1}{3}\pi\cdot 6^2\cdot 18-\dfrac{1}{3}\pi\left(\dfrac{x}{18}\cdot 6\right)^2\cdot x=216\pi-\dfrac{\pi}{27}x^3$（立方厘米），圆柱形筒中注入的溶液的体积为$\pi\cdot 5^2\cdot y=25\pi y$（立方厘米）。据题意知，$25\pi y=216\pi-\dfrac{\pi}{27}x^3$，故$y=\dfrac{1}{25}\left(216-\dfrac{x^3}{27}\right)$，于是$\dfrac{dy}{dt}=\dfrac{dy}{dx}\cdot\dfrac{dx}{dt}=-\dfrac{1}{675}\cdot 3x^2\cdot\dfrac{dx}{dt}=-\dfrac{1}{225}x^2\cdot\dfrac{dx}{dt}$。当$x=12$(厘米)时，$\dfrac{dx}{dt}=-1$(厘米/分)，于是此时圆筒中水面的上升速度为$\dfrac{dy}{dx}=-\dfrac{1}{225}\cdot 12^2(-1)=\dfrac{16}{25}=0.64$(厘米/分).

9. 图4-12所示电路中，输出功率$P=i^2R$，其中电流$i=\dfrac{U}{r+R}$.求当调整可变电阻R时，功率P的变化率$\dfrac{dP}{dR}$.

解：因$P=i^2R,i=\dfrac{U}{r+R}$，则$\dfrac{dP}{dR}=2iR\dfrac{di}{dR}+i^2=\dfrac{-2U^2R}{(r+R)^3}+\dfrac{U^2}{(r+R)^2}=\dfrac{U^2(r-R)}{(r+R)^3}$

§7. 不可导的函数举例

1. 求下列函数在所示点x_0的左导数$f'_-(x_0)$和右导数$f'_+(x_0)$：

(1) $y=\begin{cases} x^2, & x\leqslant 0, \\ xe^x, & x>0, \end{cases}\quad x_0=0$

(2) $y=\begin{cases} \dfrac{x}{1+e^{\frac{1}{x}}}, & x\neq 0, \\ 0, & x=0, \end{cases}\quad x_0=0$

(3) $y=\begin{cases} x^2\sin\dfrac{1}{x}, & x\neq 0, \\ 0, & x=0, \end{cases}\quad x_0=0$

解：

(1) $f'_+(x_0)=\lim\limits_{x\to+0}\dfrac{xe^x-0}{x}=1$；$f'_-(x_0)=\lim\limits_{x\to-0}\dfrac{x^2-0}{x}=0.$

(2) $f'_+(x_0)=\lim\limits_{x\to+0}\dfrac{\dfrac{x}{1+e^{\frac{1}{x}}}-0}{x}=\lim\limits_{x\to+0}\dfrac{1}{1+e^{\frac{1}{x}}}=0$；

$f'_-(x_0)=\lim\limits_{x\to-0}\dfrac{1}{1+e^{\frac{1}{x}}}=1.$

(3) $f'_+(x_0)=\lim\limits_{x\to+0}\dfrac{x^2\sin\dfrac{1}{x}-0}{x}=0$；$f'_-(x_0)=\lim\limits_{x\to-0}\dfrac{x^2\sin\dfrac{1}{x}-0}{x}=0.$

2. 求下列函数在导数不存在的点的左、右导数：

(1) $y=|\ln|x||$

(2) $y=|\tan x|$

(3) $y=\sqrt{1-\cos x}$

解：

(1) $y=|\ln|x||=\begin{cases} \ln(-x), & x\leqslant -1 \\ -\ln(-x), & -1<x<0 \\ -\ln x, & 0<x<1 \\ \ln x, & x\geqslant 1 \end{cases}$

由此可知，函数在$x=0,x=\pm1$处导数不存在。

$f'_+(-1)=\lim\limits_{\Delta x\to+0}\dfrac{-\ln[-(-1+\Delta x)]-\ln(-(-1))}{\Delta x}=\lim\limits_{\Delta x\to+0}\ln(1-\Delta x)^{-\frac{1}{\Delta x}}=1$；

$f'_-(-1)=\lim\limits_{\Delta x\to-0}\dfrac{-\ln[-(-1+\Delta x)]-\ln(-(-1))}{\Delta x}=\lim\limits_{\Delta x\to-0}\ln(1-\Delta x)^{-\frac{1}{\Delta x}}=-1$；

因函数在$x=0$点无意义，故$f'_+(0)$和$f'_-(0)$无意义；

$f'_+(1)=\lim\limits_{\Delta x\to+0}\dfrac{\ln(1+\Delta x)-\ln 1}{\Delta x}=\lim\limits_{\Delta x\to+0}\ln(1+\Delta x)^{\frac{1}{\Delta x}}=1$；

$f'_-(1)=\lim\limits_{\Delta x\to-0}\dfrac{-\ln(1+\Delta x)-\ln(1)}{\Delta x}=\lim\limits_{\Delta x\to-0}-\ln(1+\Delta x)^{\frac{1}{\Delta x}}=-1.$

(2) $y=|\tan x|=\begin{cases} -\tan x, & x\in\left(k\pi-\dfrac{\pi}{2},k\pi\right] \\ \tan x, & x\in\left(k\pi,k\pi+\dfrac{\pi}{2}\right) \end{cases}\quad k\in Z$

其中$x=k\pi+\dfrac{\pi}{2}(k\in Z)$时函数无定义，且为无穷间断点，故左、右导数无意义；

$x=k\pi(k\in Z)$为导数不存在的点.

$f'_+(k\pi)=\lim\limits_{\Delta x\to+0}\dfrac{\tan(k\pi+\Delta x)-(-\tan k\pi)}{\Delta x}=\lim\limits_{\Delta x\to+0}\dfrac{\tan\Delta x}{\Delta x}=1$；$f'_-(k\pi)=\lim\limits_{\Delta x\to-0}\dfrac{-\tan(k\pi+\Delta x)-(-\tan k\pi)}{\Delta x}=$
$\lim\limits_{\Delta x\to-0}-\dfrac{\tan\Delta x}{\Delta x}=-1.$

(3) 因$y'=\dfrac{\sin x}{\sqrt{1-\cos x}}$当$x\neq 2k\pi(k\in Z)$时才有定义，故$x=2k\pi(k\in Z)$为$y=\sqrt{1-\cos x}$的不可导点.

$f'_+(2k\pi)=\lim\limits_{\Delta x\to+0}\dfrac{\sqrt{1+\cos(2k\pi+\Delta x)}-\sqrt{1+2k\pi}}{\Delta x}=\lim\limits_{\Delta x\to+0}\dfrac{\sqrt{1-\cos\Delta x}}{\Delta x}=-\lim\limits_{\Delta x\to+0}\sqrt{\dfrac{1-\cos\Delta x}{\Delta x^2}}=$

$\dfrac{\sqrt{2}}{2}$；

$f'_{-}(2k\pi)=\lim\limits_{\Delta x\to -0}\dfrac{\sqrt{1+\cos(2k\pi+\Delta x)}-\sqrt{1+2k\pi}}{\Delta x}=\lim\limits_{\Delta x\to -0}\dfrac{\sqrt{1-\cos\Delta x}}{\Delta x}=-\lim\limits_{\Delta x\to -0}-\sqrt{\dfrac{1-\cos\Delta x}{\Delta x^2}}=-\dfrac{\sqrt{2}}{2}$.

3. 若

(1) $f(x)$在x_0点可导，$g(x)$在x_0点不可导，证明函数$F(x)=f(x)+g(x)$在x_0点不可导；

(2) $f(x)$和$g(x)$在x_0点都不可导，能否断定他们的和函数$F(x)=f(x)+g(x)$在x_0点不可导？

证明:

(1) 假设$F(x)=f(x)+g(x)$在x_0点可导，又$f(x)$在x_0点可导，则$g(x)=F(x)-f(x)$在x_0点可导，这与已知矛盾，故假设不成立。从而函数$F(x)=f(x)+g(x)$在x_0点不可导.

(2) 不能。例：

(i) 可导：$f(x)=\dfrac{|x|+x}{2},g(x)=\dfrac{x-|x|}{2}$在$x=0$点都不可导，但它们的和函数$F(x)=f(x)+g(x)=x$在$x=0$点可导且$F'(0)=1$；

(ii) 不可导：$f(x)=\dfrac{|x|}{2},g(x)=\dfrac{|x|}{2}$在$x=0$点都不可导，它们的和函数$F(x)=f(x)+g(x)=|x|$在$x=0$点也不可导.

4. 在上题条件下，它们的积$G(x)=f(x)\cdot g(x)$的可导情况怎样？

解:

(1) 它们的积$G(x)=f(x)\cdot g(x)$在x_0点可能可导。
例：

(i) 可导：$f(x)=x$在$x=0$点可导且$f'(0)=1$；$g(x)=|x|$在$x=0$点不可导，它们的积$G(x)=f(x)\cdot g(x)=x|x|$在$x=0$可导且$G'(0)=\lim\limits_{\Delta x\to 0}\dfrac{\Delta G(x)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{\Delta x|\Delta x|}{\Delta x}=\lim\limits_{\Delta x\to 0}|\Delta x|=0$

(ii) 不可导：$f(x)=1$在$x=0$点可导且$f'(0)=0$；$g(x)=|x|$在$x=0$点不可导，它们的积$G(x)=f(x)\cdot g(x)=|x|$在$x=0$点不可导.

(2) 它们的积$G(x)=f(x)\cdot g(x)$在x_0点可能可导。
例：

(i) 可导：$f(x)=|x|,g(x)=|x|$在$x=0$点都不可导，它们的积$G(x)=f(x)\cdot g(x)=x^2$在$x=0$可导且$G'(0)=0$

(ii) 不可导：$f(x)=x^{\frac{2}{3}},g(x)=|x^{\frac{1}{3}}|$在$x=0$点都不可导，它们的积$G(x)=f(x)\cdot g(x)=|x|$在$x=0$点不可导.

5. 若函数$f(x)$在有限区间(a,b)中有导数，且$\lim\limits_{x\to a}f(x)=\infty$，是否必有$\lim\limits_{x\to a}f'(x)=\infty$？以例子$f(x)=\dfrac{1}{x}+\cos\dfrac{1}{x}$说明之.
反之，若$f(x)$在有限区间(a,b)中有导数，且$\lim\limits_{x\to a}f'(x)=\infty$，是否必有$\lim\limits_{x\to a}f(x)=\infty$？以例子$f(x)=\sqrt[3]{x}$说明之.

解:

(1) 一般地说，不能保证有$\lim\limits_{x\to a}f'(x)=\infty$.

例：对于$\left(0,\dfrac{\pi}{2}\right)$内定义的函数$f(x)=\dfrac{1}{x}+\cos\dfrac{1}{x}$，显然有$\lim\limits_{x\to 0}f(x)=\infty$.

又$f'(x)=-\dfrac{1}{x^2}-\dfrac{1}{x^2}\left(-\sin\dfrac{1}{x}\right)=\dfrac{1}{x^2}\left(\sin\dfrac{1}{x}-1\right)$，

对于特殊的一串数$x_n=\dfrac{1}{2n\pi+\dfrac{\pi}{2}}(n=1,2,\cdots)$，有$f'(x_n)=0$，故$\lim\limits_{n\to\infty}f'(x_n)=0$；

对于$x'_n=\dfrac{1}{n\pi}(n=1,2,\cdots)$，有$f'(x'_n)=-n^2\pi^2$，故$\lim\limits_{n\to\infty}f'(x'_n)=-\infty$，故$f'(x)$在$x=0$点极限不存在，也非无穷，即$\lim\limits_{x\to 0}f'(x)=\infty$不成立.

(2) 不能保证必有$\lim\limits_{x\to a}f(x)=\infty$.

例：$f(x)=\sqrt[3]{x}$，它在$(0,b)(b>0)$上有导数，且$f'(x)=\dfrac{1}{3\sqrt[3]{x^2}}$，$\lim\limits_{x\to 0}f'(x)=\infty$，但$\lim\limits_{x\to 0}f(x)=0$.

6. 若

(1) $f(x)$在$x=g(x_0)$有导数，而$g(x)$在x_0点没有导数；

(2) $f(x)$在$x=g(x_0)$没有导数，而$g(x)$在x_0点有导数；

(3) $f(x)$在$x=g(x_0)$没有导数，而$g(x)$在x_0点也没有导数；

则复合函数$F(x)=f(g(x))$在x_0点是否可导？

解：

(1) 复合函数$F(x)=f(g(x))$在x_0点可能可导.
例：

(i) 可导：$f(u)=u^2, g(x)=|x|, x_0=0$，$f(u)=u^2$在$u_0=0=g(x_0)$可导且$f'(0)=0$，$g(x)=|x|$在$x_0=0$不可导；$F(x)=f(g(x))=|x|^2=x^2$在$x_0=0$可导且$F'(0)=0$；

(ii) 可导：$f(u)=u, g(x)=|x|, x_0=0$，$f(u)=u$在$u_0=0=g(x_0)$可导且$f'(0)=1$，$g(x)=|x|$在$x_0=0$不可导；$F(x)=f(g(x))=|x|$在$x_0=0$不可导.

(2) 复合函数$F(x)=f(g(x))$在x_0点可能可导.
例：

(i) 可导：$f(u)=|u|, g(x)=x^2, x_0=0$，$f(u)=|u|$在$u_0=0=g(x_0)$不可导，$g(x)=x^2$在$x_0=0$可导且$g'(0)=0$；$F(x)=f(g(x))=|x^2|=x^2$在$x_0=0$可导且$F'(0)=0$；

(ii) 可导：$f(u)=|u|, g(x)=x, x_0=0$，$f(u)=|u|$在$u_0=0=g(x_0)$不可导，$g(x)=x$在$x_0=0$可导且$g'(0)=1$；$F(x)=f(g(x))=|x|$在$x_0=0$不可导.

(3) 复合函数$F(x)=f(g(x))$在x_0点可能可导.
例：

(i) 可导：$f(u)=2u+|u|, g(x)=\dfrac{2}{3}x-\dfrac{|x|}{3}, x_0=0$，$f(u)=2u+|u|$在$u_0=0=g(x_0)$不可导，$g(x)=\dfrac{2}{3}x-\dfrac{|x|}{3}$在$x_0=0$不可导；$F(x)=f(g(x))=2\left(\dfrac{2}{3}x-\dfrac{|x|}{3}\right)+\left|\dfrac{2}{3}x-\dfrac{|x|}{3}\right|=\begin{cases} x, & x\geqslant 0 \\ x, & x<0 \end{cases}=x$即$F(x)=x(\forall x\in(-\infty,+\infty))$，故$F(x)$在$x_0=0$可导且$F'(0)=1$；

(ii) 可导：$f(u)=|u|, g(x)=|x|, x_0=0$，$f(u)=|u|$在$u_0=0=g(x_0)$不可导，$g(x)=|x|$在$x_0=0$不可导；$F(x)=f(g(x))=|x|$在$x_0=0$不可导.

§8. 高阶导数与高阶微分

1. $y=2x^3+x^2+x+1$，求$y',y'',y^{(3)}$和$y^{(4)}$.

   解：$y'=6x^2+2x+1,y''=12x+2,y^{(3)}=12,y^{(4)}=0$

2. $y=e^{\alpha t}$(α为常数)，求$y'',y^{(3)},y^{(n)}$.

   解：$y'=\alpha e^{\alpha t},y''=\alpha^2e^{\alpha t},y^{(3)}=\alpha^3e^{\alpha t},y^{(n)}=\alpha^ne^{\alpha t}$

3. 求下列函数的高阶导数：

   (1) $y=\dfrac{x}{\sqrt{1-x^2}}$，求$y''$

   (2) $y=x\ln x$，求y''

   (3) $y=e^{-x^2}$，求y''

   (4) $y=\dfrac{\arcsin x}{\sqrt{1-x^2}}$，求$y''$

   (5) $y=x^2\cdot e^{2x}$，求y'''

   (6) $y=a^{3x}$，求y'''

   (7) $y=x^3\sinh x$，求$y^{(30)}$

   (8) $y=x^3\cos x$，求$y^{(50)}$

   解：

   (1) $y'=\dfrac{\sqrt{1-x^2}+\dfrac{x^2}{\sqrt{1-x^2}}}{1-x^2}=\dfrac{1}{(1-x^2)^{\frac32}}=(1-x^2)^{-\frac32},y''=3x(1-x^2)^{-\frac52}$

   (2) $y'=1+\ln x,y''=\dfrac{1}{x}$

   (3) $y'=-2xe^{-x^2},y''=-2e^{-x^2}(1-2x^2)=2e^{-x^2}(2x^2-1)$

   (4) $y'=\dfrac{1+\dfrac{x\arcsin x}{\sqrt{1-x^2}}}{1-x^2}=\dfrac{1}{1-x^2}+\dfrac{x\arcsin x}{(1-x^2)^{\frac32}}$,

   $$y''=\frac{2x}{(1-x^2)^2}+\frac{\left(\arcsin x+\dfrac{x}{\sqrt{1-x^2}}\right)(1-x^2)^{\frac32}+3x(1-x^2)^{\frac12}\cdot x\arcsin x}{(1-x^2)^3}=\frac{3x}{(1-x^2)^2}+\frac{(2x^2+1)\arcsin x}{(1-x^2)^{\frac52}}$$

   (5) $y'''=(x^2e^{2x})'''=x^2(e^{2x})'''+3(x^2)'(e^{2x})''+3(x^2)''(e^{2x})'+(x^2)'''e^{2x}=4e^{2x}(2x^2+6x+3)$

   (6) $y'=3a^{3x}\ln a,y''=9\ln^2a\cdot a^{3x},y'''=27\ln^3a\cdot a^{3x}$

   (7) 因$(x^3)'=3x^2,(x^3)''=6x,(x^3)'''=6,(x^3)^{(4)}=\cdots=(x^3)^{(30)}=0;(\sinh x)^{(30)}=\sinh x,(\sinh x)^{(29)}=\cosh x,(\sinh x)^{(28)}=\sinh x,(\sinh x)^{(27)}=\cosh x$，故$y^{(30)}=(x^3\sinh x)^{(30)}=x^3(\sinh x)^{(30)}+30(x^3)'(\sinh x)^{(29)}+435(x^3)''(\sinh x)^{(28)}+4060(x^3)'''(\sin h)^{(27)}=x\sinh x(x^2+2610)+30\cosh x(3x^2+812)$

   (8) 因$(x^3)'=3x^2,(x^3)''=6x,(x^3)'''=6,(x^3)^{(4)}=\cdots=(x^3)^{(50)}=0;(\cos x)^{(50)}=-\cos x,(\cos x)^{(49)}=-\sin x,(\cos x)^{(48)}=\cos x,(\cos x)^{(47)}=\sin x$，故$y^{(50)}=(x^3\cos x)^{(50)}=x^3(\cos x)^{(50)}+50(x^3)'(\cos x)^{(49)}+1225(x^3)''(\cos x)^{(48)}+19600(x^3)'''(\cos x)^{(47)}=x\cos x(7350-x^2)+150\sin x(784-x^2)$

4. 利用数学归纳法证明下面公式：

   (1) $(a^x)^{(n)}=a^x\cdot(\ln a)^n(a>0)$

   (2) $(\cos x)^{(n)}=\cos\left(x+n\cdot\dfrac{\pi}{2}\right)$

   (3) $(\ln x)^{(n)}=\dfrac{(-1)^{n-1}\cdot(n-1)!}{x^n}$

   证明：

   (1) (i) 当$n=1$时，$(a^x)'=a^x\ln a=a^x(\ln a)^1$，则$n=1$时公式成立.

   (ii) 假设当$n=k$时公式成立，即$(a^x)^{(k)}=a^x(\ln a)^k$成立，

   则当$n=k+1$时，$(a^x)^{(k+1)}=\left[(a^x)(\ln a)^{(k)}\right]'=(\ln a)^k(a^x)'=(\ln a)^k\cdot a^x\ln a=a^x(\ln a)^{k+1}$，于是当$n=k+1$时公式也成立.

   综合上述可知，当n为任意自然数时，公式$(a^x)^{(n)}=a^x\cdot(\ln a)^n(a>0)$都成立.

(2) (i) 当$n=1$时，$(\cos x)'=-\sin x=\cos\left(x+\frac{\pi}{2}\right)$，则$n=1$时公式成立.

(ii) 假设当$n=k$时公式成立，即$(\cos x)^{(k)}=\cos\left(x+k\cdot\frac{\pi}{2}\right)$成立，

则当$n=k+1$时，$(\cos x)^{(k+1)}=\left[(\cos x)^{(k)}\right]'=\left[\cos\left(x+k\cdot\frac{\pi}{2}\right)\right]'=-\sin\left(x+k\cdot\frac{\pi}{2}\right)=\cos\left(x+(k+1)\cdot\frac{\pi}{2}\right)$，于是当$n=k+1$时公式也成立.

综合上述可知，当n为任意自然数时，公式$(\cos x)^{(n)}=\cos\left(x+n\cdot\frac{\pi}{2}\right)$都成立.

(3) (i) 当$n=1$时，$(\ln x)'=\frac{1}{x}=\frac{(-1)^{1-1}(1-1)!}{x^1}$，则$n=1$时公式成立.

(ii) 假设当$n=k$时公式成立，即$(\ln x)^{(k)}=\frac{(-1)^{k-1}\cdot(k-1)!}{x^n}$成立，

则当$n=k+1$时，$(\ln x)^{(k+1)}=\left[(\ln x)^{(k)}\right]'=\left[\frac{(-1)^{k-1}\cdot(k-1)!}{x^k}\right]'=-k\cdot\frac{(-1)^{k-1}\cdot(k-1)!}{x^{k+1}}=\frac{(-1)^k\cdot k!}{x^{k+1}}=\frac{(-1)^{k+1-1}\cdot(k+1-1)!}{x^{k+1}}$，于是当$n=k+1$时公式也成立.

综合上述可知，当n为任意自然数时，公式$(\ln x)^{(n)}=\frac{(-1)^{n-1}\cdot(n-1)!}{x^n}$都成立.

5. 求n阶导数：

(1) $y=\frac{1}{x(1-x)}$

(2) $y=\frac{1}{x^2-2x-8}$

(3) $y=a_mx^m+a_{m-1}x^{m-1}+\cdots+a_0$

(4) $y=\cos^2\omega x$

(5) $y=\frac{e^x}{x}$

(6) $y=2^x\cdot\ln x$

(7) $y=e^{ax}p_n(x)$，其中$p_n(x)$为n次多项式.

解:

(1) 因$y=\frac{1}{x(1-x)}=\frac{1}{x}+\frac{1}{1-x}$，则$y^{(n)}=\left(\frac{1}{x}-\frac{1}{1-x}\right)^{(n)}=\left(\frac{1}{x}\right)^{(n)}+\left(\frac{1}{1-x}\right)^{(n)}=(x^{-1})^{(n)}+[(1-x)^{-1}]^{(n)}=(-1)^n\cdot n!\cdot x^{-n-1}+(-1)^{2n}\cdot n!\cdot(1-x)^{-n-1}=n![(-1)^nx^{-n-1}+(1-x)^{-n-1}]$

(2) 因$y=\frac{1}{x^2-2x-8}=\frac{1}{(x+2)(x-4)}=\frac{1}{6}\left(\frac{1}{x-4}-\frac{1}{x+2}\right)$，则$y^{(n)}=\frac{1}{6}\left(\frac{1}{x-4}-\frac{1}{x+2}\right)^{(n)}=\frac{1}{6}[((x-4)^{-1})^{(n)}-((x+2)^{-1})^{(n)}]=\frac{1}{6}[(-1)^n\cdot n!(x-4)^{-n-1}-(-1)^n\cdot n!(n+2)^{-n-1}]=\frac{(-1)^n}{6}n!\left(\frac{1}{(x-4)^{n+1}}-\frac{1}{(x+2)^{n+1}}\right)$

(3) 当$n>m$时，$x^{(n)}=(x^2)^{(n)}=\cdots=(x^m)^{(n)}=0$，则$y^{(n)}=0$；
当$n=m$时，$x^{(n)}=(x^2)^{(n)}=\cdots=(x^{m-1})^{(n)}=0,(x^m)^{(n)}=(x^m)^{(m)}=m!$，则$y^{(n)}=a_m\cdot m!$；
当$n<m$时，$x^{(n)}=(x^2)^{(n)}=\cdots=(x^{n-1})^{(n)}=0,(x^n)^{(n)}=n!,\cdots,(x^m)^{(n)}=\frac{m!}{(m-n)!}x^{m-n}$，

则$y^{(n)}=a_m\cdot\frac{m!}{(m-n)!}x^{m-n}+a_{m-1}\cdot\frac{(m-1)!}{(m-n-1)!}x^{m-n-1}+\cdots+a_nn!=\sum\limits_{i=0}^{m-n}a_{m-i}\frac{(m-i)!}{(m-n-i)!}x^{m-n-i}$.

(4) 因$y'=-2\omega\cos\omega x\sin\omega x=-\omega\sin 2\omega x$，则$y^{(n)}=(y')^{(n-1)}=(-\omega\sin 2\omega x)^{(n-1)}=-\omega\sin\left(2\omega x+\frac{n-1}{2}\pi\right)\cdot(2\omega)^{n-1}=-2^{n-1}\omega^n\sin\left(2\omega x+\frac{n-1}{2}\pi\right)=2^{n-1}\omega^n\cos\left(2\omega x+\frac{n}{2}\pi\right)$

(5) $y^{(n)}=\left(\frac{e^x}{x}\right)^{(n)}=\left(e^x\cdot\frac{1}{x}\right)^{(n)}=\sum\limits_{k=0}^{n}C_n^ke^x\left(\frac{1}{x}\right)^{(k)}=e^x\left[\frac{1}{x}+\sum\limits_{k=1}^{n}(-1)^k\frac{n(n-1)\cdots(n-k+1)}{x^{k+1}}\right]$

(6) $y^{(n)} = (2^x \cdot \ln x)^{(n)} = \sum\limits_{k=0}^{n} C_n^k (2^x)^{(n-k)} (\ln x)^{(k)} =$
$\sum\limits_{k=1}^{n} C_n^k (\ln 2)^{n-k} \cdot 2^x \cdot \dfrac{(-1)^{k-1}(k-1)!}{x^k} + 2^x (\ln 2)^n \ln x =$
$2^x[(\ln 2)^n \ln x + n(\ln 2)^{n-1} x^{-1} + \cdots + (-1)^{n-2}(n-2)! \cdot n \ln 2 \cdot x^{-(n-1)} + (-1)^{n-1}(n-1)! \cdot x^{-n}]$

(7) $y^{(n)} = (e^{ax} p_n(x))^{(n)} = a^n e^{ax} p_n(x) + C_n^1 a^{n-1} e^{ax} p_n'(x) + \cdots + e^{ax} p_n^{(n)}(x) = e^{ax}[a^n p_n(x) + C_n^1 a^{n-1} p_n'(x) + \cdots + p_n^{(n)}(x)]$

6. 若$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ 证明$f^{(n)}(0) = 0$.

证明: 当$x \neq 0$时，$f'(x) = \dfrac{2}{x^3} e^{-\frac{1}{x^2}}, f''(x) = e^{-\frac{1}{x^2}} \left(-\dfrac{6}{x^4} + \dfrac{4}{x^6}\right)$，由此推断$f^{(n)}(x) = e^{-\frac{1}{x^2}} P_n\left(\dfrac{1}{x}\right) (x \neq 0)$，其中$P_n(t)$是关于$t$的多项式。

下面证明：对任意正整数n，均有命题$f^{(n)}(x) = e^{-\frac{1}{x^2}} P_n\left(\dfrac{1}{x}\right) (x \neq 0)$

当$n = 1$时，命题显然成立.

假设当$n = k$时，命题成立，即有$f^{(k)}(x) = e^{-\frac{1}{x^2}} P_k\left(\dfrac{1}{x}\right) (x \neq 0), P_k(t)$是关于$t$的多项式，

则当$n = k+1$时，$f^{(k+1)}(x) = [f^{(k)}(x)]' = \left[e^{-\frac{1}{x^2}} P_k\left(\dfrac{1}{x}\right)\right]' =$

$e^{-\frac{1}{x^2}} \left[\dfrac{2}{x^3} P_k\left(\dfrac{1}{x}\right) - \dfrac{1}{x^2} P_k'\left(\dfrac{1}{x}\right)\right] = e^{-\frac{1}{x^2}} \left[2\left(\dfrac{1}{x}\right)^3 P_k\left(\dfrac{1}{x}\right) - \left(\dfrac{1}{x}\right)^2 P_k'\left(\dfrac{1}{x}\right)\right] = e^{-\frac{1}{x^2}} P_{k+1}\left(\dfrac{1}{x}\right)$，

其中$P_{k+1}(t)$是关于t的另一个多项式.

据数学归纳法可知，命题对一切自然数n均成立.

当$n = 1$时，$f'(0) = \lim\limits_{\Delta x \to 0} \dfrac{f(0+\Delta x) - f(0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{e^{-(\frac{1}{\Delta x})^2}}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\frac{1}{\Delta x}}{e^{(\frac{1}{\Delta x})^2}} = 0$

假设当$n = k$时，$f^{(k)}(0) = 0$，则$f^{(k+1)} = \lim\limits_{\Delta x \to 0} \dfrac{f^{(k)}(0+\Delta x) - f^{(k)}(0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{e^{-(\frac{1}{\Delta x})^2} P_k\left(\frac{1}{\Delta x}\right)}{\Delta x} =$
$\lim\limits_{\Delta x \to 0} \dfrac{\frac{1}{\Delta x} P_k\left(\frac{1}{\Delta x}\right)}{e^{(\frac{1}{\Delta x})^2}} = 0$

据数学归纳法可知，$f^{(n)}(0) = 0$.

7. 设$f(x)$的各阶导数存在，求y''及y'''：

(1) $y = f(x^2)$

(2) $y = f\left(\dfrac{1}{x}\right)$

(3) $y = f(e^{-x})$

(4) $y = f(\ln x)$

解:

(1) $y' = 2xf'(x^2), y'' = 2f'(x^2) + 4x^2 f''(x^2)$,
$y''' = 12xf''(x^2) + 8x^3 f'''(x^2)$

(2) $y' = -\dfrac{1}{x^2} f'\left(\dfrac{1}{x}\right), y'' = \dfrac{2}{x^3} f'\left(\dfrac{1}{x}\right) + \dfrac{1}{x^4} f''\left(\dfrac{1}{x}\right)$,
$y''' = -\dfrac{6}{x^4} f'\left(\dfrac{1}{x}\right) - \dfrac{6}{x^5} f''\left(\dfrac{1}{x}\right) - \dfrac{1}{x^6} f'''\left(\dfrac{1}{x}\right)$

(3) $y' = -e^{-x} f'(e^{-x}), y'' = e^{-x} f'(e^{-x}) + e^{-2x} f''(e^{-x}), y''' = -e^{-x} f'(e^{-x}) - 3e^{-2x} f''(e^{-x}) - e^{-3x} f'''(e^{-x})$

(4) $y' = \dfrac{1}{x} f'(\ln x), y'' = \dfrac{1}{x^2} f''(\ln x) - \dfrac{1}{x^2} f'(\ln x) = \dfrac{1}{x^2}[f''(\ln x) - f'(\ln x)], y''' = \dfrac{1}{x^3}[2f'(\ln x) - 3f''(\ln x) + f'''(\ln x)]$

8. 设$y = e^x \sin x, z = e^x \cos x$，证明它们满足方程$y'' = 2z, z'' = -2y$.

证明: 因$y = e^x \sin x, z = e^x \cos x$，则$y' = e^x(\sin x + \cos x), y'' = 2e^x \cos x; z' = e^x(\cos x - \sin x), z'' = -2e^x \sin x$，于是$y'' = 2z, z'' = -2y$.

9. 设$y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$，$C_1,C_2,\lambda_1,\lambda_2$是常数，证明它满足方程$y''-(\lambda_1+\lambda_2)y'+\lambda_1\lambda_2 y=0$.
证明：因$y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$，$C_1,C_2,\lambda_1,\lambda_2$是常数，则$y'=C_1\lambda_1e^{\lambda_1 x}+C_2\lambda_2e^{\lambda_2 x}, y''=C_1\lambda_1^2e^{\lambda_1 x}+C_2\lambda_2^2e^{\lambda_2 x}$，
于是$y''-(\lambda_1+\lambda_2)y'+\lambda_1\lambda_2 y=C_1\lambda_1^2e^{\lambda_1 x}+C_2\lambda_2^2e^{\lambda_2 x}-(\lambda_1+\lambda_2)(C_1\lambda_1e^{\lambda_1 x}+C_2\lambda_2e^{\lambda_2 x})+\lambda_1\lambda_2(C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x})=0$即$y''-(\lambda_1+\lambda_2)y'+\lambda_1\lambda_2 y=0$.

10. 设$y=C_1\sin x+C_2\cos x$，证明y满足方程$y''+y=0$.
证明：因$y=C_1\sin x+C_2\cos x$，则$y'=C_1\cos x-C_2\sin x, y''=-C_1\sin x-C_2\cos x=-(C_1\sin x+C_2\cos x)=-y$即$y''+y=0$.

11. 若函数φ为$\varphi(x)=\dfrac{f(x)-f(a)}{f'(a)}\left[1+\dfrac{f(x)-f(a)}{f'(a)^2}\left(f'(a)-\dfrac{1}{2}f''(a)\right)\right]$，求$\varphi'(a)$及$\varphi''(a)$.
解：因$\varphi(x)=\dfrac{f(x)-f(a)}{f'(a)}\left[1+\dfrac{f(x)-f(a)}{f'(a)^2}\left(f'(a)-\dfrac{1}{2}f''(a)\right)\right]$，
则$\varphi'(x)=\dfrac{f'(x)}{f'(a)}\left[1+\dfrac{f(x)-f(a)}{f'(a)^2}\left(f'(a)-\dfrac{1}{2}f''(a)\right)\right]+$
$\dfrac{f(x)-f(a)}{f'(a)}\left[\dfrac{f'(x)}{f'(a)^2}\left(f'(a)-\dfrac{1}{2}f''(a)\right)\right]$，
$\varphi''(x)=\dfrac{f''(x)}{f'(a)}\left[1+\dfrac{f(x)-f(a)}{f'(a)^2}\left(f'(a)-\dfrac{1}{2}f''(a)\right)\right]+2\dfrac{f'(x)}{f'(a)}\left[\dfrac{f'(x)}{f'(a)^2}\left(f'(a)-\dfrac{1}{2}f''(a)\right)\right]+$
$\dfrac{f(x)-f(a)}{f'(a)}\left[\dfrac{f''(x)}{f'(a)^2}\left(f'(a)-\dfrac{1}{2}f''(a)\right)\right]$，
则$\varphi'(a)=1,\varphi''(a)=2$

12. 设$x=\varphi(y)$是$y=f(x)$的反函数，试问如何由f',f'',f'''算出$\varphi'''(y)$？
解：因$\varphi'(y)=\dfrac{1}{f'(x)}$，则$\varphi''(y)f'(x)=-\dfrac{f''(x)}{[f'(x)]^2}$，于是$\varphi''(y)=-\dfrac{f''(x)}{[f'(x)]^3}$，
则$\varphi'''(y)f'(x)=-\dfrac{f'''(x)[f'(x)]^3-3[f'(x)]^2[f''(x)]^2}{[f'(x)]^6}$，从而$\varphi'''(y)=\dfrac{3[f''(x)]^2-f'''(x)f'(x)}{[f'(x)]^5}$.

13. 试求阻尼振动$s=ae^{-\lambda t}\sin\omega t$在时刻$t$的速度和加速度，并求出速度方向的反转点.
解：速度$v=s'=ae^{-\lambda t}(-\lambda\sin\omega t+\omega\cos\omega t)$，加速度$a=v'=s''=ae^{-\lambda t}[(\lambda^2-\omega^2)\sin\omega t-2\lambda\omega\cos\omega t]$；
速度的反转点即$v=0$，则$-\lambda\sin\omega t+\omega\cos\omega t=0$，于是$\tan\omega t=\dfrac{\omega}{\lambda}(\lambda\neq 0)$.

14. 求下列参数方程的二阶导数$\dfrac{d^2y}{dx^2}$：

(1) $\begin{cases} x= & 2t-t^2 \\ y= & 3t-t^3 \end{cases}$

(2) $\begin{cases} x= & a\cos t \\ y= & a\sin t \end{cases}$

(3) $\begin{cases} x= & a(t-\sin t) \\ y= & a(1-\cos t) \end{cases}$

(4) $\begin{cases} x= & e^t\cos t \\ y= & e^t\sin t \end{cases}$

(5) $\begin{cases} x= & a\cos^3 t \\ y= & a\sin^3 t \end{cases}$

(6) $\begin{cases} x= & f'(t) \\ y= & tf'(t)-f(t) \end{cases}$

解：

(1) $\dfrac{dy}{dx}=\dfrac{3-3t^2}{2-2t}=\dfrac{3}{2}(1+t)$

$$\frac{d^2y}{dx^2}=\frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}=\frac{3}{4(1-t)}$$

(2) $\dfrac{dy}{dx}=\dfrac{a\cos t}{-a\sin t}=-\cot t$

$$\frac{d^2y}{dx^2}=\frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}=-\frac{1}{a\sin^3 t}$$

(3) $\dfrac{dy}{dx}=\dfrac{a\sin t}{a(1-\cos t)}=\cot\dfrac{t}{2}$

$\dfrac{d^2y}{dx^2}=\dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}=-\dfrac{1}{4a\sin^4\dfrac{t}{2}}$

(4) $\dfrac{dy}{dx}=\dfrac{e^t(\sin t+\cos t)}{e^t(\cos t-\sin t)}=\dfrac{\sin t+\cos t}{\cos t-\sin t}$

$\dfrac{d^2y}{dx^2}=\dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}=\dfrac{2}{e^t(\cos t-\sin t)^3}$

(5) $\dfrac{dy}{dx}=\dfrac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t}=-\tan t$

$\dfrac{d^2y}{dx^2}=\dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}=\dfrac{1}{3a\cos^4 t\sin t}$

(6) $\dfrac{dy}{dx}=\dfrac{tf''(t)}{f''(t)}=t$

$\dfrac{d^2y}{dx^2}=\dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}=\dfrac{1}{f''(t)}$

15. 求由隐函数所确定的二阶导数：

(1) $e^{x+y}-xy=0$

(2) $x^3+y^3-3axy=0$

(3) $y^2+2\ln y-x^4=0$

解:

(1) 对方程$e^{x+y}-xy=0$两端关于x求导，得

$$(1+y')e^{x+y}-y-xy'=0 \tag{4}$$

于是$y'=\dfrac{y-e^{x+y}}{e^{x+y}-x}$,

再对(4)两端关于x求导，得$y''e^{x+y}+(1+y')^2e^{x+y}-2y'-xy''=0$，则$y''=\dfrac{2y'-(1+y')^2e^{x+y}}{e^{x+y}-x}$,

将$y'=\dfrac{y-e^{x+y}}{e^{x+y}-x}$代入上式，即得$y''=\dfrac{2(y-e^{x+y})}{(e^{x+y}-x)^2}-\dfrac{(x-y)^2e^{x+y}}{(e^{x+y}-x)^3}$.

(2) 对方程$x^3+y^3-3axy=0$两端关于x求导，得

$$x^2+y^2y'-axy'-ay=0 \tag{1}$$

于是$y'=\dfrac{ay-x^2}{y^2-ax}$,

再对(1)两端关于x求导，得$2x+2y(y')^2+y^2y''-2ay'-axy''=0$，则$y''=\dfrac{2ay'-2y(y')^2-2x}{y^2-ax}$,

将$y'=\dfrac{ay-x^2}{y^2-ax}$代入上式，即得$y''=\dfrac{2a(ay-x^2)}{(y^2-ax)^2}-\dfrac{2y(ay-x^2)^2}{(y^2-ax)^3}-\dfrac{2x}{y^2-ax}$.

(3) 对方程$y^2+2\ln y-x^4=0$两端关于x求导，得

$$yy'+\frac{1}{y}y'-2x^3=0 \tag{1}$$

于是$y'=\dfrac{2x^3y}{y^2+1}$,

再对(1)两端关于x求导，得$(y')^2+yy''+\dfrac{yy''-(y')^2}{y^2}-6x^2=0$，则$y''=\dfrac{6x^2y^2+(y')^2(1-y^2)}{y(y^2+1)}$,

将$y'=\dfrac{2x^3y}{y^2+1}$代入上式，即得$y''=\dfrac{2x^2y}{(y^2+1)^3}[3(y^2+1)^2+2x^4(1-y^2)]$.

16. 求高阶微分(x是自变量)：

(1) $y=\sqrt{1+x^2}$，求d^2y

(2) $y=x^x$，求d^2y

(3) $y=x\cos 2x$，求d^3y

(4) $y=\dfrac{1}{\sqrt{x}}$，求d^3y

(5) $y=x^n\cdot e^x$，求d^ny

(6) $y=\dfrac{\ln x}{x}$，求d^ny

解:

(1) $dy=\dfrac{x}{\sqrt{1+x^2}}dx, d^2y=(1+x^2)^{-\frac{3}{2}}dx^2$

(2) $dy=x^x(\ln x+1)dx, d^2y=x^x\left[(\ln x+1)^2+\dfrac{1}{x}\right]dx^2$

(3) $d^3y=(x\cos 2x)^{(3)}dx^3=(x(\cos 2x)^{(3)}+3(\cos 2x)^{(2)})dx^3=(8x\sin 2x-12\cos 2x)dx^3$

(4) $d^3y=\left(\dfrac{1}{\sqrt{x}}\right)^{(3)}dx^3=-\dfrac{15}{8}x^{-\frac{7}{2}}dx^3$

(5) $d^ny=(x^n\cdot e^x)^{(n)}dx^n=\left(e^x\sum\limits_{k=0}^{n}C_n^k\dfrac{n!}{(n-k)!}x^{n-k}\right)dx^n$

(6) $d^ny=\left(\dfrac{\ln x}{x}\right)^{(n)}dx^n=\left(\dfrac{1}{x}\ln x\right)^{(n)}dx^n=$
$$\left[(-1)^n\frac{n!\ln x}{x^{n+1}}+\sum_{k=1}^{n}C_n^k(-1)^{n-1}\frac{(n-k)!(k-1)!}{x^{n+1}}\right]dx^n=$$
$$(-1)^n\frac{n!}{x^{n+1}}\left[\ln x-\sum_{k=1}^{n}\frac{1}{k}\right]dx^n$$

17. 对$y=e^x$求d^2y，考虑下面两种情形：

(1) 当x是自变量时；

(2) 当x是中间变量时.

解:

(1) $dy=e^xdx, d^2y=e^xdx^2$

(2) $dy=e^xdx, d^2y=e^x(dx^2+d^2x)$

18. 若u,v为x的函数，且可微分足够多次，求高阶微分：

(1) $y=u(x)\cdot v(x)$，求d^2y

(2) $y=\dfrac{u(x)}{v(x)}$，求d^2y

(3) $y=u^m(x)v^n(x)(m,n$为常数)，求d^2y

(4) $y=a^{u(x)}(a>0)$，求d^2y

(5) $y=\ln u(x)$，求d^3y

(6) $y=\sin(u(x))$，求d^3y

解:

(1) $dy=(u'(x)v(x)+u(x)v'(x))dx,$
$d^2y=[u''(x)v(x)+2u'(x)v'(x)+u(x)v''(x)]dx^2$

(2) $dy=\dfrac{u'(x)v(x)-u(x)v'(x)}{v^2(x)}dx,$
$$d^2y=\left[\frac{u''(x)}{v(x)}-\frac{u(x)v''(x)+2u'(x)v'(x)}{v^2(x)}+\frac{2u(x)(v'(x))^2}{v^3(x)}\right]dx^2$$

(3) $dy = [mu^{m-1}(x)v^n(x)u'(x) + nu^m(x)v^{n-1}(x)v'(x)]dx$,
$d^2y = [m(m-1)u^{m-2}(x)v^n(x)(u'(x))^2 + 2mnu^{m-1}(x)v^{n-1}(x)u'(x)v'(x) + mu^{m-1}(x)v^n(x)u''(x) + n(n-1)u^m(x)v^{n-2}(x)(v'(x))^2 + nu^m(x)v^{n-1}(x)v''(x)]dx^2$

(4) $dy = a^{u(x)} \ln a \cdot u'(x)dx$,
$d^2y = a^{u(x)} \ln a[\ln a(u'(x))^2 + u''(x)]dx^2$

(5) $dy = \dfrac{u'(x)}{u(x)}dx, d^2y = \left[\dfrac{u''(x)}{u(x)} - \dfrac{(u'(x))^2}{u^2(x)}\right] dx^2$,
$d^3y = \left[\dfrac{u'''(x)}{u(x)} - \dfrac{3u'(x)u''(x)}{u^2(x)} + \dfrac{2(u'(x))^3}{u^3(x)}\right] dx^3$

(6) $dy = \cos(u(x))u'(x)dx, d^2y = [\cos(u(x))u''(x) - \sin(u(x))(u'(x))^2]dx^2$,
$d^3y = [\cos(u(x))u'''(x) - 3\sin(u(x))u'(x)u''(x) - \cos(u(x))(u'(x))^3]dx^3$.

第五章　微分学的基本定理及其应用

§1.　中值定理

1. 在费尔马定理中，若x_0为区间的端点，试举例说明结论不成立.
解：例：函数$y=x$在区间$[-1,1]$上有定义，且可导，在端点$x_0=1$达到最大值，即$\forall x\in[-1,1]$，恒有$f(x)\leqslant f(x_0)=1$，然而$y'|_{x=1}=1\neq 0$.

2. 对于$x_0\in(a,b)$，若$f'(x_0)>0$，则存在它的左、右邻域$O_-(x_0,\delta),O_+(x_0,\delta)$使当$x\in O_-(x_0,\delta)$的时候$f(x_0)>f(x)$，当$x\in O_+(x_0,\delta)$的时候$f(x_0)<f(x)$.
证明：因$f'(x_0)=\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}>0$，故据极限性质，得存在$x_0$的$\delta(\delta>0)$邻域$O(x_0,\delta)\subset(a,b)$，使当$x\in O(x_0,\delta)$时，有$\dfrac{f(x)-f(x_0)}{x-x_0}>0$，从而当$x\in O_-(x_0,\delta)$即$x-x_0<0$时，有$f(x_0)>f(x)$，当$x\in O_+(x_0,\delta)$即$x-x_0>0$时，有$f(x_0)<f(x)$.

3. 证明：若$f'_+(x_0)>0,f'_-(x_0)<0$，则存在x_0的一个邻域，使得在此邻域内$f(x)\geqslant f(x_0)$.
证明：因$f'_+(x_0)=\lim\limits_{x\to x_0+0}\dfrac{f(x)-f(x_0)}{x-x_0}>0$，则由右极限性质，得必存在$x_0$的$\delta_1(\delta_1>0)$右邻域$O_+(x_0,\delta_1)$，使当$x\in O_+(x_0,\delta_1)$即$0<x-x_0<\delta_1$时，有$\dfrac{f(x)-f(x_0)}{x-x_0}>0$，从而有$f(x_0)<f(x)$；
又$f'_-(x_0)=\lim\limits_{x\to x_0-0}\dfrac{f(x)-f(x_0)}{x-x_0}<0$，则由左极限性质，得必存在$x_0$的$\delta_2(\delta_2>0)$左邻域$O_-(x_0,\delta_2)$，使当$x\in O_-(x_0,\delta_2)$即$0<x_0-x<\delta_2$时，有$\dfrac{f(x)-f(x_0)}{x-x_0}<0$，从而有$f(x_0)<f(x)$；
取$\delta=\min(\delta_1,\delta_2)$，当$x\in O(x_0,\delta)$时，总有$f(x)\geqslant f(x_0)$.

4. 若$f(x)$在$[a,b]$连续，$f(a)=f(b)=0,f'(a)\cdot f'(b)>0$，则$f(x)$在$(a,b)$内至少有一个零点.
证明：因$f'(a)\cdot f'(b)>0$，不妨设$f'(a)>0,f'(b)>0(f'(a)<0,f'(b)<0$情况同理可证)
又$f'(a)=f'_+(a)=\lim\limits_{x\to a+0}\dfrac{f(x)-f(a)}{x-a}>0$，则由右极限性质，得必存在$a$的$\delta_1(\delta_1>0)$右邻域$O_+(a,\delta_1)$，使当$x\in O_+(a,\delta_1)$即$0<x-a<\delta_1$时，有$\dfrac{f(x)-f(a)}{x-a}>0$，从而有$f(a)<f(x)$；
取定$x_1\in O_+(a,\delta_1)$，则有$f(x_1)>f(a)$
又$f(a)=0$，则$f(x_1)>0$
又$f'(b)=f'_-(b)=\lim\limits_{x\to b-0}\dfrac{f(x)-f(b)}{x-b}>0$，则由左极限性质，得必存在$b$的$\delta_2(\delta_2>0)$左邻域$O_-(b,\delta_2)$，使当$x\in O_-(b,\delta_2)$即$0<b-x<\delta_2$时，有$\dfrac{f(x)-f(b)}{x-x_0}>0$，从而有$f(b)>f(x)$；
取定$x_2\in O_-(b,\delta_1)$，则有$f(x_2)<f(b)$
又$f(b)=0$，则$f(x_2)<0$
因$f(x)$在$[a,b]$连续，故在$[x_1,x_2]$也连续，又$f(x_1)>0,f(x_2)<0$，则由零点存在定理可知，在$[x_1,x_2]$内至少有一个零点，
又$[x_1,x_2]\subset[a,b]$，从而$f(x)$在$[a,b]$内至少有一个零点.
同理，当$f'(a)<0,f'(b)<0$时，$f(x)$在$[a,b]$内至少有一个零点.

5. 由$f(x+\Delta x)-f(x)=f'(x+\theta\Delta x)\Delta x(0<\theta<1)$，求函数$\theta=\theta(x,\Delta x)$，设
(1) $f(x)=ax^2+bx+c(a\neq 0)$
(2) $f(x)=\dfrac{1}{x}$
(3) $f(x)=e^x$

解：

(1) $f'(x)=2ax+b,f'(x+\theta\Delta x)=2a(x+\theta\Delta x)+b$,
则$[2a(x+\theta\Delta x)+b]\Delta x=f(x+\Delta x)-f(x)=a(x+\Delta x)^2+b(x+\Delta x)+c-(ax^2+bx+c)=2ax\cdot\Delta x+a(\Delta x)^2+b\Delta x=\left[2a\left(x+\dfrac{1}{2}\Delta x\right)+b\right]\Delta x$，于是$\theta=\dfrac{1}{2}$

(2) $f'(x)=-\dfrac{1}{x^2},f'(x+\theta\Delta x)=-\dfrac{1}{(x+\theta\Delta x)^2}$，
则$-\dfrac{\Delta x}{(x+\theta\Delta x)^2}=f(x+\Delta x)-f(x)=\dfrac{1}{x+\Delta x}-\dfrac{1}{x}=-\dfrac{\Delta x}{x(x+\Delta x)}$，从而$\Delta x^2\theta^2+2x\cdot\Delta x\theta=0$，

于是$\theta=\dfrac{-x\pm\sqrt{x^2+x\Delta x}}{\Delta x}$，此处取正负号要视确保$\theta\in(0,1)$而定，且应有$\dfrac{\Delta x}{x}>-1(x\neq 0)$(由$x^2+x\Delta x>0$，则$\dfrac{\Delta x}{x}>-1$)

(3) $f'(x)=e^x, f'(x+\theta\Delta x)=e^{x+\theta\Delta x}$，
则$e^{x+\theta\Delta x}\Delta x=f(x+\Delta x)-f(x)=e^{x+\Delta x}-e^x=e^x(e^{\Delta x}-1)$，从而$e^{\theta\Delta x}\Delta x=e^{\Delta x}-1$，于是$\theta=\dfrac{1}{\Delta x}\ln\dfrac{e^{\Delta x}-1}{\Delta x}$，可以验证$\theta\in(0,1)$

6. 设$f(x)$在区间$[a,b]$内连续，在(a,b)可导，利用函数

$$\Phi(x)=\begin{vmatrix} x & f(x) & 1\\ b & f(b) & 1\\ a & f(a) & 1\end{vmatrix}$$

证明拉格朗日公式，并叙述函数$\Phi(x)$的几何意义.

证明：因$\Phi(x)=\begin{vmatrix} x & f(x) & 1\\ b & f(b) & 1\\ a & f(a) & 1\end{vmatrix}=(a-b)f(x)+(f(b)-f(a))x+bf(a)-af(b)$，
又$f(x)$在区间$[a,b]$内连续，则由连续函数的四则运算法则，知$\Phi(x)$在$[a,b]$连续；
又$f(x)$在(a,b)可导，则由可导函数的四则运算法则，知$\Phi(x)$在(a,b)可导.
又$\Phi(a)=\begin{vmatrix} a & f(a) & 1\\ b & f(b) & 1\\ a & f(a) & 1\end{vmatrix}=0,\Phi(b)=\begin{vmatrix} b & f(b) & 1\\ b & f(b) & 1\\ a & f(a) & 1\end{vmatrix}=0$，则由洛尔定理，得在$(a,b)$内至少有一点$\xi$，使$\Phi'(\xi)=0$.
而$\Phi'(x)=(a-b)f'(x)+f(b)-f(a)$，则$0=\Phi'(\xi)=(a-b)f'(\xi)+f(b)-f(a)$即$f'(\xi)=\dfrac{f(b)-f(a)}{b-a}$.
$\Phi(x)$的几何意义：三角形面积公式$S_\Delta=\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$，其中$(x_1,y_1),(x_2,y_2),(x_3,y_3)$表示顶点坐标，
则$\Phi(x)$表示以$A(x,f(x)),B(a,f(a)),C(b,f(b))$为顶点的三角形面积的两倍.

7. 试对下列函数写出拉格朗日公式$f(b)-f(a)=f'(c)(b-a)$，并求c.

(1) $f(x)=x^3, x\in[0,1]$

(2) $f(x)=\arctan x, x\in[0,1]$

解：

(1) 因$f'(x)=3x^2$，则$3c^2(1-0)=1^3-0^3$即$3c^2=1$，又$c\in(0,1)$，故$c=\dfrac{\sqrt{3}}{3}$.

(2) 因$f'(x)=\dfrac{1}{1+x^2}$，则$\dfrac{1}{1+c^2}(1-0)=\arctan 1-\arctan 0$即$\dfrac{1}{1+c^2}=\dfrac{\pi}{4}$，又$c\in(0,1)$，故$c=\sqrt{\dfrac{4}{\pi}-1}$.

8. 试对下列函数写出柯西公式$\dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{f'(c)}{g'(c)}$，并求$c$.

(1) $f(x)=\sin x, g(x)=\cos x, x\in\left[0,\dfrac{\pi}{2}\right]$

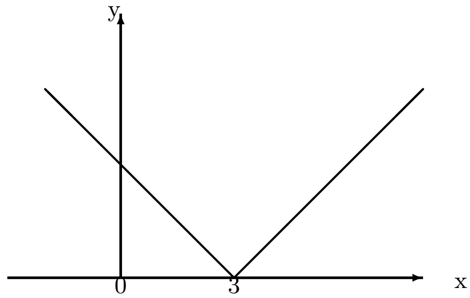
(2) $f(x)=x^2, g(x)=\sqrt{x}, x\in[1,4]$

解：

(1) 因$f'(x)=\cos x, g'(x)=-\sin x$，则$\dfrac{f\left(\dfrac{\pi}{2}\right)-f(0)}{g\left(\dfrac{\pi}{2}\right)-g(0)}=\dfrac{f'(c)}{g'(c)}$即$\dfrac{1-0}{0-1}=\dfrac{\cos c}{\sin c}$，亦即$\cot c=1$，又$c\in\left(0,\dfrac{\pi}{2}\right)$，故$c=\dfrac{\pi}{4}$.

(2) 因$f'(x)=2x, g'(x)=\dfrac{1}{2\sqrt{x}}$，则$\dfrac{f(4)-f(1)}{g(4)-g(1)}=\dfrac{f'(c)}{g'(c)}$即$\dfrac{16-1}{2-1}=\dfrac{2c}{\dfrac{1}{2\sqrt{c}}}$，亦即$4c^{\frac{3}{2}}=15$，又$c\in(1,4)$，故$c=\left(\dfrac{15}{4}\right)^{\dfrac{2}{3}}$.

9. 试作函数$y=|x-1|$在区间$[0,3]$上的图形，这里为什么没有平行于弦的切线，拉格朗日定理中哪个条件不成立？
解：函数在点$x=1$处不可导，即其图形ACB为一折线，此折线在$C(0,1)$点的切线不存在，拉格朗日定理中的第二个条件即在$(0,3)$内可导这一条件不满足.



10. 利用拉格朗日公式证明不等式：

(1) $|\sin x-\sin y|\leqslant|x-y|$

(2) 当$x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$时，$|x|\leqslant|\tan x|$(等号只有在$x=0$时成立)

(3) $n\cdot y^{n-1}(y-x)<x^n-y^n<n\cdot x^{n-1}(x-y)(n>1,x>y)$

(4) $\dfrac{x}{1+x}<\ln(1+x)<x(x>0)$

(5) 若$x\neq 0,e^x>1+x$(分$x>0,x<0$两种情况证明)

证明：

(1) 不妨设$x>y,f(t)=\sin t$在$[y,x]$上连续，在(y,x)内可导，故拉格朗日定理成立，因而有$\sin x-\sin y=\cos\xi(x-y)(\xi\in(y,x))$，则$|\sin x-\sin y|=|\cos\xi(x-y)|=|\cos\xi||(x-y)|\leqslant|x-y|(\forall(x,y\in(-\infty,+\infty))$成立.

(2) 不妨设$x\in\left(0,\dfrac{\pi}{2}\right),f(t)=\tan t$在$[0,x]$上连续，在$(0,x)$内可导，故拉格朗日定理成立，因而有$\tan x-\tan 0=\sec^2\xi(x-0)\left(\xi\in(0,x),x\in\left(0,\dfrac{pi}{2}\right)\right)$，则$x=\cos^2\xi\cdot\tan x<\tan x$
同理可证，当$x\in\left(-\dfrac{\pi}{2},0\right)$时，$-x<-\tan x$.
当$x=0$时，$|\tan x|=|x|$.
总之，当$x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$时，$|x|\leqslant|\tan x|$成立.
当$x=0$时，等号成立；当$0<|\xi|<|x|<\dfrac{\pi}{2}$时，$0<\cos^2\xi<1$，故只能成立$|x|<|\tan x|$.

(3) $f(t)=t^n$在$[y,x]$上连续，在(y,x)内可导，故拉格朗日定理成立，因而有$x^n-y^n=n\cdot\xi^{n-1}(x-y)(0<y<\xi<x)$，
又$n>1$，则$y^{n-1}<\xi^{n-1}<x^{n-1}$，故$n\cdot y^{n-1}(x-y)<n\cdot\xi^{n-1}(x-y)<n\cdot x^{n-1}(x-y)$即$n\cdot y^{n-1}(x-y)<x^n-y^n<n\cdot x^{n-1}(x-y)$成立.

(4) $f(t)=\ln(1+t)$在$[0,x]$上连续，在$(0,x)$内可导，故拉格朗日定理成立，因而有$\ln(1+x)=\ln(1+x)-\ln 1=\dfrac{1}{1+\xi}(1+x-1)=\dfrac{x}{1+\xi}(0<\xi<x)$，又$1<1+\xi<1+x$，则$\dfrac{1}{1+x}<\dfrac{1}{1+\xi}<1$，从而有$\dfrac{x}{1+x}<\dfrac{x}{1+\xi}<x(x>0)$即$\dfrac{x}{1+x}<\ln(1+x)<x(x>0)$成立.

(5) $f(t)=e^t$显然满足拉格朗日定理条件.
当$x>0$时，对$f(t)=e^t$在$[0,x]$应用拉格朗日公式，有$e^x-e^0=e^\xi(x-0)$即$e^x-1=xe^\xi(0<\xi<x)$，因$0<\xi<x$，则$e^\xi>1$，从而$e^x-1=xe^\xi>x$即$e^x>1+x$；
当$x<0$时，对$f(t)=e^t$在$[x,0]$应用拉格朗日公式，有$e^0-e^x=e^\xi(0-x)$即$1-e^x=-xe^\xi(x<\xi<0)$，因$x<\xi<0$，则$0<e^\xi<1$，从而$1-e^x=-xe^\xi<-x$即$e^x>1+x$.
总之，若$x\neq 0$，总有$e^x>1+x$.

11. 若$f'(x)\equiv k$，试证$f(x)=kx+b$.
证明：考虑$F(x)=f(x)-kx$
由于$F'(x)=f'(x)-k\equiv 0$，据拉格朗日定理的推论1知，$F(x)=f(x)-kx=b(\forall x\in(-\infty,+\infty)$，故$f(x)=kx+b$.

12. 证明方程$x^3-3x+c=0$在$[0,1]$内不含有两个不同的根.
证明：令$f(x)=x^3-3x+c$
用反证法.设$f(x)$在$[0,1]$内有两个不同根$0<x_1<x_2<1$.
此时$f(x_1)=f(x_2)=0$，据洛尔定理，必存在$\xi\in(x_1,x_2)$，使$f'(\xi)=0$即$3\xi^2-3=0$，解得$\xi=\pm1$，这与$\xi\in(x_1,x_2)\subset(0,1)$矛盾.
故假设不成立.即方程$x^3-3x+c=0$在$[0,1]$内不含有两个不同的根.

13. 若在$[a,b]$上$|f'(x)|\geqslant|\varphi'(x)|,f'(x)\neq0$，则$|\Delta f(x)|\geqslant|\Delta\varphi(x)|$.并证在$\left[\dfrac{1}{2},x\right]$上$\Delta\arctan x\leqslant\Delta\ln(1+x^2)$，由此证明在$\left[\dfrac{1}{2},1\right]$上以下的不等式成立：$\arctan x-\ln(1+x^2)\geqslant\dfrac{\pi}{4}-\ln2$.
证明：因在$[a,b]$上$|f'(x)|\geqslant|\varphi'(x)|,f'(x)\neq0$，故$f(x),\varphi(x)$在$[a,b]$上可导，从而在$[a,b]$上连续.
任取$x,x+\Delta x\in[a,b],\Delta x>0$，则$f(x),\varphi(x)$在$[x,x+\Delta x]$上连续可导且$f'(x)\neq0$.
由柯西定理，得必存在$\xi\in(x,x+\Delta x)$，使$\dfrac{\varphi(x+\Delta x)-\varphi(x)}{f(x+\Delta x)-f(x)}=\dfrac{\varphi'(\xi)}{f'(\xi)}$即$\dfrac{\Delta\varphi(x)}{\Delta f(x)}=\dfrac{\varphi'(\xi)}{f'(\xi)}$，于是$\left|\dfrac{\Delta\varphi(x)}{\Delta f(x)}\right|=\left|\dfrac{\varphi'(\xi)}{f'(\xi)}\right|\leqslant1$即$|\Delta f(x)|\geqslant|\Delta\varphi(x)|$.
因$(\arctan x)'=\dfrac{1}{1+x^2},(\ln(1+x^2))'=\dfrac{2x}{1+x^2}$，且在$\left[\dfrac{1}{2},x\right]$上，有$2x>1$，则$(\ln(1+x^2))'=\dfrac{2x}{1+x^2}>\dfrac{1}{1+x^2}=(\arctan x)'>0$，取$f(x)=\ln(1+x^2),\varphi(x)=\arctan x$，则由上面的结论知，在$\left[\dfrac{1}{2},x\right]$上，$\Delta\arctan x=|\Delta\arctan x|\leqslant|\Delta\ln(1+x^2)|=\Delta\ln(1+x^2)$.
在$\left[\dfrac{1}{2},1\right]$上任取一个$x$，在$[x,1]$上有$\arctan1-\arctan x=\Delta\arctan x\leqslant\Delta\ln(1+x^2)=\ln(1+1^2)-\ln(1+x^2)$即$\dfrac{\pi}{4}-\arctan x\leqslant\ln2-\ln(1+x^2)$，从而$\arctan x-\ln(1+x^2)\geqslant\dfrac{\pi}{4}-\ln2$.

14. 若$f(x)$在区间X(由穷或无穷)中具有有界的导数，即$|f'(x)|\leqslant M$，则$f(x)$在X中一致连续.
证明：因若$f(x)$在区间X上可导，从而也在X上连续，且$|f'(x)|\leqslant M,M>0$
任取$x_1,x_2\in X$，不妨设$x_1<x_2$，则$f(x)$在$[x_1,x_2]$上连续可导.
由拉格朗日中值定理，得$\exists\xi\in(x_1,x_2)$，使$f(x_2)-f(x_1)=f'(\xi)(x_2-x_1)$，则$|f(x_2)-f(x_1)|=|f'(\xi)|(x_2-x_1)\leqslant M(x_2-x_1)$，于是对$\forall\varepsilon>0$，取$\delta=\dfrac{\varepsilon}{M}$，则当$|x_2-x_1|<\delta=\dfrac{\varepsilon}{M}$时，$|f(x_2)-f(x_1)|\leqslant M(x_2-x_1)<\varepsilon$成立，于是$f(x)$在$X$中一致连续.

§2. 泰勒公式

1. 当$|x|$充分小时，推导下列近似公式：
$\tan x \approx x; \cos x \cdot \sin x \approx x;\ \sqrt[n]{1 \pm x} \approx 1 \pm \dfrac{x}{n}; e^x \approx 1 + x.$
证明：
   (1) 令$f(x) = \tan x$，因$|x|$充分小，用近似公式时可取$x_0 = 0$，于是$f(x_0) = 0, f'(0) = \sec^2 x\big|_{x=0} = 1$，从而$f(x) \approx f(0) + f'(0)(x - 0)$即为$\tan x \approx x$.
   (2) 令$f(x) = \cos x \cdot \sin x$，因$|x|$充分小，用近似公式时可取$x_0 = 0$，于是$f(x_0) = 0, f'(0) = (-\sin^2 x + \cos^2 x)\big|_{x=0} = 1$，从而$f(x) \approx f(0) + f'(0)(x - 0)$即为$\cos x \cdot \sin x \approx x$.
   (3) 令$f(x) = \sqrt[n]{1 \pm x}$，因$|x|$充分小，用近似公式时可取$x_0 = 0$，于是$f(x_0) = 1, f'(0) = \pm\dfrac{1}{n}(1 \pm x)^{\frac{1}{n}-1}\Big|_{x=0} = \pm\dfrac{1}{n}$，从而$f(x) \approx f(0) + f'(0)(x - 0)$即为$\sqrt[n]{1 \pm x} \approx 1 \pm \dfrac{x}{n}$.
   (4) 令$f(x) = e^x$，因$|x|$充分小，用近似公式时可取$x_0 = 0$，于是$f(x_0) = 1, f'(0) = e^x\big|_{x=0} = 1$，从而$f(x) \approx f(0) + f'(0)(x - 0)$即为$e^x \approx 1 + x$.

2. 求$\tan 4^o$的近似值.
解：由上题知，$\tan x \approx x$，故$\tan 4^o = \tan \dfrac{\pi}{45} \approx \dfrac{\pi}{45} \approx 0.0698$.

3. 求$\sqrt{37}$的近似值.
解：因$\sqrt{37} = \sqrt{36+1} = 6\sqrt{1 + \dfrac{1}{36}}$，故据第1题，得$\sqrt{37} = 6\sqrt{1 + \dfrac{1}{36}} \approx 6\left(1 + \dfrac{1}{72}\right) \approx 6.083$.

4. 图5-5所示为一凸透镜，设透镜凸面半径为R，口径为$2H$，H远比R小.
   (1) 证明：透镜厚度$D \approx \dfrac{H^2}{2R}$；
   (2) 设$2H = 50$毫米，$R = 100$毫米，求D.

解：

   (1) 因$D = R - \sqrt{R^2 - H^2}$，则$D = R\left[1 - \sqrt{1 - \left(\dfrac{H}{R}\right)^2}\right]$.
   又H远比R小，故$\left|\left(\dfrac{H}{R}\right)^2\right|$充分小，则$\sqrt{1 - \left(\dfrac{H}{R}\right)^2} \approx 1 - \dfrac{1}{2}\left(\dfrac{H}{R}\right)^2 = 1 - \dfrac{H^2}{2R^2}$，从而$D \approx R\left[1 - \left(1 - \dfrac{H^2}{2R^2}\right)\right] = \dfrac{H^2}{2R}$.
   (2) $D = R - \sqrt{R^2 - H^2} = 100 - \sqrt{100^2 - 25^2} \approx 3.175; D \approx \dfrac{H^2}{2R} = \dfrac{25^2}{200} = 3.125$

5. 测得圆钢直径为30.12毫米，已知其误差为0.05毫米.求圆钢截面积的绝对误差和相对误差.
解：因圆面积$S = \dfrac{\pi}{4}D^2$，则利用导数估计误差，S有绝对误差$|\Delta S| \approx \left|\dfrac{\pi}{2}D\Delta D\right| = \dfrac{\pi}{2} \times 30.12 \times 0.05 \approx 2.3656$(毫米^2)；相对误差为$\left|\dfrac{\Delta S}{S}\right| \approx \left|\dfrac{\frac{\pi}{2}D\Delta D}{\frac{\pi}{4}D^2}\right| = \left|\dfrac{2\Delta D}{D}\right| \approx 0.33\%$.

6. 测得金属球体的直径$D = 10.12$毫米，误差$\Delta D = 0.05$毫米.计算球体的体积及其绝对误差，相对误差.
解：因球体体积$V = \dfrac{\pi}{6}D^3$，故球体体积$V = \dfrac{\pi}{6}(10.12)^3 \approx 542.675$(毫米^3)；
利用导数误差估计，V有绝对误差$|\Delta V| \approx \left|\dfrac{\pi}{2}D^2\Delta D\right| = \dfrac{\pi}{2} \times 10.12^2 \times 0.05 \approx 8.044$(毫米^3)；
相对误差$\left|\dfrac{\Delta V}{V}\right| \approx \left|\dfrac{\frac{\pi}{2}D^2\Delta D}{\frac{\pi}{6}D^3}\right| = \left|\dfrac{3\Delta D}{D}\right| \approx 1.48\%$.

7. 求下列函数在$x = 0$点的泰勒展开式：
   (1) $f(x) = \sqrt{1+x}$
   (2) $f(x) = \dfrac{1}{1+x}$
   (3) $f(x) = e^{\sin x}$(展开直到含有x^3的项)

(4) $f(x)=\cos x$

(5) $f(x)=\ln\cos x$(展开直到含有x^6的项)

(6) $f(x)=\ln(1+x)$

解:

(1) $f(x)=\sqrt{1+x}, f'(x)=\dfrac{1}{2}(1+x)^{-\frac{1}{2}}, f''(x)=-\dfrac{1}{4}(1+x)^{-\frac{3}{2}},\cdots,f^{(n)}(x)=(-1)^{n-1}\dfrac{(2n-3)!!}{2^n}(1+x)^{\frac{1}{2}-n}$
把$x=0$依次代入上列各式，有$f(0)=1, f'(0)=\dfrac{1}{2}, f''(0)=-\dfrac{1}{4},\cdots,f^{(n)}(0)=(-1)^{n-1}\dfrac{(2n-3)!!}{2^n}$
于是得函数$f(x)=\sqrt{1+x}$在$x=0$的泰勒展开式：$f(x)=1+\dfrac{x}{2}-\dfrac{x^2}{8}+\cdots+\dfrac{(-1)^{n-1}\dfrac{(2n-3)!!}{2^n}}{n!}x^n+o(x^n)=1+\dfrac{x}{2}-\dfrac{x^2}{8}+\cdots+\dfrac{(-1)^{n-1}(2n-3)!!}{n!\cdot 2^n}+o(x^n)$

(2) $f(x)=\dfrac{1}{1+x}=(1+x)^{-1}, f'(x)=-(1+x)^{-2}, f''(x)=2(1+x)^{-3},\cdots,f^{(n)}=(-1)^n\cdot n!(1+x)^{-(n+1)}$
把$x=0$依次代入上列各式，有$f(0)=1, f'(0)=-1, f''(0)=2,\cdots,f^{(n)}(0)=(-1)^n\cdot n!$
于是得函数$f(x)=\dfrac{1}{1+x}$在$x=0$的泰勒展开式：$f(x)=1-x+x^2-\cdots+(-1)^n x^n+o(x^n)$

(3) 注意$\sin x$为x的等价无穷小.
则$e^{\sin x}=1+\sin x+\dfrac{1}{2!}\sin^2 x+\dfrac{1}{3!}\sin^3 x+o_1(\sin^3 x)=1+(x-\dfrac{x^3}{3!}+o(x^3))+\dfrac{1}{2}(x+o(x^2))^2+\dfrac{1}{6}(x+o(x^2))^3+o_1(\sin^3 x)=1+x-\dfrac{x^3}{6}+\dfrac{x^2}{2}+\dfrac{x^3}{6}+o(x^3)=1+x+\dfrac{x^2}{2}+o(x^3)$.

(4) $f(x)=\cos x, f'(x)=-\sin x, f''(x)=-\cos x,\cdots,f^{(k)}(x)=\cos\left(x+\dfrac{k}{2}\pi\right)$
把$x=0$依次代入上列各式，有$f(0)=1, f'(0)=0, f''(0)=-1,\cdots,f^{(2m)}(0)=(-1)^m, f^{(2m+1)}(0)=0,\cdots(m\in Z^+)$
于是得函数$f(x)=\cos x$在$x=0$的泰勒展开式：$f(x)=1-\dfrac{x^2}{2}+\cdots+(-1)^n\dfrac{x^{2n}}{(2n)!}+o(x^{2n})$

(5) $f(x)=\ln\cos x=\dfrac{1}{2}\ln(1-\sin^2 x)=-\dfrac{1}{2}\left(\sin^2 x+\dfrac{\sin^4 x}{2}+\dfrac{\sin^6 x}{3}+o(\sin^6 x)\right)=-\dfrac{1}{2}\left[\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+o_1(x^5)\right)^2+\dfrac{1}{2}\left(x-\dfrac{x^3}{3!}+o_2(x^3)\right)^4+\dfrac{1}{3}\left(x+o_3(x^2)\right)\right)^6+o(x^6)\Bigg]=-\dfrac{x^2}{2}-\dfrac{x^4}{12}-\dfrac{x^6}{45}+o(x^6)$

(6) $f(x)=\ln(1+x), f'(x)=\dfrac{1}{1+x}=(1+x)^{-1}, f''(x)=-(1+x)^{-2},\cdots,f^{(n)}(x)=(-1)^{n-1}!(1+x)^{-n}$
把$x=0$依次代入上列各式，有$f(0)=0, f'(0)=1, f''(0)=-1,\cdots,f^{(n)}(0)=(-1)^(n-1)!$
于是得函数$f(x)=\ln(1+x)$在$x=0$的泰勒展开式：$f(x)=x-\dfrac{x^2}{2}+\cdots+(-1)^{n-1}\dfrac{x^n}{n}+o(x^n)$

8. 求函数$\ln x$在$x=1$的泰勒展开式.
解：由上题结论，得$\ln x=\ln(1+x-1)=(x-1)-\dfrac{(x-1)^2}{2}+\cdots+(-1)^{n-1}\dfrac{(x-1)^n}{n}+o((x-1)^n)$.

9. 求函数$\sqrt{x}$在$x=1$的泰勒展开式（展开到x^3项）.
解:$f(x)=\sqrt{x}, f'(x)=\dfrac{1}{2}(1+x)^{-\frac{1}{2}}, f''(x)=-\dfrac{1}{4}(1+x)^{-\frac{3}{2}}, f'''(x)=\dfrac{3}{8}(1+x)^{-\frac{5}{2}}$
把$x=1$依次代入上列各式，有$f(1)=1, f'(1)=\dfrac{1}{2}, f''(1)=-\dfrac{1}{4}, f'''(1)=\dfrac{3}{8}$
于是得函数$f(x)=\sqrt{x}$在$x=1$的泰勒展开式：$f(x)=1+\dfrac{1}{2}(x-1)-\dfrac{1}{8}(x-1)^2+\dfrac{1}{16}(x-1)^3+o((x-1)^3)$.

10. 将多项式$P_3(x)=1+3x+5x^2-2x^3$表成$x+1$的正整数幂的多项式.
解：因$P_3(x)=1+3x+5x^2-2x^3, P_3'(x)=3+10x-6x^2, P_3''(x)=10-12x, P_3'''(x)=-12, P_3^{(4)}=\cdots=P_3^{(n)}=0$
把$x=-1$依次代入上列各式，有$P_3(-1)=5, P_3'(-1)=-13, P_3''(-1)=22, P_3'''(-1)=-12, P_3^{(4)}=\cdots=P_3^{(n)}=0$
于是得$P_3(x)=5-13(x+1)+11(x+1)^2-2(x+1)^3$.

11. 利用泰勒公式计算$\sqrt[3]{7}$至四位小数.

解: $\sqrt[3]{7}=2\left(1-\frac{1}{8}\right)^{\frac{1}{3}}\approx 2\left[1+\frac{1}{3}\left(-\frac{1}{8}\right)+\frac{1}{2!}\cdot\frac{1}{3}\left(\frac{1}{3}-1\right)\left(-\frac{1}{8}\right)^2+\frac{1}{3!}\cdot\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(-\frac{1}{8}\right)^3\right]\approx$ 1.9130

$\Delta<2\cdot\frac{1}{4!}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}\left(\frac{1}{8}\right)^4\approx 2.01\times 10^{-5}$.

12. 利用泰勒公式求下列极限：

(1) $\lim\limits_{x\to 0}\dfrac{\cos x-e^{-\frac{x^2}{2}}+\frac{1}{12}x^4}{x^6}$

(2) $\lim\limits_{x\to 0}\dfrac{e^x\sin x-x(1+x)}{x^3}$

(3) $\lim\limits_{x\to\infty}\left[x-x^2\ln\left(1+\frac{1}{x}\right)\right]$

(4) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

(5) $\lim\limits_{x\to+\infty}(\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5})$

(6) $\lim\limits_{x\to 0}\dfrac{1-\cos(\sin x)}{2\ln(1+x^2)}$

解:

(1) 利用泰勒公式，有$\cos x=1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+o(x^6), e^{-\frac{x^2}{2}}=1-\frac{x^2}{2}+\frac{x^4}{8}-\frac{x^6}{48}+ox^6$，

则$\cos x-e^{-\frac{x^2}{2}}+\frac{1}{12}x^4=\frac{7}{360}x^6+o(x^6)$，于是$\lim\limits_{x\to 0}\dfrac{\cos x-e^{-\frac{x^2}{2}}+\frac{1}{12}x^4}{x^6}=\frac{7}{360}$.

(2) 利用泰勒公式，有$e^x=1+x+\frac{x^2}{2}+o(x^2), \sin x=x-\frac{x^3}{3!}+o(x^3)$，则$e^x\sin x-x(1+x)=\frac{x^3}{3}+o(x^3)$，于是$\lim\limits_{x\to 0}\dfrac{e^x\sin x-x(1+x)}{x^3}=\frac{1}{3}$.

(3) 利用泰勒公式，有$\ln\left(1+\frac{1}{x}\right)=\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}+o\left(\frac{1}{x^3}\right)$则$x-x^2\ln\left(1+\frac{1}{x}\right)=\frac{1}{2}-\frac{1}{3x}+o\left(\frac{1}{x}\right)$，于是$\lim\limits_{x\to\infty}\left[x-x^2\ln\left(1+\frac{1}{x}\right)\right]=\frac{1}{2}$.

(4) 利用泰勒公式，有$\sin x=x-\frac{x^3}{3!}+o(x^3)$，则$\frac{1}{x}-\frac{1}{\sin x}=\frac{\sin x-x}{x\sin x}=\dfrac{-\frac{x^3}{3!}+o(x^3)}{x\left(x-\frac{x^3}{3!}+o(x^3)\right)}=$

$\dfrac{-\frac{x}{6}+o(x)}{1-\frac{x^2}{6}+o(x^2)}$，于是$\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=0$

(5) 因$\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5}=x\left(1+\frac{1}{x}\right)^{\frac{1}{6}}-x\left(1-\frac{1}{x}\right)^{\frac{1}{6}}$

利用泰勒公式，有$\left(1+\frac{1}{x}\right)^{\frac{1}{6}}=1+\frac{1}{6x}-\frac{5}{72x^2}+o\left(\frac{1}{x^2}\right), \left(1-\frac{1}{x}\right)^{\frac{1}{6}}=1-\frac{1}{6x}-\frac{5}{72x^2}+o\left(\frac{1}{x^2}\right)$，

则$\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5}=\frac{1}{3}+o\left(\frac{1}{x}\right)$，于是$\lim\limits_{x\to+\infty}(\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5})=\frac{1}{3}$.

(6) 利用泰勒公式，有$\cos(\sin x)=1-\frac{\sin^2 x}{2}+\frac{\sin^4 x}{4!}+o(\sin^4 x), \ln(1+x^2)=x^2-\frac{x^4}{2}+o(x^4)$，

则$\dfrac{1-\cos(\sin x)}{2\ln(1+x^2)}=\dfrac{\sin^2 x\left[1-\frac{1}{12}\sin^2 x+o(\sin^2 x)\right]}{4x^2\left(1-\frac{1}{2}x^2+o(x^2)\right)}$，从而$\lim\limits_{x\to 0}\dfrac{1-\cos(\sin x)}{2\ln(1+x^2)}=\frac{1}{4}$

13. 决定α,β，使$\lim\limits_{x\to+\infty}(\sqrt[4]{16x^4-8x^3+10x-7}-\alpha x-\beta)=0$.

解：因$\sqrt[4]{16x^4-8x^3+10x-7}=2x\cdot\sqrt[4]{1+\left(-\dfrac{1}{2x}+\dfrac{5}{8x^3}-\dfrac{7}{16x^4}\right)}=2x-\dfrac{1}{4}+\dfrac{5}{16x^2}-\dfrac{7}{32x^3}+\varepsilon(\lim\limits_{x\to+\infty}\varepsilon=0)$

故$\sqrt[4]{16x^4-8x^3+10x-7}-\alpha x-\beta=(2-\alpha)x-\left(\dfrac{1}{4}+\beta\right)+\dfrac{5}{16x^2}-\dfrac{7}{32x^3}+\varepsilon$

由此可知，欲使$\lim\limits_{x\to+\infty}(\sqrt[4]{16x^4-8x^3+10x-7}-\alpha x-\beta)=\lim\limits_{x\to+\infty}\left[(2-\alpha)x-\left(\dfrac{1}{4}+\beta\right)+\dfrac{5}{16x^2}-\dfrac{7}{32x^3}+\varepsilon\right]=0$，必须$\alpha=2,\beta=-\dfrac{1}{4}$.

14. 决定A，使极限$\lim\limits_{x\to0}\dfrac{\sqrt[n]{Q(x)}-A}{x}$存在，其中$Q(x)=a_0+a_1x+\cdots+a_mx^m,a_0\neq0,m$为自然数.

解：$\lim\limits_{x\to0}\dfrac{\sqrt[n]{Q(x)}-A}{x}=\lim\limits_{x\to0}\dfrac{\sqrt[n]{a_0+a_1x+\cdots+a_mx^m}-A}{x}=\lim\limits_{x\to0}\dfrac{\sqrt[n]{a_0}\left(\sqrt[n]{1+\dfrac{a_1}{a_0}x+\cdots+\dfrac{a_m}{a_0}x^m}-A\right)}{x}=$
$\lim\limits_{x\to0}\dfrac{\sqrt[n]{a_0}\left(1+\dfrac{1}{n}\left(\dfrac{a_1}{a_0}x+\cdots+\dfrac{a_m}{a_0}x^m\right)+o(x)-A\right)}{x}$存在$\Longleftrightarrow\sqrt[n]{a_0}-A=0$即$A=\sqrt[n]{a_0}$，此时原式$=\dfrac{a_1\cdot\sqrt[n]{a_0}}{na_0}$.

§3. 函数的升降、凸性与极值

1. 证明下列函数的单调性：

(1) $y=x-\sin x$

(2) $y=\left(1+\dfrac{1}{x}\right)^{x}\ (x>0)$

证明:

(1) 因$y=f(x)$在$(-\infty,+\infty)$内连续可导，故$f'(x)=1-\cos x$；又$-1\leqslant\cos x\leqslant 1$，故$f'(x)\geqslant 0$，于是$y=x-\sin x$在$(-\infty,+\infty)$单调上升.

(2) 因$y=\left(1+\dfrac{1}{x}\right)^{x}$，故$y'=\left(1+\dfrac{1}{x}\right)^{x}\left[\ln(1+x)-\ln x-\dfrac{1}{1+x}\right]$

又$x>0$，故$\left(1+\dfrac{1}{x}\right)^{x}>0$，则只需判断方括号中式子的符号.

令$f(x)=\ln x$在$[x,1+x]$(对$\forall x>0$)上应用拉格朗日定理，有$\ln(1+x)-\ln x=\dfrac{1}{\xi}(1+x-x)=\dfrac{1}{\xi}(x<\xi<1+x)$，于是$\dfrac{1}{x}>\dfrac{1}{\xi}>\dfrac{1}{1+x}$，故$\ln(1+x)-\ln x=\dfrac{1}{\xi}>\dfrac{1}{1+x}$即$\ln(1+x)-\ln x-\dfrac{1}{1+x}>0(\forall x>0)$，由此可知$y'>0$，从而$y=\left(1+\dfrac{1}{x}\right)^{x}$在$(0,+\infty)$上单调增加.

2. 单调函数的导数是否必为单调？

解: 不一定.

例：$y=x^3$在$(-\infty,+\infty)$上单调上升，但$y'=3x^2$却不单调.

3. 证明下列不等式：

(1) $x>\sin x>\dfrac{2}{\pi}x\left(0<x<\dfrac{\pi}{2}\right)$

(2) $x-\dfrac{x^3}{6}>\sin x>x(x<0)$

(3) $x-\dfrac{x^2}{2}<\ln(1+x)<x(x>0)$

(4) $\tan x>x+\dfrac{x^3}{3}\left(0<x<\dfrac{\pi}{2}\right)$

(5) $2\sqrt{x}>3-\dfrac{1}{x}(x>1)$

(6) $\dfrac{1}{2^{p-1}}\leqslant x^p+(1-x)^p\leqslant 1(0\leqslant x\leqslant 1,p>1)$

证明:

(1) 设$f(x)=x-\sin x$，由第1题，知$f(x)$在$\left(0,\dfrac{\pi}{2}\right)$内单调上升，又$f(0)=0$，故对$\forall x\in\left(0,\dfrac{\pi}{2}\right)$，有$f(x)>f(0)=0$即$x-\sin x>0$，从而$x>\sin x\left(0<x<\dfrac{\pi}{2}\right)$；

设$g(x)=\dfrac{\sin x}{x},g\left(\dfrac{\pi}{2}\right)=\dfrac{2}{\pi},g'(x)=\dfrac{x\cos x-\sin x}{x^2}\left(0<x<\dfrac{\pi}{2}\right)$

注意到$u(x)=x\cos x-\sin x\left(0<x<\dfrac{\pi}{2}\right)$且$u(0)=0$，由于$u'(x)=-x\sin x<0\left(0<x<\dfrac{\pi}{2}\right)$，故当$x\in\left(0,\dfrac{\pi}{2}\right)$时，$u(x)$单调下降即$u(x)<u(0)=0\left(0<x<\dfrac{\pi}{2}\right)$，由此得，$g'(x)<0\left(0<x<\dfrac{\pi}{2}\right)$，故$g(x)$在$\left(0,\dfrac{\pi}{2}\right)$上单调下降，于是$g(x)>g\left(\dfrac{\pi}{2}\right)=\dfrac{2}{\pi}$即$\dfrac{\sin x}{x}>\dfrac{2}{\pi},x\in\left(0<x<\dfrac{\pi}{2}\right)$，则$\sin x>\dfrac{2}{\pi}x$，从而$x>\sin x>\dfrac{2}{\pi}x\left(0<x<\dfrac{\pi}{2}\right)$

(2) 设$f(x)=x-\sin x$，由第1题，知$f(x)$在$(-\infty,0)$内单调上升，又$f(0)=0$，故对$\forall x\in(-\infty,0)$，有$f(x)<f(0)=0$即$x-\sin x>0$，从而$x<\sin x(x<0)$；

设$g(x)=x-\dfrac{x^3}{6}-\sin x,g(0)=0,g'(x)=1-\dfrac{x^2}{2}-\cos x$

再设$h(x)=1-\dfrac{x^2}{2}-\cos x(x<0)$且$h(0)=0$，由于$h'(x)=-x+\sin x>0$，故当$x\in(-\infty,0)$时，$h(x)$单调上升即$h(x)<h(0)=0(x<0)$，由此得，$g'(x)<0(x<0)$，故$g(x)$在$(-\infty,0)$上单调下降，于是$g(x)>g(0)=0$即$x-\dfrac{x^3}{6}-\sin x>0(x<0)$，则$x-\dfrac{x^3}{6}>\sin x$，从而$x-\dfrac{x^3}{6}>\sin x>x(x<0)$

(3) 设$f(x)=\ln(1+x)-x,g(x)=\ln(1+x)-x+\dfrac{x^2}{2}(x>0)$，故$f'(x)=\dfrac{1}{1+x}-1=-\dfrac{x}{1+x}<0(x>0)$，则$f(x)$在$(0,+\infty)$内单调下降，又$f(0)=0$，故对$\forall x>0$，有$f(x)<f(0)=0$即$\ln(1+x)<x(x>0)$；
$g'(x)=\dfrac{1}{1+x}-1+x=\dfrac{x^2}{1+x}>0(x>0)$
故$g(x)$在$(0,+\infty)$上单调上升，又$g(0)=0$，于是$g(x)>g(0)=0$即$\ln(1+x)>x-\dfrac{x^2}{2}(x>0)$，从而$x-\dfrac{x^2}{2}<\ln(1+x)<x(x>0)$

(4) 设$f(x)=\tan x-x-\dfrac{x^3}{3}\left(0<x<\dfrac{\pi}{2}\right)$，故$f'(x)=\sec^2 x-1-x^2=\tan^2 x-x^2=(\tan x+x)(\tan x-x)$，又因$(\tan x-x)'=\sec^2 x-1=\tan^2 x\leqslant 0\left(\forall x\in\left(0,\dfrac{\pi}{2}\right)\right)$，则$\tan x-x$在$\left(0,\dfrac{\pi}{2}\right)$内单调上升，故对$\forall x\in\left(0,\dfrac{\pi}{2}\right)$，有$\tan x-x>0\left(0<x<\dfrac{\pi}{2}\right)$；
于是$f'(x)=(\tan x+x)(\tan x-x)>0\left(0<x<\dfrac{\pi}{2}\right)$，由此可知，$f(x)$在$\left(0,\dfrac{\pi}{2}\right)$上单调上升，又$f(0)=0$，于是$f(x)>f(0)=0$即$\tan x-x-\dfrac{x^3}{3}>0\left(0<x<\dfrac{\pi}{2}\right)$，从而$\tan x>x+\dfrac{x^3}{3}\left(0<x<\dfrac{\pi}{2}\right)$

(5) 设$f(x)=2\sqrt{x}-3+\dfrac{1}{x}(x>1)$，故$f'(x)=\dfrac{1}{\sqrt{x}}-\dfrac{1}{x^2}=\dfrac{x^{\frac{3}{2}}-1}{x^2}>0(x>1)$，于是$f(x)$在$(1,+\infty)$上单调上升，又$f(1)=0$，于是$f(x)>f(1)=0$即$2\sqrt{x}-3+\dfrac{1}{x}>0(x>1)$，从而$\dfrac{1}{2^{p-1}}\leqslant x^p+(1-x)^p\leqslant 1(0\leqslant x\leqslant 1,p>1)$

(6) 设$f(x)=x^p+(1-x)^p(0\leqslant x\leqslant 1,p>1)$，故$f'(x)=px^{p-1}-p(1-x)^{p-1}$，
令$f'(x)=px^{p-1}-p(1-x)^{p-1}=0$，解得$x=\dfrac{1}{2}$，比较$f(0)=1,f(1)=1,f\left(\dfrac{1}{2}\right)=\dfrac{1}{2^{p-1}}$，由此得$\min\limits_{0\leqslant x\leqslant 1}f(x)=\dfrac{1}{2^{p-1}},\max\limits_{0\leqslant x\leqslant 1}f(x)=1$，从而$\dfrac{1}{2^{p-1}}\leqslant x^p+(1-x)^p\leqslant 1(0\leqslant x\leqslant 1,p>1)$

4. 确定下列函数的上升、下降区间：

(1) $y=x^3-6x$

(2) $y=2x^3-3x^2-12x+1$

(3) $y=x^4-2x^3$

(4) $y=x+\sin x$

(5) $y=\dfrac{2x}{1+x^2}$

(6) $y=2x^2-\sin x$

(7) $y=x^n e^{-x}(n>0,x\leqslant 0)$

解:

(1) 因$y'=3x^2-6=3(x^2-2)$，得驻点$x=\pm\sqrt{2}$
当$x<-\sqrt{2}$或$x>\sqrt{2}$时，$y'>0$，函数严格上升；当$-\sqrt{2}<x<\sqrt{2}$时，$y'<0$，函数严格下降.
从而在区间$(-\infty,-\sqrt{2})\bigcup(\sqrt{2},+\infty)$上函数严格上升；在区间$(-\sqrt{2},\sqrt{2})$上函数严格下降.

(2) 因$y'=6x^2-6x-12=6(x^2-x-2)=6(x-2)(x+1)$，得驻点$x=-1,x=2$
当$x<-1$或$x>2$时，$y'>0$，函数严格上升；当$-1<x<2$时，$y'<0$，函数严格下降.
从而在区间$(-\infty,-1)\bigcup(2,+\infty)$上函数严格上升；在区间$(-1,2)$上函数严格下降.

(3) 因$y'=4x^3-6x^2=2x^2(2x-3)$，得驻点$x=0,x=\dfrac{3}{2}$
当$x>\dfrac{3}{2}$时，$y'>0$，函数严格上升；当$x<\dfrac{3}{2}$时，$y'\leqslant 0$且仅在$x=0$处$y'=0$，函数严格下降.
从而在区间$\left(\dfrac{3}{2},+\infty\right)$上函数严格上升；在区间$\left(-\infty,\dfrac{3}{2}\right)$上函数严格下降.

(4) 因$y'=1+\cos x\leqslant 0$，故函数在$(-\infty,+\infty)$上函数上升.

(5) 因$y'=\dfrac{2(1-x^2)}{(1+x^2)^2}$，得驻点$x=\pm 1$
当$x<-1$或$x>1$时，$y'<0$，函数严格下降；当$-1<x<1$时，$y'>0$，函数严格上升.
从而在区间$(-\infty,-1)\bigcup(1,+\infty)$上函数严格下降；在区间$(-1,1)$上函数严格上升.

(6) 因$y' = 4x - \cos x, y'' = 4 + \sin x > 0$，则$y'$在$(-\infty, +\infty)$上单调上升.
又$y'(0) = -1, y'\left(\frac{\pi}{2}\right) = 2\pi$，则在$\left(0, \frac{\pi}{2}\right)$内有一个点$x_0$满足$y'(x_0) = 0$即$4x_0 = \cos x_0$
当$x > x_0$时，$y' > 0$，函数严格上升；当$x < x_0$时，$y' < 0$，函数严格下降.
从而在区间$(x_0, +\infty)$上函数严格上升；在区间$(-\infty, x_0)$上函数严格下降.

(7) 因$y' = nx^{n-1}e^{-x} - x^n e^{-x} = x^{n-1}e^{-x}(n - x)$
因$n > 0, x > 0$，故$x^{n-1} > 0, e^{-x} > 0$，则$x^{n-1}e^{-x} > 0$
当$0 < x < n$时，$y' > 0$，函数严格上升；当$x > n$时，$y' < 0$，函数严格下降.
从而在区间$(0, n)$上函数严格上升；在区间$(n, +\infty)$上函数严格下降.

5. 求下列函数的极值：

(1) $y = x - \ln(1 + x)$

(2) $y = \sqrt{x}\ln x$

(3) $y = x + \frac{1}{x}$

(4) $y = \sin^3 x + \cos^3 x$

(5) $y = \cos x + \cosh x$

解:

(1) 因$y' = 1 - \frac{1}{1+x} = \frac{x}{1+x}, y'' = \frac{1}{(1+x)^2} > 0$
此函数的定义域为$(-1, +\infty)$，则驻点为$x = 0$，函数只能在这点有极值，于是$x = 0$是函数的极小点，极小值为$y = 0$.

(2) 因$y' = \frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x}}\ln x = \frac{1}{2\sqrt{x}}(\ln x + 2), y'' = -\frac{1}{2x^{\frac{3}{2}}} + \frac{1}{2x^{\frac{3}{2}}} - \frac{1}{4x^{\frac{3}{2}}}\ln x = -\frac{\ln x}{2^{\frac{3}{2}}}$
驻点为$x = e^{-2}$，函数只能在这点有极值，又$y''|_{x=e^{-2}} > 0$，于是$x = e^{-2}$是函数的极小点，极小值为$y = -\frac{2}{e}$.

(3) 因$y' = 1 - \frac{1}{x^2}, y'' = -\frac{1}{x^3} > 0$
此函数的定义域为$(-\infty, 0)\bigcup(0, +\infty)$，则驻点为$x = \pm 1$，函数只能在这两点有极值，又$y''|_{x=1} = 1 > 0, y''|_{x=-1} = -1 < 0$，于是$x = 1$是函数的极小点，极小值为$y = 2$；$x = -1$是函数的极大点，极大值为$y = 2$.

(4) 因$y' = 3\sin x\cos x(\sin x - \cos x) = \frac{3}{2}\sin 2x(\sin x - \cos x), y'' = 3\cos 2x(\sin x - \cos x) + \frac{3}{2}\sin 2x(\cos x + \sin x)$
驻点为$x = k\pi + \frac{\pi}{4}, x = \frac{k\pi}{2}(k \in Z)$，又$y''|_{x=2k\pi} = -3 < 0, y''|_{x=2k\pi+\frac{\pi}{4}} = \frac{3}{2}\sqrt{2} > 0, y''|_{x=2k\pi+\frac{\pi}{2}} = -3 < 0, y''|_{2k\pi+\pi} = 3 > 0, y''|_{x=2k\pi+\frac{5\pi}{4}} = -\frac{3}{2}\sqrt{2} < 0, y''|_{x=2k\pi+\frac{3\pi}{2}} = 3 > 0$,
于是$x = 2k\pi$时，有极大值$y = 1$；$x = 2k\pi + \frac{\pi}{2}$时，有极大值$y = 1$；$x = 2k\pi + \frac{5\pi}{4}$时，有极大值$y = -\frac{\sqrt{2}}{2}$；
$x = 2k\pi + \frac{\pi}{4}$时，有极小值$y = \frac{\sqrt{2}}{2}$；$x = 2k\pi + \pi$时，有极小值$y = -1$；$x = 2k\pi + \frac{3\pi}{2}$时，有极小值$y = -1$.

(5) 因$y' = 1 - \sin x + \sinh x$，不易求驻点，但由$-\sin x + \frac{e^x - e^{-x}}{2} = 0$易见$x = 0$是一个驻点
由$-\sin x, \sinh x$在$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$的严格单调性，知这驻点是唯一的.
$y'' = -\cos x + \cosh x, y''(0) = 0; y''' = \sin x + \sinh x, y'''(0) = 0; y^{(4)} = \cos x + \cosh x, y^{(4)}(0) = 2 > 0$,
于是$x = 0$是函数的极小点，极小值为$y = 2$.

6. 若$f(x)$在点x_0具有直到n阶连续导数，并且$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0, f^{(n)}(x_0) \neq 0$，那么当$n$为奇数时，$f(x_0)$非极值；当$n$为偶数而$f^{(n)}(x_0) > 0$时，$f(x_0)$为极小值；当$n$为偶数而$f^{(n)}(x_0) < 0$时，$f(x_0)$为极小大值.

证明: 将$f(x)$在$x = x_0$点用泰勒公式展开：$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n)$
因$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$，故$f(x) = f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n)$

当$x\to x_0$时，$o((x-x_0)^n)\to 0$，故当x充分靠近x_0时，即当$|x-x_0|$充分小时，$f(x)-f(x_0)$与$\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$有相同的符号

若$f^{(n)}(x_0)>0$，

(1) n为奇数时，若$x>x_0$，则$(x-x_0)^n>0$，于是$\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n>0$，从而$f(x)-f(x_0)>0$即$f(x)>f(x_0)$；

若$x<x_0$，则$(x-x_0)^n<0$，于是$\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n<0$，从而$f(x)-f(x_0)<0$即$f(x)<f(x_0)$

因此$f(x_0)$不是极值.

(2) n为偶数时，只要x充分接近x_0，不论$x>x_0$，还是$x<x_0$，都有$(x-x_0)^n>0$，此时$\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n>0(x\neq x_0)$，从而$f(x)-f(x_0)>0$，即在$x_0$充分小某邻域内，恒有$f(x)>f(x_0)$，这表明$f(x_0)$是极小值.

若$f^{(n)}(x_0)<0$，

(1) n为奇数时，若$x>x_0$，则$(x-x_0)^n>0$，于是$\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n<0$，从而$f(x)-f(x_0)<0$即$f(x)<f(x_0)$；

若$x<x_0$，则$(x-x_0)^n<0$，于是$\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n>0$，从而$f(x)-f(x_0)>0$即$f(x)>f(x_0)$

因此$f(x_0)$不是极值.

(2) n为偶数时，只要x充分接近x_0，不论$x>x_0$，还是$x<x_0$，都有$(x-x_0)^n>0$，此时$\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n<0(x\neq x_0)$，从而$f(x)-f(x_0)<0$，即在$x_0$充分小某邻域内，恒有$f(x)<f(x_0)$，这表明$f(x_0)$是极大值.

7. 求下列函数在指定区间上的最大值和最小值：

(1) $y=|x^2-3x+2|,[-10,10]$

(2) $y=e^{|x-3|},[-5,5]$

解：

(1) $y=\begin{cases}(x-2)(x-1), & x\leqslant 1\\ -(x-2)(x-2), & 1<x\leqslant 2\\ (x-2)(x-1) & x>2\end{cases}$，

求导，得$y'=\begin{cases}2x-3, & x<1\\ \text{不存在}, & x=1\\ -2x+3, & -1<x<2\\ \text{不存在}, & x=2\\ 2x-3, & x>2\end{cases}$，则驻点$x=\frac{3}{2}$，导数不存在的点$x=1,x=2$

又$y(-10)=132,y(1)=0,y\left(\frac{3}{2}\right)=\frac{5}{4},y(2)=0,y(10)=72$，故函数的最大值是132，最小值是0.

(2) $y=\begin{cases}e^{x-3}, & x\geqslant 3\\ e^{3-x}, & x<3\end{cases}$，求导，得$y'=\begin{cases}e^{x-3}, & x>3\\ \text{不存在}, & x=3\\ -e^{3-x}, & x<3\end{cases}$，显然无驻点

又$y(-5)=e^8,y(3)=1,y(5)=e^2$，故函数的最大值为$e^8$，最小值为1.

8. 铁路上AB段的距离为100公里，工厂C与A相距40公里，AC垂直于AB.今要在AB中间一点D向工厂C修一条公路（图5-21），使从原料供应站B运货到工厂C所用运费最省.问D点应该设在何处？已知每一公里的铁路运费与公路运费之比是3:5.

解：设$|AD|=x$公里，则$|DB|=100-x$公里；每公里铁路运费为$3t$元，则每公里公路运费为$5t$元，总运费为yt元

则$yt=\sqrt{x^2+1600}(5t)+(100-x)(3t)$即$y=5\sqrt{x^2+1600}+3(100-x)$，于是$y'=\frac{5x-3\sqrt{x^2+1600}}{\sqrt{x^2+1600}},y''=\frac{8000}{(x^2+1600)^{\frac{3}{2}}}>0$，驻点为$x=30$，且$x=30$为极小点，故$D$点应设在距$A$30公里处.

9. 把一根圆木锯成矩形木条.问矩形的长和宽取多大时，截面积最大？

解：设圆木截面半径为R，矩形的长、宽分别为x,y，则$\sqrt{x^2+y^2}=2R$，于是$y=\sqrt{4R^2-x^2}$，从而$S=$

$xy=x\sqrt{4R^2-x^2}$

则$S'=\dfrac{4R^2-2x^2}{\sqrt{4R^2-x^2}},S''=\dfrac{2x^3-12R^2x}{(4R^2-x^2)^{\frac{3}{2}}}$，驻点为$x=\sqrt{2}R$，此时$S''<0$，则$x=\sqrt{2}R$为极大点，此时$x=y=\sqrt{2}R$，故矩形的长、宽均取$\sqrt{2}R$时，截面积最大.

10. 设$S=(x-a_1)^2+(x-a_2)^2+\cdots+(x-a_n)^2$.问$x$多大时，$S$最小？

解：$S'=2[nx-(a_1+a_2+\cdots+a_n)],S''=2n>0$，驻点为$x=\dfrac{a_1+a_2+\cdots+a_n}{n}$，且$x$为极小点，即当$x=\dfrac{a_1+a_2+\cdots+a_n}{n}$时，$S$最小.

11. 做一个圆柱形锅炉，已知其容积为V，两端面材料的每单位面积价格为a元，侧面材料的每单位价格为b元，问锅炉的直径和高的比等于多少时，造价最省？

解：设此圆柱形锅炉的直径为D，高为H，则$V=\dfrac{1}{4}\pi D^2H$，于是$H=\dfrac{4V}{\pi D^2}$

造价$G=2a\left(\dfrac{\pi}{4}D^2\right)+b\pi DH=\dfrac{\pi}{2}aD^2+b\dfrac{4V}{D}$，则$G'=\pi aD-\dfrac{4bV}{D^2}$，驻点为$D=\sqrt[3]{\dfrac{4bV}{a\pi}}$.当$D<\sqrt[3]{\dfrac{4bV}{a\pi}}$时，$G'<0$；当$D>\sqrt[3]{\dfrac{4bV}{a\pi}}$时，$G'>0$，则$D=\sqrt[3]{\dfrac{4bV}{a\pi}}$是唯一极小点，从而是最小点.

于是$\dfrac{D}{H}=\dfrac{D}{\dfrac{4V}{\pi D^2}}=\dfrac{\pi D^3}{4V}=\dfrac{b}{a}$即当锅炉的直径与高的比为$\dfrac{b}{a}$时，造价最省.

12. 用一块半径为R的圆形铁皮，剪去一块圆心角为α的圆扇形做成一个漏斗.问α为多大时，漏斗的容积最大？

解：由题设知，余下部分的圆心角为$x=2\pi-\alpha$，漏斗底周长为$Rx=R(2\pi-\alpha)$，底半径为$\dfrac{Rx}{2\pi}$，其高为$h=\sqrt{R^2-\left(\dfrac{Rx}{2\pi}\right)^2}=\dfrac{R}{2\pi}\sqrt{4\pi^2-x^2}(x>0)$，于是漏斗的容积为$V=\dfrac{1}{3}\pi\left(\dfrac{Rx}{2\pi}\right)^2\cdot\dfrac{R}{2\pi}\sqrt{4\pi^2-x^2}=\dfrac{R^3}{24\pi^2}x^2\sqrt{4\pi^2-x^2}(x>0)$

按题设，只需考虑当x为何值时，函数$f(x)=x^4(4\pi^2-x^2)$的值最大.

$f'(x)=16\pi^2x^3-6x^5,f''(x)=48\pi^2x^2-30x^4$，驻点为$x=2\pi\sqrt{\dfrac{2}{3}}$，且$f''\left(2\pi\sqrt{\dfrac{2}{3}}\right)<0$，故$x=2\pi\sqrt{\dfrac{2}{3}}$为极大点，因而剪去的圆心角应为$\alpha=2\pi\left(1-\sqrt{\dfrac{2}{3}}\right)$，所做漏斗的容积最大.

13. 底为a，高为h的三角形，试求其内接最大矩形的面积.

解：设其内接矩形的长、宽分别为b,c

则由已知，得$\dfrac{b}{a}=\dfrac{h-c}{h}$即$b=\dfrac{h-c}{h}a$，于是$S=bc=ac\dfrac{h-c}{h}=\dfrac{ahc-ac^2}{h}$，则$S'=\dfrac{ah-2ac}{h},S''=-\dfrac{2a}{h}<0$，驻点为$c=\dfrac{h}{2}$，于是$c=\dfrac{h}{2}$为极大点，此时$b=\dfrac{a}{2}$，从而最大面积为$S=bc=\dfrac{ah}{4}$.

14. 给定长为l的线段，试把它分为两段，使以这两段为边所围成的矩形的面积最大.

解：设此矩形的长为x，则宽为$l-x$

$S=x(l-x)=lx-x^2$，则$S'=l-2x,S''=-2<0$，驻点为$x=\dfrac{l}{2}$，且$x=\dfrac{l}{2}$为极大点，因此当$x=\dfrac{l}{2}$时，矩形面积最大，且$S=\dfrac{l^2}{4}$。

15. 设内接于椭圆$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$，而边平行于轴的最大矩形.

解：由已知设所求矩形与x正半轴交于$(x,0)$，则其与y正半轴交于$\left(0,\dfrac{b}{a}\sqrt{a^2-x^2}\right)$

此矩形的面积为S，则$\dfrac{1}{4}S=x\cdot\dfrac{b}{a}\sqrt{a^2-x^2}$，从而$S=\dfrac{4b}{a}\sqrt{a^2-x^2}$，

则$s'=\dfrac{4b}{a}\cdot\dfrac{a^2-2x^2}{\sqrt{a^2-x^2}},S''=\dfrac{4b}{a}\cdot\dfrac{2x^3-3a^2x}{(a^2-x^2)^{\frac{3}{2}}}$，驻点为$x=\dfrac{\sqrt{2}}{2}a$，此时$S''<0$，则$x=\dfrac{\sqrt{2}}{2}a$为$S$的极大值点，于是$x=\dfrac{\sqrt{2}}{2}a$时矩形面积最大，最大面积为$S=2ab$.

16. 求点$M(p,p)$到抛物线$y^2=2px$的最短距离.

解：点$M(p,p)$到抛物线$y^2=2px$上任意点(x,y)的距离为$d=\sqrt{(x-p)^2+(y-p)^2}=\sqrt{\left(\dfrac{y^2}{2p}-p\right)^2+(y-p)^2}=\sqrt{\dfrac{y^4}{4p^2}+2p^2-2py}$

设$f(y)=\dfrac{y^4}{4p^2}+2p^2-2py$，则$f'(y)=\dfrac{1}{p^2}(y^3-2p^3),f''(y)=\dfrac{3y^2}{p^2}>0$，驻点为$y=\sqrt[3]{2}p$，且它就是$f(y)$的极小值点，因此所求最短距离为$d=\sqrt{f(\sqrt[3]{2}p)}=|p|\sqrt{2+2^{-\frac{2}{3}}-2^{\frac{3}{4}}}$.

17. 甲船以$u=20$浬/小时的速度向东航行，正午时在其正北面$h=82$浬处有乙船以$v=16$浬/小时的速度向正南航行，问何时两船距离最近？
解:设x小时后两船距离最近，两船相距S浬，则$S=\sqrt{(82-16x)^2+(20x)^2}=\sqrt{656x^2-2624x+6724}$
令$f(x)=656x^2-2624x+6724$，求其最小值。则$f'(x)=1312x-2624,f''(x)=1312>0$，驻点为$x=2$且它为$f(x)$的极小值点，则2小时后两船距离最近，此时$S=10\sqrt{41}$.

18. 平地上放一重物，重量为P公斤.已知物体与地面的摩擦系数为μ。现加一力F，使物体开始移动.问此力与水平方向的夹角φ为多大时，用力最省？（图5-22）？
解：据题设，有$F\cos\varphi=\mu(PG-F\sin\varphi)$即$F=\dfrac{\mu PG}{\cos\varphi+\mu\sin\varphi}$
令$y=\cos\varphi+\mu\sin\varphi$，为使$F$最小，只要使$y$最大
由$y'=-\sin\varphi+\mu\cos\varphi,y''=-\cos\varphi-\mu\sin\varphi$，驻点为$\varphi=\arctan\mu$，此时$y''<0$，表明当$\varphi=\arctan\mu$时，$y$取最大值，从而$F$取最小值，即用力最省.

19. 如图5-23所示，有甲、乙两生产队合用一变压器，问变压器M应设在何处，所用输电线最省？
解：设M应设在与甲的垂直位置距离为x公里处，所用输电线l最省
由已知，得$l=\sqrt{1+x^2}+\sqrt{2.25^2+(3-x)^2}$，则$l'=\dfrac{x}{\sqrt{1+x^2}}+\dfrac{x-3}{\sqrt{2.25^2+(3-x)^2}},l''=\dfrac{1}{(1+x^2)^{\frac{3}{2}}}+\dfrac{2.25}{(2.25^2+(3-x)^2)^{\frac{3}{2}}}>0$，驻点为$x=1.2$，且为最小值点，即当$x=1.2$公里时，所用输电线最省.

20. 椭圆$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$的切线与两坐标轴分别交于$A,B$两点，
   (1) 求AB两点间的距离的最小值；
   (2) 求ΔOAB的最小面积.

解：设切点为(x,y)，则切线斜率为$k=-\dfrac{b^2x}{a^2y}$，于是切线方程为$Y-y=-\dfrac{b^2x}{a^2y}(X-x)$，
不失一般性，可设点$M(x.y)$在第一象限，切线在两坐标轴上的截距分别为$\dfrac{a^2}{x},\dfrac{b^2}{y}$，则

   (1) 所求AB两点间的距离为$d=\sqrt{\dfrac{a^4}{x^2}+\dfrac{b^4}{y^2}}=a\sqrt{\dfrac{a^2}{x^2}+\dfrac{b^2}{a^2-x^2}}$
   令$f(x)=\dfrac{a^2}{x^2}+\dfrac{b^2}{a^2-x^2}$，要求$d$的最小值，只需求$f(x)$的最小值.
   由$f'(x)=-\dfrac{2a^2}{x^3}+\dfrac{2b^2x}{(a^2-x^2)^2},f''(x)=\dfrac{6a^2}{x^4}+\dfrac{2a^2b^2+6b^2x^2}{(a^2-x^2)^3}>0$，且由于$x\in[0,a],x^2\leqslant a^2$，则驻点满足$x^2=\dfrac{a^3}{a+b}$且此时$f(x)$取最小值，即$d$取最小值，最短距离为$d=a\sqrt{f(x)}=a+b$.

   (2) 按题设，有$S=\dfrac{1}{2}\cdot\dfrac{a^2}{x}\cdot\dfrac{ab}{\sqrt{a^2-x^2}}=\dfrac{a^3b}{2x\sqrt{a^2-x^2}}$，考虑函数$g(x)=x^2(a^2-x^2)$
   要求S的最小值，只要求$g(x)$的最大值
   由$g'(x)=2a^2x-4x^3,g''(x)=2a^2-12x^2$，驻点为$x=\dfrac{a}{\sqrt{2}}$且此时$g''(x)<0$，即当$x=\dfrac{a}{\sqrt{2}}$时$g(x)$取最大值，从而$S$取最小值，最小面积为$S=ab$.

21. 讨论函数$x^\alpha(\alpha>1$及$0<\alpha<1),e^x,\ln x,x\ln x$在$(0,+\infty)$内的凸性.
解：$f(x)=x^\alpha,f'(x)=\alpha x^{\alpha-1},f''(x)=\alpha(\alpha-1)x^{\alpha-2}$
当$\alpha>1$时，$f''(x)>0$，则x^α在$(0,+\infty)$内下凸；当$0<\alpha<1$时，$f''(x)<0$，则x^α在$(0,+\infty)$内上凸.
$f(x)=e^x,f'(x)=e^x,f''(x)=e^x>0(x>0)$，则$e^x$在$(0,+\infty)$内下凸
$f(x)=\ln x,f'(x)=\dfrac{1}{x},f''(x)=-\dfrac{1}{x^2}<0(x>0)$，则$\ln x$在$(0,+\infty)$内上凸
$f(x)=x\ln x,f'(x)=1+\ln x,f''(x)=\dfrac{1}{x}>0(x>0)$，则$x\ln x$在$(0,+\infty)$内下凸

22. 讨论下列函数的凸性和拐点：
   (1) $y=3x^2-x^3$
   (2) $y=\dfrac{a^2}{a^2+x^2}(a>0)$

(3) $y = x + \sin x$

(4) $y = \sqrt{1 + x^2}$

解:

(1) $y' = 6x - 3x^2, y'' = 6 - 6x$，$y'' = 0$的根为$x = 1$，列表如下：

x	$(-\infty, 1)$	$(1, +\infty)$
y''符号	+	-
y	下凸	上凸

，拐点坐标为$(1, 2)$

(2) $y' = -\dfrac{2ax}{(a^2+x^2)^2}, y'' = \dfrac{2a^2(3x^2 - a^2)}{(a^2+x^2)^3}$，$y'' = 0$的根为$x = \pm\dfrac{\sqrt{3}}{3}a$，列表如下：

x	$\left(-\infty, -\dfrac{\sqrt{3}}{3}a\right)$	$\left(-\dfrac{\sqrt{3}}{3}a, \dfrac{\sqrt{3}}{3}a\right)$	$\left(\dfrac{\sqrt{3}}{3}a, +\infty\right)$
y''符号	+	-	+
y	下凸	上凸	下凸

，拐点坐标为$\left(-\dfrac{\sqrt{3}}{3}a, \dfrac{3}{4}\right), \left(\dfrac{\sqrt{3}}{3}a, \dfrac{3}{4}\right)$

(3) $y' = 1 + \cos x, y'' = -\sin x$，$y'' = 0$的根为$x = k\pi(k \in Z)$，列表如下：

x	$((2k-1)\pi, 2k\pi)$	$(2k\pi, (2k+1)\pi)$	$((2k+1)\pi, 2(k+1)\pi)$
y''符号	+	-	+
y	下凸	上凸	下凸

，拐点坐标为$(k\pi, k\pi)$

(4) $y' = \dfrac{x}{\sqrt{1+x^2}}, y'' = \dfrac{1}{(1+x^2)^{\frac{3}{2}}}$，则$y'' > 0$，故函数是下凸的，从而无拐点.

23. 证明曲线$y = \dfrac{x+1}{x^2+1}$有位于同一直线上的三个拐点.

证明: $y' = \dfrac{1 - 2x - x^2}{(x^2+1)^2}, y'' = \dfrac{2(x-1)(x+2-\sqrt{3})(x+2+\sqrt{3})}{(x^2+1)^3}$

令$y'' = 0$，得$x_1 = 1, x_2 = -2 + \sqrt{3}, x_3 = -2 - \sqrt{3}$

当$x < -2 - \sqrt{3}$时，$y'' < 0$；当$-2 - \sqrt{3} < x < -2 + \sqrt{3}$时，$y'' > 0$；当$-2 + \sqrt{3} < x < -1$时，$y'' < 0$；当$x > -1$时，$y'' > 0$

于是曲线在x_1, x_2, x_3处有三个拐点$A(1,1), B\left(-2 + \sqrt{3}, \dfrac{\sqrt{3}+1}{4}\right), C\left(-(2+\sqrt{3}), \dfrac{1-\sqrt{3}}{4}\right)$

过A, B的直线方成为$y = \dfrac{1}{4}x + \dfrac{3}{4}$，将$C$点坐标代入上述方程，得$\dfrac{1-\sqrt{3}}{4} = \dfrac{-2-\sqrt{3}}{4} + \dfrac{3}{4} = \dfrac{1-\sqrt{3}}{4}$即$C$满足此方程，则曲线$y = \dfrac{x+1}{x^2+1}$有位于同一直线上的三个拐点.

24. 若$f(x)$是下凸函数（或严格下凸函数），$f'(x_0)$存在，则

$$\left.\begin{array}{l} f(x) \geqslant f(x_0) + f'(x_0)(x - x_0) \\ f(x) > f(x_0) + f'(x_0)(x - x_0) \end{array}\right\} (x \neq x_0).$$

证明: 设x为$f(x)$定义域内任一点，$x \neq x_0$（不妨设$x > x_0$）

令$x_1 = \dfrac{x + x_0}{2}$，由$f(x)$为下凸函数，则$\dfrac{f(x) - f(x_0)}{x - x_0} \geqslant \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$；$x_2 = \dfrac{x_1 + x_0}{2}$，由$f(x)$为下凸函数，则$\dfrac{f(x_1) - f(x_0)}{x - x_0} \geqslant \dfrac{f(x_2) - f(x_0)}{x_2 - x_0}$；$x_3 = \dfrac{x_2 + x_0}{2}$，由$f(x)$为下凸函数，则$\dfrac{f(x_2) - f(x_0)}{x - x_0} \geqslant \dfrac{f(x_3) - f(x_0)}{x_3 - x_0}$

如此进行下去，可得数列$\{x_n\}$，$|x_n - x_0| = \dfrac{|x - x_0|}{2^n} \to 0(n \to \infty)$，则$x_n \to x_0(n \to \infty)$，且$\dfrac{f(x_n) - f(x_0)}{x_n - x_0} \geqslant \dfrac{f(x_{n+1}) - f(x_0)}{x_{n+1} - x_0}$

又$f'(x_0)$存在，则$\lim\limits_{n\to\infty} \dfrac{f(x_n) - f(x_0)}{x_n - x_0} = \lim\limits_{x_n \to x_0} \dfrac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0)$

又$\dfrac{f(x) - f(x_0)}{x - x_0} \geqslant \dfrac{f(x_n) - f(x_0)}{x_n - x_0}$，则由极限性质，得$\dfrac{f(x) - f(x_0)}{x - x_0} \geqslant f'(x_0)$，从而$f(x) \geqslant f(x_0) + f'(x_0)(x - x_0)$

同理可证，若$f(x)$是严格下凸函数，则$f(x) > f(x_0) + f'(x_0)(x - x_0)$.

25. 若$f(x)$是下凸函数，则$-f(x)$是上凸函数.

证明: 因$f(x)$是下凸函数，则$f(x)$在$[a, b]$上连续，对$[a, b]$中任意两点x_1, x_2，恒有$f\left(\dfrac{x_1 + x_2}{2}\right) \leqslant \dfrac{f(x_1) + f(x_2)}{2}$，于是$-f\left(\dfrac{x_1 + x_2}{2}\right) \geqslant -\dfrac{f(x_1) + f(x_2)}{2} = \dfrac{(-f(x_1)) + (-f(x_2))}{2}$，从而$-f(x)$是上凸函数.

26. (1) 若$f_n(x)$是下凸函数，问$F(x) = \min\limits_{n}\{f_n(x)\}$是不是下凸函数？

(2) 若$f(x),g(x)$是下凸函数，问$f(x)+g(x)$是不是下凸函数？

(3) 说明三次函数不是下凸函数.

解:

(1) 不一定.
例：

当$f_1(x)=\dfrac{1}{x},f_2(x)=x^2(x>0)$时，$f_1(x),f_2(x)$都是下凸函数，但$F(x)=\min\left\{\dfrac{1}{x},x^2\right\}$在$(1,1)$点不满足下凸函数定义，即$F(x)$不是下凸函数.
当$f_1(x)=x^2,f_2(x)=\dfrac{x^2}{2}$时，$f_1(x),f_2(x)$都是下凸函数，且$F(x)=\min\left\{x^2,\dfrac{x^2}{2}\right\}=\dfrac{x^2}{2}$是下凸函数.

(2) $f(x)+g(x)$是下凸函数.
因$f(x),g(x)$是下凸函数，则$f\left(\dfrac{x_1+x_2}{2}\right)\leqslant\dfrac{f(x_1)+f(x_2)}{2}$,
$g\left(\dfrac{x_1+x_2}{2}\right)\leqslant\dfrac{g(x_1)+g(x_2)}{2}$，于是$(f+g)\left(\dfrac{x_1+x_2}{2}\right)=f\left(\dfrac{x_1+x_2}{2}\right)+g\left(\dfrac{x_1+x_2}{2}\right)\leqslant\dfrac{f(x_1)+f(x_2)}{2}+\dfrac{g(x_1)+g(x_2)}{2}=\dfrac{1}{2}[(f+g)(x_1)+(f+g)(x_2)]$即$f(x)+g(x)$是下凸函数.

(3) 设$f(x)=ax^3+bx^2+cx+d(a\neq 0)$，则$f'(x)=3ax^2+2bx+c,f''(x)=6ax+2b$
于是，

$a>0$时，当$x>-\dfrac{b}{3a}$时，$f''(x)>0$，$f(x)$是下凸函数；当$x<-\dfrac{b}{3a}$时，$f''(x)<0$，$f(x)$是上凸函数
$a<0$时，当$x>-\dfrac{b}{3a}$时，$f''(x)<0$，$f(x)$是上凸函数；当$x<-\dfrac{b}{3a}$时，$f''(x)>0$，$f(x)$是下凸函数

则$f(x)$不是下凸函数，在$x=-\dfrac{b}{3a}$处有拐点.

27. 如何选择参数$h>0$，方能使曲线$y=\dfrac{h}{\sqrt{\pi}}e^{-h^2x^2}$在$x=\pm\sigma(\sigma>0,\sigma$为已给定的常数)处有拐点.

解: $y'=-\dfrac{2h^3}{\sqrt{\pi}}xe^{-h^2x^2},y''=\dfrac{2h^3}{\sqrt{\pi}}e^{-h^2x^2}(2h^2x^2-1)$
令$y''=0$，则$x=\pm\dfrac{1}{\sqrt{2}h}$
当$x<-\dfrac{1}{\sqrt{2}h}$时，$y''>0$，曲线下凸；当$-\dfrac{1}{\sqrt{2}h}<x<\dfrac{1}{\sqrt{2}h}$时，$y''<0$，曲线上凸；当$x>\dfrac{1}{\sqrt{2}h}$时，$y''>0$，曲线下凸
则在$x=\pm\dfrac{1}{\sqrt{2}h}$处有两个拐点，于是$\pm\dfrac{1}{\sqrt{2}h}=\pm\sigma$，又$h,\sigma>0$，则$h=\dfrac{1}{\sqrt{2}\sigma}$.

28. 求$y=\dfrac{x^2}{x^2+1}$的极值及拐点，并求拐点处的切线方程.

解: $y'=\dfrac{2x}{(1+x^2)^2},y''=\dfrac{2-6x^2}{(1+x^2)^3}$

驻点为$x=0$，列出下表：

x	$(-\infty,0)$	0	$(0,+\infty)$
y'符号	-	0	+
y	↘	极小值0	↗

则当$x=0$，y有极小值$y=0$.
令$y''=0$，则$x=\pm\dfrac{\sqrt{3}}{3}$，列出下表：

x	$\left(-\infty,-\dfrac{\sqrt{3}}{3}\right)$	$\left(-\dfrac{\sqrt{3}}{3},\dfrac{\sqrt{3}}{3}\right)$	$\left(\dfrac{\sqrt{3}}{3},+\infty\right)$
y''符号	-	+	-
y	上凸	下凸	上凸

故拐点为$\left(-\dfrac{\sqrt{3}}{3},\dfrac{1}{4}\right),\left(\dfrac{\sqrt{3}}{3},\dfrac{1}{4}\right)$.

在拐点$\left(-\dfrac{\sqrt{3}}{3},\dfrac{1}{4}\right)$处的切线方程为$y-\dfrac{1}{4}=\dfrac{-\dfrac{2\sqrt{3}}{3}}{\left(\dfrac{1}{3}+1\right)^2}\left(x+\dfrac{\sqrt{3}}{3}\right)$

即$3\sqrt{3}x+8y+1=0$;

在拐点$\left(\dfrac{\sqrt{3}}{3},\dfrac{1}{4}\right)$处的切线方程为$y-\dfrac{1}{4}=\dfrac{\dfrac{2\sqrt{3}}{3}}{\left(\dfrac{1}{3}+1\right)^2}\left(x-\dfrac{\sqrt{3}}{3}\right)$

即$3\sqrt{3}x-8y-1=0.$

29. 作出下列函数的图形：

(1) $y=x^3-6x$

(2) $y=\dfrac{3x}{1+x^2}$

(3) $y=5e^{-x^2}$

(4) $y=\dfrac{e^x+e^{-x}}{2}$

(5) $y=\dfrac{1}{x^2-1}$

(6) $y=\ln\dfrac{1+x}{1-x}$

(7) $y=(x-1)^2(x+2)^3$

(8) $y=\dfrac{(x-1)^3}{(x+1)^3}$

(9) $y=\dfrac{x^2-2x-3}{x^2+1}$

(10) $y=x+\arctan x$

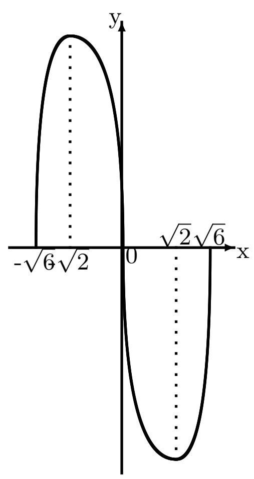
解:

(1) (i) 定义域$(-\infty,+\infty)$，是奇函数，曲线关于原点对称.

(ii) $y'=3x^2-6, y''=6x$，当$x=\pm\sqrt{2}$时，$y'=0$；当$x=0$时，$y''=0$.

(iii) 列表讨论如下：

x	$(-\infty,-\sqrt{2})$	$-\sqrt{2}$	$(-\sqrt{2},0)$	0	$(0,\sqrt{2})$	$\sqrt{2}$	$(\sqrt{2},+\infty)$
y'	+	0	-	-	-	0	+
y''	-	-	-	0	+	+	+
y	上凸↗	极大值$4\sqrt{2}$	上凸↘	0	下凸↘	极小值$-4\sqrt{2}$	下凸↗



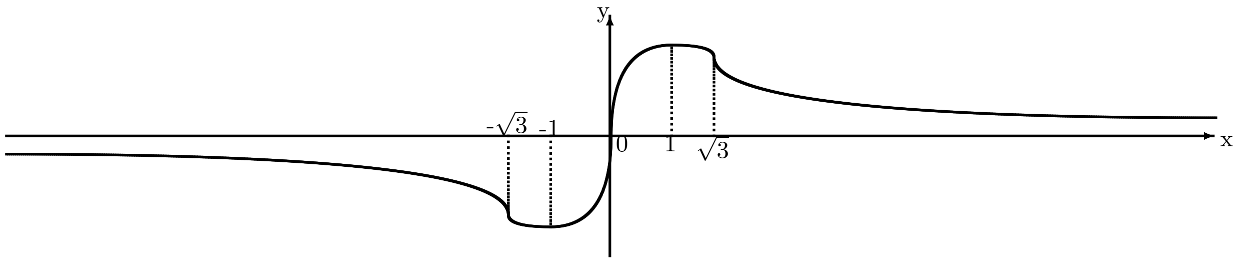
(2) (i) 定义域$(-\infty,+\infty)$，是奇函数，曲线关于原点对称.

(ii) $y'=\dfrac{3(1-x^2)}{(1+x^2)^2}, y''=\dfrac{6x(x^2-3)}{(1+x^2)^3}$，当$x=\pm1$时，$y'=0$；当$x=0, x=\pm\sqrt{3}$时，$y''=0$.

(iii) 列表讨论如下：

x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},+\infty)$
y'	-	-	-	0	+	+	+	0	-	-	-
y''	-	0	+	+	+	0	-	-	-	0	+
y	上凸↘	$-\dfrac{3}{4}\sqrt{3}$	下凸↘	极小值 $-\dfrac{3}{2}$	下凸↗	0	上凸↗	极大值 $\dfrac{3}{2}$	上凸↘	$\dfrac{3}{4}\sqrt{3}$	下凸↘

(iv) 当$x \to \infty$时，$y \to 0$，故$y = 0$是曲线的一条水平渐近线.



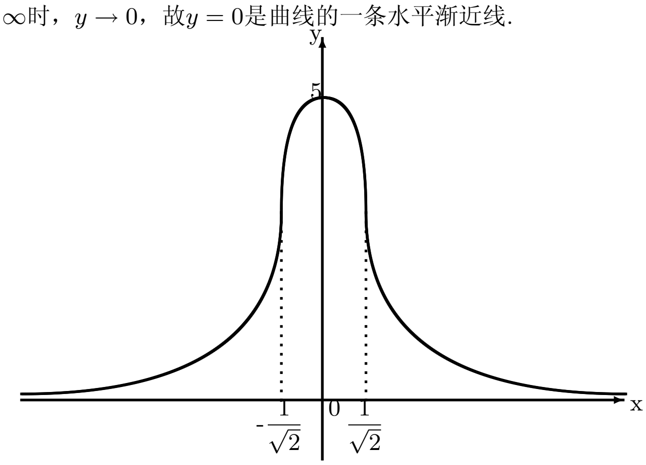
(3) (i) 定义域$(-\infty, +\infty)$，是偶函数，曲线关于y轴对称.

(ii) $y' = -10xe^{-x^2}, y'' = 10e^{-x^2}(2x^2-1)$，当$x = 0$时，$y' = 0$；当$x = \pm\dfrac{1}{\sqrt{2}}$时，$y'' = 0$.

(iii) 列表讨论如下：

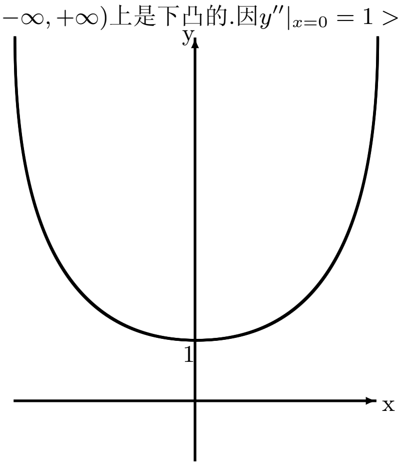
x	$(-\infty, -\dfrac{1}{\sqrt{2}})$	$-DF1\sqrt{2}$	$(-\dfrac{1}{\sqrt{2}}, 0)$	0	$(0, \dfrac{1}{sqrt2})$	$\dfrac{1}{\sqrt{2}}$	$(\dfrac{1}{\sqrt{2}}, +\infty)$
y'	+	+	+	0	-	-	-
y''	+	0	-	-	-	0	+
y	下凸↗	$\dfrac{5}{\sqrt{e}}$	上凸↗	极大值 5	上凸↘	$\dfrac{5}{\sqrt{e}}$	下凸↘

(iv) 当$x \to \infty$时，$y \to 0$，故$y = 0$是曲线的一条水平渐近线.



(4) (i) 定义域$(-\infty, +\infty)$，是偶函数，曲线关于y轴对称（这是双曲余弦函数$\cosh x = \dfrac{e^x + e^{-x}}{2}$）.

(ii) $y' = \sinh x, y'' = \cosh x$，当$x = 0$时，$y' = 0$；
由于$y'' > 0(x \in (-\infty, +\infty)$，故$y$在$(-\infty, +\infty)$上是下凸的.因$y''|_{x=0} = 1 > 0$，故$y_{\min} = 1$.



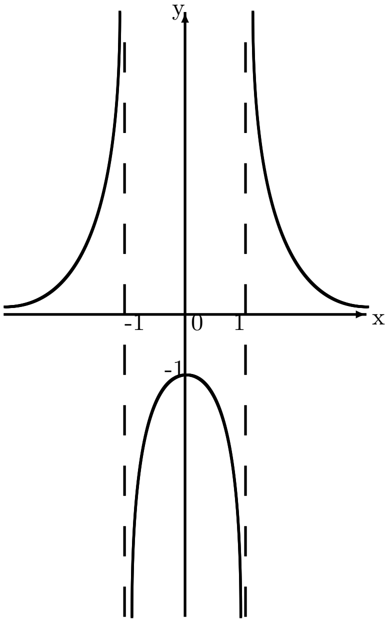
(5) (i) 定义域$(-\infty, -1)\bigcup(-1, 1)\bigcup(1, +\infty)$，是偶函数，曲线关于$y$轴对称.

(ii) $y' = -\dfrac{2x}{(x^2-1)^2}, y'' = \dfrac{2(x^2+1)}{(x^2-1)^3}$，当$x = 0$时，$y' = 0$；当$x = \pm 1$时，$y'$不存在；当$x = \pm 1$时，$y''$不存在.

(iii) 列表讨论如下：

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,+\infty)$
y'	$+$	不存在	$+$	0	-	不存在	-
y''	$+$	不存在	-	-	-	不存在	$+$
y	下凸↗	无定义	上凸↗	极大值 -1	上凸↘	无定义	下凸↘

(iv) 当$x\to\infty$时，$y\to 0$，故$y=0$是曲线的一条水平渐近线；当$x\to\pm1$时，$y\to\infty$，故$x=\pm1$是曲线的一条垂直渐近线.



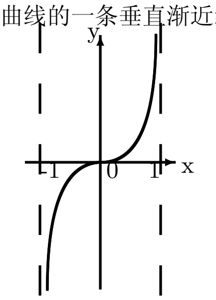
(6) (i) 定义域$(-1,1)$，是奇函数，曲线关于原点对称.

(ii) $y'=\dfrac{2}{1-x^2},y''=\dfrac{4x}{(1-x^2)^2}$，$y'=0$无解；当$x=0$时，$y''=0$.

(iii) 列表讨论如下：

x	$(-1,0)$	0	$(0,1)$
y'	$+$	$+$	$+$
y''	-	0	$+$
y	上凸↗	0	下凸↗

(iv) 当$x\to1^-$时，$y\to+\infty$，故$x=1$是曲线的一条垂直渐近线；
当$x\to-1^+$时，$y\to-\infty$，故$x=-1$是曲线的一条垂直渐近线.



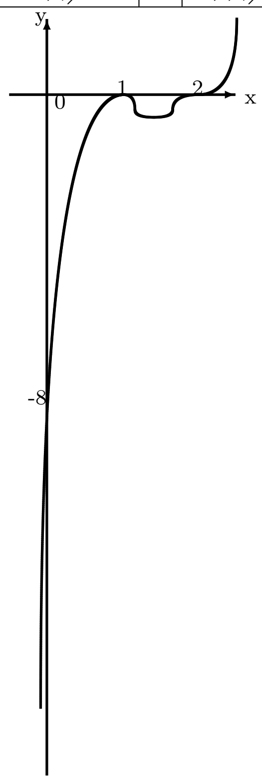
(7) (i) 定义域$(-\infty,+\infty)$.

(ii) $y'=(x-1)(x-2)^2(5x-7),y''=2(x-2)(10x^2-28x+19)$，当$x=1,x=2,x=\dfrac{7}{5}=1.4$时，$y'=0$；当$x=2,x=\dfrac{14\pm\sqrt{6}}{10}$时，$y''=0$.

(iii) 列表讨论如下：

x	$(-\infty,1)$	1	$\left(1,-\dfrac{14-\sqrt{6}}{10}\right)$	$\dfrac{14-\sqrt{6}}{10}$	$\left(-\dfrac{14-\sqrt{6}}{,}1.4\right)$	1.4
y'	$+$	0	-	-	-	0
y''	-	-	-	0	$+$	$+$
y	上凸↗	极大值 0	上凸↘	-0.0154	下凸↘	极小值 -0.0346

x	$\left(1.4,\dfrac{14+\sqrt{6}}{10}\right)$	$\dfrac{14+\sqrt{6}}{10}$	$\left(\dfrac{14+\sqrt{6}}{,}2\right)$	2	$(2,+\infty)$
y'	$+$	$+$	$+$	0	$+$
y''	$+$	0	-	0	$+$
y	下凸↗	-0.0186	上凸↗	0	下凸↗



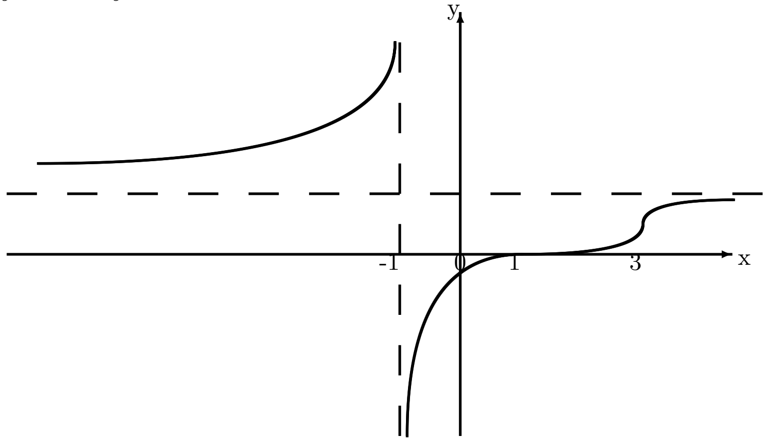
(8) (i) 定义域$(-\infty,-1)\bigcup(-1,+\infty)$.

(ii) $y'=\dfrac{6(x-1)^2}{(x+1)^4}, y''=-\dfrac{12(x-1)(x-3)}{(x+1)^5}$，当$x=1$时，$y'=0$；当$x=1,x=3$时，$y''=0$；当$x=-1$时，$y',y''$均不存在.

(iii) 列表讨论如下：

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,3)$	3	$(3,+\infty)$
y'	$+$	不存在	$+$	0	$+$	$+$	$+$
y''	-	不存在	-	0	$+$	0	-
y	上凸↗	无定义	上凸↗	0	下凸↗	$\frac{1}{8}$	上凸↗

(iv) 当$x\to-1^-$时，$y\to+\infty$，故$x=-1$是曲线的一条垂直渐近线；当$x\to\infty$时，$y\to1$，故$y=1$是曲线的一条水平渐近线.



(9) (i) 定义域$(-\infty,+\infty)$.

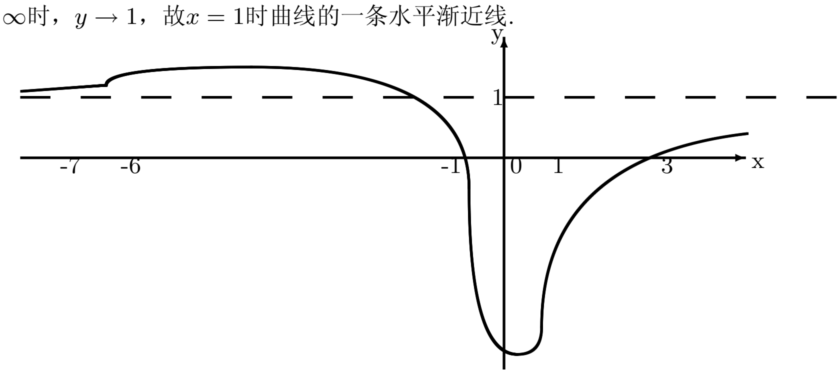
(ii) $y' = \dfrac{2(x^2+4x-1)}{(x^2+1)^2}, y'' = -\dfrac{4(x^3+6x^2-3x-2)}{(x^2+1)^3}$，当$x=-2\pm\sqrt{5}$时，$y'=0$；$y''=0$的根为$x_1,x_2,x_3$，其中$x_1\in(-7,-6),x_2\in(-1,0),x_3\in\left(\dfrac{1}{2},1\right)$.

(iii) 列表讨论如下：

x	$(-\infty,x_1)$	x_1	$(x_1,-2-\sqrt{5})$	$-2-\sqrt{5}$	$(-2-\sqrt{5},x_2)$	x_2
y'	+	+	+	0	-	-
y''	+	0	-	-	-	0
y	下凸↗	拐点	上凸↗	极大值 $\sqrt{5}-1$	上凸↘	拐点

x	$(x_2,-2+\sqrt{5})$	$-2+\sqrt{5}$	$(-2+\sqrt{5},x_3)$	x_3	$(x_3,+\infty)$
y'	-	0	+	+	+
y''	+	+	+	0	-
y	下凸↘	极小值 $-\sqrt{5}-1$	下凸↗	拐点	上凸↗

(iv) 当$x\to\infty$时，$y\to 1$，故$x=1$时曲线的一条水平渐近线.

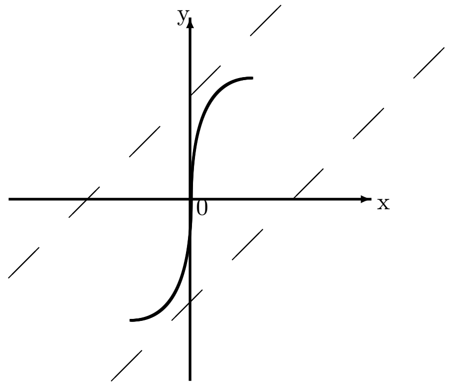


(10) (i) 定义域$(-\infty,+\infty)$，是奇函数，曲线关于原点对称且当$x=0$时，$y=0$.

(ii) $y'=1+\dfrac{1}{1+x^2}>0$，故曲线单调上升，无极值点.

$y''=-\dfrac{2x}{(1+x^2)^2}$，当$x=0$时，$y''=0$且当$x>0$时，$y''<0$；当$x<0$时，$y''>0$，则$(0,0)$为拐点.

(iii) $k=\lim\limits_{x\to\infty}\dfrac{y}{x}=1, b_1=\lim\limits_{x\to-\infty}(y-kx)=-\dfrac{\pi}{2}, b_2=\lim\limits_{x\to+\infty}(y-kx)=\dfrac{\pi}{2}$，故曲线有两条斜渐近线：$y=x+\dfrac{\pi}{2}, y=x-\dfrac{\pi}{2}$.



30. 试作下列函数的图形：$y=\begin{cases}\dfrac{9x+x^4}{x-x^3}, & x\neq 0\\ 9, & x=0\end{cases}$

解:

(1) 定义域$(-\infty,-1)\bigcup(-1,1)\bigcup(1,+\infty)$.

(2) $y'=\begin{cases}\dfrac{-x^4+3x^2+18x}{(1-x^2)^2}, & x\neq 0\\ 0, & x=0\end{cases}, y''=\begin{cases}-\dfrac{2(x^3+27x^2+3x+9}{(x^2-1)^2}, & x\neq 0\\ 18, & x=0\end{cases}$

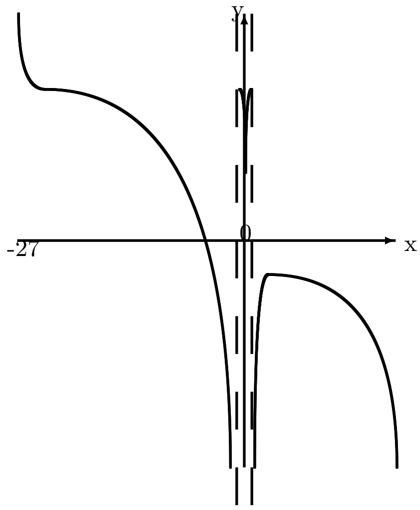
，当$x=0,x=3$时，$y'=0$；$y''=0$的根为x_1，其中$x_1\in(-27,-26)$；当$x=\pm1$时，y',y''均不存在.

(3) 列表讨论如下：

x	$(-\infty,x_1)$	x_1	$(x_1,-1)$	-1	$(-1,0)$	0
y'	-	-	-	不存在	-	0
y''	$+$	0	-	无定义	-	-
y	下凸↘	拐点	上凸↘	无定义	上凸↘	极小值 9

x	$(0,1)$	1	$(1,3)$	3	$(3,+\infty)$
y'	$+$	不存在	$+$	0	-
y''	-	不存在	-	-	-
y	上凸↗	无定义	上凸↗	极大值 $-\frac{9}{2}$	上凸↘

(4) 当$x\to\pm1$时，$y\to\infty$，故$x=\pm1$是曲线的垂直渐近线.



§4. 平面曲线的曲率

1. 求曲线$y=4x-x^2$的曲率以及在点$(2,4)$的曲率半径.

解：因$y=4x-x^2$，故$y'=4-2x, y''=-2$，则曲率$K=\left|\dfrac{y''}{(1+y'^2)^{\frac{3}{2}}}\right|=\dfrac{2}{[1+4(2-x)^2]^{\frac{3}{2}}}$，于是曲率半径$\rho=\dfrac{1}{K}=\dfrac{1}{2}[1+4(x-2)^2]^{\frac{3}{2}}$，从而在点$(2,4)$的曲率半径$\rho=\dfrac{1}{2}$.

2. 求下列曲线的曲率与曲率半径：

(1) 悬链线$y=a\cosh\dfrac{x}{a}(a>0)$

(2) 抛物线$y^2=2px(p>0)$

(3) 旋轮线$x=a(t-\sin t), y=a(1-\cos t)(a>0)$

(4) 心脏线$\rho=a(1+\cos\theta)(a>0)$

(5) 双纽线$\rho^2=2a^2\cos 2\theta(a>0)$

(6) 对数螺线$\rho=ae^{\lambda\theta}(\lambda>0)$

解：

(1) 因$y=a\cosh\dfrac{x}{a}$，故$y'=\sinh\dfrac{x}{a}, y''=\dfrac{1}{a}\cosh\dfrac{x}{a}$，则曲率$K=\left|\dfrac{y''}{(1+y'^2)^{\frac{3}{2}}}\right|=\dfrac{1}{a\cosh^2\dfrac{x}{a}}$，于是曲率半径$\rho=\dfrac{1}{K}a\cosh^2\dfrac{x}{a}$.

(2) 因$y^2=2px$，则$2yy'=2p$即$y'=\dfrac{p}{y}$，故$y''=-\dfrac{p}{y^2}y'=-\dfrac{p^2}{y^3}$，则曲率$K=\left|\dfrac{y''}{(1+y'^2)^{\frac{3}{2}}}\right|=\dfrac{p^2}{(y^2+p^2)^{\frac{3}{2}}}=\dfrac{p^2}{(2px+p^2)^{\frac{3}{2}}}=\dfrac{1}{p\left(1+\dfrac{2x}{p}\right)^{\frac{3}{2}}}$，于是曲率半径$\rho=\dfrac{1}{K}=p\left(1+\dfrac{2x}{p}\right)^{\frac{3}{2}}\left(\text{或}\dfrac{(y^2+p^2)^{\frac{3}{2}}}{p^2}\right)$.

(3) 因$x=a(t-\sin t), y=a(1-\cos t)$，故$x'=a(1-\cos t), x''=a\sin t; y'=a\sin t y''=a\cos t$，则曲率$K=\left|\dfrac{x'y''-x''y'}{(x'^2+y'^2)^{\frac{3}{2}}}\right|=\dfrac{1}{4a\left|\sin\dfrac{t}{2}\right|}$，于是曲率半径$\rho=\dfrac{1}{K}=4a\left|\sin\dfrac{t}{2}\right|(=2\sqrt{2ay})$.

(4) 因$\rho=a(1+\cos\theta)$，故$\rho'=-a\sin\theta, \rho''=-a\cos\theta$，则曲率$K=\left|\dfrac{\rho^2+2\rho'^2-\rho\rho''}{(\rho^2+\rho'^2)^{\frac{3}{2}}}\right|=\dfrac{3}{2\sqrt{2a\rho}}$，于是曲率半径$R=\dfrac{2\sqrt{2a\rho}}{3}$.

(5) 因$\rho^2=2a^2\cos 2\theta$，则$2\rho\rho'=-4a^2\sin 2\theta$，故$\rho'=\dfrac{-2a^2\sin 2\theta}{\rho}, \rho''=-\dfrac{4a^4+\rho^4}{\rho^3}$，则曲率$K=\left|\dfrac{\rho^2+2\rho'^2-\rho\rho''}{(\rho^2+\rho'^2)^{\frac{3}{2}}}\right|=\dfrac{3\rho}{2a^2}$，于是曲率半径$R=\dfrac{2a^2}{3\rho}$.

(6) 因$\rho=ae^{\lambda\theta}$，故$\rho'=\lambda ae^{\lambda\theta}=\lambda\rho, \rho''=a\lambda^2e^{\lambda\theta}=\lambda^2\rho$，则曲率$K=\left|\dfrac{\rho^2+2\rho'^2-\rho\rho''}{(\rho^2+\rho'^2)^{\frac{3}{2}}}\right|=\dfrac{1}{|\rho|(1+\lambda^2)^{\frac{1}{2}}}$，于是曲率半径$R=|\rho|\sqrt{1+\lambda^2}$.

3. 求曲线$y=2(x-1)^2$的最小曲率半径.

解：因$y=2(x-1)^2$，故$y'=4(x-1), y''=4$，则曲率半径$R=\dfrac{1}{K}=\left|\dfrac{(1+y'^2)^{\frac{3}{2}}}{y''}\right|=\dfrac{[1+16(x-1)^2]^{\frac{3}{2}}}{4}$

要使R最小，则必有$[1+16(x-1)^2]^{\frac{3}{2}}$最小，即当$x=1$时，$R_{\min}=\dfrac{1}{4}$.

4. 一飞机沿抛物线路径$y=\dfrac{x^2}{4000}$（单位为米）作俯冲飞行，在坐标原点O的速度$v=140$米/秒，飞行员体重$G=70$公斤.求此时座椅对飞行员的反力.

解：由物理学知识知，作匀速圆周运动的物体所受的向心力为$F=\dfrac{mv^2}{R}$，其中m为物体的质量，v为它的速

度，R为圆的半径.

所求座椅对飞行员的反力大小应为$F = Gg + \dfrac{mv^2}{R}$，其方向应指向圆心.

据题意，先求曲率半径，$y' = \dfrac{x}{2000}, y'' = \dfrac{1}{2000}$，则曲率半径$R = \dfrac{1}{K} = \left|\dfrac{(2000^2 + x^2)^{\frac{3}{2}}}{2000^2}\right|$，于是在坐标原点$O$的$R = 2000$（米），又在坐标原点$O$的速度$v = 140$米/秒，从而$F = 1372(N)$.

5. 一起车重量是P，以等速v驶过拱桥（图5-32），桥面ACB是一抛物线，其尺寸如图示.求汽车过C点时对桥面的压力.

解：以O为原点，AB为x轴，CO为y轴建立坐标系，则抛物线方程$y = -\dfrac{4\delta}{l^2}x^2 + \delta$

由物理学知道，汽车过C点时对桥面的压力为$F = \dfrac{mv^2}{R}\cos\theta + mg$

据题意，先求曲率半径，$y' = -\dfrac{8\delta}{l^2}x, y'' = -\dfrac{8\delta}{l^2}$，则曲率半径$R = \dfrac{1}{K} = \left|\dfrac{(l^2 + 8\delta x)^{\frac{3}{2}}}{8l\delta}\right|$，于是在点$C$的$R = \dfrac{l^2}{8\delta}$，又在点$C$的$\theta = \pi$，从而$F = Pg + \dfrac{Pv^2}{R}\cos\theta = \dfrac{gl^2 - 8\delta v^2}{l^2}P$.

§5. 待定型

1. 利用洛必达法则求下列极限：

(1) $\lim\limits_{x\to 0}\dfrac{\tan ax}{\sin bx}$

(2) $\lim\limits_{x\to 0}\dfrac{1-\cos x^2}{x^3\sin x}$

(3) $\lim\limits_{x\to\infty}\dfrac{\dfrac{\pi}{2}-\arctan x}{\sin\dfrac{1}{x}}$

(4) $\lim\limits_{x\to\infty}\dfrac{x^b}{e^{ax}}$

(5) $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$

(6) $\lim\limits_{x\to\pi}(\pi-x)\tan\dfrac{x}{2}$

(7) $\lim\limits_{x\to 0}\dfrac{\ln(\cos ax)}{\ln(\cos bx)}$

(8) $\lim\limits_{x\to 0}\dfrac{\cos(\sin x)-\cos x}{x^4}$

(9) $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{x}$

(10) $\lim\limits_{x\to 1}\dfrac{x-1}{\ln x}$

(11) $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x-a}(a>0)$

(12) $\lim\limits_{x\to\frac{\pi}{6}}\dfrac{1-2\sin x}{\cos 3x}$

(13) $\lim\limits_{x\to 0}\dfrac{\ln x}{\cot x}$

(14) $\lim\limits_{x\to+\infty}\dfrac{\ln^c x}{x^b}(b,c>0)$

(15) $\lim\limits_{x\to 0}\dfrac{(1+x)^{\frac{1}{x}}-e}{x}$

(16) $\lim\limits_{x\to 0}x^b\ln^c x(b,c>0)$

(17) $\lim\limits_{x\to 0}x^{\sin x}$

(18) $\lim\limits_{x\to 1}x^{\frac{1}{1-x}}$

(19) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$

(20) $\lim\limits_{x\to+0}\left(\ln\dfrac{1}{x}\right)^x$

解:

(1) $\lim\limits_{x\to 0}\dfrac{\tan ax}{\sin bx}=\lim\limits_{x\to 0}\dfrac{a\sec^2 ax}{b\cos bx}=\dfrac{a}{b}$

(2) $\lim\limits_{x\to 0}\dfrac{1-\cos x^2}{x^3\sin x}=\lim\limits_{x\to 0}\dfrac{1-\cos x^2}{x^4}=\lim\limits_{x\to 0}\dfrac{2x\sin x^2}{4x^3}\lim\limits_{x\to 0}\dfrac{\sin x^2}{2x^2}=\dfrac{1}{2}$

(3) $\lim\limits_{x\to\infty}\dfrac{\dfrac{\pi}{2}-\arctan x}{\sin\dfrac{1}{x}}=\lim\limits_{x\to\infty}\dfrac{-\dfrac{1}{1+x^2}}{-\dfrac{1}{x^2}\cos\dfrac{1}{x}}=\lim\limits_{x\to\infty}\dfrac{x^2}{(1+x^2)\cos\dfrac{1}{x}}=1$

(4) 当b为正整数，$\lim\limits_{x\to\infty}\dfrac{x^b}{e^{ax}}=\lim\limits_{x\to\infty}\dfrac{bx^{b-1}}{ae^{ax}}=\cdots=\lim\limits_{x\to\infty}\dfrac{b!}{a^be^{ax}}=0$

当b不为正整数，则$[b]\leqslant b<[b]+1$，于是$\dfrac{|x|^{[b]}}{e^{ax}}\leqslant\dfrac{|x|^b}{e^{ax}}<\dfrac{|x|^{[b]+1}}{e^{ax}}(|x|>1)$，而左、右两端当$x\to\infty$时，上面已证明它们的极限为0，因此，中间的极限也为0.

从而，对任意a,b，均有$\lim\limits_{x\to\infty}\dfrac{x^b}{e^{ax}}=0$

(5) $\lim\limits_{x\to1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)=\lim\limits_{x\to1}\dfrac{x-1-\ln x}{(x-1)\ln x}=\lim\limits_{x\to1}\dfrac{1-\dfrac{1}{x}}{\ln x+\dfrac{x-1}{x}}=\lim\limits_{x\to1}\dfrac{x-1}{x\ln x+x-1}=\lim\limits_{x\to1}\dfrac{1}{\ln x+1+1}=\dfrac{1}{2}$

(6) $\lim\limits_{x\to\pi}(\pi-x)\tan\dfrac{x}{2}=\lim\limits_{x\to\pi}\dfrac{\pi-x}{\cot\dfrac{x}{2}}=\lim\limits_{x\to\pi}\dfrac{-1}{-\dfrac{1}{2}\csc^2\dfrac{x}{2}}=2$

(7) $\lim\limits_{x\to0}\dfrac{\ln(\cos ax)}{\ln(\cos bx)}=\lim\limits_{x\to0}\dfrac{-a\tan ax}{-b\tan bx}=\dfrac{a}{b}\lim\limits_{x\to0}\dfrac{\tan ax}{\tan bx}=\dfrac{a}{b}\lim\limits_{x\to0}\dfrac{a\sec^2ax}{b\sec^2bx}=\dfrac{a^2}{b^2}(b\neq0)$

(8) $\lim\limits_{x\to0}\dfrac{\cos(\sin x)-\cos x}{x^4}=\lim\limits_{x\to0}\dfrac{-\sin(\sin x)\cos x+\sin x}{4x^3}$

$=\lim\limits_{x\to0}\dfrac{-\cos(\sin x)\cos^2x+\sin(\sin x)\sin x+\cos x}{12x^2}=$

$\lim\limits_{x\to0}\dfrac{\sin(\sin x)\cos^3x+\dfrac{3}{2}\cos(\sin x)\sin2x+\sin(\sin x)\cos x-\sin x}{24x}=$

$\lim\limits_{x\to0}\left[\dfrac{\cos(\sin x)\cos^4x-3\sin(\sin x)\sin2x\cos x+3\cos(\sin x)\cos2x}{24}+\dfrac{\cos(\sin x)\cos^2x-\sin(\sin x)\sin x-\cos x}{24}\right]=\dfrac{1}{6}$

(9) $\lim\limits_{x\to0}\dfrac{a^x-b^x}{x}=\lim\limits_{x\to0}\dfrac{a^x\ln a-b^x\ln b}{1}=\ln a-\ln b=\ln\dfrac{a}{b}(a\neq0,b\neq0)$

(10) $\lim\limits_{x\to1}\dfrac{x-1}{\ln x}=\lim\limits_{x\to1}\dfrac{1}{\dfrac{1}{x}}=1$

(11) $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x-a}=\lim\limits_{x\to a}\dfrac{a^x\ln a-ax^{a-1}}{1}=a^a(\ln a-1)$

(12) $\lim\limits_{x\to\frac{\pi}{6}}\dfrac{1-2\sin x}{\cos3x}=\lim\limits_{x\to\frac{\pi}{6}}\dfrac{-2\cos x}{-3\sin3x}=\dfrac{\sqrt{3}}{3}$

(13) $\lim\limits_{x\to0}\dfrac{\ln x}{\cot x}=\lim\limits_{x\to0}\dfrac{\dfrac{1}{x}}{-\csc^2x}=-\lim\limits_{x\to0}\dfrac{\sin^2x}{x}=0$

(14) 令$y=\ln x$,则$x=e^y$，于是$\lim\limits_{x\to+\infty}\dfrac{\ln^cx}{x^b}=\lim\limits_{y\to+\infty}\dfrac{y^c}{e^{by}}=0$(由(4)得)

(15) $\lim\limits_{x\to0}\dfrac{(1+x)^{\frac{1}{x}}-e}{x}=\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}\left[\dfrac{1}{x(1+x)}-\dfrac{1}{x^2}\ln(1+x)\right]=e\lim\limits_{x\to0}\dfrac{x-(1+x)\ln(1+x)}{x^2}=e\lim\limits_{x\to0}\dfrac{1-1-\ln(1+x)}{2x}=-\dfrac{e}{2}$

(16) 令$y=\ln x$,则$x=e^y$，于是$\lim\limits_{x\to0}x^b\ln^cx=\lim\limits_{y\to-\infty}e^{by}y^c=\lim\limits_{y\to-\infty}\dfrac{y^c}{e^{-by}}=0$(由(4)得)

(17) $\lim\limits_{x\to0}x^{\sin x}=e^{\lim\limits_{x\to0}\sin x\ln x}$，而$\lim\limits_{x\to0}\sin x\ln x=\lim\limits_{x\to0}\dfrac{\ln x}{\dfrac{1}{x}}=\lim\limits_{x\to0}\dfrac{\dfrac{1}{x}}{-\dfrac{1}{x^2}}=-\lim\limits_{x\to0}x=0$，于是$\lim\limits_{x\to0}x^{\sin x}=1$

(18) $\lim\limits_{x\to1}x^{\frac{1}{1-x}}=e^{\lim\limits_{x\to1}\frac{\ln x}{1-x}}$，而$\lim\limits_{x\to1}\frac{\ln x}{1-x}=-\lim\limits_{x\to1}\dfrac{1}{x}=-1$，于是$\lim\limits_{x\to1}x^{\dfrac{1}{1-x}}=\dfrac{1}{e}$

(19) $\lim\limits_{x\to0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)=\lim\limits_{x\to0}\dfrac{e^x-x-1}{x(e^x-1)}=\lim\limits_{x\to0}=\dfrac{e^x-1}{e^x-1+xe^x}=\lim\limits_{x\to0}\dfrac{e^x}{2e^x+xe^x}=\dfrac{1}{2}$

(20) $\lim\limits_{x\to+0}\left(\ln\frac{1}{x}\right)^x=e^{\lim\limits_{x\to+0}x\ln\left(\ln\frac{1}{x}\right)}$

令$y=\frac{1}{x}$，则$\lim\limits_{x\to+0}x\ln(\ln\frac{1}{x})=\lim\limits_{y\to+\infty}\frac{\ln(\ln y)}{y}=\lim\limits_{y\to+\infty}\frac{1}{y\ln y}=0$，从而$\lim\limits_{x\to+0}\left(\ln\frac{1}{x}\right)^x=1$

2. 试说明下列函数不能用洛必达法则求极限：

(1) $\lim\limits_{x\to0}\dfrac{x^2\sin\frac{1}{x}}{\sin x}$

(2) $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x-\cos x}$

(3) $\lim\limits_{x\to\infty}\dfrac{2x+\sin 2x}{(2x+\sin x)e^{\sin x}}$

(4) $\lim\limits_{x\to1}\dfrac{(x^2-1)\sin x}{\ln\left(1+\sin\frac{\pi}{2}x\right)}$

解:

(1) 因$\dfrac{x^2\sin\frac{1}{x}}{\sin x}$的分子、分母同时对$x$求导数，得$\dfrac{2x\sin\frac{1}{x}-\cos\frac{1}{x}}{\cos x}$，而$\dfrac{\cos\frac{1}{x}}{\cos x}$当$x\to0$时极限不存在，因此洛必达法则不能适用，但是原极限是存在的。事实上，有$\lim\limits_{x\to0}\dfrac{x^2\sin\frac{1}{x}}{\sin x}=\lim\limits_{x\to0}\dfrac{x}{\sin x}\cdot x\sin\dfrac{1}{x}=0$

(2) 因$\dfrac{x+\sin x}{x-\cos x}$的分子、分母同时对$x$求导数，得$\dfrac{1+\cos x}{1+\sin x}$，当$x\to\infty$时此函数极限不存在，因此洛必达法则不能适用，但是原极限是存在的。事实上，有$\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x-\cos x}=\lim\limits_{x\to\infty}\dfrac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}}=1$

(3) 对于不同的序列：$x_n'=2n\pi+\dfrac{\pi}{2}$及$x_n''=2n\pi(n=1,2,\cdots)$，当$n\to\infty$时，则取不同的极限$\dfrac{1}{e}$及1，从而原极限不存在.

用洛必达法则求解，有$\lim\limits_{x\to\infty}\dfrac{2x+\sin 2x}{(2x+\sin x)e^{\sin x}}=$

$\lim\limits_{x\to\infty}\dfrac{2+2\cos 2x}{(2+\cos x+2x\cos x+\sin x\cos x)e^{\sin x}}=$

$\lim\limits_{x\to\infty}\dfrac{4\cos^2 x}{[2+\cos x(1+2x+\sin x)]e^{\sin x}}=$

$\lim\limits_{x\to\infty}\dfrac{4}{\left[\frac{2}{\cos^2 x}+\frac{1}{\cos x}(1+2x+\sin x)\right]e^{\sin x}}$，因$e^{\sin x}\geqslant e^{-1}, 1+2x+\sin x\geqslant 2x$，则$\left|[\dfrac{2}{\cos^2 x}+\dfrac{1}{\cos x}(1+2x+\sin x)]e^{\sin x}\right|\geqslant$

$e^{-1}(-2+2|x|)\to+\infty(x\to\infty)$，则$\lim\limits_{x\to\infty}\dfrac{2x+\sin 2x}{(2x+\sin x)e^{\sin x}}=0.$

(4) 直接求极限可得$\lim\limits_{x\to1}\dfrac{(x^2-1)\sin x}{\ln\left(1+\sin\frac{\pi}{2}x\right)}=0$，但此极限不符合用洛必达法则求极限的条件.

§6. 方程的近似解

1. 求方程$x^3-x-4=0$的正根，使误差不超过0.0001.
   解：设$f(x)=x^3-x-4$，在$[1,2]$间，$f(1)=-4<0,f(2)=2>0$即$f(1)f(2)<0$且$f'(x)=3x^2-1>0,f''(x)=6x>0$
   因$f(2)f''(2)=24>0$，则从点$(2,f(2))$即点$(2,2)$开始作切线，取$x_0=2$作初值.
   于是$x_1=2-\dfrac{f(2)}{f'(2)}\approx 1.81818,x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}\approx 1.79663,x_3=x_2-\dfrac{f(x_2)}{f'(x_2)}\approx 1.79632,x_4=x_3-\dfrac{f(x_3)}{f'(x_3)}\approx 1.79632$
   x_3与x_4的前5位数相同，这表示已接近于根的精确值。为了说明精确度，用1.7963试一下，有$f(1.7963)\approx -0.00019<0$，而$f(1.79632)\approx 0.00002>0$，故若取1.7963作为根的近似值，则误差不超过0.0001.
2. 求方程$x^3-x-4=0$的正根，使误差不超过0.0001.
   解：设$f(x)=x^3-5x^2+6x-1$，$f(0)=-1<0,f(1)=1>0,f'(x)=3x^2-10x+6$，此时$f'(0)=6>0,f'(1)=-1<0$，故在$(0,1)$内$f'(x)$有零点$\dfrac{5-\sqrt{7}}{3}$，此时$f'(x)$在$\left(0,\dfrac{5-\sqrt{7}}{3}\right)$内为正；$f'(x)$在$\left(\dfrac{5-\sqrt{7}}{3},1\right)$内为负.
   现分别考虑$f(x)$在$(0,0.7)$与$(0.7,1)$中的根
   因$f(0.7)=1.093>0$，故在$(0,0.7)$中必有实根ξ，但在$(0.7,1)$中无根.
   现求$\xi,f''(x)=6x-10<0(\forall x\in(0,0.7))$，因$f(0)=-1,f''(0)=-10$，故取$x_0=0$作初值.
   于是$x_1=0-\dfrac{f(0)}{f'(0)}\approx 0.16667,x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}\approx 0.19706,x_3=x_2-\dfrac{f(x_2)}{f'(x_2)}\approx 0.19806,x_4=x_3-\dfrac{f(x_3)}{f'(x_3)}\approx 0.19806$
   x_3与x_4的前5位数相同，这表示已接近于根ξ的精确值。为了说明精确度，用0.1980试一下，有$f(0.1980)\approx -0.00026<0$，而$f(0.1981)\approx 0.01397>0$，故若取0.1980作为根的近似值，则误差不超过0.0001.

第二部分　单变量积分学

第六章　不定积分

§1.　不定积分的概念及运算法则

1. 证明：若$\int f(t)dt = F(t)+C$，则$\int f(ax+b)dx = \frac{1}{a}F(ax+b)+C$.

证明：因$\int f(t)dt = F(t)+C$，故$[F(t)+C]' = f(t)$，则$\left[\frac{1}{a}T(ax+b)\right]' = \frac{1}{a}[F(ax+b)]' = f(ax+b)$，于是$\int f(ax+b)dx = \frac{1}{a}F(ax+b)+C$.

2. 求下列不定积分：

(1) $\int (2-\sec^2 x)\,\mathrm{d}x$

(2) $\int \left(x^4 - 2x^3 + \frac{\sqrt{x}}{2}\right)\mathrm{d}x$

(3) $\int \left(\sqrt{x} + \sqrt[3]{x} + \frac{2}{\sqrt{x}} + \frac{2}{\sqrt[3]{x}} - 2\right)\mathrm{d}x$

(4) $\int \left(e^x + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}\right)\mathrm{d}x$

(5) $\int \left(2\cos x + \frac{1}{2}\sin x\right)\mathrm{d}x$

(6) $\int \left(\cos x - \frac{2}{1+x^2} + \frac{1}{4\sqrt{1-x^2}}\right)\mathrm{d}x$

(7) $\int \left(\frac{1}{2}\cos x + \sin x + 1\right)\mathrm{d}x$

(8) $\int \left(2^x + \left(\frac{1}{3}\right)^x - \frac{e^x}{5}\right)\mathrm{d}x$

(9) $\int (3-x^2)^3\,\mathrm{d}x$

(10) $\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,\mathrm{d}x$

解：

(1) $\int (2-\sec^2 x)\,\mathrm{d}x = 2x - \tan x + C$

(2) $\int \left(x^4 - 2x^3 + \frac{\sqrt{x}}{2}\right)\mathrm{d}x = \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^{\frac{3}{2}} + C$

(3) $\int \left(\sqrt{x} + \sqrt[3]{x} + \frac{2}{\sqrt{x}} + \frac{2}{\sqrt[3]{x}} - 2\right)\mathrm{d}x = \frac{2}{3}x^{\frac{3}{2}} + \frac{3}{4}x^{\frac{4}{3}} - 2x + 3x^{\frac{2}{3}} + 4x^{\frac{1}{2}} + C$

(4) $\int \left(e^x + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}\right)\mathrm{d}x = e^x + \ln|x| - \frac{1}{x} - \frac{1}{2x^2} + C$

(5) $\int \left(2\cos x + \frac{1}{2}\sin x\right)\mathrm{d}x = 2\sin x - \frac{1}{2}\cos x + C$

(6) $\int \left(\cos x - \frac{2}{1+x^2} + \frac{1}{4\sqrt{1-x^2}}\right)\mathrm{d}x = \sin x - 2\arctan x + \frac{1}{4}\arcsin x + C$

(7) $\int \left(\frac{1}{2}\cos x + \sin x + 1\right)\mathrm{d}x = \frac{1}{2}\sin x - \cos x + x + C$

(8) $\int \left(2^x + \left(\frac{1}{3}\right)^x - \frac{e^x}{5}\right)\mathrm{d}x = \frac{1}{\ln 2}2^x - \frac{1}{\ln 3}\left(\frac{1}{3}\right)^x - \frac{e^x}{5} + C$

(9) $\int (3-x^2)^3\,\mathrm{d}x = \int (27 - 27x^2 + 9x^4 - x^6)\,\mathrm{d}x = 27x - 9x^3 + \frac{9}{5}x^5 - \frac{1}{7}x^7 + C$

(10) $\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,\mathrm{d}x = \int (x^{\frac{3}{4}} - x^{-\frac{5}{4}})\,\mathrm{d}x = \frac{4}{7}x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + C$

§2. 不定积分的计算

1. 求下列不定积分：

(1) $\displaystyle\int \frac{\mathrm{d}x}{5x-7}$

(2) $\displaystyle\int \cos(\omega t-\varphi)\,\mathrm{d}t$

(3) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1-\left(\frac{x}{2}+3\right)^2}}$

(4) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1-2x^2}}$

(5) $\displaystyle\int \tan^{10} x\sec^2 x\,\mathrm{d}x$

(6) $\displaystyle\int e^{\alpha x}\cdot 2^x\,\mathrm{d}x$

(7) $\displaystyle\int (2^x+3^x)^2\,\mathrm{d}x$

(8) $\displaystyle\int \tan x\,\mathrm{d}x$

(9) $\displaystyle\int \tan\sqrt{1+x^2}\cdot\frac{x\,\mathrm{d}x}{\sqrt{1+x^2}}$

(10) $\displaystyle\int (\alpha x^2+\beta)^\mu x\,\mathrm{d}x(\mu\neq -1)$

(11) $\displaystyle\int \frac{\mathrm{d}x}{1-\cos x}$

(12) $\displaystyle\int \frac{\mathrm{d}x}{A^2\sin^2 x+B^2\cos^2 x}$

(13) $\displaystyle\int \frac{\sin x\cdot\cos x}{1+\sin^4 x}\,\mathrm{d}x$

(14) $\displaystyle\int \frac{\mathrm{d}x}{\sin^2\left(x+\frac{\pi}{4}\right)}$

(15) $\displaystyle\int x^2\sqrt[8]{1+x^3}\,\mathrm{d}x$

(16) $\displaystyle\int \frac{\sin^2 x\cos x}{1+\sin^3 x}\,\mathrm{d}x$

(17) $\displaystyle\int \frac{1-2\sin x}{\cos^2 x}\,\mathrm{d}x$

(18) $\displaystyle\int \frac{\mathrm{d}x}{e^x+e^{-x}}$

(19) $\displaystyle\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}\,\mathrm{d}x$

(20) $\displaystyle\int \frac{1+\sin 2x}{\sin^2 x}\,\mathrm{d}x$

(21) $\displaystyle\int \sqrt{\frac{\ln(x+\sqrt{1+x^2})}{1+x^2}}\,\mathrm{d}x$

(22) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+e^{2x}}}$

(23) $\displaystyle\int \frac{\mathrm{d}x}{x^2-2x+2}$

(24) $\displaystyle\int \frac{\mathrm{d}x}{(\arcsin x)^2\sqrt{1-x^2}}$

(25) $\displaystyle\int \frac{x^2+7}{x^2-2x-3}\,\mathrm{d}x$

(26) $\displaystyle\int \frac{x^2-1}{x^4+1}\,\mathrm{d}x$

解：

(1) $\displaystyle\int \frac{\mathrm{d}x}{5x-7} = \frac{1}{5}\int \frac{\mathrm{d}(5x-7)}{5x-7} = \frac{1}{5}\ln|5x-7|+C$

(2) $\displaystyle\int \cos(\omega t-\varphi)\,\mathrm{d}t = \frac{1}{\omega}\int \cos(\omega t-\varphi)\,\mathrm{d}(\omega t-\varphi) = \frac{1}{\omega}\sin(\omega t-\varphi)+C$

(3) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1-\left(\frac{x}{2}+3\right)^2}} = 2\int \frac{\mathrm{d}\left(\frac{x}{2}+3\right)}{\sqrt{1-\left(\frac{x}{2}+3\right)^2}} = 2\arcsin\left(\frac{x}{2}+3\right)+C$

(4) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1-2x^2}} = \frac{\sqrt{2}}{2}\int \frac{\mathrm{d}(\sqrt{x})}{\sqrt{1-(\sqrt{2}x)^2}} = \frac{\sqrt{2}}{2}\arcsin(\sqrt{2}x)+C$

(5) $\displaystyle\int \tan^{10}x\sec^2 x\,\mathrm{d}x = \int \tan^{10}x\,\mathrm{d}(\tan x) = -\frac{1}{11}\tan^{11}x+C$

(6) $\displaystyle\int e^{\alpha x}\cdot 2^x\,\mathrm{d}x = \int (2e^{\alpha})^x\,\mathrm{d}x = \frac{(2e^{\alpha})^x}{\ln(2e^{\alpha})}+C$

(7) $\displaystyle\int (2^x+3^x)^2\,\mathrm{d}x = \int (4^x+2\cdot 6^x+9^x)\,\mathrm{d}x = \frac{4^x}{\ln 4}+\frac{2}{\ln 6}6^x+\frac{9^x}{\ln 9}+C$

(8) $\displaystyle\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x = -\int \frac{\mathrm{d}(\cos x)}{\cos x} = -\ln|\cos x|+C = \ln|\sec x|+C$

(9) $\displaystyle\int \tan\sqrt{1+x^2}\cdot\frac{x\,\mathrm{d}x}{\sqrt{1+x^2}} = \int \tan\sqrt{1+x^2}\,\mathrm{d}(\sqrt{1+x^2}) = \ln|\sec\sqrt{1+x^2}|+C$

(10) $\displaystyle\int (\alpha x^2+\beta)^{\mu}x\,\mathrm{d}x = \frac{1}{2\alpha}\int (\alpha x^2+\beta)^{\mu}\,\mathrm{d}(\alpha x^2+\beta) = \frac{(\alpha x^2+\beta)^{\mu+1}}{2\alpha(\mu+1)}+C$

(11) $\displaystyle\int \frac{\mathrm{d}x}{1-\cos x} = \int \csc^2\frac{x}{2}\,\mathrm{d}\left(\frac{x}{2}\right) = -\cot\frac{x}{2}+C$

(12) $\displaystyle\int \frac{\mathrm{d}x}{A^2\sin^2 x+B^2\cos^2 x} = \frac{1}{AB}\int \frac{1}{1+\left(\frac{A}{B}\right)^2\tan^2 x}\,\mathrm{d}\frac{A\tan x}{B} = \frac{1}{AB}\arctan\left(\frac{A}{B}\tan x\right)+C$

(13) $\displaystyle\int \frac{\sin x\cdot\cos x}{1+\sin^4 x}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{1+(\sin^2 x)^2}\,\mathrm{d}(\sin^2 x) = \frac{1}{2}\arctan(\sin^2 x)+C$

(14) $\displaystyle\int \frac{\mathrm{d}x}{\sin^2\left(x+\frac{\pi}{4}\right)} = \int \csc^2\left(x+\frac{\pi}{4}\right)\,\mathrm{d}\left(x+\frac{\pi}{4}\right) = -\cot\left(x+\frac{\pi}{4}\right)+C$

(15) $\displaystyle\int x^2\sqrt[8]{1+x^3}\,\mathrm{d}x = \frac{1}{3}\int \sqrt[8]{1+x^3}\,\mathrm{d}(1+x^3) = \frac{8}{27}(1+x^3)^{\frac{9}{8}}+C$

(16) $\displaystyle\int \frac{\sin^2 x\cos x}{1+\sin^3 x}\,\mathrm{d}x = \frac{1}{3}\int \frac{\mathrm{d}(1+\sin^3 x)}{1+\sin^3 x} = \frac{1}{3}\ln(1+\sin^3 x)+C$

(17) $\displaystyle\int \frac{1-2\sin x}{\cos^2 x}\,\mathrm{d}x = \int \sec^2 x\,\mathrm{d}x + 2\int \frac{\mathrm{d}\cos x}{\cos^2 x} = \tan x-2\sec x+C$

(18) $\displaystyle\int \frac{\mathrm{d}x}{e^x+e^{-x}} = \int \frac{\mathrm{d}e^x}{e^{2x}+1} = \arctan(e^x)+C$

(19) $\displaystyle\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}\,\mathrm{d}x = \int \frac{\mathrm{d}(\sin x-\cos x)}{\sqrt[3]{\sin x-\cos x}} = \frac{3}{2}(\sin x-\cos x)^{\frac{2}{3}}+C$

(20) $\displaystyle\int \frac{1+\sin 2x}{\sin^2 x}\,\mathrm{d}x = \int \csc^2 x\,\mathrm{d}x + \int \frac{\mathrm{d}(\sin^2 x)}{\sin^2 x} = -\cot x+\ln(\sin^2 x)+C = -\cot x+2\ln|\sin x|+C$

(21) $\displaystyle\int \sqrt{\frac{\ln(x+\sqrt{1+x^2})}{1+x^2}}\,\mathrm{d}x =$

$$\int \sqrt{\ln(x+\sqrt{1+x^2})}\,\mathrm{d}(\ln(x+\sqrt{1+x^2}) = \frac{2}{3}[\ln(x+\sqrt{1+x^2}]^{\frac{3}{2}}+C$$

(22) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+e^{2x}}} = -\int \frac{\mathrm{d}e^{-x}}{\sqrt{1+e^{-2x}}} = -\ln(e^{-x}+\sqrt{1+e^{-2x}})+C$

(23) $\displaystyle\int \frac{\mathrm{d}x}{x^2-2x+2} = \int \frac{\mathrm{d}(x-1)}{(x-1)^2+1} = \arctan(x-1)+C$

(24) $\displaystyle\int \frac{\mathrm{d}x}{(\arcsin x)^2\sqrt{1-x^2}} = \int \frac{\mathrm{d}(\arcsin x)}{(\arcsin x)^2} = -\frac{1}{\arcsin x}+C$

(25) $\displaystyle\int \frac{x^2+7}{x^2-2x-3}\,\mathrm{d}x = \int \left(1+\frac{2x+10}{(x+1)(x-3)}\right)\mathrm{d}x =$
$\displaystyle\int \left(1-\frac{2}{x+1}+\frac{4}{x-3}\right)\mathrm{d}x = x-2\ln|x+1|+4\ln|x-3|+C = x+2\ln\frac{(x-3)^2}{|x+1|}+C$

(26) $\displaystyle\int \frac{x^2-1}{x^4+1}\,\mathrm{d}x = \int \frac{1-x^{-2}}{x^2+x^{-2}}\,\mathrm{d}x = \int \frac{\mathrm{d}\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2} =$

$\displaystyle\frac{\sqrt{2}}{4}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+C = \frac{\sqrt{2}}{4}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|+C$

2. 求下列不定积分：

(1) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$

(2) $\displaystyle\int \frac{(2\sqrt{u}+1)^2}{u^2}\,\mathrm{d}u$

(3) $\displaystyle\int e^{\sqrt{x+1}}\,\mathrm{d}x$

(4) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,\mathrm{d}x$

(5) $\displaystyle\int \sqrt{x^2+a^2}\,\mathrm{d}x$

(6) $\displaystyle\int \sqrt{x^2-a^2}\,\mathrm{d}x$

(7) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}}$

(8) $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{\alpha x^2+\beta}}$

(9) $\displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{5+x-x^2}}$

(10) $\displaystyle\int \sqrt{2+x-x^2}\,\mathrm{d}x$

解:

(1) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x = -2\int \sin\sqrt{x}\,\mathrm{d}(\sqrt{x}) = -2\cos\sqrt{x}+C$

(2) $\displaystyle\int \frac{(2\sqrt{u}+1)^2}{u^2}\,\mathrm{d}u = \int \left(\frac{4}{u}+\frac{4}{u^{\frac{3}{2}}}+\frac{1}{u^2}\right)\mathrm{d}u = 4\ln|u|-8u^{-\frac{1}{2}}-\frac{1}{u}+C$

(3) 令$\sqrt{1+x}=t$，则$x=t^2-1$, $\mathrm{d}x=2t\,\mathrm{d}t$，于是$\displaystyle\int e^{\sqrt{x+1}}\,\mathrm{d}x = 2\int te^t\,\mathrm{d}t = 2(t-1)e^t+C = 2(\sqrt{1+x}-1)e^{\sqrt{1+x}}+C$

(4) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,\mathrm{d}x = -\int \frac{4-x^2-4}{\sqrt{4-x^2}}\,\mathrm{d}x = -\int \sqrt{4-x^2}\,\mathrm{d}x + 4\int \frac{\mathrm{d}x}{\sqrt{4-x^2}} = -\frac{x}{2}\sqrt{4-x^2}-2\arcsin\frac{x}{2}+4\arcsin\frac{x}{2}+C = 2\arcsin\frac{x}{2}-\frac{x}{2}\sqrt{4-x^2}+C$

(5) $I=\int\sqrt{x^2+a^2}\,\mathrm{d}x=x\sqrt{x^2+a^2}-\int\dfrac{x^2}{\sqrt{x^2+a^2}}\,\mathrm{d}x=x\sqrt{x^2+a^2}-\int\sqrt{x^2+a^2}\,\mathrm{d}x+\int\dfrac{a^2}{\sqrt{x^2+a^2}}\,\mathrm{d}x=$ $x\sqrt{x^2+a^2}-I+a^2\ln|x+\sqrt{x^2+a^2}|+C_1$，

于是$2I=x\sqrt{x^2+a^2}+a^2\ln|x+\sqrt{x^2+a^2}|+C_1$，从而$I=\dfrac{x}{2}\sqrt{x^2+a^2}+\dfrac{a^2}{2}\ln(x+\sqrt{x^2+a^2})+C(C=\dfrac{C_1}{2})$

(6) $I=\int\sqrt{x^2-a^2}\,\mathrm{d}x=x\sqrt{x^2-a^2}-\int\dfrac{x^2}{\sqrt{x^2-a^2}}\,\mathrm{d}x=x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}\,\mathrm{d}x-\int\dfrac{a^2}{\sqrt{x^2-a^2}}\,\mathrm{d}x=$ $x\sqrt{x^2-a^2}-I-a^2\ln|x+\sqrt{x^2-a^2}|+C_1$，

于是$2I=x\sqrt{x^2-a^2}-a^2\ln|x+\sqrt{x^2-a^2}|+C_1$，从而$I=\dfrac{x}{2}\sqrt{x^2-a^2}-\dfrac{a^2}{2}\ln(x+\sqrt{x^2-a^2})+C(C=\dfrac{C_1}{2})$

(7) $\int\dfrac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}}=\int\dfrac{\mathrm{d}x}{\sqrt{-[x^2-(a+b)x]-ab}}=$ $\int\dfrac{\mathrm{d}\left(x-\dfrac{a+b}{2}\right)}{\sqrt{-\left(x-\dfrac{a+b}{2}\right)^2+\left(\dfrac{a-b}{2}\right)^2}}=\arcsin\dfrac{x-\dfrac{a+b}{2}}{\dfrac{a-b}{2}}+C=\arcsin\dfrac{2x-a-b}{a-b}+C$（其中$a<b$）

(8) $\int\dfrac{\mathrm{d}x}{x^2\sqrt{\alpha x^2+\beta}}=\int\dfrac{\mathrm{d}x}{x^3\sqrt{\alpha+\dfrac{\beta}{x^2}}}=-\dfrac{1}{2}\int\dfrac{\mathrm{d}\dfrac{1}{x^2}}{\sqrt{\alpha+\dfrac{\beta}{x^2}}}=-\dfrac{1}{\beta}\sqrt{\alpha+\dfrac{\beta}{x^2}}+C$

(9) $\int\dfrac{x\,\mathrm{d}x}{\sqrt{5+x-x^2}}=\int\dfrac{x\,\mathrm{d}x}{\sqrt{\dfrac{21}{4}-\left(x-\dfrac{1}{2}\right)^2}}=\int\dfrac{x-\dfrac{1}{2}}{\sqrt{\dfrac{21}{4}-\left(x-\dfrac{1}{2}\right)^2}}+\dfrac{1}{2}\int\dfrac{\mathrm{d}x}{\sqrt{\dfrac{21}{4}-\left(x-\dfrac{1}{2}\right)^2}}=-\sqrt{\dfrac{21}{4}-\left(x-\dfrac{1}{2}\right)^2}+$ $\dfrac{1}{2}\arcsin\dfrac{x-\dfrac{1}{2}}{\dfrac{\sqrt{21}}{2}}+C=-\sqrt{5+x-x^2}+\dfrac{1}{2}\arcsin\dfrac{2x-1}{\sqrt{21}}+C$

(10) $\int\sqrt{2+x-x^2}\,\mathrm{d}x=\int\sqrt{\dfrac{9}{4}-\left(x-\dfrac{1}{2}\right)^2}\,\mathrm{d}x=\dfrac{x-\dfrac{1}{2}}{2}\sqrt{\dfrac{9}{4}-\left(x-\dfrac{1}{2}\right)^2}+\dfrac{9}{8}\arcsin\dfrac{x-\dfrac{1}{2}}{\dfrac{3}{2}}+C=\dfrac{2x-1}{4}\sqrt{2+x-x^2}+$ $\dfrac{9}{8}\arcsin\dfrac{2x-1}{3}+C$

3. 求下列不定积分：

(1) $\int x^2\cos x\,\mathrm{d}x$

(2) $\int x^3\ln x\,\mathrm{d}x$

(3) $\int \ln x\,\mathrm{d}x$

(4) $\int x^n\ln x\,\mathrm{d}x$($n$为正整数)

(5) $\int\dfrac{\arcsin x}{\sqrt{1-x}}\,\mathrm{d}x$

(6) $\int\csc x\,\mathrm{d}x$

(7) $\int\cos(\ln x)\,\mathrm{d}x$

(8) $\int\dfrac{x\,\mathrm{d}x}{\sin^2 x}$

(9) $\int x\cos^2 x\,\mathrm{d}x$

(10) $\int x\sin^2 x\,\mathrm{d}x$

(11) $\int \arccos x\,\mathrm{d}x$

(12) $\int (\arcsin x)^2\,\mathrm{d}x$

(13) $\int e^{ax}\cos bx\,\mathrm{d}x$

(14) $\int \ln(x+\sqrt{1+x^2})\,\mathrm{d}x$

解：

(1) $\int x^2\cos x\,\mathrm{d}x = x^2\sin x - 2\int x\sin x\,\mathrm{d}x = x^2\sin x + 2x\cos x - 2\int\cos x\,\mathrm{d}x = x^2\sin x + 2x\cos x - 2\sin x + C$

(2) $\int x^3\ln x\,\mathrm{d}x = \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,\mathrm{d}x = \frac{1}{4}x^4\ln x - \frac{x^4}{16} + C$

(3) $\int \ln x\,\mathrm{d}x = x\ln x - \int\,\mathrm{d}x = x\ln x - x + C$

(4) $\int x^n\ln x\,\mathrm{d}x = \frac{x^{n+1}}{x+1}\ln x - \frac{1}{n+1}\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$

(5) $\int \frac{\arcsin x}{\sqrt{1-x}}\,\mathrm{d}x = -2\arcsin x\cdot\sqrt{1-x} + 2\int\frac{1}{\sqrt{1+x}}\,\mathrm{d}x = -2\sqrt{1-x}\arcsin x + 4\sqrt{1+x} + C$

(6) $\int \csc x\,\mathrm{d}x = \int\frac{\mathrm{d}x}{\sin x} = \int\frac{\frac{1}{2\cos(\frac{x}{2})}}{\tan\frac{x}{2}}\,\mathrm{d}x = \int\frac{\mathrm{d}\left(\tan\frac{x}{2}\right)}{\tan\frac{x}{2}} = \ln\left|\tan\frac{x}{2}\right| + C$

(7) 因$I = \int\cos(\ln x)\,\mathrm{d}x = x\cos\ln x + \int\sin(\ln x)\,\mathrm{d}x = x\cos(\ln x) + x\sin(\ln x) - \int\cos(\ln x)\,\mathrm{d}x = x\cos(\ln x) + x\sin(\ln x) - I + C_1$，故$I = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C\left(C = \frac{C_1}{2}\right)$

(8) $\int\frac{x\,\mathrm{d}x}{\sin^2 x} = \int x\csc^2 x\,\mathrm{d}x = -x\cot x + \int\cot x\,\mathrm{d}x = -x\cot x + \ln|\sin x| + C$

(9) $\int x\cos^2 x\,\mathrm{d}x = \frac{1}{2}\int x(1+\cos 2x)\,\mathrm{d}x = \frac{x^2}{4} + \frac{1}{2}\int x\cos 2x\,\mathrm{d}x = \frac{x^2}{4} + \frac{x}{4}\sin 2x - \frac{1}{4}\int\sin 2x\,\mathrm{d}x = \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + C$

(10) $\int x\sin^2 x\,\mathrm{d}x = \int x(1-\cos^2 x)\,\mathrm{d}x = \frac{x^2}{2} - \int x\cos^2 x\,\mathrm{d}x = \frac{x^2}{2} - \left(\frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x\right) + C = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x$

(11) $\int\arccos x\,\mathrm{d}x = x\arccos x + \int\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = x\arccos x - \sqrt{1-x^2} + C$

(12) $\int(\arcsin x)^2\,\mathrm{d}x = x(\arcsin x)^2 - \int\frac{2x\arcsin x}{\sqrt{1-x^2}}\,\mathrm{d}x = x(\arcsin x)^2 + 2\arcsin x\cdot\sqrt{1-x^2} - 2\int\,\mathrm{d}x = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$

(13) $I = \int e^{ax}\cos bx\,\mathrm{d}x = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,\mathrm{d}x = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}\int e^{ax}\cos bx\,\mathrm{d}x = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}I + C_1$，则$I = \frac{a\cos bx + b\sin bx}{a^2+b^2}e^{ax} + C\left(C = \frac{a^2}{a^2+b^2}C_1\right)$

(14) $\int\ln(x+\sqrt{1+x^2})\,\mathrm{d}x = x\ln(x+\sqrt{1+x^2}) - \int\frac{x}{\sqrt{1+x^2}}x = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C$

4. 求下列不定积分：

(1) $\int\frac{x^3+1}{x^3-5x^2+6x}\,\mathrm{d}x$

(2) $\displaystyle\int \frac{\mathrm{d}x}{(x+1)(x+2)^2}$

(3) $\displaystyle\int \frac{\mathrm{d}x}{(x+1)(x+2)^2(x+3)^3}$

(4) $\displaystyle\int \frac{x^2+5x+4}{x^4+5x^2+4}\,\mathrm{d}x$

(5) $\displaystyle\int \frac{\mathrm{d}x}{(x^2-4x+4)(x^2-4x+5)}$

(6) $\displaystyle\int \frac{\mathrm{d}x}{x^4+x^2+1}$

(7) $\displaystyle\int \frac{\mathrm{d}x}{(x+1)(x^2+1)}$

(8) $\displaystyle\int \frac{\mathrm{d}x}{x^3+1}$

(9) $\displaystyle\int \frac{x^2\,\mathrm{d}x}{1-x^4}$

(10) $\displaystyle\int \frac{x^6+x^4-4x^2-2}{x^3(x^2+1)^2}\,\mathrm{d}x$

解：

(1) 因$\displaystyle\frac{x^3+1}{x^3-5x^2+6x}=1+\frac{1}{6x}-\frac{9}{2(x-2)}+\frac{28}{3(x-3)}$，故$\displaystyle\int\frac{x^3+1}{x^3-5x^2+6x}\,\mathrm{d}x=x+\frac{1}{6}\ln|x|-\frac{9}{2}\ln|x-2|+\frac{28}{3}\ln|x-3|+C$

(2) 因$\displaystyle\frac{1}{(x+1)(x+2)^2}=\frac{1}{x+1}-\frac{1}{x+2}-\frac{1}{(x+2)^2}$，故$\displaystyle\int\frac{\mathrm{d}x}{(x+1)(x+2)^2}=\ln|x+1|-\ln|x+2|+\frac{1}{x+2}+C=\ln\left|\frac{x+1}{x+2}\right|+\frac{1}{x+2}+C$

(3) 因$\displaystyle\frac{1}{(x+1)(x+2)^2(x+3)^3}=\frac{1}{8(x+1)}+\frac{2}{x+2}-\frac{1}{(x+2)^2}-\frac{17}{8(x+3)}-\frac{5}{4(x+3)^2}-\frac{1}{2(x+3)^3}$，
故$\displaystyle\int\frac{\mathrm{d}x}{(x+1)(x+2)^2(x+3)^3}=\frac{1}{8}\ln|x+1|+2\ln|x+2|+\frac{1}{x+2}-\frac{17}{8}\ln|x+3|+\frac{5}{4(x+3)}+\frac{1}{4(x+3)^2}+C=\frac{1}{8}\ln|x+1|+2\ln|x+2|-\frac{17}{8}\ln|x+3|+\frac{9x^2+50x+68}{4(x+2)(x+3)^2}+C$

(4) 因$\displaystyle\frac{x^2+5x+4}{x^4+5x^2+4}=\frac{\frac{5}{3}x+1}{x^2+1}+\frac{-\frac{5}{3}x}{x^2+4}$，故$\displaystyle\int\frac{x^2+5x+4}{x^4+5x^2+4}\,\mathrm{d}x=\frac{5}{6}\ln(x^2+1)+\arctan x-\frac{5}{6}\ln(x^2+4)+C=\frac{5}{6}\ln\left(\frac{x^2+1}{x^2+4}\right)+\arctan x+C$

(5) 因$\displaystyle\frac{1}{(x^2-4x+4)(x^2-4x+5)}=\frac{1}{x^2-4x+4}-\frac{1}{x^2-4x+5}=\frac{1}{(x-2)^2}-\frac{1}{(x-2)^2+1}$，故$\displaystyle\int\frac{\mathrm{d}x}{(x^2-4x+4)(x^2-4x+5)}=-\frac{1}{x-2}-\arctan(x-2)+C$

(6) 因$\displaystyle\frac{1}{x^4+x^2+1}=\frac{x+1}{2(x^2+x+1)}-\frac{x-1}{2(x^2-x+1)}$，故$\displaystyle\int\frac{\mathrm{d}x}{x^4+x^2+1}=\frac{1}{4}\ln(x^2+x+1)+\frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}}{3}(2x+1)\right)-\frac{1}{4}\ln(x^2-x+1)+\frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}}{3}(2x-1)\right)$

(7) 因$\displaystyle\frac{1}{(x+1)(x^2+1)}=\frac{1}{2(x+1)}-\frac{x-1}{2(x^2+1)}$，故$\displaystyle\int\frac{\mathrm{d}x}{(x+1)(x^2+1)}=\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\arctan x+C=\frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}+\frac{1}{2}\arctan x+C$

(8) 因$\displaystyle\frac{1}{x^3+1}=\frac{1}{3(x+1)}+\frac{-x+2}{3(x^2-x+1)}$，
故$\displaystyle\int\frac{\mathrm{d}x}{x^3+1}=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln|x^2-x+1|+\frac{\sqrt{3}}{3}\arctan\frac{\sqrt{3}(2x-1)}{3}+C=\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1}+\frac{\sqrt{3}}{3}\arctan\frac{\sqrt{3}(2x-1)}{3}+C$

(9) 因$\dfrac{x^2}{1-x^4}=\dfrac{1}{2(1-x^2)}-\dfrac{1}{2(1+x^2)}=\dfrac{1}{4(1-x)}+\dfrac{1}{4(1+x)}-\dfrac{1}{2(x^2+1)}$，故$\displaystyle\int\frac{x^2\,\mathrm{d}x}{1-x^4}=-\frac{1}{4}\ln|1-x|+\frac{1}{4}\ln|1+x|-\frac{1}{2}\arctan x+C=\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|+\frac{1}{2}\arctan x+C$

(10) 因$\dfrac{x^6+x^4-4x^2-2}{x^3(x^2+1)^2}=\dfrac{x^4(x^2+1)}{x^3(x^2+1)^2}-\dfrac{4x^2+2}{x^3(x^2+1)^2}=\dfrac{x}{x^2+1}-2\dfrac{(x^2+1)^2-x^4}{x^3(x^2+1)^2}=\dfrac{x}{x^2+1}-\dfrac{2}{x^3}+\dfrac{2x}{(x^2+1)^2}$，故$\displaystyle\int\frac{x^6+x^4-4x^2-2}{x^3(x^2+1)^2}\,\mathrm{d}x=\frac{1}{x^2}+\frac{1}{2}\ln(x^2+1)-\frac{1}{x^2+1}+C$

5. 求下列不定积分：

(1) $\displaystyle\int\frac{\mathrm{d}x}{4+5\cos x}$

(2) $\displaystyle\int\frac{\mathrm{d}x}{\sin x+\tan x}$

(3) $\displaystyle\int\frac{x\,\mathrm{d}x}{\sqrt{5+x-x^2}}$

(4) $\displaystyle\int\frac{1}{x\sqrt[4]{1+x^4}}\,\mathrm{d}x$

(5) $\displaystyle\int\frac{x\,\mathrm{d}x}{\sqrt{2+4x}}$

(6) $\displaystyle\int\frac{\cos x}{1+\sin x}\,\mathrm{d}x$

(7) $\displaystyle\int\frac{\mathrm{d}x}{x(1+2\sqrt{x}+\sqrt[3]{x})}$

(8) $\displaystyle\int\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\,\mathrm{d}x$

(9) $\displaystyle\int\frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2(x-1)^4}}$

(10) $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{x}(1+\sqrt[4]{x})^3}$

(11) $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}}(a>0)$

(12) $\displaystyle\int\frac{x\,\mathrm{d}x}{\sqrt[4]{x^3(a-x)}}$

(13) $\displaystyle\int x\sqrt{x^4+2x^2-1}\,\mathrm{d}x$

(14) $\displaystyle\int\sqrt{2+x-x^2}\,\mathrm{d}x$

(15) $\displaystyle\int\frac{x^2\,\mathrm{d}x}{\sqrt{1+x-x^2}}$

(16) $\displaystyle\int\frac{x^2+1}{x\sqrt{x^4+1}}\,\mathrm{d}x$

(17) $\displaystyle\int\sin^6 x\,\mathrm{d}x$

(18) $\displaystyle\int\sin^2 x\cos^4 x\,\mathrm{d}x$

(19) $\displaystyle\int\sin^4 x\cos^4 x\,\mathrm{d}x$

(20) $\displaystyle\int\frac{\cos^4 x}{\sin^3 x}\,\mathrm{d}x$

(21) $\displaystyle\int\frac{\mathrm{d}x}{\sin^3 x\cos^5 x}$

(22) $\displaystyle\int\tan x\cdot\tan(x+a)\,\mathrm{d}x$

(23) $\int \sin 5x \cos x \,\mathrm{d}x$

(24) $\int \dfrac{\sin^2 x}{1+\sin^2 x}\,\mathrm{d}x$

(25) $\int \dfrac{\mathrm{d}x}{\sin(x+a)\sin(x+b)}$

(26) $\int xe^x \cos x \,\mathrm{d}x$

(27) $\int \dfrac{\mathrm{d}x}{(2+\cos x)\sin x}$

(28) $\int \ln(x+\sqrt{1+x^2})^2 \,\mathrm{d}x$

(29) $\int \dfrac{\sin x \cos x}{\sin x + \cos x}\,\mathrm{d}x$

(30) $\int \dfrac{\ln x}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x$

(31) $\int xe^x \sin x \,\mathrm{d}x$

(32) $\int \dfrac{x^3 \arccos x}{\sqrt{1-x^2}}\,\mathrm{d}x$

(33) $\int (x+|x|)^2 \,\mathrm{d}x$

(34) $\int x^2 e^x \cos x \,\mathrm{d}x$

(35) $\int \dfrac{xe^x}{(1+x)^2}\,\mathrm{d}x$

(36) $\int \sqrt{x}\ln^2 x \,\mathrm{d}x$

(37) $\int \dfrac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}}$

(38) $\int x \ln \dfrac{1+x}{1-x}\,\mathrm{d}x$

(39) $\int x \arctan x \cdot \ln(1+x^2)\,\mathrm{d}x$

(40) $\int \sinh^2 x \cosh^2 x \,\mathrm{d}x$

解:

(1) 令$\tan\dfrac{x}{2}=t$，则$\cos x=\dfrac{1-t^2}{1+t^2}$, $\mathrm{d}x=\dfrac{2\,\mathrm{d}t}{1+t^2}$,

于是$\displaystyle\int \frac{\mathrm{d}x}{4+5\cos x}=\int \frac{2}{(3-t)(3+t)}\,\mathrm{d}t=\frac{1}{3}\ln\left|\frac{3+t}{3-t}\right|+C=\frac{1}{3}\ln\left|\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right|+C$

(2) 令$\tan\dfrac{x}{2}=t$，则$\sin x=\dfrac{2t}{1+t^2}$, $\tan x=\dfrac{2t}{1-t^2}$, $\mathrm{d}x=\dfrac{2\,\mathrm{d}t}{1+t^2}$,

于是$\displaystyle\int \frac{\mathrm{d}x}{\sin x+\tan x}=\int \frac{1-t^2}{2t}\,\mathrm{d}t=\frac{1}{2}\ln|t|-\frac{t^2}{4}+C=\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|-\frac{1}{4}\left(\tan\frac{x}{2}\right)^2+C$

(3) $$\int \frac{x\,\mathrm{d}x}{\sqrt{5+x-x^2}}=\int \frac{x\,\mathrm{d}x}{\sqrt{\frac{21}{4}-\left(x-\frac{1}{2}\right)^2}}=\int \frac{x-\frac{1}{2}}{\sqrt{\frac{21}{4}\,\mathrm{d}x-\left(x-\frac{1}{2}\right)^2}}+\frac{1}{2}\int \frac{\mathrm{d}x}{\sqrt{\frac{21}{4}-\left(x-\frac{1}{2}\right)^2}}=-\sqrt{\frac{21}{4}-\left(x-\frac{1}{2}\right)^2}+$$

$$\frac{1}{2}\arcsin\frac{x-\frac{1}{2}}{\frac{\sqrt{21}}{2}}+C=-\sqrt{5+x-x^2}+\frac{1}{2}\arcsin\frac{2x-1}{\sqrt{21}}+C$$

(4) 令$t=\sqrt[4]{1+x^4}$，则$x=\sqrt[4]{t^4-1}, dx=t^3(t^4-1)^{-\frac{3}{4}}\,\mathrm{d}t$

于是$\displaystyle\int\frac{1}{x\sqrt[4]{1+x^4}}\,\mathrm{d}x=\int\frac{t^2}{t^4-1}=\frac{1}{4}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)\,\mathrm{d}t+\frac{1}{2}\int\frac{1}{1+t^2}\,\mathrm{d}t=\frac{1}{4}\ln\left|\frac{t-1}{t+1}\right|+\frac{1}{2}\arctan t+C=\frac{1}{4}\ln\left|\frac{\sqrt[4]{1+x^4}-1}{\sqrt[4]{1+x^4}+1}\right|+\frac{1}{2}\arctan(\sqrt[4]{1+x^4})+C$

(5) $\displaystyle\int\frac{x\,\mathrm{d}x}{\sqrt{2+4x}}=\frac{1}{2}\int x\,\mathrm{d}\sqrt{4x+2}=\frac{1}{2}x\sqrt{2+4x}-\frac{1}{2}\int(2+4x)^{\frac{1}{2}}\,\mathrm{d}x=\frac{x}{2}\sqrt{2+4x}-\frac{1}{12}(2+4x)^{\frac{3}{2}}+C$

(6) $\displaystyle\int\frac{\cos x}{1+\sin x}\,\mathrm{d}x=\int\frac{\mathrm{d}\sin x}{1+\sin x}=\ln(1+\sin x)+C$

(7) 令$\sqrt[6]{x}=t$，则$x=t^6$, $\mathrm{d}x=6t^5\,\mathrm{d}t$

于是$\displaystyle\int\frac{\mathrm{d}x}{x(1+2\sqrt{x}+\sqrt[3]{x})}=6\int\frac{\mathrm{d}t}{t(1+2t^3+t^2)}=6\int\left[\frac{1}{t}-\frac{1}{4(t+1)}-\frac{6t-1}{4(2t^2-t+1)}\right]\,\mathrm{d}t$

又$\displaystyle\int\frac{6t-1}{4(2t^2-t+1)}\,\mathrm{d}t=\frac{3}{8}\int\frac{\mathrm{d}(2t^2-t+1)}{2t^2-t+1}+\frac{1}{8}\int\frac{\mathrm{d}t}{2t^2-t+1}=\frac{3}{8}\ln|2t^2-t+1|+\frac{1}{4\sqrt{7}}\arctan\frac{4t-1}{\sqrt{7}}+C_1$，

从而$\displaystyle\int\frac{\mathrm{d}x}{x(1+2\sqrt{x}+\sqrt[3]{x})}=6\ln|t|-\frac{3}{2}\ln|t+1|-\frac{9}{4}\ln|2t^2-t+1|-\frac{3}{2\sqrt{7}}\arctan\frac{4t-1}{\sqrt{7}}+C=6\ln|\sqrt[6]{x}|-\frac{3}{2}\ln|\sqrt[6]{x}+1|-\frac{9}{4}\ln|2\sqrt[3]{x}-\sqrt[6]{x}+1|-\frac{3}{2\sqrt{7}}\arctan\frac{4\sqrt[6]{x}-1}{\sqrt{7}}+C=\frac{3}{4}\ln\frac{x\sqrt[3]{x}}{(1+\sqrt[6]{x})^2(2\sqrt[3]{x}-\sqrt[6]{x}+1)^3}-\frac{3}{2\sqrt{7}}\arctan\frac{4\sqrt[6]{x}-1}{\sqrt{7}}+C$

(8) $\displaystyle\int\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\,\mathrm{d}x=\int\frac{(\sqrt{x+1}-\sqrt{x-1})^2}{(\sqrt{x+1}+\sqrt{x-1})(\sqrt{x+1}-\sqrt{x-1})}\,\mathrm{d}x=\int(x-\sqrt{x^2-1})\,\mathrm{d}x=\frac{x^2}{2}-\frac{x}{2}\sqrt{x^2-1}+\frac{1}{2}\ln|x+\sqrt{x^2-1}|+C$

(9) 令$\displaystyle\sqrt[3]{\frac{x+1}{x-1}}=t$，则$x=\dfrac{t^3+1}{t^3-1}$, $\mathrm{d}x=-\dfrac{6t^2}{(t^3-1)^2}\,\mathrm{d}t$

于是$\displaystyle\int\frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2(x-1)^4}}=-\frac{3}{2}\int\,\mathrm{d}t=-\frac{3}{2}t+C=-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C$

(10) 令$\sqrt[4]{x}=t$，则$x=t^4$, $\mathrm{d}x=4t^3\,\mathrm{d}t$

于是$\displaystyle\int\frac{\mathrm{d}x}{\sqrt{x}(1+\sqrt[4]{x})^3}=4\int\frac{t}{(1+t)^3}\,\mathrm{d}t=4\int\left[\frac{1}{(1+t)^2}-\frac{1}{(1+t)^3}\right]\,\mathrm{d}t=-\frac{4}{1+t}+\frac{2}{(1+t)^2}+C=-\frac{4}{1+\sqrt[4]{x}}+\frac{2}{(1+\sqrt[4]{x})^2}+C$

(11) $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int\frac{\mathrm{d}\left(\sqrt{a}\left(x+\frac{b}{2a}\right)\right)}{\left[\sqrt{a}\left(x+\frac{b}{2a}\right)\right]^2+\frac{4ac-b^2}{4a}}=\frac{1}{\sqrt{a}}\ln\left|\sqrt{a}\left(x+\frac{b}{2a}\right)+\sqrt{ax^2+bx+c}\right|+C$

(12) 令$\displaystyle\sqrt[4]{\frac{a-x}{x}}=t$

于是$\displaystyle\int\frac{x\,\mathrm{d}x}{\sqrt[4]{x^3(a-x)}}=-\int\frac{4at^2}{(1+t^4)^2}\,\mathrm{d}t=-4a\int\left[\frac{t}{(t^2+\sqrt{2}t+1)(t^2-\sqrt{2}t+1)}\right]^2\,\mathrm{d}t=-\frac{a}{2}\int\frac{\mathrm{d}t}{(t^2-\sqrt{2}t+1)^2}-\frac{a}{2}\int\frac{\mathrm{d}t}{(t^2+\sqrt{2}t+1)^2}+a\int\frac{\mathrm{d}t}{t^4+1}$

又$\displaystyle\int\frac{\mathrm{d}t}{(t^2-\sqrt{2}t+1)^2}=\int\frac{\mathrm{d}\left(t-\frac{\sqrt{2}}{2}\right)}{\left[\left(t-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}\right]^2}=\frac{2t-\sqrt{2}}{2(t^2-\sqrt{2}t+1)}+\sqrt{2}\arctan(\sqrt{2}t-1)+C_1$

$\displaystyle\int\frac{\mathrm{d}t}{(t^2+\sqrt{2}t+1)^2}=\frac{2t+\sqrt{2}}{2(t^2+\sqrt{2}t+1)}+\sqrt{2}\arctan(\sqrt{2}t+1)+C_2$

$\displaystyle\int\frac{1}{t^4+1}\,\mathrm{d}t=\frac{1}{2}\int\frac{t^2+1}{t^4+1}\,\mathrm{d}t-\frac{1}{2}\int\frac{t^2-1}{t^4+1}\,\mathrm{d}t$

而$\displaystyle\int \frac{t^2+1}{t^4+1}\,\mathrm{d}t = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,\mathrm{d}t = \int \frac{\mathrm{d}\left(t-\frac{1}{t}\right)}{\left(t-\frac{1}{t}\right)^2+2} = \frac{1}{\sqrt{2}}\arctan\frac{t^2-1}{\sqrt{2}t}+C_3$,

$\displaystyle\int \frac{t^2-1}{t^4+1}\,\mathrm{d}t = \frac{1}{2\sqrt{2}}\ln\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}+C_4$

从而$\displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt[4]{x^3(a-x)}} = -\frac{at^3}{1+t^4}-\frac{a\sqrt{2}}{2}\arctan\frac{\sqrt{2}t}{1-t^2}+\frac{a}{4\sqrt{2}}\ln\left|\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}\right|+\frac{a}{2\sqrt{2}}\arctan\frac{t^2-1}{\sqrt{2}t}+C$,

其中$t=\sqrt[4]{\dfrac{a-x}{x}}$.

(13) $\displaystyle\int x\sqrt{x^4+2x^2-1}\,\mathrm{d}x = \frac{1}{2}\int\sqrt{(x^2+1)^2-2}\,\mathrm{d}x^2 = \frac{x^2+1}{4}\sqrt{x^4+2x^2-1}-\frac{1}{2}\ln(x^2+1+\sqrt{x^4+2x^2-1})+C$

(14) $\displaystyle\int\sqrt{2+x-x^2}\,\mathrm{d}x = \int\sqrt{\frac{9}{4}-\left(x-\frac{1}{2}\right)^2}\,\mathrm{d}x = \frac{2x-1}{4}\sqrt{2+x-x^2}+\frac{9}{8}\arcsin\frac{2x-1}{3}+C$

(15) $\displaystyle\int\frac{x^2\,\mathrm{d}x}{\sqrt{1+x-x^2}} = -\int\sqrt{1+x-x^2}\,\mathrm{d}x+\int\frac{x+1}{\sqrt{1+x-x^2}}\,\mathrm{d}x = -\frac{2x-1}{4}\sqrt{1+x-x^2}-\frac{5}{8}\arcsin\frac{2x-1}{\sqrt{5}}-\sqrt{1+x-x^2}+\frac{3}{2}\arcsin\frac{2x-1}{\sqrt{5}}+C = -\frac{2x+3}{4}\sqrt{1+x-x^2}+\frac{7}{8}\arcsin\frac{2x-1}{\sqrt{5}}+C$

(16) $\displaystyle\int\frac{x^2+1}{x\sqrt{x^4+1}}\,\mathrm{d}x = \int\frac{1+\frac{1}{x^2}}{\sqrt{x^2+\frac{1}{x^2}}}\,\mathrm{d}x = \int\frac{\mathrm{d}\left(x-\frac{1}{x}\right)}{\sqrt{\left(x-\frac{1}{x}\right)^2+2}} = \ln\left(x-\frac{1}{x}+\sqrt{x^2+\frac{1}{x^2}}\right)+C = \ln\left|\frac{x^2-1+\sqrt{x^4+1}}{x}\right|+C$

(17) $\displaystyle\int\sin^6 x\,\mathrm{d}x = \int\left(\frac{1-\cos 2x}{2}\right)^3\mathrm{d}x = \frac{1}{8}\int(1-3\cos 2x+3\cos^2 2x-\cos^3 2x)\,\mathrm{d}x = \frac{1}{8}x-\frac{3}{16}\sin 2x+\frac{3}{16}\int(1+\cos 4x)\,\mathrm{d}x-\frac{1}{16}\int\cos^2 2x\,\mathrm{d}\sin 2x = \frac{1}{8}x-\frac{3}{16}\sin 2x+\frac{3}{16}x+\frac{3}{64}\sin 4x-\frac{1}{16}\sin 2x+\frac{1}{48}\sin^3 2x+C = \frac{5}{16}x-\frac{1}{4}\sin 2x+\frac{3}{64}\sin 4x+\frac{1}{48}\sin^3 2x+C$

(18) $\displaystyle\int\sin^2 x\cos^4 x\,\mathrm{d}x = -\frac{1}{5}\int\sin x\,\mathrm{d}\cos^5 x = -\frac{1}{5}\sin x\cos^5 x+\frac{1}{5}\int\cos^6 x = -\frac{1}{5}\sin x\cos^5 x+\frac{1}{5}\left[\frac{5}{16}x-\frac{1}{4}\sin 2x+\frac{3}{64}\sin 4x+\frac{1}{48}\sin^3 2x\right]+C = \frac{1}{16}x-\frac{1}{20}\sin 2x+\frac{3}{320}\sin 4x+\frac{1}{240}\sin^3 2x-\frac{1}{5}\sin x\cos^5 x+C$

(19) $\displaystyle\int\sin^4 x\cos^4 x\,\mathrm{d}x = \int\left(\frac{\sin 2x}{2}\right)^4\mathrm{d}x = \frac{1}{16}\int\left(\frac{1-\cos 4x}{2}\right)^2\mathrm{d}x = \frac{1}{64}\int(1-2\cos 4x+\cos^2 4x)\,\mathrm{d}x = \frac{3}{128}x-\frac{\sin 4x}{128}+\frac{1}{1024}\sin 8x+C$

(20) $\displaystyle\int\frac{\cos^4 x}{\sin^3 x}\,\mathrm{d}x = -\frac{1}{2}\int\cos^3 x\,\mathrm{d}\frac{1}{\sin^2 x} = -\frac{1}{2}\cdot\frac{\cos^3 x}{\sin^2 x}-\frac{3}{2}\int\frac{\cos^2 x}{\sin x}\,\mathrm{d}x = -\frac{\cos^3 x}{2\sin^2 x}-\frac{3}{2}\int\frac{\mathrm{d}x}{\sin x}+\frac{3}{2}\int\sin x\,\mathrm{d}x = -\frac{\cos^3 x}{2\sin^2 x}-\frac{3}{2}\int\frac{\sec^2\frac{x}{2}}{\tan\frac{x}{2}}\,\mathrm{d}\frac{x}{2}-\frac{3}{2}\cos x = -\frac{\cos^3 x}{2\sin^2 x}-\frac{3}{2}\ln\left|\tan\frac{x}{2}\right|-\frac{3}{2}\cos x+C$

(21) $\displaystyle\int\frac{\mathrm{d}x}{\sin^3 x\cos^5 x} = \int\frac{\sin^2 x+\cos^2 x}{\sin^3 x\cos^5 x}\,\mathrm{d}x = \int\frac{\mathrm{d}x}{\sin x\cos^5 x}+\int\frac{\mathrm{d}x}{\sin^3 x\cos^3 x} = \int\frac{\sin^2 x+\cos^2 x}{\sin x\cos^5 x}\,\mathrm{d}x+\int\frac{\sin^2 x+\cos^2 x}{\sin^3 x\cos^3 x}\,\mathrm{d}x = \int\frac{\sin x}{\cos^5 x}\,\mathrm{d}x+2\int\frac{\mathrm{d}x}{\sin x\cos^3 x}+\int\frac{\mathrm{d}x}{\sin^3 x\cos x} = \frac{1}{4}\cos^{-4} x+2\int\frac{\sin^2 x+\cos^2 x}{\sin x\cos^3 x}\,\mathrm{d}x+\int\frac{\sin^2 x+\cos^2 x}{\sin^3 x\cos x}\,\mathrm{d}x = \frac{1}{4}\sec^4 x+2\int\frac{\sin x}{\cos^3 x}\,\mathrm{d}x+3\int\frac{\mathrm{d}x}{\sin x\cos x}+\int\frac{\cos x}{\sin^3 x}\,\mathrm{d}x = \frac{1}{4}\sec^4 x+\sec^2 x+3\int\frac{\mathrm{d}\tan x}{\tan x}-\frac{1}{2}\csc^2 x = \frac{1}{4}\sec^4 x+\sec^2 x+3\ln|\tan x|-\frac{1}{2}\csc^2 x+C_1 = \frac{1}{4}\tan^4 x+\frac{3}{2}\tan^2 x-\frac{1}{2}\cot^2 x+3\ln|\tan x|+C$

(22) $\displaystyle\int\tan x\cdot\tan(x+a)\,\mathrm{d}x = \int\tan x\cdot\frac{\tan x+\tan a}{1-\tan x\tan a}\,\mathrm{d}x = \int\frac{\tan^2 x+\tan x\tan a+1-1}{1-\tan x\tan a}\,\mathrm{d}x = \int\frac{1+\tan^2 x}{1-\tan x\tan a}\,\mathrm{d}x-\int\mathrm{d}x = \int\frac{\mathrm{d}\tan x}{1-\tan x\tan a}-x = -\cot a\ln|1-\tan x\tan a|-x+C_1 = \cot a\ln\left|\frac{\cos x}{\cos(x+a)}\right|-x+C$

(23) $\int \sin 5x\cos x\,\mathrm{d}x = \frac{1}{2}\int(\sin 6x+\sin 4x)\,\mathrm{d}x = -\frac{1}{12}\cos 6x - \frac{1}{8}\cos 4x + C$

(24) $\int \frac{\sin^2 x}{1+\sin^2 x}\,\mathrm{d}x = \int \frac{1}{\csc^2 x+1}\,\mathrm{d}x = \int\left(1-\frac{\csc^2 x}{1+\csc^2 x}\right)\,\mathrm{d}x = x+\int\frac{\mathrm{d}\cot x}{2+\cot^2 x} = x+\frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}\cot x\right)+ C$

(25) 设$\sin(a-b)\neq 0$，
则$\int\frac{\mathrm{d}x}{\sin(x+a)\sin(x+b)} = \frac{1}{\sin(a-b)}\int\frac{\sin[(x+a)-(x+b)]}{\sin(x+a)\sin(x+b)}\,\mathrm{d}x = \frac{1}{\sin(a-b)}\int\left[\frac{\cos(x+b)}{\sin(x+b)}-\frac{\cos(x+a)}{\sin(x+a)}\right]\mathrm{d}x = \frac{1}{\sin(a-b)}\ln\left|\frac{\sin(x+b)}{\sin(x+a)}\right|+C$

(26) $I=\int xe^x\cos x\,\mathrm{d}x = xe^x\cos x-\int e^x(\cos x-x\sin x)\,\mathrm{d}x = xe^x\cos x-\int e^x\cos x\,\mathrm{d}x+\int xe^x\sin x\,\mathrm{d}x = xe^x\cos x-\frac{\sin x+\cos x}{2}e^x+xe^x\sin x-\int e^x(\sin x+x\cos x)\,\mathrm{d}x = xe^x\cos x-\frac{\sin x+\cos x}{2}e^x+xe^x\sin x-\frac{\sin x-\cos x}{2}e^x-\int xe^x\cos x\,\mathrm{d}x+C_1 = e^x(x\cos x+x\sin x-\sin x)-I+C_1$，
则$I=\int xe^x\cos x\,\mathrm{d}x = \frac{e^x}{2}(x\cos x+x\sin x-\sin x)+C$

(27) 令$tan\frac{x}{2}=t$，则$\sin x=\frac{2t}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2}$，$\mathrm{d}x=\frac{2\,\mathrm{d}t}{1+t^2}$
于是$\int\frac{\mathrm{d}x}{(2+\cos x)\sin x} = \int\frac{1+t^2}{t(3+t^2)}\,\mathrm{d}t = \int\left(\frac{1}{3t}+\frac{2t}{3(3+t^2)}\right)\,\mathrm{d}t = \frac{1}{3}\ln|t(t^2+3)|+C = \frac{1}{3}\ln\left|\tan\frac{x}{2}\left(\tan^2\frac{x}{2}+3\right)\right|+C = \frac{1}{6}\ln\frac{(1-\cos x)(2+\cos x)^2}{(1+\cos x)^3}+C$

(28) $\int\ln(x+\sqrt{1+x^2})^2\,\mathrm{d}x = x\ln(x+\sqrt{1+x^2})^2 - \int x\cdot\frac{1}{(x+\sqrt{1+x^2})^2}\cdot 2(x+\sqrt{1+x^2})\cdot\left(1+\frac{x}{\sqrt{1+x^2}}\right)\,\mathrm{d}x = x\ln(x+\sqrt{1+x^2})^2-\int\frac{\mathrm{d}(1+x^2)}{\sqrt{1+x^2}} = x\ln(x+\sqrt{1+x^2})^2-2\sqrt{1+x^2}+C$

(29) $\int\frac{\sin x\cos x}{\sin x+\cos x}\,\mathrm{d}x = \int\frac{\sin^2\left(x+\frac{\pi}{4}\right)-\frac{1}{2}}{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}\,\mathrm{d}x = \frac{\sqrt{2}}{2}\int\sin\left(x+\frac{\pi}{4}\right)\,\mathrm{d}x-\frac{1}{2\sqrt{2}}\int\frac{\mathrm{d}x}{\sin\left(x+\frac{\pi}{4}\right)} = -\frac{\sqrt{2}}{2}\cos\left(x+\frac{\pi}{4}\right)-\frac{1}{2\sqrt{2}}\int\frac{\mathrm{d}\left(\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right)}{\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)} = \frac{1}{2}(\sin x-\cos x)-\frac{\sqrt{2}}{4}\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|+C$

(30) $\int\frac{\ln x}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x = \int\ln x\,\mathrm{d}\left(\frac{x}{\sqrt{1+x^2}}\right) = \frac{x\ln x}{\sqrt{1+x^2}}-\int\frac{\mathrm{d}x}{\sqrt{1+x^2}} = \frac{x\ln x}{\sqrt{1+x^2}}-\ln(x+\sqrt{1+x^2})+C$

(31) $\int xe^x\sin x\,\mathrm{d}x = xe^x\sin x-\int e^x(\sin x+x\cos x)\,\mathrm{d}x = xe^x\sin x-\int e^x\sin x\,\mathrm{d}x-\int xe^x\cos x\,\mathrm{d}x = xe^x\sin x-\frac{\sin x-\cos x}{2}e^x-\frac{e^x}{2}(x\cos x+x\sin x-\sin x)+C = \frac{e^x}{2}(x\sin x-x\cos x+\cos x)+C$

(32) $\int\frac{x^3\arccos x}{\sqrt{1-x^2}}\,\mathrm{d}x = -x^2\arccos x\sqrt{1-x^2}+2\int x\arccos x\sqrt{1-x^2}\,\mathrm{d}x-\int x^2\,\mathrm{d}x = -x^2\sqrt{1-x^2}\arccos x-\frac{x^3}{3}-\frac{2}{3}(1-x^2)^{\frac{3}{2}}\arccos x-\frac{2}{3}\int(1-x^2)\,\mathrm{d}x = -x^2\sqrt{1-x^2}\arccos x-\frac{x^3}{3}-\frac{2}{3}(1-x^2)^{\frac{3}{2}}\arccos x-\frac{2}{3}\left(x-\frac{x^3}{3}\right)+C = -\frac{6x+x^3}{9}-\frac{2+x^2}{3}\sqrt{1-x^2}\arccos x+C$

(33) $\int(x+|x|)^2\,\mathrm{d}x = \int(2x^2+2x|x|)\,\mathrm{d}x = \frac{2}{3}x^3+2\int x\cdot sgnx\cdot x\,\mathrm{d}x = \frac{2}{3}x^3+\frac{2}{3}x^3 sgnx+C = \frac{2}{3}x^3+\frac{2}{3}x^2|x|+C$

(34) $\int x^2e^x\cos x\,\mathrm{d}x = x^2e^x\cos x-\int e^x(2x\cos x-x^2\sin x)\,\mathrm{d}x = x^2e^x\cos x-e^x(x\cos x+x\sin x-\sin x)+x^2e^x\sin x-\int e^x(2x\sin x+x^2\cos x)\,\mathrm{d}x = x^2e^x(\cos x+\sin x)-e^x(x\cos x+x\sin x-\sin x)-e^x(x\sin x-x\cos x+\cos x)-\int x^2e^x\cos x\,\mathrm{d}x+C_1 = e^x(x^2\cos x+x^2\sin x-2x\sin x+\sin x-\cos x)-I+C_1$，
则$I=\int x^2e^x\cos x\,\mathrm{d}x = \frac{e^x}{2}(x^2\cos x+x^2\sin x-2x\sin x+\sin x-\cos x)+C$

(35) $\int \frac{xe^x}{(1+x)^2}\,\mathrm{d}x = -\int xe^x\,\mathrm{d}\frac{1}{1+x} = -\frac{xe^x}{1+x} + \int e^x\,\mathrm{d}x = -\frac{xe^x}{1+x} + e^x + C = \frac{e^x}{x+1} + C$

(36) $\int \sqrt{x}\ln^2 x\,\mathrm{d}x = \frac{2}{3}\ln^2 x\cdot x^{\frac{3}{2}} - \frac{4}{3}\int x^{\frac{1}{2}}\ln x\,\mathrm{d}x = \frac{2}{3}\ln^2 x\cdot x^{\frac{3}{2}} - \frac{8}{9}x^{\frac{3}{2}}\ln x + \frac{8}{9}\int\sqrt{x}\,\mathrm{d}x = \frac{2}{3}\ln^2 x\cdot x^{\frac{3}{2}} - \frac{8}{9}x^{\frac{3}{2}}\ln x + \frac{16}{27}x^{\frac{3}{2}} + C = \frac{2}{27}x^{\frac{3}{2}}(9\ln^2 x - 12\ln x + 8) + C$

(37) 令$t = \sqrt{\frac{b-x}{x-a}}$，则$x = \frac{b+at^2}{1+t^2}, x-a = \frac{b-a}{1+t^2}, \mathrm{d}x = -\frac{2(b-a)t}{(1+t^2)^2}\,\mathrm{d}t$，
于是$\int \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}} = -2\int\frac{\mathrm{d}t}{1+t^2} = -2\arctan t + C = -2\arctan\sqrt{\frac{b-x}{x-a}} + C$

(38) $\int x\ln\frac{1+x}{1-x}\,\mathrm{d}x = \frac{x^2}{2}\ln\frac{1+x}{1-x} - \int\frac{x^2}{1-x^2}\,\mathrm{d}x = \frac{x^2}{2}\ln\frac{1+x}{1-x} - \int\frac{\mathrm{d}x}{1-x^2} + \int\mathrm{d}x = \frac{1}{2}x^2\ln\frac{1+x}{1-x} - \frac{1}{2}\ln\frac{1+x}{1-x} + x + C$

(39) $\int x\arctan x\cdot\ln(1+x^2)\,\mathrm{d}x = \frac{x^2}{2}\arctan x\ln(1+x^2) - \frac{1}{2}\int x^2\left[\frac{\ln(1+x^2)}{1+x^2} + \frac{2x\arctan x}{1+x^2}\right]\mathrm{d}x = \frac{x^2}{2}\arctan x\ln(1+x^2) - \frac{1}{2}\int\ln(1+x^2)\,\mathrm{d}x + \frac{1}{2}\int\frac{\ln(1+x^2)}{1+x^2}\,\mathrm{d}x - \int x\arctan x\,\mathrm{d}x + \int\frac{x\arctan x}{1+x^2}\,\mathrm{d}x = \frac{x^2}{2}\arctan x\ln(1+x^2) - \frac{x}{2}\ln(1+x^2) + \int\frac{x^2}{1+x^2}\,\mathrm{d}x + \frac{1}{2}\arctan x\ln(1+x^2) - \int\frac{x\arctan x}{1+x^2}\,\mathrm{d}x + \int\frac{x\arctan x}{1+x^2}\,\mathrm{d}x - \frac{x^2}{2}\arctan x + \frac{1}{2}\int\frac{x^2}{1+x^2}\,\mathrm{d}x = \frac{x^2}{2}\arctan x\ln(1+x^2) - \frac{x}{2}\ln(1+x^2) + \frac{3}{2}x - \frac{3}{2}\arctan x + \frac{1}{2}\arctan x\ln(1+x^2) - \frac{x^2}{2}\arctan x + C = \frac{1}{2}\arctan x[x^2\ln(1+x^2) + \ln(1+x^2) - x^2 - 3] - \frac{x}{2}\ln(1+x^2) + \frac{3}{2}x + C$

(40) $\int\sinh^2 x\cosh^2 x\,\mathrm{d}x = \frac{1}{4}\int\sinh^2 2x\,\mathrm{d}x = \frac{1}{8}\int(\cosh 4x - 1)\,\mathrm{d}x = \frac{1}{32}\sinh 4x - \frac{x}{8} + C$

第七章　定积分

§1　定积分的概念

利用定积分的定义计算积分：

(1) $\int_0^l f(x)\,\mathrm{d}x$，其中$f(x)=ax+b, a, b$是常数

(2) $\int_{-1}^2 x^2\,\mathrm{d}x$

(3) $\int_0^1 a^x\,\mathrm{d}x$

解：

(1) 因$f(x)$在$[0,l]$上连续，故定积分必存在，据定积分定义，将区间$[0,l]$$n$等分，则每一子区间的长为$\Delta x_i=\dfrac{l}{n}$，取$\xi_i$为每个子区间$[x_{i-1},x_i]$的右端点，即$\xi_i=\dfrac{i}{n}l(i=1,2,\cdots,n)$.

作积分和$\sum\limits_{i=1}^n f(\xi_i)\Delta x_i=\sum\limits_{i=1}^n(a\xi_i+b)\Delta x_i=\sum\limits_{i=1}^n\left(\dfrac{ia}{n}l+b\right)\dfrac{l}{n}=\sum\limits_{i=1}^n(nb+ia)\dfrac{l}{n^2}=bl+\dfrac{n+1}{2n}al^2$，

于是$\int_0^l f(x)\,\mathrm{d}x=\lim\limits_{||x||=\frac{1}{n}\to 0}\sum\limits_{i=1}^n(a\xi_i+b)\Delta x_i=\lim\limits_{n\to\infty}\left(bl+\dfrac{n+1}{2n}al^2\right)=bl+\dfrac{a}{2}l^2$

(2) 因x^2在$[-1,2]$上连续，故定积分必存在，据定积分定义，将区间$[-1,2]$$n$等分，则每一子区间的长为$\Delta x_i=\dfrac{3}{n}$，取$\xi_i$为每个子区间$[x_{i-1},x_i]$的右端点，即$\xi_i=-1+\dfrac{3i}{n}=\dfrac{3i-n}{n}(i=1,2,\cdots,n)$.

作积分和$\sum\limits_{i=1}^n f(\xi_i)\Delta x_i=\sum\limits_{i=1}^n\xi_i^2\Delta x_i=\sum\limits_{i=1}^n\left(\dfrac{3i-n}{n}\right)^2\dfrac{3}{n}=\sum\limits_{i=1}^n\dfrac{3}{n^3}(9i^2-6ni+n^2)=\dfrac{9}{2}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right)-9\left(1+\dfrac{1}{n}\right)+3$，

于是$\int_{-1}^2 x^2\,\mathrm{d}x=\lim\limits_{||x||=\frac{1}{n}\to 0}\sum\limits_{i=1}^n\xi_i^2\Delta x_i=\lim\limits_{n\to\infty}\left[\dfrac{9}{2}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right)-9\left(1+\dfrac{1}{n}\right)+3\right]=3$

(3) 因a^x在$[0,1]$上连续，故定积分必存在，据定积分定义，将区间$[0,1]$$n$等分，则每一子区间的长为$\Delta x_i=\dfrac{1}{n}$，取$\xi_i$为每个子区间$[x_{i-1},x_i]$的右端点，即$\xi_i=\dfrac{i}{n}(i=1,2,\cdots,n)$.

作积分和$\sum\limits_{i=1}^n f(\xi_i)\Delta x_i=\sum\limits_{i=1}^n a^{\xi_i}\Delta x_i=\sum\limits_{i=1}^n\dfrac{1}{n}a^{\frac{i}{n}}=\begin{cases}\dfrac{a^{\frac{1}{n}}(1-a)}{n(1-a^{\frac{1}{n}})}, & a\neq 1\\ 1, & a=1\end{cases}$ 于是$\int_0^1 a^x\,\mathrm{d}x=\lim\limits_{||x||=\frac{1}{n}\to 0}\sum\limits_{i=1}^n a^{\xi_i}\Delta x_i=\begin{cases}\lim\limits_{n\to\infty}\dfrac{a^{\frac{1}{n}}(1-a)}{n(1-a^{\frac{1}{n}})}=\dfrac{a-1}{\ln a}, & a\neq 1\\ 1, & a=1\end{cases}$

§2 定积分存在的条件

1. 判断下列函数的可积性：

   (1) $f(x)$在$[-2,2]$上有界，它的不连续点是$\dfrac{1}{n}(n=1,2,3,\cdots)$

   (2) $f(x)=sgn\left(\sin\dfrac{\pi}{x}\right)$，在$[0,1]$.

   解

   (1) $f(x)$在$[-2,2]$上是可积的.
   因$f(x)$在$[-2,2]$上有界，故$\exists M>0$，使$|f(x)|\leqslant M,\forall x\in[-2,2]$，从而其振幅$\omega(f)\leqslant 2M$.
   $\forall\varepsilon>0$，取自然数N满足$N=\left[\dfrac{2M}{\varepsilon}\right]+1$，于是在$\left[\dfrac{1}{N},2\right]$上$f(x)$只有有限多个不连续点，因而$f(x)$在$\left[\dfrac{1}{N},2\right]$上可积.
   在$\left[0,\dfrac{1}{N}\right]$上，将其分割为部分区间$\Delta x_i$，第$i$个小区间$\Delta x_i$上的振幅设为$\omega_i(f)\leqslant\omega(f)\leqslant 2M$，则$\sum\limits_{i=1}^{n}\omega_i(f)\Delta x_i\leqslant 2M\sum\limits_{i=1}^{n}\Delta x_i\leqslant\dfrac{2M}{N}<2M\cdot\dfrac{\varepsilon}{2M}=\varepsilon$，故$f(x)$在$\left[0,\dfrac{1}{N}\right]$上也是可积的.
   又由于$f(x)$在$[-2,0]$上连续，当然可积.
   据积分关于区间可加性，得$f(x)$在$[-2,2]$上可积.

   (2) 补充定义$f(0)=0$.
   由于$f(x)$在$[0,1]$上有界，又$sgn\left(\sin\dfrac{\pi}{x}\right)$只在$x=0,\dfrac{1}{n}(n=1,2,\cdots)$间断，故由本题（1），得$f(x)$在$[0,1]$上可积.

2. 若函数$f(x)$在$[a,b]$上可积，其积分是I，今在$[a,b]$内有限个点上改变$f(x)$的值使它成为另一个函数$f^*(x)$，证明$f^*(x)$也在$[a,b]$上可积，并且其积分仍为I.
   证明：令$F(x)=f(x)-f^*(x)$，则$F(x)$在$[a,b]$上除改变了$f(x)$的函数值的有限个点外均为0，即除这有限个点外，函数连续，从而$F(x)$在$[a,b]$上可积，且积分为0.
   又$f^*(x)=f(x)-F(x)$，据可积函数的差仍可积，
   有$\displaystyle\int_a^b f^*(x)\,\mathrm{d}x=\int_a^b f(x)\,\mathrm{d}x-\int_a^b F(x)\,\mathrm{d}x=\int_a^b f(x)\,\mathrm{d}x=I.$

3. 讨论$f,f^2,|f|$三者间可积性的关系.
   解：

   (1) 若$f(x)$在$[a,b]$上可积，则$f^2(x)$在$[a,b]$上也可积.
   因$f(x)$在$[a,b]$上可积，故$f(x)$在$[a,b]$上必有界. 设$f(x)\leqslant M,M$为常数$(x\in[a,b])$
   在区间$[x_{i-1},x_i]$上任取两点x',x''，考虑$f^2(x'')-f^2(x')=[f(x'')-f(x')][f(x'')+f(x')]$
   取ω_i表示$f(x)$在$[x_{i-1},x_i]$上的幅度，则$|f^2(x'')-f^2(x')|\leqslant 2\omega_i M$
   若$f^2(x)$在$[x_{i-1},x_i]$上的幅度为Ω_i，就有$\Omega_i\leqslant 2\omega_i M$，从而有$\sum\limits_i\Omega_i\Delta x_i\leqslant 2M\sum\limits_i\omega_i\Delta x_i$
   由于$f(x)$可积，有$\sum\limits_i\omega_i\Delta x_i\to 0(\lambda(\Delta)\to 0)$，则$\sum\limits_i\Omega_i\Delta x_i\to 0(\lambda(\Delta)\to 0)$，这就说明了$f^2(x)$在$[a,b]$上的可积性.

   (2) 若$f(x)$在$[a,b]$上可积，则$|f(x)|$在$[a,b]$上也可积.
   分别把函数$f(x)$与$|f(x)|$在区间$[x_{i-1},x_i]$上的幅度记为ω_i,ω_i^*.
   因对属于$[x_{i-1},x_i]$的任意两点x',x''，有$||f(x')|-|f(x'')||\leqslant|f(x')-f(x'')|$，故有$\omega_i^*\leqslant\omega_i$，于是$\sum\limits_i\omega_i^*\Delta x_i\leqslant\sum\limits_i\omega_i\Delta x_i$
   由于$\sum\limits_i\omega_i\Delta x_i\to 0(\lambda(\Delta)\to 0)$，就可以推得$\lambda(\Delta)\to 0$时，也有$\sum\limits_i\omega_i^*\Delta x_i\to 0$，这就说明了$|f(x)$在$[a,b]$上的可积性.

   (3) 若$|f(x)|$在$[a,b]$上可积，不能肯定$f(x)$在$[a,b]$上也可积.
   例：$f(x)=\begin{cases}1, & x\text{为有理数}\\ -1, & x\text{为无理数}\end{cases}$
   $|f(x)|=1$在任何闭区间上可积，但$f(x)$却在任何闭区间上都不可积.

(4) 若$f^2(x)$在$[a,b]$上可积，不能肯定$f(x)$在$[a,b]$上也可积.

例：$f(x)=\begin{cases}1, & x\text{为有理数}\\ -1, & x\text{为无理数}\end{cases}$

$f^2(x)=1$在任何闭区间上可积，但$f(x)$却在任何闭区间上都不可积.

(5) 若$|f(x)|$在$[a,b]$上可积，则由(1)，得$f^2(x)=|f(x)|^2$在$[a,b]$上一定可积.

(6) 若$f^2(x)$在$[a,b]$上可积，则由$|f(x)|=\sqrt{f^2(x)}$，外层函数连续，里层函数可积，得此函数必可积.

4. 若函数$f(x)$在$[a,b]$上可积，证明存在折线函数列$\varphi_n(x)(n=1,2,3,\cdots)$使得$\int_a^b f(x)\,\mathrm{d}x=\lim\limits_{n\to\infty}\int_a^b \varphi_n(x)\,\mathrm{d}x$

证明：将$[a,b]n$等分，设分点为$a=x_0^{(n)}<x_1^{(n)}<\cdots<x_{n-1}^{(n)}<x_n^{(n)}=b$即$x_i^{(n)}=a+\dfrac{i}{n}(b-a),i=0,1,\cdots,n$

在$[x_{i-1}^{(n)},x_i^{(n)}]$上令$\varphi_n(x)$为过点$(x_{i-1}^{(n)},f(x_{i-1}^{(n)}))$及$(x_i^{(n)},f(x_i^{(n)}))$的直线，即当$x\in(x_{i-1}^{(n)},x_i^{(n)})$时，令$\varphi_n(x)=f(x_{i-1}^{(n)})+\dfrac{x-x_{i-1}^{(n)}}{x_i^{(n)}-x_{i-1}^{(n)}}(f(x_i^{(n)})-f(x_{i-1}^{(n)}))$

则$\varphi_n(x)$是$[a,b]$上的一个折线函数列，当然是连续函数列，因此$\int_a^b\varphi_n(x)\,\mathrm{d}x$有定义.

若令$m_i^{(n)},M_i^{(n)}$及$\omega_i^{(n)}$分别表示函数$f(x)$在$[x_{i-1}^{(n)},x_i^{(n)}]$上的下确界、上确界及振幅，则当$x\in[x_{i-1}^{(n)},x_i^{(n)}]$时，$m_i^{(n)}\leqslant\varphi_n(x)\leqslant M_i^{(n)},m_i^{(n)}\leqslant f(x)\leqslant M_i^{(n)}$，从而$|\varphi_n(x)-f(x)|\leqslant\omega_i^{(n)}$

于是，有$\left|\int_a^b f(x)\,\mathrm{d}x-\int_a^b\varphi_n(x)\,\mathrm{d}x\right|\leqslant\int_a^b|f(x)-\varphi_n(x)|\,\mathrm{d}x=\sum\limits_{i=1}^n\int_{x_{i-1}^{(n)}}^{x_i^{(n)}}|f(x)-\varphi_n(x)|\,\mathrm{d}x\leqslant\sum\limits_{i=1}^n\omega_i^{(n)}\Delta x_i^{(n)}$

由于$f(x)$在$[a,b]$上可积，故当$\max|\Delta x_i^{(n)}|=\dfrac{b-a}{n}\to 0$时，必有$\sum\limits_{i=1}^n\omega_i^{(n)}\Delta x_i^{(n)}\to 0$，因而$\int_a^b f(x)\,\mathrm{d}x=\lim\limits_{n\to\infty}\int_a^b\varphi_n(x)\,\mathrm{d}x$.

5. 若函数$f(x)$在$[A,B]$可积，证明$\lim\limits_{h\to 0}\int_a^b|f(x+h)-f(x)|\,\mathrm{d}x=0$，其中$A<a<b<B$(这一性质称为积分的连续性)

证明：因$f(x)$在$[A,B]$可积，由上题结论，对$\forall\varepsilon>0$，存在$[A,B]$上的连续函数$\varphi(x)$，使$\int_A^B|f(x)-\varphi(x)|\,\mathrm{d}x<\dfrac{\varepsilon}{4}$

因$\varphi(x)$在$[A,B]$连续，从而一致连续，则对上述$\varepsilon>0$，存在$\delta>0$，对$[A,B]$中任意两点x',x''，只要$|x'-x''|<\delta$，就有$|\varphi(x')-\varphi(x'')|<\dfrac{\varepsilon}{2(b-a)}$

于是当$|h|<\delta$时，有$\int_a^b|f(x+h)-f(x)|\,\mathrm{d}x=\int_a^b|f(x+h)-\varphi(x+h)+\varphi(x+h)-\varphi(x)+\varphi(x)-f(x)|\,\mathrm{d}x\leqslant\int_a^b|f(x+h)-\varphi(x+h)|\,\mathrm{d}x+\int_a^b|\varphi(x+h)-\varphi(x)|\,\mathrm{d}x+\int_a^b|f(x)-\varphi(x)|\,\mathrm{d}x\leqslant\int_A^B|f(x+h)-\varphi(x+h)|\,\mathrm{d}x+\int_a^b|\varphi(x+h)-\varphi(x)|\,\mathrm{d}x+\int_A^B|f(x)-\varphi(x)|\,\mathrm{d}x<\dfrac{\varepsilon}{4}+\int_a^b\dfrac{\varepsilon}{2(b-a)}\,\mathrm{d}x+\dfrac{\varepsilon}{4}=\varepsilon$

从而$\lim\limits_{h\to 0}\int_a^b|f(x+h)-f(x)|\,\mathrm{d}x=0$

§3 定积分的性质

1. 若$f(x),g(x)$在$[a,b]$可积，证明$f(x)+g(x)$也在$[a,b]$可积，并且$\int_a^b (f(x)+g(x))\,\mathrm{d}x=\int_a^b f(x)\,\mathrm{d}x+\int_a^b g(x)\,\mathrm{d}x$.

证明：因$f(x),g(x)$在$[a,b]$可积，即$\int_a^b f(x)\,\mathrm{d}x,\int_a^b g(x)\,\mathrm{d}x$存在，故对任意分法$\Delta: a=x_0<x_1<\cdots<x_n=b$以及$[x_{i-1},x_i]$中任意$\xi_i$，有$\lim\limits_{\lambda(\Delta)\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i=\int_a^b f(x)\,\mathrm{d}x$

由分法Δ及ξ_i的任意性，得$g(x)$在此任意分法下，对上述$[x_{i-1},x_i]$中的ξ_i，也有$\lim\limits_{\lambda(\Delta)\to 0}\sum\limits_{i=1}^{n} g(\xi_i)\Delta x_i=\int_a^b g(x)\,\mathrm{d}x$，

于是$\int_a^b f(x)\,\mathrm{d}x+\int_a^b g(x)\,\mathrm{d}x=\lim\limits_{\lambda(\Delta)\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i+\lim\limits_{\lambda(\Delta)\to 0}\sum\limits_{i=1}^{n} g(\xi_i)\Delta x_i=\lim\limits_{\lambda(\Delta)\to 0}\sum\limits_{i=1}^{n}(f(\xi_i)+g(\xi_i))\Delta x_i=\int_a^b (f(x)+g(x))\,\mathrm{d}x$

从而$f(x)+g(x)$也在$[a,b]$可积，并且$\int_a^b (f(x)+g(x))\,\mathrm{d}x=\int_a^b f(x)\,\mathrm{d}x+\int_a^b g(x)\,\mathrm{d}x$.

2. 设$f(x)=\begin{cases} 1, & \text{当}x\text{为有理数} \\ -1, & \text{当}x\text{为无理数} \end{cases}$

证明：$|f(x)|$在任何区间$[a,b]$上可积，但$f(x)$在$[a,b]$不可积.

证明：因$f(x)=\begin{cases} 1, & \text{当}x\text{为有理数} \\ -1, & \text{当}x\text{为无理数} \end{cases}$，故$|f(x)|=1$，

则$|f(x)|$在$[a,b]$上连续，从而$|f(x)|$在$[a,b]$上可积

对于函数$f(x)$，在$[a,b]$的任一部分区间$[x_{i-1},x_i](i=1,2,\cdots,n)$上$\omega_i=2$，故$\sum\limits_{i=1}^{n}\omega_i\Delta x_i=2\sum\limits_{i=1}^{n}\Delta x_i=2(b-a)\nrightarrow 0(n\to\infty)$，于是函数$f(x)$在$[a,b]$上不可积.

3. 设$f(x)$在$[a,b]$连续，$f(x)\geqslant 0$，$f(x)$不恒为零，证明$\int_a^b f(x)\,\mathrm{d}x>0$.

证明：因$f(x)\geqslant 0$且不恒为零，则必存在$x_0\in[a,b]$，使得$f(x_0)>0$

由连续函数的局部保号性，存在$0<\delta\leqslant\min\left(\dfrac{x_0-a}{2},\dfrac{b-x_0}{2}\right)$，使当$x\in[x_0-\delta,x_0+\delta]$时，$f(x)>\dfrac{f(x_0)}{2}>0$，于是有$\int_a^b f(x)\,\mathrm{d}x\geqslant\int_{x_0-\delta}^{x_0+\delta} f(x)\,\mathrm{d}x\geqslant\int_{x_0-\delta}^{x_0+\delta}\dfrac{f(x_0)}{2}\,\mathrm{d}x=f(x_0)\delta>0$.

4. 比较下列各题中积分的大小：

(1) $\int_0^1 x\,\mathrm{d}x,\int_0^1 x^2\,\mathrm{d}x$

(2) $\int_0^{\frac{\pi}{2}} x\,\mathrm{d}x,\int_0^{\frac{\pi}{2}}\sin x\,\mathrm{d}x$

(3) $\int_{-2}^{-1}\left(\dfrac{1}{3}\right)^x\mathrm{d}x,\int_0^1 3^x\,\mathrm{d}x$

解：

(1) 因$x\in(0,1)$时，$x>x^2$，则$\int_0^1 x\,\mathrm{d}x>\int_0^1 x^2\,\mathrm{d}x$

(2) 因$x\in\left(0,\dfrac{\pi}{2}\right)$时，$x>\sin x$，则$\int_0^{\frac{\pi}{2}} x\,\mathrm{d}x>\int_0^{\frac{\pi}{2}}\sin x\,\mathrm{d}x$

(3) 因$\int_{-2}^{-1}\left(\dfrac{1}{3}\right)^x\mathrm{d}x=\int_0^1\left(\dfrac{1}{3}\right)^{x-2}\mathrm{d}x=\int_0^1 3^{2-x}\,\mathrm{d}x$且当$x\in(0,1)$时，$2-x>x$，故$3^{2-x}>3^x$，

从而$\int_{-2}^{-1}\left(\dfrac{1}{3}\right)^x\mathrm{d}x>\int_0^1 3^x\,\mathrm{d}x$

5. 设$f(x)$在$[a,b]$连续，$\int_a^b f^2(x)\,\mathrm{d}x=0$，证明$f(x)$在$[a,b]$上恒为零.

证明：用反证法.假设$f(x)$在$[a,b]$上不恒为零，则$f^2(x)\geqslant 0$且不恒为零.

又$f(x)$在$[a,b]$连续，故$f^2(x)$在$[a,b]$连续，
则据第3题可知$\int_a^b f^2(x)\,\mathrm{d}x>0$，这与已知$\int_a^b f^2(x)\,\mathrm{d}x=0$矛盾.
于是假设错误，从而$f(x)$在$[a,b]$上恒为零.

6. 举例说明：$f^2(x)$在$[a,b]$可积，但$f(x)$在$[a,b]$不可积.
解：例：$f(x)=\begin{cases}1, & \text{当}x\text{为有理数}\\ -1, & \text{当}x\text{为无理数}\end{cases}$，故$f^2(x)=1$，
则$f^2(x)$在$[a,b]$上连续，从而$f^2(x)$在$[a,b]$上可积
又由第二题可知，函数$f(x)$在$[a,b]$上不可积.

7. 设$f(x),g(x)$在$[a,b]$连续，证明$\lim\limits_{\max(\Delta x_i)\to 0}\sum\limits_{i=1}^n f(\xi_i)g(\theta_i)\Delta x_i=\int_a^b f(x)g(x)\,\mathrm{d}x$，其中$x_{i-1}\leqslant\xi_i\leqslant x_i, x_{i-1}\leqslant\theta_i\leqslant x_i(i=1,2,\cdots,n),\Delta x_i=x_i-x_{i-1}(x_0=a,x_n=b)$.
证明：因$f(x),g(x)$在$[a,b]$连续，则$f(x)\cdot g(x)$在$[a,b]$连续，故$f(x)\cdot g(x)$在$[a,b]$可积，即$\lim\limits_{\max(\Delta x_i)\to 0}\sum\limits_{i=1}^n f(\xi_i)g(\xi_i)\Delta x_i=\int_a^b f(x)g(x)\,\mathrm{d}x$
而$\lim\limits_{\max(\Delta x_i)\to 0}\sum\limits_{i=1}^n f(\xi_i)g(\theta_i)\Delta x_i=\lim\limits_{\max(\Delta x_i)\to 0}\sum\limits_{i=1}^n f(\xi_i)g(\xi_i)\Delta x_i+\lim\limits_{\max(\Delta x_i)\to 0}\left[\sum\limits_{i=1}^n f(\xi_i)g(\theta_i)\Delta x_i-\sum\limits_{i=1}^n f(\xi_i)g(\xi_i)\Delta x_i\right]$
又$\left|\sum\limits_{i=1}^n f(\xi_i)g(\theta_i)\Delta x_i-\sum\limits_{i=1}^n f(\xi_i)g(\xi_i)\Delta x_i\right|=\left|\sum\limits_{i=1}^n f(\xi_i)(g(\theta_i)-g(\xi_i))\Delta x_i\right|\leqslant\sum\limits_{i=1}^n|f(\xi_i)||g(\theta_i)-g(\xi_i)|\Delta x_i\leqslant\sum\limits_{i=1}^n M(f)\omega_i(g)\Delta x_i=M(f)\sum\limits_{i=1}^n\omega_i(g)\Delta x_i$，其中$M(f)$表示$|f|$在$[a,b]$上的上界，$\omega_i(g)$表示$g$在$[x_{i-1},x_i]$上的振幅.
由f的连续性和g的可积性，当$\max(\Delta x_i)\to 0$时，上面不等式右端$M(f)\sum\limits_{i=1}^n\omega_i(g)\Delta x_i\to 0$，从而$\lim\limits_{\max(\Delta x_i)\to 0}\sum\limits_{i=1}^n f(\xi_i)g(\theta_i)\Delta x_i=\int_a^b f(x)g(x)\,\mathrm{d}x$.

8. 设$y=\varphi(x)(x\geqslant 0)$是严格单调增加的连续函数，$\varphi(0)=0,x=\psi(x)$是它的反函数，证明$\int_0^a\varphi(x)\,\mathrm{d}x+\int_a^b\psi(y)\,\mathrm{d}y\geqslant ab(a\geqslant 0,b\geqslant 0)$.
证明：由$y=\varphi(x)$也是严格单调增加的连续函数，$\varphi(0)=0$知其反函数$x=\psi(y)$是严格单调增加的连续函数，且$\psi(0)=0$，因而$\int_0^a\varphi(x)\,\mathrm{d}x,\int_0^b\psi(y)\,\mathrm{d}y$有定义
令$g(x)=bx-\int_0^x\varphi(t)\,\mathrm{d}t$，特别地，有

$$g(a)=ab-\int_0^a\varphi(t)\,\mathrm{d}t \tag{1}$$

而且$g'(x)=b-\varphi(x)$
由$\varphi(x)$是严格单调增加的连续函数，因此当$0<x<\psi(b)$时，有$g'(x)>0$；当$x>\psi(b)$时，有$g'(x)<0$；当$x=\psi(b)$时，有$g'(x)=0$，因此当$x=\psi(b)$时，$g(x)$取最大值，即有

$$g(a)\leqslant\max g(x)=g(\psi(b)) \tag{2}$$

分部积分，得$\int_0^{\psi(b)}x\varphi'(x)\,\mathrm{d}x=b\psi(b)-\int_0^{\psi(b)}\varphi(x)\,\mathrm{d}x=g(\psi(b))$，用变量代换$y=\varphi(x)$，则$x=\psi(y)$，于是

$$g(\psi(b))=\int_0^{\psi(b)}x\varphi'(x)\,\mathrm{d}x=\int_0^b\psi(y)\,\mathrm{d}y \tag{3}$$

将(**??**)、(**??**)代入(**??**)就得到$\int_0^a\varphi(x)\,\mathrm{d}x+\int_a^b\psi(y)\,\mathrm{d}y\geqslant ab(a\geqslant 0,b\geqslant 0)$.

§4 定积分的计算

1. 计算下列定积分：

(1) $\displaystyle\int_1^2 \frac{(x+1)(x^2-3)}{3x^2}\,\mathrm{d}x$

(2) $\displaystyle\int_1^{\frac{\pi}{2}} (a\sin x + b\cos x)\,\mathrm{d}x$

(3) $\displaystyle\int_0^1 \left(\frac{x-1}{x+1}\right)^4 \mathrm{d}x$

(4) $\displaystyle\int_0^1 \frac{x^2+1}{x^4+1}\,\mathrm{d}x$

(5) $\displaystyle\int_0^{\frac{1}{\sqrt{5}}} x^3(1-5x^2)^{10}\,\mathrm{d}x$

(6) $\displaystyle\int_0^1 x^2(2-3x^2)^2\,\mathrm{d}x$

(7) $\displaystyle\int_{-\frac{1}{5}}^{\frac{1}{5}} x\sqrt{2-5x}\,\mathrm{d}x$

(8) $\displaystyle\int_0^{\frac{\pi}{2}} \sin mx\cos nx\,\mathrm{d}x$

(9) $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{18x-4}{\sqrt{9x^2+6x+5}}\,\mathrm{d}x$

(10) $\displaystyle\int_0^{\sqrt{\ln 2}} x^3 e^{-x^2}\,\mathrm{d}x$

(11) $\displaystyle\int_0^1 x\arctan x\,\mathrm{d}x$

(12) $\displaystyle\int_0^{2\pi} x\cos^2 x\,\mathrm{d}x$

(13) $\displaystyle\int_{-\pi}^{\pi} x^2\cos x\,\mathrm{d}x$

(14) $\displaystyle\int_0^{\frac{2\pi}{\omega}} \sin\omega t\sin(\omega t+\varphi)\,\mathrm{d}t$

(15) $\displaystyle\int_0^3 \frac{x\,\mathrm{d}x}{1+\sqrt{1+x}}$

(16) $\displaystyle\int_0^4 x(x+\sqrt{x})\,\mathrm{d}x$

(17) $\displaystyle\int_{-\pi}^{\pi} \sin mx\sin nx\,\mathrm{d}x$

(18) $\displaystyle\int_{-\pi}^{\pi} \sin mx\cos nx\,\mathrm{d}x$

(19) $\displaystyle\int_{-1}^{0} (x+1)\sqrt{1-x-x^2}\,\mathrm{d}x$

(20) $\displaystyle\int_0^{0.75} \frac{\mathrm{d}x}{(x+1)\sqrt{x^2+1}}$

解：

(1) $\displaystyle\int_1^2 \frac{(x+1)(x^2-3)}{3x^2}\,\mathrm{d}x = \int_1^2 \frac{x^3+x^2-3x+3}{3x^2}\,\mathrm{d}x =$
$\displaystyle\frac{1}{3}\int_1^2 \left(x+1-\frac{3}{x}-\frac{3}{x^2}\right)\mathrm{d}x = \frac{1}{3}\left(\frac{x^2}{2}+x-3\ln x+\frac{3}{x}\right)\Big|_1^2 = \frac{1}{3}-\ln 2$

(2) $\int_1^{\frac{\pi}{2}}(a\sin x+b\cos x)\,\mathrm{d}x=(-a\cos x+b\sin x)|_0^{\frac{\pi}{2}}=a+b$

(3) 令$y=x+1$，则$\int_0^1\left(\frac{x-1}{x+1}\right)^4\mathrm{d}x=\int_1^2\left(\frac{y-2}{y}\right)^4\mathrm{d}y=\int_1^2\left(1-\frac{8}{y}+\frac{24}{y^2}-\frac{32}{y^3}+\frac{16}{y^4}\right)\mathrm{d}y=$
$\left(y-8\ln y-\frac{24}{y}+\frac{16}{y^2}-\frac{16}{3y^3}\right)\Big|_1^2=\frac{17}{3}-8\ln 2$

(4) $\int_0^1\frac{x^2+1}{x^4+1}\mathrm{d}x=\frac{1}{2}\int_0^1\left[\frac{1}{x^2+\sqrt{2}x+1}+\frac{1}{x^2-\sqrt{2}x+1}\right]\mathrm{d}x=$
$\frac{1}{2}\left[\sqrt{2}\arctan(\sqrt{2}x+1)+\sqrt{2}\arctan(\sqrt{2}x-1)\right]\Big|_0^1=\frac{\sqrt{2}}{2}(\arctan(\sqrt{2}+1)+\arctan(\sqrt{2}-1))=\frac{\sqrt{2}}{4}\pi$

(5) 令$x=\frac{1}{\sqrt{5}}\sin u$, $\mathrm{d}x=\frac{1}{\sqrt{5}}\cos u\,\mathrm{d}u$，
则$\int_0^{\frac{1}{\sqrt{5}}}x^3(1-5x^2)^{10}\,\mathrm{d}x=\frac{1}{25}\int_0^{\frac{\pi}{2}}\sin^3 u\cos^{21}u\,\mathrm{d}u=-\frac{1}{25}\int_0^{\frac{\pi}{2}}(1-\cos^2 u)\cos^{21}u\,\mathrm{d}\cos u=-\frac{1}{25}\left(\frac{\cos^{22}u}{22}-\frac{\cos^{24}u}{24}\right)\Big|_0^{\frac{\pi}{2}}=$
$\frac{1}{6600}$

(6) $\int_0^1 x^2(2-3x^2)^2\,\mathrm{d}x=\int_0^1(4x^2-12x^4+9x^6)\,\mathrm{d}x=\frac{23}{105}$

(7) 令$u=\sqrt{2-5x}$
则$\int_{-\frac{1}{5}}^{\frac{1}{5}}x\sqrt{2-5x}\,\mathrm{d}x=\frac{2}{25}\int_1^{\sqrt{3}}(2u^2-u^3)\,\mathrm{d}u=\frac{2}{375}(3\sqrt{3}-7)$

(8) 当$m\neq\pm n$时，$\int_0^{\frac{\pi}{2}}\sin mx\cos nx\,\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{2}}[\sin(m+n)x+\sin(m-n)x]\,\mathrm{d}x=\frac{m}{m^2-n^2}-\frac{\cos\frac{m+n}{2}\pi}{2(m+n)}-$
$\frac{\cos\frac{m-n}{2}\pi}{2(m-n)}$；
当$m=\pm n$且$m\neq 0$时，$\int_0^{\frac{\pi}{2}}\sin mx\cos nx\,\mathrm{d}x=\pm\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin 2nx\,\mathrm{d}x=-\frac{1}{4n}\cos 2nx\Big|_0^{\frac{\pi}{2}}=\pm\frac{1}{4n}(1-$
$\cos n\pi)=\pm\frac{1}{4n}[1-(-1)^n](n\in Z$且$n\neq 0)$；
当$m=0$时，$\int_0^{\frac{\pi}{2}}\sin mx\cos nx\,\mathrm{d}=0$

(9) $\int_{-\frac{1}{3}}^{\frac{1}{3}}\frac{18x-4}{\sqrt{9x^2+6x+5}}\mathrm{d}x=\int_{-\frac{1}{3}}^{\frac{1}{3}}\frac{\mathrm{d}(9x^2+6x+5)}{\sqrt{9x^2+6x+5}}-\frac{10}{3}\int_{-\frac{1}{3}}^{\frac{1}{3}}\frac{\mathrm{d}x}{\sqrt{1+\left(\frac{3x+1}{2}\right)^2}}=$
$\left[2\sqrt{9x^2+6x+5}-\frac{10}{3}\ln\left(\frac{3x+1}{2}+\sqrt{1+\left(\frac{3x+1}{2}\right)^2}\right)\right]\Bigg|_{-\frac{1}{3}}^{\frac{1}{3}}=4(\sqrt{2}-1)-\frac{10}{3}\ln(\sqrt{2}+1)$

(10) 令$u=x^2$
$\int_0^{\sqrt{\ln 2}}x^3e^{-x^2}\,\mathrm{d}x=\frac{1}{2}\int_0^{\ln 2}ue^{-u}\,\mathrm{d}u=\frac{1}{2}\left(-ue^{-u}\Big|_0^{\ln 2}+\int_0^{\ln 2}e^{-u}\,\mathrm{d}u\right)=\frac{1}{4}(1-\ln 2)$

(11) $\int_0^1 x\arctan x\,\mathrm{d}x=\frac{x^2}{2}\arctan x\Big|_0^1-\frac{1}{2}\int_0^1\frac{x^2}{1+x^2}\mathrm{d}x=\frac{\pi}{8}-\frac{1}{2}+\frac{1}{2}\arctan x\Big|_0^1=\frac{\pi-2}{4}$

(12) $\int_0^{2\pi}x\cos^2 x\,\mathrm{d}x=\frac{1}{2}\int_0^{2\pi}x(1+\cos 2x)\,\mathrm{d}x=\frac{x^2}{4}\Big|_0^{2\pi}+\frac{1}{4}x\sin 2x\Big|_0^{2\pi}-\frac{1}{4}\int_0^{2\pi}\sin 2x\,\mathrm{d}x=\pi^2$

(13) $\int_{-\pi}^{\pi}x^2\cos x\,\mathrm{d}x=2\int_0^{\pi}x^2\cos x\,\mathrm{d}x=2x^2\sin x\Big|_0^{\pi}-4\int_0^{\pi}x\sin x\,\mathrm{d}x=4x\cos x\Big|_0^{\pi}-4\int_0^{\pi}\cos x\,\mathrm{d}x=-4\pi$

(14) $\int_0^{\frac{2\pi}{\omega}}\sin\omega t\sin(\omega t+\varphi)\,\mathrm{d}t=\frac{1}{2}\int_0^{\frac{2\pi}{\omega}}[\cos\varphi-\cos(2\omega t+\varphi)]\,\mathrm{d}t=\frac{\pi}{\omega}\cos\varphi-\frac{1}{4\omega}\sin(2\omega t+\varphi)\Big|_0^{\frac{2\pi}{\omega}}=\frac{\pi}{\omega}\cos\varphi$

(15) $\int_0^3\frac{x\,\mathrm{d}x}{1+\sqrt{1+x}}=\int_0^3\frac{x(1-\sqrt{1+x})}{-x}\mathrm{d}x=\int_0^3(\sqrt{1+x}-1)\,\mathrm{d}x=\frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_0^3-3=\frac{5}{3}$

(16) $\int_0^4 x(x+\sqrt{x})\,\mathrm{d}x=\int_0^4(x^2+x^{\frac{3}{2}})\,\mathrm{d}x=\frac{512}{15}$

(17) 当$m \neq \pm n(m,n\in Z)$时，$\int_{-\pi}^{\pi}\sin mx\sin nx\,\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}[\cos(m-n)x-\cos(m+n)x]\,\mathrm{d}x=\left.\frac{\sin(m-n)x}{2(m-n)}\right|_{-\pi}^{\pi}-\left.\frac{\sin(m+n)x}{2(m+n)}\right|_{-\pi}^{\pi}=0$；

当$m=\pm n(m,n\in Z)$且$m\neq 0$时，$\int_{-\pi}^{\pi}\sin mx\sin nx\,\mathrm{d}x=\pm\int_{-\pi}^{\pi}\sin^2 mx\,\mathrm{d}x=\pm\int_{0}^{\pi}(1-\cos 2mx)\,\mathrm{d}x=\pm\pi$；

当$m=0$时，$\int_{-\pi}^{\pi}\sin mx\sin nx\,\mathrm{d}x=0$

(18) $\int_{-\pi}^{\pi}\sin mx\cos nx\,\mathrm{d}x=0$

(19) $\int_{-1}^{0}(x+1)\sqrt{1-x-x^2}\,\mathrm{d}x=-\frac{1}{2}\int_{-1}^{0}(1-x-x^2)^{\frac{1}{2}}\,\mathrm{d}(1-x-x^2)+\frac{1}{2}\int_{-1}^{0}\sqrt{1-x-x^2}\,\mathrm{d}x=\frac{1}{2}\int_{-1}^{0}\sqrt{1-x-x^2}\,\mathrm{d}x=\frac{1}{4}+\frac{5}{8}\arcsin\frac{\sqrt{5}}{5}$

(20) 令$x=\tan t$

则$\int_{0}^{0.75}\frac{\mathrm{d}x}{(x+1)\sqrt{x^2+1}}=\int_{0}^{\arctan 0.75}\frac{\mathrm{d}t}{\sin t+\cos t}=$

$$\int_{0}^{\arctan 0.75}\frac{\mathrm{d}\left(t+\frac{\pi}{4}\right)}{\sqrt{2}\sin\left(t+\frac{\pi}{4}\right)}=\left.\frac{\sqrt{2}}{2}\ln\tan\left(\frac{t}{2}+\frac{\pi}{8}\right)\right|_{0}^{\arctan 0.75}=\frac{1}{\sqrt{2}}\ln\frac{\tan\left(\frac{\arctan 0.75}{2}+\frac{\pi}{8}\right)}{\tan\frac{\pi}{8}}=\frac{1}{\sqrt{2}}\ln\frac{9+4\sqrt{2}}{7}$$

2. 计算下列积分：

(1) $\int_{0}^{\frac{\pi}{2}}\sin^7 x\,\mathrm{d}x$

(2) $\int_{0}^{\frac{\pi}{2}}\cos^4 x\,\mathrm{d}x$

(3) $\int_{0}^{\pi}\sin^5 x\,\mathrm{d}x$

(4) $\int_{0}^{2\pi}\cos^6 x\,\mathrm{d}x$

(5) $\int_{0}^{a}(a^2-x^2)^n\,\mathrm{d}x$

(6) $\int_{0}^{1}(1-x^2)^6\,\mathrm{d}x$

解：

(1) $\int_{0}^{\frac{\pi}{2}}\sin^7 x\,\mathrm{d}x=\frac{6!!}{7!!}$

(2) $\int_{0}^{\frac{\pi}{2}}\cos^4 x\,\mathrm{d}x=\frac{3!!}{4!!}\cdot\frac{\pi}{2}=\frac{3\pi}{16}$

(3) $\int_{0}^{\pi}\sin^5 x\,\mathrm{d}x=\int_{0}^{\frac{\pi}{2}}\sin^5 x\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}\sin^5 x\,\mathrm{d}x$

在后一积分中，令$x=\pi-y$，则$\sin x=\sin y,\ \mathrm{d}x=-\mathrm{d}y$，于是$\int_{\frac{\pi}{2}}^{\pi}\sin^5 x\,\mathrm{d}x=-\int_{\frac{\pi}{2}}^{0}\sin^5 y\,\mathrm{d}y=\int_{0}^{\frac{\pi}{2}}\sin^5 y\,\mathrm{d}y=I_5$

从而$\int_{0}^{\pi}\sin^5 x\,\mathrm{d}x=2I_5=2\cdot\frac{4!!}{5!!}=\frac{16}{15}$

(4) $\int_{0}^{2\pi}\cos^6 x\,\mathrm{d}x=\int_{-\pi}^{\pi}\cos^6 x\,\mathrm{d}x=2\int_{0}^{\pi}\cos^6 x\,\mathrm{d}x=4\int_{0}^{\frac{\pi}{2}}\cos^6 x\,\mathrm{d}x=4\cdot\frac{5!!}{6!!}\cdot\frac{\pi}{2}=\frac{15}{24}\pi$

(5) 令$x=a\cos t,\ \mathrm{d}x=-a\sin t\,\mathrm{d}t$，

则$\int_{0}^{a}(a^2-x^2)^n\,\mathrm{d}x=-a^{2n+1}\int_{\frac{\pi}{2}}^{0}\sin^{2n+1}t\,\mathrm{d}t=a^{2n+1}\int_{0}^{\frac{\pi}{2}}\sin^{2n+1}t\,\mathrm{d}t=\frac{(2n)!!}{(2n+1)!!}a^{2n+1}$

(6) 在上题中，令$a=1,n=6$，则$\int_0^1 (1-x^2)^6\,\mathrm{d}x=\frac{12!!}{13!!}$

3. 设$f(x)$是周期函数，周期是T，证明$\int_a^{a+nT} f(x)\,\mathrm{d}x=n\int_0^T f(x)\,\mathrm{d}x$，此处$n$是正整数.

证明: $\int_a^{a+nT} f(x)\,\mathrm{d}x=\int_a^0 f(x)\,\mathrm{d}x+\int_0^T f(x)\,\mathrm{d}x+\cdots+\int_{(n-1)T}^{nT} f(x)\,\mathrm{d}x+\int_{nT}^{a+nT} f(x)\,\mathrm{d}x$

对上述等式的最后一个积分，设$x-nT=t$，则$\int_{nT}^{a+nT} f(x)\,\mathrm{d}x=\int_0^a f(t+nT)\,\mathrm{d}t=\int_0^a f(t)\,\mathrm{d}t$

对$1<i<n$，考虑积分$\int_{(i-1)T}^{iT} f(x)\,\mathrm{d}x$，设$x-(i-1)T=t$，则$\int_{(i-1)T}^{iT} f(x)\,\mathrm{d}x=\int_0^T f(t+(n-1)T)\,\mathrm{d}t=$
$\int_0^T f(t)\,\mathrm{d}t$

从而$\int_a^{a+nT} f(x)\,\mathrm{d}x=n\int_0^T f(x)\,\mathrm{d}x$

4. 证明：(m,n为正整数)

(1) $\int_{-\pi}^{\pi} \sin^2 mx\,\mathrm{d}x=\pi,\int_{-\pi}^{\pi} \cos^2 mx\,\mathrm{d}x=\pi$

(2) $\int_{-\pi}^{\pi} \cos mx\cos nx\,\mathrm{d}x=0(m\neq n)$

证明:

(1) $\int_{-\pi}^{\pi} \sin^2 mx\,\mathrm{d}x=\frac{1}{m}\int_{-\pi}^{\pi} \sin^2 mx\,\mathrm{d}mx=\frac{1}{m}\int_0^{\pi}(1-\cos 2mx)\,\mathrm{d}mx=\pi-\left.\frac{1}{2m}\sin 2mx\right|_0^{\pi}=\pi$

同理可得，$\int_{-\pi}^{\pi} \cos^2 mx\,\mathrm{d}x=\pi$

(2) $\int_{-\pi}^{\pi} \cos mx\cos nx\,\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}[\cos(m+n)x+\cos(m-n)x]\,\mathrm{d}x=\left.\frac{\sin(m+n)x}{(m+n)}\right|_0^{\pi}+\left.\frac{\sin(m-n)x}{(m-n)}\right|_0^{\pi}=0$

5. 证明若函数$f(x)$在闭区间$[0,1]$连续，则:

(1) $\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x=\int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x$

(2) $\int_0^{\pi} xf(\sin x)\,\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x$

由此计算$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x$

证明:

(1) 令$\frac{\pi}{2}-t=x$，则$\mathrm{d}x=-\mathrm{d}t,f(\sin x)=f(\cos x)$，

于是$\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x=-\int_{\frac{\pi}{2}}^0 f(\cos t)\,\mathrm{d}t=\int_0^{\frac{\pi}{2}} f(\cos t)\,\mathrm{d}t$

即$\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x=\int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x$

(2) 设$t=\pi-x$，则$\mathrm{d}x=-\mathrm{d}t$且$xf(\sin x)=(\pi-t)f(\sin t)$

于是$\int_0^{\pi} xf(\sin x)\,\mathrm{d}x=-\int_{\pi}^0(\pi-t)f(\sin t)\,\mathrm{d}t=\pi\int_0^{\pi} f(\sin t)\,\mathrm{d}t-\int_0^{\pi} tf(\sin t)\,\mathrm{d}t$，则$2\int_0^{\pi} xf(\sin x)\,\mathrm{d}x=$
$\pi\int_0^{\pi} f(\sin x)\,\mathrm{d}x$，从而$\int_0^{\pi} xf(\sin x)\,\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x$

$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x=-\frac{\pi}{2}\int_0^{\pi}\frac{\mathrm{d}\cos x}{1+\cos^2 x}=-\left.\frac{\pi}{2}\arctan(\cos x)\right|_0^{\pi}=\frac{\pi^2}{4}$

6. 证明奇函数的一切原函数皆为偶函数，偶函数的原函数中有一为奇函数.

证明: 设$f(x)$在$[-l,l]$上有定义，且$F(x)$是$f(x)$ 的一个原函数

当$f(x)$为奇函数即当$f(-x)=-f(x)$时，由于$f(x)=\frac{\mathrm{d}}{\mathrm{d}x}F(x)$,

$f(-x)=-\frac{\mathrm{d}}{\mathrm{d}x}F(-x)$，故有$\frac{\mathrm{d}}{\mathrm{d}x}[F(x)-F(-x)]=0$，从而可得$F(x)-F(-x)=C_1$且$C_1=0$，于是$F(x)=F(-x)$，则$f(x)$的一个原函数$F(x)$为偶函数，从而$f(x)$的任一个原函数$F(x)+C$($C$为任意常数)也为偶函数

当$f(x)$为偶函数即当$f(-x)=f(x)$时，类似可得$F(x)+F(-x)=C_2$且$C_2=2F(0)$，于是$f(x)$有一个原函数$F(x)-F(0)$是奇函数.

7. 若$f(x)$关于$x=T$对称，且$a<T<b$，则$\int_a^b f(x)\,\mathrm{d}x = 2\int_T^b f(x)\,\mathrm{d}x + \int_a^{2T-b} f(x)\,\mathrm{d}x$

证明：$\int_a^b f(x)\,\mathrm{d}x = \int_a^{2T-b} f(x)\,\mathrm{d}x + \int_{2T-b}^{T} f(x)\,\mathrm{d}x + \int_T^b f(x)\,\mathrm{d}x$

对上述等式右端的第二个积分，设$x=2T-t$，则$\int_{2T-b}^{T} f(x)\,\mathrm{d}x = -\int_b^T f(2T-t)\,\mathrm{d}t$

又$f(x)$关于$x=T$对称，则$f(2T-t)=f(t)$

于是$\int_{2T-b}^{T} f(x)\,\mathrm{d}x = -\int_b^T f(2T-t)\,\mathrm{d}t = -\int_b^T f(t)\,\mathrm{d}t = \int_T^b f(t)\,\mathrm{d}t$，从而$\int_a^b f(x)\,\mathrm{d}x = 2\int_T^b f(x)\,\mathrm{d}x + \int_a^{2T-b} f(x)\,\mathrm{d}x$

8. 证明：$\int_0^a x^3 f(x^2)\,\mathrm{d}x = \frac{1}{2}\int_0^{a^2} xf(x)\,\mathrm{d}x (a>0)$

证明：令$t=x^2$，则$2x\,\mathrm{d}x = \mathrm{d}t$，于是$\int_0^a x^3 f(x^2)\,\mathrm{d}x = \frac{1}{2}\int_0^{a^2} tf(t)\,\mathrm{d}t = \frac{1}{2}\int_0^{a^2} xf(x)\,\mathrm{d}x$

9. 利用分部积分证明：$\int_0^x f(u)(x-u)\,\mathrm{d}u = \int_0^x \left\{\int_0^u f(x)\,\mathrm{d}x\right\}\mathrm{d}u$

证明：$\int_0^x \left\{\int_0^u f(x)\,\mathrm{d}x\right\}\mathrm{d}u = u\int_0^u f(x)\,\mathrm{d}x\Big|_0^x - \int_0^x uf(u)\,\mathrm{d}u = x\int_0^x f(t)\,\mathrm{d}t - \int_0^x uf(u)\,\mathrm{d}u = \int_0^x xf(u)\,\mathrm{d}u - \int_0^x uf(u)\,\mathrm{d}u = \int_0^x f(u)(x-u)\,\mathrm{d}u$

10. 一长度为l的横梁，所受载荷按规律$p(x)=a+bx+cx^2$分布，试由下述条件决定系数a,b,c；总载荷是$P=\int_0^l p(x)\,\mathrm{d}x$，极大载荷位于$\frac{2}{3}l$处，且在极大点的左右两边各承受总载荷的一半.

解：由已知，得

(1) $P=\int_0^l p(x)\,\mathrm{d}x = al+\frac{b}{2}l^2+\frac{c}{3}l^3$

(2) 令$P(x)=\int_0^x p(t)\,\mathrm{d}t$，则$P'\left(\frac{2}{3}l\right)=p\left(\frac{2}{3}l\right)=a+\frac{2}{3}bl+\frac{4}{9}cl^2=0$

(3) $\int_0^{\frac{2}{3}l} p(x)\,\mathrm{d}x = \frac{2}{3}al+\frac{2}{9}bl+\frac{8}{81}cl^3=\frac{P}{2}$

联立方程组$\begin{cases} al+\frac{b}{2}l^2+\frac{c}{3}l^3=P \\ \frac{2}{3}al+\frac{2}{9}bl+\frac{8}{81}cl^3=\frac{P}{2} \\ a+\frac{2}{3}bl+\frac{4}{9}cl^2=0 \end{cases}$

求解，得$\begin{cases} a=\frac{4}{l}P \\ b=-\frac{69}{4l^2}P \\ c=\frac{135}{8l^3}P \end{cases}$

11. 若$f(x)$连续，求：

(1) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^b f(t)\,\mathrm{d}t\right)$

(2) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{x^2} f(t)\,\mathrm{d}t\right)$

解：

(1) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^b f(t)\,\mathrm{d}t\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_b^x f(t)\,\mathrm{d}t\right) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_b^x f(t)\,\mathrm{d}t\right) = -f(x)$

(2) 因$\frac{\mathrm{d}}{\mathrm{d}x^2}\left(\int_0^{x^2} f(t)\,\mathrm{d}t\right)=f(x^2)$，故$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{x^2} f(t)\,\mathrm{d}t\right) = \frac{\mathrm{d}}{\mathrm{d}x^2}\left(\int_0^{x^2} f(t)\,\mathrm{d}t\right)\cdot\frac{\mathrm{d}x^2}{\mathrm{d}x} = 2xf(x^2)$

12. 求极限：

(1) $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n-1}{n^2}\right)$

(2) $\lim\limits_{n\to\infty}\frac{1^p+2^p+\cdots+n^p}{n^{p+1}}(p>0)$

(3) $\lim\limits_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\cdots+\sqrt{1+\frac{n-1}{n}}\right)$

(4) $\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$

解：

(1) 函数$f(x)=x$在$[0,1]$连续，因而可积.将$[0,1]n$等分，分点为$\frac{i}{n},\Delta x_i=\frac{1}{n}(i=0,1,\cdots,n-1)$，在每个小区间$[x_i,x_{i+1}]=\left[\frac{i}{n},\frac{i+1}{n}\right]$，取$\xi_i=\frac{i}{n}$，则$f(\xi_i)=\frac{i}{n}$，

于是$\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n-1}{n^2}\right)=\lim\limits_{n\to\infty}\frac{1}{n}\left(\frac{1}{n}+\frac{2}{n}+\cdots+\frac{n-1}{n}\right)=\lim\limits_{n\to\infty}\sum\limits_{i=0}^{n-1}\frac{i}{n}\cdot\frac{1}{n}=\int_0^1 x\,\mathrm{d}x=\frac{1}{2}$

(2) 函数$f(x)=x^p$在$[0,1]$连续，因而可积.将$[0,1]n$等分，分点为$\frac{i}{n},\Delta x_i=\frac{1}{n}(i=1,2,\cdots,n)$，在每个小区间$[x_{i-1},x_i]=\left[\frac{i-1}{n},\frac{i}{n}\right]$，取$\xi_i=\frac{i}{n}$，则$f(\xi_i)=\left(\frac{i}{n}\right)^p$，

于是$\lim\limits_{n\to\infty}\frac{1^p+2^p+\cdots+n^p}{n^{p+1}}=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\frac{i}{n}\right)^p\cdot\frac{1}{n}=\int_0^1 x^p\,\mathrm{d}x=\frac{1}{p+1}$

(3) 函数$f(x)=\sqrt{1+x}$在$[0,1]$连续，因而可积.将$[0,1]n$等分，分点为$\frac{i}{n},\Delta x_i=\frac{1}{n}(i=0,1,\cdots,n-1)$，在每个小区间$[x_i,x_{i+1}]=\left[\frac{i}{n},\frac{i+1}{n}\right]$，取$\xi_i=\frac{i}{n}$，则$f(\xi_i)=\sqrt{1+\frac{i}{n}}$，

于是$\lim\limits_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\cdots+\sqrt{1+\frac{n-1}{n}}\right)=\lim\limits_{n\to\infty}\sum\limits_{i=0}^{n-1}\sqrt{1+\frac{i}{n}}\cdot\frac{1}{n}-\lim\limits_{n\to\infty}\frac{1}{n}=\int_0^1\sqrt{x+1}\,\mathrm{d}x-0=\frac{2}{3}(2\sqrt{2}-1)$

(4) 因$\frac{\sqrt[n]{n!}}{n}=\sqrt[n]{\frac{n!}{n^n}}=\left(\frac{1}{n}\right)^{\frac{1}{n}}\left(\frac{2}{n}\right)^{\frac{1}{n}}\cdots\left(\frac{n}{n}\right)^{\frac{1}{n}}$，故$\ln\frac{\sqrt[n]{n!}}{n}=\frac{1}{n}\left(\ln\frac{1}{n}+\ln\frac{2}{n}+\cdots+\ln\frac{n}{n}\right)$

又函数$f(x)=\ln x$在$(0,1]$连续，考虑$\lim\limits_{\xi\to+0}\int_\xi^1\ln x\,\mathrm{d}x$.将$(0,1]n$等分，分点为$\frac{i}{n},\Delta x_i=\frac{1}{n}(i=1,2,\cdots,n)$，在每个小区间$[x_{i-1},x_i]=\left[\frac{i-1}{n},\frac{i}{n}\right]$，取$\xi_i=\frac{i}{n}$，则$f(\xi_i)=\left(\frac{i}{n}\right)^p$，

于是$\lim\limits_{n\to\infty}\ln\frac{\sqrt[n]{n!}}{n}=\lim\limits_{n\to\infty}\frac{1}{n}\left(\ln\frac{1}{n}+\ln\frac{2}{n}+\cdots+\ln\frac{n}{n}\right)=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\ln\frac{i}{n}\cdot\frac{1}{n}=\lim\limits_{\xi\to+0}\int_\xi^1\ln x\,\mathrm{d}x=\lim\limits_{\xi\to+0}(x\ln x-x)|_\xi^1=-1$

从而$\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=e^{-1}=\frac{1}{e}$

13. 根据例7有$\lim\limits_{n\to\infty}\frac{I_{2n}}{I_{2n+1}}=1$，由此推证$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdots$

证明：因$I_{2n}=\int_0^{\frac{\pi}{2}}\sin^{2n}x\,\mathrm{d}x=\frac{(2n-1)(2n-3)\cdots3\cdot1}{2n(2n-2)\cdots4\cdot2}\cdot\frac{\pi}{2},I_{2n+1}=\int_0^{\frac{\pi}{2}}\sin^{2n+1}x\,\mathrm{d}x=\frac{2n(2n-2)\cdots4\cdot2}{(2n+1)(2n-1)\cdots5\cdot3}$，则$\frac{I_{2n+1}}{I_{2n}}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi}$

当$0\leqslant x\leqslant\frac{\pi}{2}$时，$0\leqslant\sin x\leqslant1,\sin^{2n+1}x\leqslant\sin^{2n}x\leqslant\sin^{2n-1}x$，则$\int_0^{\frac{\pi}{2}}\sin^{2n+1}x\,\mathrm{d}x\leqslant\int_0^{\frac{\pi}{2}}\sin^{2n}x\,\mathrm{d}x\leqslant\int_0^{\frac{\pi}{2}}\sin^{2n-1}x\,\mathrm{d}x$即$I_{2n+1}\leqslant I_{2n}\leqslant I_{2n+1}$，于是$1\leqslant\frac{I_{2n}}{I_{2n+1}}\leqslant\frac{I_{2n-1}}{I_{2n+1}}$

又由递推公式$I_n=\frac{n-1}{n}I_{n-2},I_{2n+1}=\frac{2n}{2n+1}I_{2n-1}$即$\frac{I_{2n+1}}{I_{2n-1}}=\frac{2n}{2n+1}$，故$\lim\limits_{n\to\infty}\frac{I_{2n+1}}{I_{2n-1}}=\lim\limits_{n\to\infty}\frac{2n}{2n+1}=1$，于是$\lim\limits_{n\to\infty}\frac{I_{2n}}{I_{2n+1}}=1$，从而$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdots$

14. 设$f(x)$与$g(x)$都在$[a,b]$可积，证明

$$\left[\int_a^b f(x)g(x)\,\mathrm{d}x\right]^2 \leqslant \int_a^b f^2(x)\,\mathrm{d}x\cdot\int_a^b g^2(x)\,\mathrm{d}x$$

又等式在何时成立？

证明：对任何实数h，因$[hf(x)-g(x)]^2=h^2f^2(x)-2hf(x)g(x)+g^2(x)\geqslant 0$

由积分的性质，得$\int_a^b(h^2f^2(x)-2hf(x)g(x)+g^2(x))\,\mathrm{d}x\geqslant 0$即$h^2\int_a^b f^2(x)\,\mathrm{d}x-2h\int_a^b f(x)g(x)\,\mathrm{d}x+\int_a^b g^2(x)\,\mathrm{d}x\geqslant 0$

由二次三项式非负的条件，得$\left(2\int_a^b f(x)g(x)\,\mathrm{d}x\right)^2-4\int_a^b f^2(x)\,\mathrm{d}x\cdot\int_a^b g^2(x)\,\mathrm{d}x\leqslant 0$即$\left[\int_a^b f(x)g(x)\,\mathrm{d}x\right]^2\leqslant \int_a^b f^2(x)\,\mathrm{d}x\cdot\int_a^b g^2(x)\,\mathrm{d}x$

要使等号成立，只要$\left(2\int_a^b f(x)g(x)\,\mathrm{d}x\right)^2-4\int_a^b f^2(x)\,\mathrm{d}x\cdot\int_a^b g^2(x)\,\mathrm{d}x=0$即$h^2\int_a^b f^2(x)\,\mathrm{d}x-2h\int_a^b f(x)g(x)\,\mathrm{d}x+\int_a^b g^2(x)\,\mathrm{d}x=0$有重根.

不妨设h_0为方程的重根，则$h_0^2\int_a^b f^2(x)\,\mathrm{d}x-2h_0\int_a^b f(x)g(x)\,\mathrm{d}x+\int_a^b g^2(x)\,\mathrm{d}x=0$即$\int_a^b[h_0f(x)-g(x)]^2\,\mathrm{d}x=0$

而当$g(x)=h_0f(x)$时，$\int_a^b[h_0f(x)-g(x)]^2\,\mathrm{d}x=0$(其中$h_0$为方程$h^2\int_a^b f^2(x)\,\mathrm{d}x-2h\int_a^b f(x)g(x)\,\mathrm{d}x+\int_a^b g^2(x)\,\mathrm{d}x=0$的重根)

第八章　定积分的应用和近似计算

§1　平面图形的面积

1. 求由下列各曲线所围成的图形面积：

(1) $y^2 = 4(x+1), y^2 = 4(1-x)$

(2) $y = |\ln x|, y = 0, (0.1 \leqslant x \leqslant 10)$

(3) $y = x, y = x + \sin^2 x, (0 \leqslant x \leqslant \pi)$

(4) $y^2 = 1 - x, 2y = x + 2$

(5) 蚶线$r = a\cos\theta + b(b \geqslant a)$，当$b = a$时即为心脏线

(6) $r = 3\cos\theta, r = 1 + \cos\theta$

(7) 旋轮线$x = a(t - \sin t), y = a(1 - \cos t)(0 \leqslant t \leqslant 2\pi)$以及$x$轴

(8) 星形线$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

解：

(1) 两条曲线$x = \dfrac{y^2-4}{4}$与$x = \dfrac{-y^2+4}{4} = -\dfrac{y^2-4}{4}$的交点的纵坐标分别为$-2$及$2$，
于是$A = \displaystyle\int_{-2}^{2}\left[-\frac{y^2-4}{4} - \frac{y^2-4}{4}\right]\mathrm{d}y = -\int_0^2 (y^2-4)\,\mathrm{d}y = \frac{16}{3}$

(2) 两条曲线$y = |\ln x|$与$y = 0$的交点的横坐标为1，
于是$A = \displaystyle\int_{0.1}^{10}[\ln|x| - 0]\,\mathrm{d}x = \int_{0.1}^{1}(-\ln x)\,\mathrm{d}x + \int_1^{10}\ln x\,\mathrm{d}x = -(x\ln x - x)\Big|_{0.1}^{1} + (x\ln x - x)\Big|_1^{10} =$
$9.9\ln 10 - 8.1 \approx 14.69559$

(3) $A = \displaystyle\int_0^{\pi}(x + \sin^2 x - x)\,\mathrm{d}x = \int_0^{\pi}\sin^2 x\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi}(1 - \cos 2x)\,\mathrm{d}x = \frac{\pi}{2} - \frac{\sin 2x}{4}\Big|_0^{\pi} = \frac{\pi}{2}$

(4) 两条曲线的交点分别为$(0,1), (-8,-3)$，
于是$A = \displaystyle\int_{-3}^{1}[1 - y^2 - (2y-2)]\,\mathrm{d}y = \int_{-3}^{1}(3 - y^2 - 2y)\,\mathrm{d}y = \frac{32}{3}$

(5) 所求面积为：$A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} r^2\,\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi}(a\cos\theta + b)^2\,\mathrm{d}\theta = \frac{\pi}{2}a^2 + \pi b^2$

(6) 所求面积为：$A = \pi\left(\dfrac{3}{2}\right)^2 - A_1 = \dfrac{9}{4}\pi - A_1$
其中$A_1 = \dfrac{1}{2}\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}[9\cos^2\theta - (1+\cos\theta)^2]\,\mathrm{d}\theta = \int_0^{\frac{\pi}{3}}[8\cos^2\theta - 1 - 2\cos\theta)\,\mathrm{d}\theta = \pi$，
从而$A = \dfrac{5}{4}\pi$

(7) 所求面积为：$A = \displaystyle\int_0^{2\pi} y\,\mathrm{d}x = \int_0^{2\pi} a(1-\cos t)\cdot a(1-\cos t)\,\mathrm{d}t = a^2\int_0^{2\pi}(1-\cos t)^2\,\mathrm{d}t = 3\pi a^2$

(8) 设$x = a\cos^3 t, y = a\sin^3 t$，其中$0 \leqslant t \leqslant \dfrac{\pi}{3}$，它对应于四分之一的面积，所求面积为其四倍
即$A = 4\displaystyle\int_0^a y\,\mathrm{d}x = 4\int_{\frac{\pi}{2}}^{0}(-3a^2\sin^4 t\cos^2 t)\,\mathrm{d}t = 12a^2\int_0^{\frac{\pi}{2}}(\sin^4 x - \sin^6 x)\,\mathrm{d}x = \frac{3\pi}{8}a^2$

2. 直线$y = x$把椭圆$x^2 + 3y^2 = 6y$的面积分成两部分A(小的一块)和B(大的一块)，求$\dfrac{A}{B}$之值.

解：由已知，得椭圆方程为$\dfrac{x^2}{3} + (y-1)^2 = 1$，则椭圆面积为$S = \pi ab = \sqrt{3}\pi$
又$y = x$与椭圆$x^2 + 3y^2 = 6y$的交点的纵坐标为$0, \dfrac{3}{2}$
于是$A = \displaystyle\int_0^{\frac{3}{2}}(\sqrt{6y - 3y^2} - y)\,\mathrm{d}y = \sqrt{3}\int_0^{\frac{3}{2}}\sqrt{1 - (y-1)^2}\,\mathrm{d}y - \frac{y^2}{2}\Big|_0^{\frac{3}{2}} = \frac{\sqrt{3}}{3}\pi - \frac{3}{4}$，则$B = S - A =$
$\dfrac{2}{3}\sqrt{3}\pi + \dfrac{3}{4}$，
从而$\dfrac{A}{B} = \dfrac{4\sqrt{3}\pi - 9}{8\sqrt{3}\pi + 9}$.

3. 求曲线$y=\sqrt{1-x^2}+\arccos x$与x轴及$x=-1$所围的面积.

解：因$y_1=\sqrt{1-x^2}$的定义域为$[-1,1]$，值域为$[0,1]$；

$y_2=\arccos x$的定义域为$[-1,1]$，值域为$[0,\pi]$

则面积$A=\int_{-1}^{1} y_1\,\mathrm{d}x+\int_{-1}^{1} y_2\,\mathrm{d}x=\int_{-1}^{1}\sqrt{1-x^2}\,\mathrm{d}x+\int_{-1}^{1}\arccos x\,\mathrm{d}x=\dfrac{3}{2}\pi$

§2 曲线的弧长

求下列曲线的弧长：

1. $y=x^{\frac{3}{2}}(0\leqslant x\leqslant 4)$
2. $x=\frac{1}{4}y^2-\frac{1}{2}\ln y(1\leqslant y\leqslant e)$
3. 星形线$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}(a>0)$
4. 旋轮线$x=a(t-\sin t),y=a(1-\cos t)(0\leqslant t\leqslant 2\pi)$
5. 圆的渐开线$x=a(\cos t+t\sin t),y=a(\sin t-t\cos t)(0\leqslant t\leqslant 2\pi)$
6. 心脏线$r=a(1+\cos\theta)(0\leqslant\theta\leqslant 2\pi)$

解：

1. 所求弧长为$s=\int_0^4\sqrt{1+[(x^{\frac{3}{2}})']^2}\,\mathrm{d}x=\int_0^4\sqrt{1+\frac{9}{4}x}\,\mathrm{d}x=\frac{8}{27}(10\sqrt{10}-1)$
2. 所求弧长为$s=\int_1^e\sqrt{1+(x')^2}\,\mathrm{d}y=\int_1^e\sqrt{1+\left(\frac{y}{2}-\frac{1}{2y}\right)^2}\,\mathrm{d}y=\int_1^e\sqrt{\left(\frac{y}{2}+\frac{1}{2y}\right)^2}\,\mathrm{d}y=\int_1^e\left(\frac{y}{2}+\frac{1}{2y}\right)\mathrm{d}y=$
$\frac{e^2+1}{4}$
3. 由已知可设$x=a\cos^3 t,y=a\sin^3 t(0\leqslant t\leqslant 2\pi)$
则所求弧长为$s=4\int_0^{\frac{\pi}{2}}\sqrt{[(a\cos^3 t)']^2+[(a\sin^3 t)']^2}\,\mathrm{d}t=$
$4a\int_0^{\frac{\pi}{2}}\sqrt{(-3\cos^2 t\sin t)^2+(3\sin^2 t\cos t)^2}\,\mathrm{d}t=12a\int_0^{\frac{\pi}{2}}\sin t\cos t\,\mathrm{d}t=6a$
4. $s=\int_0^{2\pi}\sqrt{[(a(t-\sin t))']^2+[(a(1-\cos t))']^2}\,\mathrm{d}t=$
$|a|\int_0^{2\pi}\sqrt{(1-\cos t)^2+\sin^2 t}\,\mathrm{d}t=2|a|\int_0^{2\pi}\sqrt{\frac{1-\cos t}{2}}\,\mathrm{d}t=2|a|\int_0^{2\pi}\sin\frac{t}{2}\,\mathrm{d}t=8|a|$
5. $s=\int_0^{2\pi}\sqrt{[(a(\cos t+t\sin t))']^2+[(a(\sin t-t\cos t))']^2}\,\mathrm{d}t=$
$|a|\int_0^{2\pi}\sqrt{(t\cos t)^2+(t\sin t)^2}\,\mathrm{d}t=|a|\int_0^{2\pi}t\,\mathrm{d}t=2\pi^2|a|$
6. $s=\int_0^{2\pi}\sqrt{r^2+(r')^2}\,\mathrm{d}\theta=\int_0^{2\pi}\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,\mathrm{d}\theta=4|a|\int_0^{\pi}\sqrt{\frac{1+\cos\theta}{2}}\,\mathrm{d}\theta=4|a|\int_0^{\pi}\cos\frac{\theta}{2}\,\mathrm{d}\theta=$
$8|a|$

§3　体积

1. 求出由下列各曲面所围成的几何体体积：

(1) 求截锥体体积，其上下底皆为椭圆，椭圆的轴长分别等于A, B和a, b，而高为h；

(2) 求椭球体体积：$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$；

(3) 求由下列两曲面：$x^2+y^2+z^2=a^2, x^2+y^2=ax$所围成的体积；

(4) 求用通过底面直径的平面从直圆柱上切下的弓形体体积，设圆柱的底半径为a，底面方程为$x^2+y^2\leqslant a^2$，截面通过x轴上的直径且与底面成α角.

解:

(1) 作一平行于上、下底且距离下底为x的截面，此截面为椭圆，其半轴分别为：$a'=A+\left(1-\dfrac{x}{h}\right)(a-A)$，$b'=B+\left(1-\dfrac{x}{h}\right)(b-B)$

于是此截面面积为：$A(x)=\pi a'b'=$

$\pi\left[AB+(a-A)(b-B)\left(1-\dfrac{x}{h}\right)^2+(A(b-B)+B(a-A))\left(1-\dfrac{x}{h}\right)\right]$

从而所求体积为$V=\displaystyle\int_0^h A(x)\,\mathrm{d}x=\frac{\pi}{6}[(2a+A)b+(a+2A)B]$

(2) 用垂直于Ox轴的平面截椭球得截痕为一椭圆，它在yOz平面上的投影为$\dfrac{y^2}{b^2\left(1-\dfrac{x^2}{a^2}\right)}+\dfrac{z^2}{c^2\left(1-\dfrac{x^2}{a^2}\right)}=1$

由此可见其半轴分别为$b\sqrt{1-\dfrac{x^2}{a^2}}$及$c\sqrt{1-\dfrac{x^2}{a^2}}$，从而得此椭圆面积为$A(x)=\pi bc\left(1-\dfrac{x^2}{a^2}\right)$

于是，所求的椭球体体积为：

$V=\displaystyle\int_{-a}^{a}A(x)\,\mathrm{d}x=2\int_0^a\pi bc\left(1-\frac{x^2}{a^2}\right)\,\mathrm{d}x=\frac{4}{3}abc$

(3) $z=\sqrt{a^2-x^2-y^2}$(上半面)，其变化范围为$-\sqrt{ax-x^2}\leqslant y\leqslant\sqrt{ax-x^2}$

其截面积为$A(x)=2\displaystyle\int_0^{\sqrt{ax-x^2}}\sqrt{a^2-x^2-y^2}\,\mathrm{d}y=a^{\frac{3}{2}}x^{\frac{1}{2}}-a^{\frac{1}{2}}x^{\frac{3}{2}}+(a^2-x^2)\arcsin\sqrt{\frac{x}{a+x}}$

于是，所求体积为：

$V=2\displaystyle\int_0^a A(x)\,\mathrm{d}x=$

$2\displaystyle\int_0^a\left[a^{\frac{3}{2}}x^{\frac{1}{2}}-a^{\frac{1}{2}}x^{\frac{3}{2}}+(a^2-x^2)\arcsin\sqrt{\frac{x}{a+x}}\right]\,\mathrm{d}x=\frac{2}{3}a^3\left(\pi-\frac{4}{3}\right)$

(4) $y=\sqrt{a^2-x^2}, z=\sqrt{a^2-x^2}\tan\alpha$，

则$A(x)=\dfrac{1}{2}\sqrt{a^2-x^2}\cdot\sqrt{a^2-x^2}\tan\alpha=\dfrac{1}{2}(a^2-x^2)\tan\alpha$

从而所求体积为：

$V=\displaystyle\int_{-a}^{a}A(x)\,\mathrm{d}x=\int_0^a(a^2-x^2)\tan\alpha\,\mathrm{d}x=\frac{2}{3}a^3\tan\alpha$

2. 求旋转体的体积：

(1) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$绕$X$轴；

(2) $y=\sin x, y=0(0\leqslant x\leqslant\pi)$

(i) 绕x轴

(ii) 绕y轴

(3) $x=a\sin^3 t, y=b\cos^3 t(0\leqslant t\leqslant 2\pi)$

(i) 绕x轴

(ii) 绕y轴

(4) 证明由$a\leqslant x\leqslant b, 0\leqslant y\leqslant y(x)$(其中$y(x)$是连续函数)所围成的面积绕$y$轴旋转所成的旋转体的体积为：

$V=\displaystyle\int_a^b 2\pi xy(x)\,\mathrm{d}x$

(5) $x=a(t-\sin t), y=a(1-\cos t)(0\leqslant t\leqslant 2\pi, y=0)$

(i) 绕x轴

(ii) 绕y轴

(iii) 绕直线$y=2a$

解:

(1) $V=\pi\int_{-a}^{a}y^2\,\mathrm{d}x=\pi\int_{-a}^{a}\left[b^2\left(1-\frac{x^2}{a^2}\right)\right]\mathrm{d}x=\frac{4}{3}\pi ab^2$

(2) (i) $V=\pi\int_0^{\pi}\sin^2 x\,\mathrm{d}x=\frac{\pi^2}{2}$

(ii) $V=2\pi\int_0^{\pi}x\sin x\,\mathrm{d}x=2\pi^2$

(3) (i) $V=2\pi\int_0^{\frac{\pi}{2}}y^2\,\mathrm{d}x=2\pi\int_0^{\frac{\pi}{2}}b^2\cos^6 t\cdot 3a\sin^2 t\cos t\,\mathrm{d}t=6a\int_0^{\frac{\pi}{2}}ab^2\sin^2 t\cos^7 t\,\mathrm{d}t=6\pi ab^2\int_0^{\frac{\pi}{2}}(\cos^7 t-\cos^9 t)\,\mathrm{d}t=\frac{32}{105}\pi ab^2$

(ii) 利用对称性，只需将上式答案中a,b对调，即得
$V=\frac{32}{105}\pi a^2 b$

(4) **证明**: 作$[a,b]$的任意分法：$a=x_0<x_1<\cdots<x_n=b$
在$[x_{i-1},x_i]$中任取一点ξ_i，对应的函数值为$y(\xi_i)$；$A_i\approx y(\xi_i)\Delta x_i,\Delta V_i\approx 2\pi\xi_i y(\xi_i)\Delta x_i$，则$V=\lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n}2\pi\xi_i y(\xi_i)\Delta x_i$，
从而$V=\int_a^b 2\pi xy(x)\,\mathrm{d}x$

(5) (i) $V=\pi\int_0^{2\pi}y^2\,\mathrm{d}x=\pi\int_0^{2\pi}a^3(1-\cos t)^3\,\mathrm{d}t=5\pi^2a^3$

(ii) $V=2\pi\int_0^{2\pi}a^3(1-\cos t)^2(t-\sin t)\,\mathrm{d}t=6\pi^3a^3$

(iii) 作平移$y=\overline{y}+2a, x=\overline{x}$，则曲线方程为$\overline{x}=a(t-\sin t),\overline{y}=-a(1+\cos t)$及$\overline{y}=-2a$
于是$V=\pi\int_0^{2\pi}[4a^2-a^2(1+\cos t)^2]a(1-\cos t)\,\mathrm{d}t=\pi a^3\int_0^{2\pi}(3-2\cos t-\cos^2 t)(1-\cos t)\,\mathrm{d}t=7\pi^2a^3$

3. 证明把面积$0\leqslant\alpha\leqslant\theta\leqslant\beta\leqslant\pi, 0\leqslant r\leqslant r(\theta)$($r(\theta)$在$[\alpha,\beta]$上连续)绕极轴旋转所成的体积等于：$V=\frac{2\pi}{3}\int_{\alpha}^{\beta}r^3(\theta)\sin\theta\,\mathrm{d}\theta$，并求出$r=a(1+\cos\theta)$绕极轴旋转所成的体积.

(1) **证明**: 用微元法.
因以r为半径，与极线成θ角的扇形绕极轴旋转一周所得的体积为：
$V=\frac{\pi}{3}(r\sin\theta)^2r\cos\theta+\pi\int_{r\cos\theta}^{r}(r^2-x^2)\,\mathrm{d}x=\frac{2}{3}\pi r^3(1-\cos\theta)$
作$[\alpha,\beta]$的任意分法：$\alpha=\theta_0<\theta_1<\cdots<\theta_n=\beta,\Delta\theta_i=\theta_i-\theta_{i-1},\lambda=\max\limits_i\{\Delta\theta_i\}$
在每个$[\theta_{i-1},\theta_i]$都存在θ_i'，使$\cos\theta_{i-1}-\cos_i=-\sin\theta_i'(\theta_{i-1}-\theta_i)=\sin\theta_i'\Delta\theta_i$
以$r(\theta_i')$作小扇形A_i的半径，则扇形绕极轴旋转一周后所得的体积为：
$\Delta V_i=\frac{2}{3}\pi r^3(\theta_i')(1-\cos\theta_i)-\frac{2}{3}\pi r^3(\theta_i')(1-\cos\theta_{i-1})=$
$\frac{2}{3}\pi r^3(\theta_i')(\cos\theta_{i-1}-\cos\theta_i)=\frac{2}{3}\pi r^3(\theta_i')\sin\theta_i'\Delta\theta_i$，则整个曲边扇形绕极轴旋转得$\sum\limits_{i=1}^{n}\frac{2}{3}\pi r^3(\theta_i')\sin\theta_i'\Delta\theta_i$
从而$V=\lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n}\frac{2}{3}\pi r^3(\theta_i')\sin\theta_i'\Delta\theta_i=\frac{2\pi}{3}\int_{\alpha}^{\beta}r^3(\theta)\sin\theta\,\mathrm{d}\theta.$

(2) **解**: $V=\frac{2}{3}\pi\int_0^{\pi}a^3(1+\cos\theta)^3\sin\theta\,\mathrm{d}\theta=\frac{8}{3}\pi a^3$

4. 把抛物线$y=x(x-a)$在横坐标0与$c(c>a>0)$之间的弧段绕x轴旋转，问c为何值时，该旋转体的体积V等于以弦OP绕x轴旋转所成的锥体体积？(图8-14)
解: 因抛物线$y=x(x-a), x_P=c$，故$P(c,c(c-a))$
则以弦OP绕x轴旋转所成的锥体体积为：$V_1=\frac{1}{3}\pi c[c(c-a)]^2=\frac{\pi}{3}c^3(c-a)^2$

所求的旋转体体积为：$V_2 = \pi \int_0^c [x(x-a)]^2 \, \mathrm{d}x = \pi \left(\frac{c^5}{5} - \frac{a}{2} c^4 + \frac{a^2}{3} c^3 \right)$

又$V_1 = V_2$，故$\frac{\pi}{3} c^3 (c-a)^2 = \pi \left(\frac{c^5}{5} - \frac{a}{2} c^4 + \frac{a^2}{3} c^3 \right)$，

从而$c = \frac{5}{4} a$

5. 把曲线$y = \frac{\sqrt{x}}{1+x^2}$绕$x$轴旋转得一旋转体，它在点$x=0$与$x=\xi$之间的体积记作$V(\xi)$，求$a$等于何值时，能使$V(a) = \frac{1}{2} \lim\limits_{\xi \to \infty} V(\xi)$.

解：因$V(\xi) = \pi \int_0^\xi \left(\frac{\sqrt{x}}{1+x^2} \right)^2 \mathrm{d}x = \frac{\xi^2}{2(1+\xi^2)} \pi$，则$V(a) = \frac{a^2}{2(1+a^2)} \pi$

又$V(a) = \frac{1}{2} \lim\limits_{\xi \to \infty} V(\xi) = \frac{1}{2} \lim\limits_{\xi \to \infty} \frac{\xi^2}{2(1+\xi^2)} \pi = \frac{\pi}{4}$，于是$a^2 = 1$

又$a > 0$，故$a = 1$.

6. 椭圆$b^2x^2 + a^2y^2 = a^2b^2$绕x轴旋转得一旋转椭球体，把它沿x轴方向打一穿心的圆孔，使剩下的环形体体积等于椭球体体积的一半，决定钻空的半径ρ(图8-15).

解：设题中剩下的环形体体积为V，椭球体体积为V_1

因$b^2x^2 + a^2y^2 = a^2b^2$，则$y = \frac{\sqrt{a^2b^2 - b^2x^2}}{a}$

则$V_1 = \pi \int_{-a}^a y^2 \, \mathrm{d}x = \pi \int_{-a}^a \frac{a^2b^2 - b^2x^2}{a^2} \, \mathrm{d}x = \frac{4}{3} \pi a b^2$

$V = V_1 - 2\pi\rho^2 \frac{\sqrt{a^2b^2 - a^2\rho^2}}{b} - 2\pi \int_{\frac{\sqrt{a^2b^2 - a^2\rho^2}}{b}}^a \left(\frac{\sqrt{a^2b^2 - b^2x^2}}{a} \right)^2 \mathrm{d}x =$

$\frac{4}{3} \pi a b \sqrt{b^2 - \rho^2} - \frac{4\pi a}{3b} \rho^2 \sqrt{b^2 - \rho^2} = \frac{4}{3} \pi \frac{a}{b} (b^2 - \rho^2)^{\frac{3}{2}}$

由题意，得$V = \frac{1}{2} V_1$即$\frac{4}{3} \pi \frac{a}{b} (b^2 - \rho^2)^{\frac{3}{2}} = \frac{2}{3} \pi a b^2$，解此方程，得$\rho = b\sqrt{1 - 2^{-\frac{2}{3}}}$

§4 旋转曲面的面积

1. 求下列旋转曲面的面积：

(1) $x^2 = 2py + a(0 \leqslant x \leqslant a, a > 1)$绕$x$轴及$y$轴

(2) $y = \sin x(0 \leqslant x \leqslant \pi)$绕$x$轴

(3) 椭圆$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$绕$y$轴

(4) 旋轮线$x = a(t - \sin t), y = a(1 - \cos t)(0 \leqslant t \leqslant 2\pi)$绕$x$轴

(5) 双纽线$r^2 = 2a^2 \cos 2\theta$

(i) 绕极线

(ii) 绕轴$\theta = \dfrac{\pi}{2}$

(iii) 绕轴$\theta = \dfrac{\pi}{4}$

解：

(1) $y = \dfrac{x^2 - a}{2p}, -\dfrac{a}{2p} \leqslant y \leqslant \dfrac{a^2 - a}{2p}(p > 0)$

(i) $F_x = 2\pi \displaystyle\int_0^a \frac{x^2 - a}{2p} \sqrt{1 + \left[\left(\frac{x^2 - a}{2p}\right)'\right]^2} \,\mathrm{d}x = \frac{\pi}{p^2} \int_0^a (x^2 - a^2)\sqrt{x^2 + p^2} \,\mathrm{d}x =$

$\left[\dfrac{a(2a^2 - 4a + p^2)}{8p^2}\sqrt{p^2 + a^2} - \dfrac{p^2 + 4a}{8} \ln\left|\dfrac{a + \sqrt{a^2 + p^2}}{p}\right|\right] \pi$

(ii) $\sqrt{1 + (x')^2} = \dfrac{\sqrt{p^2 + 2py + a}}{\sqrt{2py + a}}$

则$F_y = 2\pi \displaystyle\int_{-\frac{a}{2p}}^{\frac{a^2-a}{2p}} \sqrt{2py + a} \frac{\sqrt{p^2 + 2py + a}}{\sqrt{2py + a}} \,\mathrm{d}y =$

$\dfrac{\pi}{p} \displaystyle\int_{-\frac{a}{2p}}^{\frac{a^2-a}{2p}} \sqrt{2py^2 + p^2 + a} \,\mathrm{d}(2py + p^2 + a) = \frac{2\pi}{3p}(p^2 + a^2)^{\frac{3}{2}} - \frac{2}{3}p^2\pi$

(2) $F = 2\pi \displaystyle\int_0^\pi \sin x \cdot \sqrt{1 + [(\sin x)']^2} \,\mathrm{d}x = 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x} \,\mathrm{d}x = 2\sqrt{2}\pi + 2\pi \ln(\sqrt{2} + 1)$

(3) $F = 2\pi \displaystyle\int_{-b}^b x\sqrt{1 + (x')^2} \,\mathrm{d}y = 2\pi \int_{-b}^b \frac{a}{b}\sqrt{b^2 - y^2} \cdot \sqrt{1 + \left(\frac{a(-y)}{b\sqrt{b^2 - y^2}}\right)^2} \,\mathrm{d}y = 2\pi \frac{a}{b} \int_{-b}^b \sqrt{b^2 + \frac{a^2 - b^2}{b^2}y^2} \,\mathrm{d}y =$

$\dfrac{4a\pi}{b} \displaystyle\int_0^b \sqrt{b^2 + \frac{c^2}{b^2}y^2} \,\mathrm{d}y = 2a\pi \left(a + \frac{b^2}{c} \ln \frac{a + c}{b}\right).$

(4) 因$\mathrm{d}S = \sqrt{(x')^2 + (y')^2} \,\mathrm{d}t = 2a \sin \dfrac{t}{2} \,\mathrm{d}t$，则$F = 2\pi \displaystyle\int_0^{2\pi} y \,\mathrm{d}s = 2\pi \int_0^{2\pi} a(1 - \cos t) \cdot 2a \sin \frac{t}{2} \,\mathrm{d}t =$

$16a^2\pi \displaystyle\int_0^{2\pi} \sin^3 \frac{t}{2} \,\mathrm{d}\frac{t}{2} = \frac{64}{3}\pi a^2.$

(5) (i) $y = \sqrt{2}a\sqrt{\cos 2\theta} \sin \theta, \mathrm{d}S = \dfrac{\sqrt{2}a}{\sqrt{\cos 2\theta}} \,\mathrm{d}\theta \left(-\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{4}\right)$

由对称性，得$F = 2 \times 2\pi \displaystyle\int_0^{\frac{\pi}{4}} 2a^2 \sin \theta \,\mathrm{d}\theta = 4\pi a^2(2 - \sqrt{2})$

(ii) $x = \sqrt{2}a\sqrt{\cos 2\theta} \cos \theta, \mathrm{d}S = \dfrac{\sqrt{2}a}{\sqrt{\cos 2\theta}} \,\mathrm{d}\theta \left(-\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{4}\right)$

则$F = 2\pi \displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2a^2 \cos \theta \,\mathrm{d}\theta = 4\sqrt{2}\pi a^2$

(iii) $x = \sqrt{2}a\sqrt{\cos 2\theta} \cos \theta, y = \sqrt{2}a\sqrt{\cos 2\theta} \sin \theta,$

$\mathrm{d}S = \dfrac{\sqrt{2}a}{\sqrt{\cos 2\theta}} \,\mathrm{d}\theta \left(-\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{4}\right)$

注意到在$-\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{4}$内，恒有$x - y \geqslant 0$，

于是，所求的表面积为$F = 2 \times 2\pi \displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x - y}{\sqrt{2}} \cdot \frac{\sqrt{2}a}{\sqrt{\cos 2\theta}} \,\mathrm{d}\theta = 4\sqrt{2}\pi a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos \theta - \sin \theta) \,\mathrm{d}\theta = 8\pi a^2$

2. 证明由$x=\varphi(t), y=\psi(t), z=\chi(t)(t_0 \leqslant t \leqslant T)$与$Oxy$平面间所限的柱面面积等于$S=\int_{t_0}^{T} \chi(t) \sqrt{\varphi'^2(t)+\psi'^2(t)}\,\mathrm{d}t$

证明: 设曲线CD在Oxy平面的投影为AB，则AB的方程为$\left\{\begin{array}{l} x=\varphi(t) \\ y=\psi(t) \end{array}\right. \quad (t_0 \leqslant t \leqslant T)$

在AB上取分点：$A=M_0, M_1, \cdots, M_{i-1}, M_i, \cdots, M_n=B$

在CD对应的分点：$C=N_0, N_1, \cdots, N_{i-1}, N_i, \cdots, N_n=D$

对应的参数：$t_0, t_1, \cdots, t_{i-1}, t_i, \cdots, t_n$

设M_i的坐标为$\chi_i=\varphi(t_i), y_i=\psi(t_i)$，则$\overline{M_i N_i}=\chi(t_i)$

直角梯形$M_i M_{i-1} N_{i-1} N_i$的面积：

$S_i=\dfrac{\overline{M_i N_i}+\overline{M_{i-1} N_{i-1}}}{2} \cdot \overline{M_{i-1} M_i}=\dfrac{\chi(t_i)+\chi(t_{i-1})}{2} \sqrt{[\varphi(t_i)-\varphi(t_{i-1})]^2+[\psi(t_i)-\psi(t_{i-1})]^2}=$
$\chi(t_i) \sqrt{[\varphi(t_i)-\varphi(t_{i-1})]^2+[\psi(t_i)-\psi(t_{i-1})]^2}-$
$\dfrac{\chi(t_i)-\chi(t_{i-1})}{2} \sqrt{[\varphi(t_i)-\varphi(t_{i-1})]^2+[\psi(t_i)-\psi(t_{i-1})]^2}$

由微分中值定理：$\varphi(t_i)-\varphi(t_{i-1})=\varphi'(\xi_i) \Delta t_i, \psi(t_i)-\psi(t_{i-1})=\psi'(\eta_i) \Delta t_i, \chi(t_i)-\chi(t_{i-1})=\chi'(\zeta_i) \Delta t_i$

代入，得$S_i=\chi(t_i) \sqrt{\varphi'^2(\xi_i)+\psi_i'^2(\eta_i)} \Delta t_i-\dfrac{1}{2} \chi'(\zeta_i) \sqrt{\varphi'^2(\xi_i)+\psi_i'^2(\eta_i)}(\Delta t_i)^2$

从而柱面面积为：

$S=\lim\limits_{\lambda \rightarrow 0} \sum\limits_{i=1}^{n} \chi(t_i) \sqrt{\varphi'^2(\xi_i)+\psi_i'^2(\eta_i)} \Delta t_i-\lim\limits_{\lambda \rightarrow 0} \dfrac{1}{2} \sum\limits_{i=1}^{n} \chi'(\zeta_i) \sqrt{\varphi'^2(\xi_i)+\psi_i'^2(\eta_i)}(\Delta t_i)^2$

$=\int_{t_0}^{T} \chi(t) \sqrt{\varphi'^2(t)+\psi'^2(t)}\,\mathrm{d}t-0=\int_{t_0}^{T} \chi(t) \sqrt{\varphi'^2(t)+\psi'^2(t)}\,\mathrm{d}t$

§5 质心

1. 求下列曲线段的质心坐标：

(1) 半径为a，弧长为$\frac{1}{2}a\alpha(\alpha \leqslant \pi)$的均匀圆弧；

(2) 以$A(0,0),B(0,1),C(2,1),D(2,0)$为顶点的矩形周界，曲线上任一点的密度等于该点到原点距离的二倍；

(3) 对数螺线$r=ae^{k\theta}(a>0,k>0)$上由点$(0,a)$到点(θ,r)的均匀弧段.

解：

(1) 以原点为圆心，弧半经的起始边所在直线为x轴建立直角坐标系，则圆弧方程为$x=a\cos\alpha, y=a\sin\alpha$，于是$x'=-a\sin\alpha, y'=a\cos\alpha$，

从而$\overline{x}=\dfrac{\int_0^{\frac{\alpha}{2}} a\cos\alpha\sqrt{x'^2+y'^2}\,\mathrm{d}\alpha}{s}=\dfrac{a^2\int_0^{\frac{\alpha}{2}}\cos\alpha\,\mathrm{d}\alpha}{s}=\dfrac{a^2\sin\frac{\alpha}{2}}{\frac{1}{2}a\alpha}=\dfrac{2a\sin\frac{\alpha}{2}}{\alpha}$

$$\overline{y}=\frac{\int_0^{\frac{\alpha}{2}} a\sin\alpha\sqrt{x'^2+y'^2}\,\mathrm{d}\alpha}{s}=\frac{a^2\int_0^{\frac{\alpha}{2}}\sin\alpha\,\mathrm{d}\alpha}{s}=\frac{2a}{\alpha}\left(1-\cos\frac{\alpha}{2}\right)$$

(2) 先求出密度函数.

AB的方程为$\begin{cases} x=0 \\ y=y \end{cases}$ $y\in[0,1]$，其上任一点的密度为

$\rho=2y$

BC的方程为$\begin{cases} x=x \\ y=1 \end{cases}$ $x\in[0,2]$，其上任一点的密度为

$\rho=2\sqrt{1+x^2}$

CD的方程为$\begin{cases} x=2 \\ y=y \end{cases}$ $y\in[0,1]$，其上任一点的密度为

$\rho=2\sqrt{4+y^2}$

DA的方程为$\begin{cases} x=x \\ y=0 \end{cases}$ $x\in[0,2]$，其上任一点的密度为

$\rho=2x$

于是$m_{AB}=\int_0^1 2y\sqrt{0^2+1^2}\,\mathrm{d}y=1$,

$$\overline{x}_{AB}=0,\overline{y}_{AB}=\frac{\int_0^1 y\cdot 2y\sqrt{0^2+1^2}\,\mathrm{d}y}{m_{AB}}=\frac{2}{3};$$

$$m_{BC}=\int_0^2 2\sqrt{1+x^2}\sqrt{1^2+0^2}\,\mathrm{d}x=2\sqrt{5}+\ln(2+\sqrt{5}),$$

$$\overline{x}_{BC}=\frac{\int_0^2 x\cdot 2\sqrt{1+x^2}\sqrt{1^2+0^2}\,\mathrm{d}x}{m_{BC}}=\frac{2(5\sqrt{5}-1)}{6\sqrt{5}+3\ln(2+\sqrt{5})},$$

$\overline{y}_{BC}=1$；

$$m_{CD}=\int_0^1 2\sqrt{4+y^2}\sqrt{0^2+1^2}\,\mathrm{d}y=\sqrt{5}+4\ln\frac{1+\sqrt{5}}{2},$$

$$\overline{x}_{CD}=2,\overline{y}_{CD}=\frac{\int_0^1 y\cdot 2\sqrt{4+y^2}\sqrt{0^2+1^2}\,\mathrm{d}y}{m_{CD}}=\frac{2(5\sqrt{5}-8)}{3\left(\sqrt{5}+4\ln\frac{1+\sqrt{5}}{2}\right)};$$

$$m_{DA}=\int_0^2 2x\sqrt{1^2+0^2}\,\mathrm{d}x=4,$$

$$\overline{x}_{DA}=\frac{\int_0^2 x\cdot 2x\sqrt{1^2+0^2}\,\mathrm{d}x}{m_{DA}}=\frac{4}{3},\overline{y}_{DA}=0$$

故此矩形周界的质心坐标为：

$$\overline{x}=\frac{m_{AB}\overline{x}_{AB}+m_{BC}\overline{x}_{BC}+m_{CD}\overline{x}_{CD}+m_{DA}\overline{x}_{DA}}{m_{AB}+m_{BC}+m_{CD}+m_{DA}}=$$

$$\frac{16\sqrt{5}+14+24\ln\dfrac{1+\sqrt{5}}{2}}{9\sqrt{5}+15+3\ln(2+\sqrt{5})+12\ln\dfrac{1+\sqrt{5}}{2}}$$

$$\overline{y}=\frac{m_{AB}\overline{y}_{AB}+m_{BC}\overline{y}_{BC}+m_{CD}\overline{y}_{CD}+m_{DA}\overline{y}_{DA}}{m_{AB}+m_{BC}+m_{CD}+m_{DA}}=$$

$$\frac{16\sqrt{5}-14+3\ln(2+\sqrt{5})}{9\sqrt{5}+15+3\ln(2+\sqrt{5})+12\ln\dfrac{1+\sqrt{5}}{2}}$$

(3) 重心的直角坐标为：

$$\overline{x}=\frac{\displaystyle\int_0^\theta x\sqrt{x'^2+y'^2}\,\mathrm{d}x}{s}=\frac{\displaystyle\int_0^\theta r\cos\theta\sqrt{[(r\cos\theta)']^2+[(r\sin\theta)']^2}\,\mathrm{d}\theta}{\displaystyle\int_0^\theta\sqrt{[(r\cos\theta)']^2+[(r\sin\theta)']^2}\,\mathrm{d}\theta}=$$

$$\frac{\displaystyle\int_0^\theta r\cos\theta\sqrt{a^2(1+k^2)}e^{k\theta}\,\mathrm{d}\theta}{\displaystyle\int_0^\theta\sqrt{a^2(1+k^2)}e^{k\theta}\,\mathrm{d}\theta}=\frac{a\displaystyle\int_0^\theta e^{2k\theta}\cos\theta\,\mathrm{d}\theta}{\displaystyle\int_0^\theta e^{k\theta}\,\mathrm{d}\theta}=\frac{ake^{2k\theta}(\sin\theta+2k\cos\theta)-2ak^2}{(4k^2+1)(e^{k\theta}-1)}$$

同法可得$\overline{y}=\dfrac{ake^{2k\theta}(2k\sin\theta-\cos\theta)+ak}{(4k^2+1)(e^{k\theta}-1)}$

于是，重心的极坐标为：

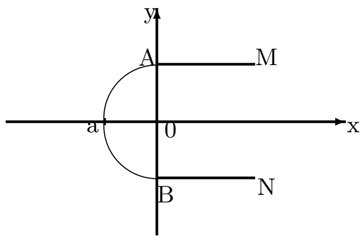
$$\overline{r}=\sqrt{\overline{x}^2+\overline{y}^2}=\frac{ak}{(4k^2+1)(e^{k\theta}-1)}\sqrt{(e^{4k\theta}+1-2e^{2k\theta}\cos\theta)(4k^2+1)}$$

$$\tan\theta_0=\frac{\overline{y}}{\overline{x}}=\frac{e^{2k\theta}(\sin\theta+2k\cos\theta)-2k}{e^{2k\theta}(2k\sin\theta-\cos\theta)+1}$$

且质心坐标为$(\overline{r},\theta_0)$

2. 用一根密度均匀的金属丝弯成半径为a的半圆弧，在两端用同样的金属丝接上两条切线（图8-19），问切线长b为多少时，方能使金属丝$MABN$的质心正好在圆心O？

解:



设金属丝的密度为μ，半圆弧的质量为：$m=\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\mu\,\mathrm{d}s=\pi a\mu$

半圆弧：$x=a\cos\theta,y=a\sin\theta\left(\dfrac{\pi}{2}\leqslant\theta\leqslant\dfrac{3}{2}\pi\right)$, $\mathrm{d}s=\sqrt{x'^2+y'^2}\,\mathrm{d}\theta=a\,\mathrm{d}\theta$

则半圆弧的质心坐标为

$$\overline{x}=\frac{\displaystyle\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}x\mu\,\mathrm{d}s}{m}=\frac{\mu a^2\displaystyle\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\cos\theta\,\mathrm{d}\theta}{\pi a\mu}=-\frac{2a}{\pi};$$

$$\overline{y}=\frac{\displaystyle\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}y\mu\,\mathrm{d}s}{m}=\frac{\mu a^2\displaystyle\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\sin\theta\,\mathrm{d}\theta}{\pi a\mu}=0$$

又两条切线的质心坐标为：$\overline{x}=\dfrac{b}{2},\overline{y}=0$，质量为：$2b\mu$

于是由已知，得质点系质心坐标为：$\overline{x}=\dfrac{-\dfrac{2a}{\pi}\cdot\pi a\mu+\dfrac{b}{2}\cdot 2b\mu}{\pi a\mu+2b\mu}=0$，从而$b=\sqrt{2}a$

3. 轴长10米，密度分布为$\rho=\rho(x)=(6+0.3x)$千克/米，其中x为距轴的一个端点的距离，求轴的质量.

解: $m=\displaystyle\int_0^{10}\rho(x)\,\mathrm{d}x=\int_0^{10}(6+0.3x)\,\mathrm{d}x=75(\text{千克})$

4. 已知一抛物线段$y=x^2(-1\leqslant x\leqslant 1)$，曲线段上任一点处的密度与该点到$y$轴的距离成正比，$x=1$处密度为5，求此曲线段的质量.

解：由已知，得$\rho(x) = k|x|$
因$x = 1$时，$\rho(1) = 5$，则$k = 5$，于是$\rho(x) = 5|x|$
又 $\mathrm{d}s = \sqrt{1 + (y')^2}\,\mathrm{d}x = \sqrt{1 + 4x^2}\,\mathrm{d}x$，
则$m = \displaystyle\int_{-1}^{1} \rho(x)\,\mathrm{d}s = 2\int_{0}^{1} 5x\sqrt{1 + 4x^2}\,\mathrm{d}x = \frac{25}{6}\sqrt{5} - \frac{5}{6}$

§6　平均值、功

1. 已知整流电路中电阻R两端的电压最大值为U_m，圆频率为ω，计算消耗在R上的平均功率(分半波整流和全波整流两种情况讨论).

解：半波整流时，消耗在R上的平均功率为：$\overline{P}_1=\dfrac{2}{T}\displaystyle\int_0^{\frac{T}{2}}P(t)\,\mathrm{d}t=\dfrac{2\omega}{2\pi}\int_0^{\frac{\pi}{\omega}}\dfrac{U_m^2}{R}\cos^2\omega t\,\mathrm{d}t=\dfrac{U_m^2}{2R}$

全波整流时，消耗在R上的平均功率为：$\overline{P}_2=\dfrac{1}{T}\displaystyle\int_0^{T}P(t)\,\mathrm{d}t=\dfrac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\dfrac{U_m^2}{R}\cos^2\omega t\,\mathrm{d}t=\dfrac{U_m^2}{2R}$

2. 计算交流电压$u=U_m\cos\omega t$在$\left[0,\dfrac{\pi}{\omega}\right]$和$\left[-\dfrac{\pi}{2\omega},\dfrac{\pi}{2\omega}\right]$内的平均值.

解：在$\left[0,\dfrac{\pi}{\omega}\right]$内的平均值为：$\overline{u}=\dfrac{\omega}{\pi}\displaystyle\int_0^{\frac{\pi}{\omega}}U_m\cos\omega t\,\mathrm{d}t=0$；

在$\left[-\dfrac{\pi}{2\omega},\dfrac{\pi}{2\omega}\right]$内的平均值为：$\overline{u}=\dfrac{\omega}{\pi}\displaystyle\int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}}U_m\cos\omega t\,\mathrm{d}t=\dfrac{2}{\pi}U_m$.

3. 求下列函数在给定区间内的平均值：

(1) $y=\sin x,[0,\pi]$

(2) $y=xe^x,[0,1]$

解：

(1) $\overline{y}=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\sin x\,\mathrm{d}x=\dfrac{2}{\pi}$

(2) $\overline{y}=\displaystyle\int_0^{1}xe^x\,\mathrm{d}x=1$

4. 把弹簧拉长所需的力与弹簧的伸长成正比.已知一公斤的力能使弹簧伸长1厘米，问把弹簧拉长10厘米要作多少功？

解：由胡克定理知，弹性恢复力F与伸长量x成正比即$F=kx$.

由条件，知$k=1$，因而$F=x$，于是所求的功为$W=\displaystyle\int_0^{10}F\,\mathrm{d}x=\int_0^{10}x\,\mathrm{d}x$=50(千克·厘米）=5J

5. 修建大桥桥墩时要先下围图.设一圆柱形围图的直径为20米，水深27米，围图高出水面3米，要把水抽尽，计算克服重力所作的功.

解：因$\Delta W=\pi r^2\cdot\Delta x\cdot x\cdot 10^3g=10^5g\pi x\Delta x$

则$W=10^5g\displaystyle\int_3^{30}\pi x\,\mathrm{d}x=4.37\times10^8\pi(\mathrm{J})$

6. 某水库的闸门是一梯形，上底6米，下底2米，高10米，求水灌满时闸门所受的力.设水的比重为1吨/米^2.

解：因$\Delta F=2xyg\Delta x=2gx\left(3-\dfrac{x}{5}\right)\Delta x$

则$F=2g\displaystyle\int_0^{10}x\left(3-\dfrac{x}{5}\right)\,\mathrm{d}x=1.63\times10^6(\mathrm{N})$

7. 物体按规律$x=ct^3(c>0)$作直线运动，x表示在时间t内物体移动的距离，设介质的阻力与速度平方成正比，求物体从$x=0$到$x=a$时阻力所作的功.

解：因$x=ct^3(c>0)$，故$v=x'=3ct^2$

又介质阻力与速度的平方成正比，则设$f=kv^2$(k为常数)，于是$f=9kc^2t^4$

又当x从$x=0$到$x=a$时，t从$t=0$到$t=\left(\dfrac{a}{c}\right)^{\frac{1}{3}}$，

则$W=\displaystyle\int_0^{\left(\frac{a}{c}\right)^{\frac{1}{3}}}9kc^2t^4\cdot3ct^2\,\mathrm{d}t=27kc^3\int_0^{\left(\frac{a}{c}\right)^{\frac{1}{3}}}t^6\,\mathrm{d}t=\dfrac{27}{7}kc^{\frac{2}{3}}a^{\frac{7}{3}}$

8. 半径为r的球沉入水中，它与水面相接，球的比重为1，现将球从水中取出，要作多少功？

解：因$\Delta W=1\cdot\pi(\sqrt{r^2-(x-r)^2})^2\Delta x(2r-x)=\pi(4r^2x-4rx^2+x^3)\Delta x$

则$W=\pi\displaystyle\int_0^{2r}(4r^2x-4rx^2+x^3)\,\mathrm{d}x=\dfrac{4}{3}\pi r^4$

§7　定积分的近似计算

1. 用抛物线形公式求$\int_0^1 \frac{\mathrm{d}x}{1+x^2}$的近似值(取$n=3$).

   解：取$n=3$，计算到4位小数，可得：

   $x_0=0, y_0=1.0000; x_1=\frac{1}{6}, 4y_1=3.8919; x_2=\frac{1}{3}, 2y_2=1.8000; x_3=\frac{1}{2}, 4y_3=3.2000; x_4=\frac{2}{3}, 2y_4=1.3846; x_5=\frac{5}{6}, 4y_5=2.3607; x_6=1, y_6=0.5000$

   利用抛物线形公式，有

   $\int_0^1 \frac{\mathrm{d}x}{1+x^2} \approx \frac{1}{18}[y_0+2(y_2+y_4)+4(y_1+y_3+y_5)+y_6]=0.7854$

2. 求某翼型的面积.翼型如图8-24所示，x轴是它的对称轴，OA长2米，10等分，测得数据如下（单位：厘米）：

x	0	20	40	60	80	100	120	140	160	180	200
y	0	8.5	11.0	11.5	10.5	10.0	8.0	6.5	4.5	2.5	0

   解：利用抛物线形公式，有

   $A \approx \frac{200}{6\times 5}[0+0+4(8.5+11.5+10.0+6.5+2.5)+2(11.0+10.5+8.0+4.5)]=\frac{20}{3}\times 224 \approx 1493.3(\mathrm{cm}^2)$

3. 在宽为20米的河面上，测量河流横截面的面积.如果从河的一岸向对岸每隔2米，测得河水深度如下表所列：

x	0	2	4	6	8	10	12	14	16	18	20
y(水深)	0.4	0.8	1.4	2.0	2.4	2.1	1.9	1.6	1.3	0.8	0.4

   (水深单位：米)求此河流横截面的面积（图8-25）

   解：利用抛物线形公式，有

   $A \approx \frac{20}{6\times 5}[0.4+0.4+4(0.8+2.0+2.1+1.6+0.8)+2(1.4+2.4+1.9+1.3)]=\frac{2}{3}\times 44 \approx 29.3(\mathrm{m}^2)$

第三篇　　级数论

第一部分　　数项级数和广义积分

第九章　　数项级数

§1.　　预备知识：上极限和下极限

1. 证明：

(1) $\overline{\lim\limits_{n\to\infty}}(x_n+y_n)\leqslant\overline{\lim\limits_{n\to\infty}}x_n+\overline{\lim\limits_{n\to\infty}}y_n$

(2) $\underline{\lim\limits_{n\to\infty}}(x_n+y_n)\geqslant\underline{\lim\limits_{n\to\infty}}x_n+\underline{\lim\limits_{n\to\infty}}y_n$

证明:

(1) 设$\overline{\lim\limits_{n\to\infty}}x_n,\overline{\lim\limits_{n\to\infty}}y_n$均为有限数，则$\{x_n\},\{y_n\}$上有界.
令$\alpha_k=\sup\limits_{n>k}\{x_n\},\beta_k=\sup\limits_{n>k}\{y_n\}$。于是，当$n>k$时，有$x_n+y_n\leqslant\alpha_k+\beta_k$
从而$\sup\limits_{n>k}\{x_n+y_n\}\leqslant\alpha_k+\beta_k$
故$\overline{\lim\limits_{n\to\infty}}(x_n+y_n)=\lim\limits_{k\to\infty}\sup\limits_{n>k}\{x_n+y_n\}\leqslant\lim\limits_{k\to\infty}\alpha_k+\lim\limits_{k\to\infty}\beta_k=\overline{\lim\limits_{n\to\infty}}x_n+\overline{\lim\limits_{n\to\infty}}y_n$
注：若$\overline{\lim\limits_{n\to\infty}}x_n,\overline{\lim\limits_{n\to\infty}}y_n$之一为$+\infty$.例如：$\overline{\lim\limits_{n\to\infty}}x_n=+\infty$，为使关系式右边加法运算有意义，则$\overline{\lim\limits_{n\to\infty}}y_n$不得为$-\infty$.这时$\{x_n\}$无上界，$\{x_n+y_n\}$亦无上界，上述关系式显然成立；若$\overline{\lim\limits_{n\to\infty}}x_n=-\infty$，为使关系式右边加法运算有意义，则$\overline{\lim\limits_{n\to\infty}}y_n$不得为$+\infty$.于是$\{y_n\}$上有界，从而$\overline{\lim\limits_{n\to\infty}}(x_n+y_n)=-\infty$，上述关系式显然成立.

(2) 因$x_n\geqslant\inf\{x_n\},y_n\geqslant\inf\{y_n\}$，故$x_n+y_n\geqslant\inf\{x_n\}+\inf\{y_n\}$
据下确界为下界中最大的，则$\inf\{x_n+y_n\}\geqslant\inf\{x_n\}+\inf\{y_n\}$，
从而$\inf\limits_{n>k}\{x_n+y_n\}\geqslant\inf\limits_{n>k}\{x_n\}+\inf\limits_{n>k}\{y_n\}$
则$\lim\limits_{k\to\infty}\inf\limits_{n>k}\{x_n+y_n\}\geqslant\lim\limits_{k\to\infty}\left(\inf\limits_{n>k}\{x_n\}+\inf\limits_{n>k}\{y_n\}\right)=\lim\limits_{k\to\infty}\inf\limits_{n>k}\{x_n\}+\lim\limits_{k\to\infty}\inf\limits_{n>k}\{y_n\}$
即$\underline{\lim\limits_{n\to\infty}}(x_n+y_n)\geqslant\underline{\lim\limits_{n\to\infty}}x_n+\underline{\lim\limits_{n\to\infty}}y_n$.

2. 设$x_n\geqslant0,y_n\geqslant0$，证明：

(1) $\overline{\lim\limits_{n\to\infty}}x_ny_n\leqslant\overline{\lim\limits_{n\to\infty}}x_n\cdot\overline{\lim\limits_{n\to\infty}}y_n$

(2) $\underline{\lim\limits_{n\to\infty}}x_ny_n\geqslant\underline{\lim\limits_{n\to\infty}}x_n\cdot\underline{\lim\limits_{n\to\infty}}y_n$

证明:

(1) 因$0\leqslant x_n\leqslant\sup\{x_n\},0\leqslant y_n\leqslant\sup\{y_n\}$，则$0\leqslant x_ny_n\leqslant\sup\{x_n\}\cdot\sup\{y_n\}$
据上确界是上界中最小的，则有$0\leqslant\sup\{x_n\cdot y_n\}\leqslant\sup\{x_n\}\cdot\sup\{y_n\}$
从而$0\leqslant\sup\limits_{n>k}\{x_n\cdot y_n\}\leqslant\sup\limits_{n>k}\{x_n\}\cdot\sup\limits_{n>k}\{y_n\}$
则$\lim\limits_{k\to\infty}\sup\limits_{n>k}\{x_n\cdot y_n\}\leqslant\lim\limits_{k\to\infty}\left(\sup\limits_{n>k}\{x_n\}\cdot\sup\limits_{n>k}\{y_n\}\right)=\lim\limits_{k\to\infty}\sup\limits_{n>k}\{x_n\}\cdot\lim\limits_{k\to\infty}\sup\limits_{n>k}\{y_n\}$
即$\overline{\lim\limits_{n\to\infty}}x_ny_n\leqslant\overline{\lim\limits_{n\to\infty}}x_n\cdot\overline{\lim\limits_{n\to\infty}}y_n$.

(2) 因$x_n\geqslant\inf\{x_n\}\geqslant0,y_n\geqslant\inf\{y_n\}\geqslant0$，则$x_ny_n\geqslant\inf\{x_n\}\cdot\inf\{y_n\}\geqslant0$
据下确界是下界中最大的，则有$\inf\{x_n\cdot y_n\}\geqslant\inf\{x_n\}\cdot\inf\{y_n\}\geqslant0$
从而$\inf\limits_{n>k}\{x_n\cdot y_n\}\geqslant\inf\limits_{n>k}\{x_n\}\cdot\inf\limits_{n>k}\{y_n\}\geqslant0$
则$\lim\limits_{k\to\infty}\inf\limits_{n>k}\{x_n\cdot y_n\}\geqslant\lim\limits_{k\to\infty}\left(\inf\limits_{n>k}\{x_n\}\cdot\inf\limits_{n>k}\{y_n\}\right)=\lim\limits_{k\to\infty}\inf\limits_{n>k}\{x_n\}\cdot\lim\limits_{k\to\infty}\inf\limits_{n>k}\{y_n\}$
即$\underline{\lim\limits_{n\to\infty}}x_ny_n\geqslant\underline{\lim\limits_{n\to\infty}}x_n\cdot\underline{\lim\limits_{n\to\infty}}y_n$

3. 若$\lim\limits_{n\to\infty}x_n$存在，则对任何数列$\{y_n\}$成立：

(1) $\overline{\lim\limits_{n\to\infty}}(x_n+y_n)=\lim\limits_{n\to\infty}x_n+\overline{\lim\limits_{n\to\infty}}y_n$

(2) $\overline{\lim\limits_{n\to\infty}}(x_n\cdot y_n)=\lim\limits_{n\to\infty}x_n\cdot\overline{\lim\limits_{n\to\infty}}y_n$，若$\lim\limits_{n\to\infty}x_n>0$

证明：设$\lim\limits_{n\to\infty}x_n=\alpha$

若$\overline{\lim\limits_{n\to\infty}}y_n=+\infty$(或$-\infty$)，则(1)显然成立. 因$\alpha>0$，则(2)显然成立.

故不妨设$\overline{\lim\limits_{n\to\infty}}y_n=\beta$为有限数

因$\overline{\lim\limits_{n\to\infty}}y_n=\beta$，故存在$\{y_n\}$的子列$\{y_{n_k}\}$，使$\lim\limits_{k\to\infty}y_{n_k}=\beta$且$\beta$为所有收敛子列的极限中的最大者.

又$\lim\limits_{n\to\infty}x_n=\alpha$，则$\lim\limits_{k\to\infty}x_{n_k}=\alpha$，故$\lim\limits_{k\to\infty}(x_{n_k}+y_{n_k})=\alpha+\beta$, $\lim\limits_{k\to\infty}(x_{n_k}\cdot y_{n_k})=\alpha\beta$

下证$\alpha+\beta$为$\{x_n+y_n\}$之一切收敛子列的极限中的最大者(用反证法)

假设$\{x_n+y_n\}$的一个收敛子列$\{x_{n_{k'}}+y_{n_{k'}}\}$，使$\lim\limits_{k'\to\infty}\left(x_{n_{k'}}+y_{n_{k'}}\right)=\gamma>\alpha+\beta$

则$\lim\limits_{k'\to\infty}y_{n_{k'}}=\lim\limits_{k'\to\infty}\left(x_{n_{k'}}+y_{n_{k'}}\right)-\lim\limits_{k'\to\infty}x_{n_{k'}}=\gamma-\alpha>\beta$

这与β为$\{y_n\}$的所有收敛子列的极限中的最大值矛盾.

于是$\alpha+\beta$就是$\{x_n+y_n\}$所有收敛子列极限的最大值.

同理可证，当$\alpha>0$时，$\alpha+\beta$为$\{x_n+y_n\}$的一切收敛子列的极限中的最大值.

从而$\overline{\lim\limits_{n\to\infty}}(x_n+y_n)=\alpha+\beta=\lim\limits_{n\to\infty}x_n+\overline{\lim\limits_{n\to\infty}}y_n$

$\overline{\lim\limits_{n\to\infty}}(x_n\cdot y_n)=\alpha\beta=\lim\limits_{n\to\infty}x_n\cdot\overline{\lim\limits_{n\to\infty}}y_n$，若$\lim\limits_{n\to\infty}x_n>0$

4. 求下列数列的上极限与下极限：

(1) $a_n=\dfrac{1}{2^{-n}+(-1)^n}\,(n=1,2,\cdots)$

(2) $a_n=(-1)^n\left(1+\dfrac{1}{n}\right)\,(n=1,2,\cdots)$

(3) $a_n=\dfrac{(-1)^n}{n}\,(n=1,2\cdots)$

(4) $a_n=\sin\dfrac{n\pi}{5}\,(n=1,2,\cdots)$

解：

(1) 它只有两个具极限的子数列：$a_{2k}\to1, a_{2k+1}\to-1\ (k\to\infty)\ (k=1,2,3\cdots)$
于是$\overline{\lim\limits_{n\to\infty}}a_n=1$, $\underline{\lim\limits_{n\to\infty}}a_n=-1$.

(2) 它只有两个具极限的子数列：$a_{2k}\to1, a_{2k+1}\to-1\ (k\to\infty)\ (k=1,2,3\cdots)$
于是$\overline{\lim\limits_{n\to\infty}}a_n=1$, $\underline{\lim\limits_{n\to\infty}}a_n=-1$.

(3) 因$\lim\limits_{n\to\infty}a_n=0$，故$\overline{\lim\limits_{n\to\infty}}a_n=0$, $\underline{\lim\limits_{n\to\infty}}a_n=0$.

(4) $-\sin\dfrac{2}{5}\pi\leqslant\sin\dfrac{n\pi}{5}\leqslant\sin\dfrac{2}{5}\pi$

当$n=10k+2\ (k=1,2,\cdots)$时，$a_{10k+2}\to\sin\dfrac{2}{5}\pi\ (k\to\infty)$

当$n=10k-2\ (k=1,2,\cdots)$时，$a_{10k-2}\to-\sin\dfrac{2}{5}\pi\ (k\to\infty)$

于是$\overline{\lim\limits_{n\to\infty}}a_n=\sin\dfrac{2}{5}\pi$, $\underline{\lim\limits_{n\to\infty}}a_n=-\sin\dfrac{2}{5}\pi$.

5. 若$\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}=\alpha$，则$\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_{k_0+n}|}=\alpha$
此处k_0是任意固定的整数.

证明

(1) 因$|a_{k_0+n}|^{\frac{1}{n}}=|a_{k_0+n}|^{\frac{1}{k_0+n}}\left(|a_{k_0+n}|^{\frac{1}{k_0+n}}\right)^{\frac{k_0}{n}}$

又$\overline{\lim\limits_{n\to\infty}}\sqrt[k_0+n]{|a_{k_0+n}|}=\alpha$，且当$\alpha>0$时，$\lim\limits_{n\to\infty}\dfrac{k_0}{n}\ln|a_{k_0+n}|^{\frac{1}{k_0+n}}=0$，故$\lim\limits_{n\to\infty}\left(|a_{k_0+n}|^{\frac{1}{k_0+n}}\right)^{\frac{k_0}{n}}=1$

由第2题(1)，得$\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_{k_0+n}|}\leqslant\alpha$

(2) 因$\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}=\alpha$，故存在子列$\{a_{n_k}\}$，使得$\lim\limits_{k\to\infty}|a_{n_k}|^{\frac{1}{n_k}}=\alpha$，

且当$\alpha>0$时，有$\lim\limits_{k\to\infty}|a_{n_k}|^{\frac{1}{n_k-k_0}}=\lim\limits_{k\to\infty}|a_{n_k}|^{\frac{1}{n_k}}\cdot\lim\limits_{k\to\infty}\left(|a_{n_k}|^{\frac{1}{n_k}}\right)^{\frac{k_0}{n_k-k_0}}=\alpha$

从而$\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_{k_0+n}|} \geqslant \alpha$

综合(1)(2)，得当$\alpha > 0$时，结论成立.

(3) 若$\alpha = 0$，则显然有$\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 0$，从而$\lim\limits_{n\to\infty} \sqrt[n]{|a_{k_0+n}|} = \lim\limits_{n\to\infty} \left(|a_{k_0+n}|^{\frac{1}{k_0+n}}\right)^{\frac{k_0+n}{n}} = 0$

于是得此结论正确.

6. 若$\overline{\lim\limits_{n\to\infty}} a_n = a < b$，证明：必存在$N$，当$n > N$时，有$a_n < b$. 又若$\underline{\lim\limits_{n\to\infty}} a_n = a < b$.情况如何？

证明:

(1) 取$\varepsilon = \dfrac{b-a}{2}$，由§1定理1，得$\{a_n\}$中至多只有有限项属于$(a+\varepsilon, +\infty) = \left(\dfrac{a+b}{2}, +\infty\right)$

令这有限项的足标最大者为N，则当$n > N$时，有$a_n < a + \varepsilon = \dfrac{a+b}{2} < \dfrac{b+b}{2} = b$

(2) 若$\underline{\lim\limits_{n\to\infty}} a_n = a < b$，结论未必成立.

例：$a_n = 1 + (-1)^n, n = 1, 2, \cdots$，这个数列为$0, 2, 0, 2, \cdots$，显然$\overline{\lim\limits_{n\to\infty}} a_n = 2$, $\underline{\lim\limits_{n\to\infty}} a_n = 0$，

而$\underline{\lim\limits_{n\to\infty}} a_n = 0 < 1$，但有无穷多项$a_{2n} = 2 > 1 \ (n = 1, 2, \cdots)$

§2. 级数的收敛性及其基本性质

1. 讨论下列级数的敛散性：

(1) $\dfrac{1}{1\cdot 6}+\dfrac{1}{6\cdot 11}+\cdots+\dfrac{1}{(5n-4)(5n+1)}+\cdots$

(2) $1+\dfrac{2}{3}+\dfrac{3}{5}+\cdots+\dfrac{n}{2n-1}+\cdots$

(3) $\left(\dfrac{1}{2}+\dfrac{1}{3}\right)+\left(\dfrac{1}{2^2}+\dfrac{1}{3^2}\right)+\cdots+\left(\dfrac{1}{2^n}+\dfrac{1}{3^n}\right)+\cdots$

(4) $\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\cdots+\dfrac{1}{(3n-2)(3n+1)}+\cdots$

(5) $\cos\dfrac{\pi}{3}+\cos\dfrac{\pi}{4}+\cos\dfrac{\pi}{5}+\cdots$

解：

(1) 因$S_n=\dfrac{1}{1\cdot 6}+\dfrac{1}{6\cdot 11}+\cdots+\dfrac{1}{(5n-4)(5n+1)}=\dfrac{1}{5}\left[1-\dfrac{1}{6}+\dfrac{1}{6}-\dfrac{1}{11}+\cdots+\dfrac{1}{5n-4}-\dfrac{1}{5n+1}\right]=\dfrac{1}{5}\left(1-\dfrac{1}{5n+1}\right)$

则$\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\dfrac{1}{5}\left(1-\dfrac{1}{5n+1}\right)=\dfrac{1}{5}$

于是据定义知，级数$\dfrac{1}{1\cdot 6}+\dfrac{1}{6\cdot 11}+\cdots+\dfrac{1}{(5n-4)(5n+1)}+\cdots$收敛.

(2) 因$\lim\limits_{n\to\infty}\dfrac{n}{2n-1}=\dfrac{1}{2}\neq 0$，故级数发散.

(3) 由于$\sum\limits_{n=1}^{\infty}\dfrac{1}{2^n}$与$\sum\limits_{n=1}^{\infty}\dfrac{1}{3^n}$均为收敛的几何级数，

故由数列级数性质2，知$\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{2^n}+\dfrac{1}{3^n}\right)=\sum\limits_{n=1}^{\infty}\dfrac{1}{2^n}+\sum\limits_{n=1}^{\infty}\dfrac{1}{2^n}=1+\dfrac{1}{2}=\dfrac{3}{2}$

(4) 因$S_n=\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\cdots+\dfrac{1}{(3n-2)(3n+1)}=\dfrac{1}{3}\left[1-\dfrac{1}{4}+\dfrac{1}{4}-\dfrac{1}{7}+\cdots+\dfrac{1}{3n-2}-\dfrac{1}{3n+1}\right]=\dfrac{1}{3}\left(1-\dfrac{1}{3n+1}\right)$

则$\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\dfrac{1}{3}\left(1-\dfrac{1}{3n+1}\right)=\dfrac{1}{3}$

于是据定义知，级数$\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\cdots+\dfrac{1}{(3n-2)(3n+1)}+\cdots$收敛.

(5) 因$\lim\limits_{n\to\infty}\cos\dfrac{\pi}{n+2}=1\neq 0$，故级数发散.

2. 利用柯西收敛原理判别下列级数是收敛还是发散.

(1) $a_0+a_1q+a_2q^2+\cdots+a_nq^n+\cdots,|q|<1,|a_n|\leqslant A,(n=0,1,2,\cdots)$

(2) $1+\dfrac{1}{2}-\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{5}-\dfrac{1}{6}+\cdots$

证明：

(1) 因对任何自然数p，

$|S_{n+p}-S_n|=\left|a_nq^n+a_{n+1}q^{n+1}+\cdots+a_{n+p-1}q^{n+p-1}\right|\leqslant\left|a_n\right|\left|q^n\right|+\left|a_{n+1}\right|\left|q^{n+1}\right|+\cdots+\left|a_{n+p-1}\right|\left|q^{n+p-1}\right|\leqslant A|q|^n\dfrac{1-|q|^p}{1-|q|}$

又$|q|<1$，则$0<1-|q|^p<1$，于是$|S_{n+p}-S_n|<A\cdot\dfrac{|q|^n}{1-|q|}$

从而对$\forall\varepsilon>0$，取$N=\left[\ln\dfrac{(1-|q|)\varepsilon}{A}/\ln|q|\right]$，当$n>N$时，对任何$p=1,2,3,\cdots$，

总成立$|S_{n+p}-S_n|<\varepsilon$

按收敛原理，级数$a_0+a_1q+a_2q^2+\cdots+a_nq^n+\cdots$收敛.

(2) 此级数为$\sum\limits_{n=0}^{\infty}\left(\frac{1}{3n+1}+\frac{1}{3n+2}-\frac{1}{3n+3}\right)$

取$0<\varepsilon_0<\frac{1}{6}$，不论$n$多大，若令$p=n$，则有

$$|S_{n+p}-S_n|=|S_{2n}-S_n|=\frac{1}{3n+1}+\frac{1}{3n+2}-\frac{1}{3n+3}+\cdots+\frac{1}{6n-2}+\frac{1}{6n-1}-\frac{1}{6n}>\frac{1}{3n+3}+\frac{1}{3n+3}-\frac{1}{3n+3}+\cdots+\frac{1}{6n}+\frac{1}{6n}-\frac{1}{6n}=\frac{1}{3}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)>\frac{1}{3}\left(\underbrace{\frac{1}{2n}+\frac{1}{2n}+\cdots+\frac{1}{2n}}_{n\text{项}}\right)=\frac{1}{6}>\varepsilon_0$$

因此级数$\sum\limits_{n=0}^{\infty}\left(\frac{1}{3n+1}+\frac{1}{3n+2}-\frac{1}{3n+3}\right)$发散.

3. 设有正项级数$\sum\limits_{n=1}^{\infty}a_n$(即每一项$a_n>0$)，试证明若对其项加括号后所组成的级数收敛，则$\sum\limits_{n=1}^{\infty}a_n$亦收敛.

证明: 设$\sum\limits_{n=1}^{\infty}a_n$部分和数列为$\{S_n\}$，加括号后所组成的级数为$\sum\limits_{n=1}^{\infty}A_n$

其中$A_n=a_{i_{n-1}+1}+a_{i_{n-1}+2}+\cdots+a_{i_n}$，显然$\sum\limits_{n=1}^{\infty}A_n$仍为正项级数.

设其部分和数列为$\{S_n{}'\}$，其中$S_n{}'=(a_1+a_2+\cdots+a_{i_1})+(a_{i_1+1}+\cdots+a_{i_2})+\cdots+(a_{i_{n-1}+1}+\cdots+a_{i_n})$
显然$S_n{}'\geqslant S_n$

又$\sum\limits_{n=1}^{\infty}A_n$收敛，由基本定理，得$\{S_n{}'\}$有上界，即存在$M>0$，使$S_n{}'\leqslant M$，从而$S_n\leqslant S_n{}'\leqslant M$，说明$\{S_n\}$有上界

则由基本定理，得$\sum\limits_{n=1}^{\infty}a_n$收敛.

4. 确定使下列级数收敛的x的范围.

(1) $\sum\limits_{n=0}^{\infty}\frac{1}{(1+x)^n}$

(2) $\sum\limits_{n=1}^{\infty}(\ln x)^n$

解:

(1) 此级数是公比为$\frac{1}{1+x}$的等比级数，故当$\left|\frac{1}{1+x}\right|<1$时级数收敛

从而收敛域为$x<-2$或$x>0$.

(2) 此级数是公比为$\ln x$的等比级数，故当$|\ln x|<1$时级数收敛

从而收敛域为$\frac{1}{e}<x<e$.

§3. 正项级数

1. 判断下列级数的收敛和发散.

(1) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

(2) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)\cdot 2^{2n-1}}$

(3) $\sum_{n=1}^{\infty} \frac{n-\sqrt{n}}{2n-1}$

(4) $\sum_{n=1}^{\infty} \sin\frac{\pi}{2^n}$

(5) $\sum_{n=1}^{\infty} \frac{1}{1+a^n}, (a>1)$

(6) $\sum_{n=1}^{\infty} \frac{1}{n\cdot\sqrt[n]{n}}$

(7) $\sum_{n=1}^{\infty} \left(\frac{1}{2n+1}\right)^n$

(8) $\sum_{n=1}^{\infty} \frac{1}{[\ln(n+1)]^n}$

(9) $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^n}$

(10) $\sum_{n=1}^{\infty} 2^n\sin\frac{\pi}{3^n}$

(11) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(12) $\sum_{n=1}^{\infty} \frac{x^n}{(1+x)(1+x^2)\cdots(1+x^n)}, (x\geqslant 0)$

(13) $\sum_{n=1}^{\infty} \left(\frac{b}{a_n}\right)^n$，其中$a_n\to a, a_n, b, a$皆正数，$a\neq 0$

解:

(1) 因$\lim\limits_{n\to\infty}\dfrac{\frac{1}{\sqrt{n^2+n}}}{\frac{1}{n}}=\lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+n}}=1$，而级数$\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$是发散的

则由比较判别法，得级数$\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n^2+n}}$亦发散.

(2) 因$\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\dfrac{\frac{1}{(2n+1)2^{2n+1}}}{\frac{1}{(2n-1)2^{2n-1}}}=\lim\limits_{n\to\infty}\dfrac{2n-1}{4(2n+1)}=\dfrac{1}{4}<1$

则由达朗贝尔判别法，得级数$\sum\limits_{n=1}^{\infty}\dfrac{1}{(2n-1)\cdot 2^{2n-1}}$收敛.

(3) 因$\lim\limits_{n\to\infty}\dfrac{n-\sqrt{n}}{2n-1}=\dfrac{1}{2}\nrightarrow 0$，故级数$\sum\limits_{n=1}^{\infty}\dfrac{n-\sqrt{n}}{2n-1}$发散.

(4) 因$\sin\dfrac{\pi}{2^n}\leqslant\dfrac{\pi}{2^n}$，而$\sum\limits_{n=1}^{\infty}\dfrac{\pi}{2^n}$收敛，故级数$\sum\limits_{n=1}^{\infty}\sin\dfrac{\pi}{2^n}$收敛.

(5) 因$\dfrac{1}{1+a^n} \leqslant \left(\dfrac{1}{a}\right)^n$，而$\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{a}\right)^n$收敛，故级数$\sum\limits_{n=1}^{\infty}\dfrac{1}{1+a^n}$收敛.

(6) 因$\lim\limits_{x\to+0} x^x = \lim\limits_{x\to+0} e^{\ln x^x} = \lim\limits_{x\to+0} e^{x\ln x} = e^{\lim\limits_{x\to+0} x\ln x} = 1$，故$\lim\limits_{n\to\infty}\dfrac{1}{\sqrt[n]{n}} = \lim\limits_{n\to\infty}\left(\dfrac{1}{n}\right)^{\frac{1}{n}} = 1$

又$\lim\limits_{n\to\infty}\dfrac{\dfrac{1}{n\cdot\sqrt[n]{n}}}{\dfrac{1}{n}} = \lim\limits_{n\to\infty}\dfrac{1}{\sqrt[n]{n}} = 1$，而级数$\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$是发散的

则由比较判别法，得级数$\sum\limits_{n=1}^{\infty}\dfrac{1}{n\cdot\sqrt[n]{n}}$发散.

(7) 因$\lim\limits_{n\to\infty}\sqrt[n]{\left(\dfrac{1}{2n+1}\right)^n} = \lim\limits_{n\to\infty}\dfrac{1}{2n+1} = 0 < 1$，故级数$\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{2n+1}\right)^n$收敛.

(8) 因$\lim\limits_{n\to\infty}\sqrt[n]{\dfrac{1}{[\ln(n+1)]^n}} = \lim\limits_{n\to\infty}\dfrac{1}{\ln(n+1)} = 0 < 1$，故级数$\sum\limits_{n=1}^{\infty}\dfrac{1}{[\ln(n+1)]^n}$收敛.

(9) 因 $\dfrac{2+(-1)^n}{2^n} \leqslant \dfrac{3}{2^n}$ 且级数 $\sum\limits_{n=1}^{\infty}\dfrac{3}{2^n}$ 收敛

则据比较判别法，得级数$\sum\limits_{n=1}^{\infty}\dfrac{2+(-1)^n}{2^n}$ 收敛.

(10) 因 $0 < 2^n\sin\dfrac{\pi}{3^n} \leqslant \pi\left(\dfrac{2}{3}\right)^n$ 且级数 $\sum\limits_{n=1}^{\infty}\left(\dfrac{2}{3}\right)^n$ 收敛

则据比较判别法，得级数$\sum\limits_{n=1}^{\infty}2^n\sin\dfrac{\pi}{3^n}$ 收敛.

(11) 因$\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\dfrac{\dfrac{(n+1)^{n+1}}{(n+1)!}}{\dfrac{n^n}{n!}} = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e > 1$，故级数$\sum\limits_{n=1}^{\infty}\dfrac{n^n}{n!}$发散.

(12) 因$\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\dfrac{x^{n+1}/[(1+x)(1+x^2)\cdots(1+x^n)(1+x^{n+1})]}{x^n/[(1+x)(1+x^2)\cdots(1+x^n)]} = \lim\limits_{n\to\infty}\dfrac{x}{1+x^{n+1}} = \begin{cases} 0<1, & x>1\text{或}x=0 \\ \frac{1}{2}<1, & x=1 \\ x<1, & 0<x<1 \end{cases}$

则据达朗贝尔判别法，得级数$\sum\limits_{n=1}^{\infty}\dfrac{x^n}{(1+x)(1+x^2)\cdots(1+x^n)}$收敛.

(13) 因$\lim\limits_{n\to\infty}\sqrt[n]{\left(\dfrac{b}{a_n}\right)^n} = \lim\limits_{n\to\infty}\dfrac{b}{a_n} = \dfrac{b}{a}$

则当$\dfrac{b}{a} < 1$即$b < a$时，级数$\sum\limits_{n=1}^{\infty}\left(\dfrac{b}{a_n}\right)^n$收敛；

当$\dfrac{b}{a} > 1$即$b > a$时，级数$\sum\limits_{n=1}^{\infty}\left(\dfrac{b}{a_n}\right)^n$发散；

当$\dfrac{b}{a} = 1$即$b = a$时，需进一步判断。例如：级数$\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{\sqrt[n]{n}}\right)^n = \sum\limits_{n=1}^{\infty}\dfrac{1}{n}$发散；而级数$\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{\sqrt[n]{n^2}}\right)^n = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$收敛.

2. 若正项级数$\sum\limits_{n=1}^{\infty}u_n$收敛，证明$\sum\limits_{n=1}^{\infty}u_n^2$也收敛，其逆如何？

证明：因$\sum\limits_{n=1}^{\infty}u_n$收敛，则$\lim\limits_{n\to\infty}u_n = 0$

取$\varepsilon_0 = 1$，则存在正整数N，当$n > N$时，有$|u_n| < \varepsilon_0 = 1$即$0 \leqslant u_n < 1$，于是$0 \leqslant u_n^2 < u_n (n > N)$，

从而由比较判别法，得$\sum\limits_{n=1}^{\infty} u_n^2$收敛.

其逆不真.例：$\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$收敛，但$\sum\limits_{n=1}^{\infty} \frac{1}{n}$发散；$\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$收敛，$\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$收敛.

3. 设$\sum\limits_{n=1}^{\infty} u_n$和$\sum\limits_{n=1}^{\infty} v_n$为两正项级数，$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 0$，证明：当$\sum\limits_{n=1}^{\infty} v_n$收敛时，$\sum\limits_{n=1}^{\infty} u_n$也收敛.又若$\sum\limits_{n=1}^{\infty} v_n$发散时，$\sum\limits_{n=1}^{\infty} u_n$如何？若$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \infty$，那么$\sum\limits_{n=1}^{\infty} u_n$和$\sum\limits_{n=1}^{\infty} v_n$的敛散性之间有什么关系？

证明：

(1) 因$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 0$，$\sum\limits_{n=1}^{\infty} u_n$和$\sum\limits_{n=1}^{\infty} v_n$为两正项级数

取$\varepsilon_0 = 1$，则存在正整数N，当$n > N$时，有$\left|\frac{u_n}{v_n}\right| < \varepsilon_0 = 1$即$0 \leqslant \frac{u_n}{v_n} < 1$，于是$u_n < v_n (n > N)$

又$\sum\limits_{n=1}^{\infty} v_n$收敛，则由比较判别法，得$\sum\limits_{n=1}^{\infty} u_n$收敛

若$\sum\limits_{n=1}^{\infty} v_n$发散，则$\sum\limits_{n=1}^{\infty} u_n$可能收敛，也可能发散

例：$\sum\limits_{n=1}^{\infty} \frac{1}{n}$发散，$\lim\limits_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n}} = 0$，但$\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$收敛；

$\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$发散，$\lim\limits_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}}} = 0$，且$\sum\limits_{n=1}^{\infty} \frac{1}{n}$发散.

(2) 因$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \infty$，$\sum\limits_{n=1}^{\infty} u_n$和$\sum\limits_{n=1}^{\infty} v_n$为两正项级数

取$G_0 = 1$，则存在正整数N，当$n > N$时，有$\frac{u_n}{v_n} > G_0 = 1$，于是$u_n > v_n (n > N)$

若$\sum\limits_{n=1}^{\infty} u_n$收敛，则由比较判别法，得$\sum\limits_{n=1}^{\infty} v_n$收敛；若$\sum\limits_{n=1}^{\infty} v_n$发散，则$\sum\limits_{n=1}^{\infty} u_n$发散；若$\sum\limits_{n=1}^{\infty} u_n$发散，则$\sum\limits_{n=1}^{\infty} v_n$敛散性不定.

4. 若两正项级数$\sum\limits_{n=1}^{\infty} u_n$和$\sum\limits_{n=1}^{\infty} v_n$发散，$\sum\limits_{n=1}^{\infty} \max(u_n, v_n)$，$\sum\limits_{n=1}^{\infty} \min(u_n, v_n)$两级数如何？

解：因两正项级数$\sum\limits_{n=1}^{\infty} u_n$和$\sum\limits_{n=1}^{\infty} v_n$发散，$u_n \leqslant \max(u_n, v_n)$

则由比较判别法，得$\sum\limits_{n=1}^{\infty} \max(u_n, v_n)$发散.

对于$\sum\limits_{n=1}^{\infty} \min(u_n, v_n)$敛散性不定.

例：$\sum\limits_{n=1}^{\infty} \frac{1}{n}$，$\sum\limits_{n=1}^{\infty} \frac{1}{2n}$都发散，$\sum\limits_{n=1}^{\infty} \min\left(\frac{1}{n}, \frac{1}{2n}\right) = \sum\limits_{n=1}^{\infty} \frac{1}{2n}$也发散；

$\sum\limits_{n=1}^{\infty} \frac{1+(-1)^n}{2}$，$\sum\limits_{n=1}^{\infty} \frac{1-(-1)^n}{2}$都发散，但$\sum\limits_{n=1}^{\infty} \min\left(\frac{1+(-1)^n}{2}, \frac{1-(-1)^n}{2}\right) = 0 + 0 + \cdots + 0 + \cdots$却收敛.

5. 利用级数收敛的必要条件证明：

(1) $\lim\limits_{n\to\infty} \frac{n^n}{(n!)^2} = 0$

(2) $\lim\limits_{n\to\infty} \frac{(2n)!}{a^{n!}} = 0\,(a > 1)$

证明：

(1) $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

因 $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{[(n+1)!]^2}}{\frac{n^n}{(n!)^2}} = \lim_{n\to\infty} \frac{1}{n+1}\left(1+\frac{1}{n}\right)^n = 0 < 1$

则据达朗贝尔判别法的极限形式，得$\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$收敛，从而由级数收敛的必要条件，得$\lim_{n\to\infty} \frac{n^n}{(n!)^2} = 0$

(2) $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(2n)!}{a^{n!}}$

因$0 < \frac{u_{n+1}}{u_n} = \frac{\frac{[2(n+1)]!}{a^{(n+1)!}}}{\frac{(2n)!}{a^{n!}}} = \frac{(2n+2)(2n+1)}{a^{n^2(n-1)!}} < \frac{4(n+1)^2}{a^{n+1}} (a > 1)$

而$\lim_{n\to\infty} \frac{n^k}{a^n} = 0 (a > 1, k \in N)$，于是$\lim_{n\to\infty} \frac{4(n+1)^2}{a^{n+1}} = 0$，从而$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0$

则据达朗贝尔判别法的极限形式，得$\sum_{n=1}^{\infty} \frac{(2n)!}{a^{n!}}$收敛，从而由级数收敛的必要条件，得$\lim_{n\to\infty} \frac{(2n)!}{a^{n!}} = 0$

6. 讨论下列级数的收敛性：

(1) $\sum_{n=1}^{\infty} \frac{1}{n\cdot(\ln n)^p}$

(2) $\sum_{n=1}^{\infty} \frac{1}{n\cdot\ln n\cdot\ln\ln n}$

(3) $\sum_{n=1}^{\infty} \frac{1}{n\cdot(\ln n)^{1+\sigma}\ln\ln n}$

(4) $\sum_{n=1}^{\infty} \frac{1}{n\cdot(\ln n)^p(\ln\ln n)^q}$

解:

(1) 由于不论p为何数，当x充分大时，函数$f(x) = \frac{1}{x(\ln x)^p}$ 都是非负递减的，且

$$\lim_{n\to\infty} \int_2^n \frac{\mathrm{d}x}{x(\ln x)^p} = \begin{cases} \frac{1}{p-1}(\ln 2)^{1-p}, & p > 1 \\ \infty, & p \leqslant 1 \end{cases}$$

故当$p > 1$时，级数收敛；当$p \leqslant 1$时，级数发散.

(2) 设$f(x) = \frac{1}{x\ln x\ln\ln x}$，$f(x)$当$x \geqslant 3$是正值递减函数.

$\lim_{n\to\infty} \int_3^n \frac{\mathrm{d}x}{x\ln x\ln\ln x} = \lim_{n\to\infty} (\ln\ln\ln n - \ln\ln\ln 2) = \infty$，则级数$\sum_{n=1}^{\infty} \frac{1}{n\cdot\ln n\cdot\ln\ln n}$发散.

(3) 因$\lim_{n\to\infty} \int_2^n \frac{\mathrm{d}x}{x(\ln x)^{1+\sigma}} = \lim_{n\to\infty} \frac{1}{\sigma}\left(\frac{1}{(\ln 2)^\sigma} - \frac{1}{(\ln n)^\sigma}\right) = \frac{1}{\sigma(\ln 2)^\sigma}$ $(\sigma > 0)$

故级数$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\sigma}}$收敛.

又$\frac{1}{n\cdot(\ln n)^{1+\sigma}\ln\ln n} \leqslant \frac{1}{n(\ln n)^{1+\sigma}}$，则由比较判别法，得级数$\sum_{n=1}^{\infty} \frac{1}{n\cdot(\ln n)^{1+\sigma}\ln\ln n}$ 收敛.

(4) 令$f(x) = \frac{1}{x(\ln x)^p(\ln\ln x)^q}$，当$n \leqslant 3$时是正值递减函数.

又因为$\int_3^{+\infty} \frac{\mathrm{d}x}{x(\ln x)^p(\ln\ln x)^q} = \int_{\ln\ln 3}^{+\infty} \frac{\mathrm{d}t}{e^{(p-1)t}t^q}$

对任何q，当$p-1 > 0$时，积分收敛，当$p-1 < 0$时，积分发散；当$p = 1$时，若$q > 1$，积分收敛，若$q \leqslant 1$，积分发散.

由柯西积分判别法知，原级数敛散性与积分敛散性条件一致

则原级数当$p>1$时收敛；当$p<1$时发散；当$p=1$时，$q>1$时级数收敛；$q\leqslant 1$时级数发散.

7. 若$\sum\limits_{n=1}^{\infty}u_n$是收敛的正项级数，并且数列$\{u_n\}$单调下降，证明$\lim\limits_{n\to\infty}nu_n=0$.

证明：因$\sum\limits_{n=1}^{\infty}u_n$收敛，设$S=\sum\limits_{n=1}^{\infty}u_n$，$S_n=\sum\limits_{k=1}^{n}u_k$

则$\lim\limits_{n\to\infty}S_n=S=\lim\limits_{n\to\infty}S_{2n}$，于是$\lim\limits_{n\to\infty}(S_{2n}-S_n)=0$

又$\{u_n\}$单调下降，则$S_{2n}-S_n=u_{n+1}+u_{n+2}+\cdots+u_{2n}\geqslant u_{2n}+u_{2n}+\cdots+u_{2n}=nu_{2n}$

又$u_n\geqslant 0$，则$0\leqslant nu_{2n}\leqslant S_{2n}-S_n$，于是得$\lim\limits_{n\to\infty}nu_{2n}=0$，从而$\lim\limits_{n\to\infty}(2n)u_{2n}=0$

又因$u_{2n+1}\leqslant u_{2n},u_n\geqslant 0$，则$0\leqslant(2n+1)u_{2n+1}\leqslant(2n+1)u_{2n}=\dfrac{2n+1}{2n}(2nu_{2n})\to 0(n\to\infty)$

于是$\lim\limits_{n\to\infty}(2n+1)u_{2n+1}=0$，从而$\lim\limits_{n\to\infty}nu_n=0$

8. 证明达朗贝尔判别法及其极限形式.

证明：

(1) 达朗贝尔判别法：

因$n>N$时，有$\dfrac{u_{n+1}}{u_n}\leqslant q<1$，则$\dfrac{u_{N+2}}{u_{N+1}}\leqslant q,u_{N+2}\leqslant qu_{N+1}$；$\dfrac{u_{N+3}}{u_{N+2}}\leqslant q,u_{N+3}\leqslant qu_{N+2}$；$\cdots$；$\dfrac{u_{N+k+1}}{u_{N+k}}\leqslant q,u_{N+k+1}\leqslant qu_{N+k}\leqslant\cdots\leqslant q^Ku_{N+1}$

因$q<1$，则$\sum\limits_{k=1}^{\infty}q^k$收敛，于是由收敛级数的性质1知，$\sum\limits_{k=1}^{\infty}q^ku_{N+1}$也收敛，从而由比较判别法，得$\sum\limits_{k=2}^{\infty}u_{N+k}$也收敛

再由收敛级数的性质5知，添加有限项$u_1,u_2,\cdots,u_{N+1}$后得到的新级数$\sum\limits_{n=1}^{\infty}u_n$也收敛.

若$n>N$时，$\dfrac{u_{n+1}}{u_n}\geqslant 1$，则$\dfrac{u_{N+1}}{u_N}\geqslant 1,u_{N+1}\geqslant u_N$，这说明$\{u_N\}$是单调增加的

又$u_n\geqslant 0$，则$u_n\nrightarrow 0(n\to\infty)$，于是$\sum\limits_{n=1}^{\infty}u_n$发散.

(2) 达朗贝尔判别法的极限形式：

(i) 若$\varlimsup\limits_{n\to\infty}\dfrac{u_n}{u_{n-1}}=\bar{r}<1$

由实数的稠密性知必存在$\varepsilon_0>0$，使得$\bar{r}<\bar{r}+\varepsilon_0<1$

由上极限的定理1的证明中，知$\left\{\dfrac{u_{n+1}}{u_n}\right\}$只有有限项大于$\bar{r}+\varepsilon_0$，于是定存在一个正整数$N$(只要取有限项中下标最大的做$N$即可)，使得当$n>N$时，有$\dfrac{u_{n+1}}{u_n}<\bar{r}+\varepsilon_0<1$，故由达朗贝尔判别法知级数收敛.

(ii) 若$\varliminf\limits_{n\to\infty}\dfrac{u_n}{u_{n-1}}=\underline{r}>1$

由实数的稠密性知必存在$\varepsilon_0>0$，使得$\bar{r}>\underline{r}-\varepsilon_0>1$

由上极限的定理2的证明中，知$\left\{\dfrac{u_{n+1}}{u_n}\right\}$只有有限项小于$\underline{r}+\varepsilon_0$，于是定存在一个正整数$N$(只要取有限项中下标最大的做$N$即可)，使得当$n>N$时，有$\dfrac{u_{n+1}}{u_n}>\underline{r}+\varepsilon_0>1$，故由达朗贝尔判别法知级数发散.

(iii) 举例说明：$\sum\limits_{n=1}^{\infty}u_n=\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$，$\varlimsup\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=\varliminf\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=1$，但$\sum\limits_{n=1}^{\infty}u_n=\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$发散；

$\sum\limits_{n=1}^{\infty}u_n=\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$，$\varlimsup\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=\varliminf\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=1$，但$\sum\limits_{n=1}^{\infty}u_n=\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$收敛.

§4. 任意项级数

1. 讨论下列级数的收敛性（包括条件收敛或绝对收敛）：

(1) $\frac{1}{2}-\frac{3}{10}+\frac{1}{2^2}-\frac{3}{10^3}+\frac{1}{2^3}-\frac{3}{10^5}+\cdots$

(2) $1-\frac{1}{2}+\frac{1}{3!}-\frac{1}{4}+\frac{1}{5!}-\cdots$

(3) $\sum\limits_{n=2}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$

(4) $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{n^3}{2^n}$

(5) $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(n+1)^2}$

(6) $\sum\limits_{n=1}^{\infty}(-1)^{n}\sin\frac{x}{n}\ (x\neq 0)$

(7) $\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\cdots+\frac{1}{\sqrt{n}-1}-\frac{1}{\sqrt{n}+1}+\cdots$

解：

(1) 因$\sum\limits_{n=1}^{\infty}\frac{1}{2^n}$收敛，$\sum\limits_{n=1}^{\infty}\frac{1}{10^{2n-1}}$收敛，则$\sum\limits_{n=1}^{\infty}\frac{3}{10^{2n-1}}$收敛

于是$\sum\limits_{n=1}^{\infty}\frac{1}{2^n}+\sum\limits_{n=1}^{\infty}\frac{1}{10^{2n-1}}$收敛，即$\frac{1}{2}+\frac{3}{10}+\frac{1}{2^2}+\frac{3}{10^3}+\frac{1}{2^3}+\frac{3}{10^5}+\cdots$收敛，从而原级数绝对收敛.

(2) 因$\sum\limits_{n=1}^{\infty}\frac{1}{n}$发散，则$\sum\limits_{n=1}^{\infty}\left(-\frac{1}{2n}\right)$发散

又对级数$\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)!}$

因$\lim\limits_{n\to\infty}\frac{\frac{1}{(2n+1)!}}{\frac{1}{(2n-1)!}}=\lim\limits_{n\to\infty}\frac{1}{2n(2n+1)}=0<1$，则由达朗贝尔判别法的极限形式，得级数$\sum\limits_{n=1}^{\infty}\frac{1}{(2n_1)!}$收敛

于是原级数发散.

(3) 因$\sum\limits_{n=2}^{\infty}\left|(-1)^{n-1}\frac{\ln n}{n}\right|=\sum\limits_{n=2}^{\infty}\frac{\ln n}{n}$

又$\lim\limits_{n\to\infty}\frac{\frac{\ln n}{n}}{\frac{1}{n}}=\lim\limits_{n\to\infty}\ln n=+\infty$且$\sum\limits_{n=1}^{\infty}\frac{1}{n}$发散，则由比较判别法，得$\sum\limits_{n=2}^{\infty}\frac{\ln n}{n}$发散

又设$f(x)=\frac{\ln x}{x}(x\geqslant 3)$，则$f'(x)=\left(\frac{\ln x}{x}\right)'=\frac{1-\ln x}{x^2}<0\ (x\geqslant 3)$，于是$f(x)=\frac{\ln x}{x}$单调下降，从而$\left\{\frac{\ln n}{n}\right\}$在$n\geqslant 3$时单调下降

又$\lim\limits_{x\to+\infty}\frac{\ln x}{x}=\lim\limits_{x\to+\infty}\frac{1}{x}=0$，则$\lim\limits_{n\to\infty}\frac{\ln n}{n}=0$

于是据莱布尼兹定理，得$\sum\limits_{n=2}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$条件收敛.

(4) $\sum\limits_{n=1}^{\infty}\left|(-1)^{n-1}\frac{n^3}{2^n}\right|=\sum\limits_{n=1}^{\infty}\frac{n^3}{2^n}$

因$\lim\limits_{n\to\infty}\frac{\frac{(n+1)^3}{2^{n+1}}}{\frac{n^3}{2^n}}=\lim\limits_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^3=\frac{1}{2}<1$，则据达朗贝尔判别法，得$\sum\limits_{n=1}^{\infty}\frac{n^3}{2^n}$收敛

从而$\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{n^3}{2^n}$绝对收敛.

(5) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(n+1)^2}=\sum_{n=1}^{\infty}\frac{n}{(n+1)^2}$

因$\lim_{n\to\infty}\frac{\frac{n}{(n+1)^2}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^2}{(n+1)^2}=1$且$\sum_{n=1}^{\infty}\frac{1}{n}$发散，则$\sum_{n=1}^{\infty}\frac{n}{(n+1)^2}$发散

设$f(x)=\frac{x}{(x+1)^2}(x\geqslant 2)$，则$f'(x)=\frac{1-x}{(x+1)^3}<0\,(x\geqslant 2)$，于是当$x\geqslant 2$时，$f(x)$单调下降，从而$\left\{\frac{n}{(n+1)^2}\right\}$当$n\geqslant 2$时单调下降

又$\lim_{n\to\infty}\frac{n}{(n+1)^2}=0$，则据莱布尼兹定理，得$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(n+1)^2}$收敛

从而$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(n+1)^2}$条件收敛.

(6) $\sum_{n=1}^{\infty}(-1)^n\sin\frac{x}{n}=\sum_{n=1}^{\infty}\left|\sin\frac{x}{n}\right|$

因$\left|\frac{\sin\frac{x}{n}}{\frac{1}{n}}\right|\to|x|\neq 0(n\to\infty)$且$\sum_{n=1}^{\infty}\frac{1}{n}$发散，则$\sum_{n=1}^{\infty}\left|\sin\frac{x}{n}\right|$发散

又对$\forall x\in R,x\neq 0$，因$\frac{x}{n}\to 0(n\to\infty)$，则存在$N\in Z^+$，当$n>N$时，有$0<\left|\frac{x}{n}\right|<\frac{\pi}{2}$，于是当$n>N$时，$\sin\frac{x}{n}$与$x$有相同的符号且$\left|\sin\frac{x}{n}\right|$随$n$增大而减小到0，则由莱布尼兹判别法，得$\sum_{n=1}^{\infty}(-1)^n\sin\frac{x}{n}$收敛

从而$\sum_{n=1}^{\infty}(-1)^n\sin\frac{x}{n}$条件收敛.

(7) 设部分和数列为$\{S_n\}$，则$S_{2n}=\sum_{k=2}^{n+1}\left(\frac{1}{\sqrt{k}-1}-\frac{1}{\sqrt{k}+1}\right)=\sum_{k=2}^{n+1}\frac{2}{k-1}=2\sum_{k=1}^{n}\frac{1}{k}$

于是$\lim_{n\to\infty}S_{2n}=+\infty$，则此级数加括号后发散，从而原级数发散.

2. 证明：若级数的项加括号后所作成的级数收敛，并且在同一个括号内项的符号相同，那末去掉括号后，此级数亦收敛；并由此考察级数$\sum_{n=1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n}$的收敛性.

证明:

(1) 已知新级数$\sum_{n=1}^{\infty}u_n'=(u_1+\cdots+u_{n_1})+(u_{n_1+1}+\cdots+u_{n_2})+\cdots+(u_{n_{k-1}+1}+\cdots+u_{n_k})+\cdots$收敛且在同一括号内的符号相同

设$\sum_{k=1}^{n}u_k=S_n,\sum_{k=1}^{n}u_k'=S_n'$，则$S_1{}'=S_{n_1},S_2{}'=S_{n_2},\cdots,S_k{}'=S_{n_k},\cdots$

当原级数的下标n从n_{k-1}到n_k时，$\sum_{n=1}^{\infty}u_n$的部分和单调变化，即

当$u_{n_{k-1}+1},\cdots,u_{n_k}$均为正时，有$S_{k-1}{}'=S_{n_{k-1}}<S_n<S_{n_k}=S_k{}'$

当$u_{n_{k-1}+1},\cdots,u_{n_k}$均为负时，有$S_{k-1}{}'=S_{n_{k-1}}>S_n>S_{n_k}=S_k{}'$

已知$\sum_{n=1}^{\infty}u_n'$收敛，即$\lim_{k\to\infty}S_k{}'=\lim_{k\to\infty}S_{k-1}{}'=S'$，则$\lim_{n\to\infty}S_n=S'$，于是$\sum_{n=1}^{\infty}u_n$收敛.

(2) 考虑$\sum_{n=1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n}$

当$n=k^2,k^2+1,\cdots,k^2+2k(k=1,2,\cdots)$时，诸$a_n$同号.记$A_k=\frac{1}{k^2}+\frac{1}{k^2+1}+\cdots+\frac{1}{k^2+2k}$，$\sum_{k=1}^{\infty}(-1)^kA_k$是交错级数

因$\int_{k^2-1}^{k^2+2k}\frac{\mathrm{d}x}{x}\leqslant\frac{1}{k^2}+\frac{1}{k^2+1}+\cdots+\frac{1}{k^2+2k}\leqslant\int_{k^2}^{(k+1)^2}\frac{\mathrm{d}x}{x}$即$\ln\frac{k^2+2k}{k^2-1}\leqslant A_k\leqslant\ln\frac{(k+1)^2}{k^2}$，从而当$k\to\infty$时，$A_k\to 0$

又$A_k - A_{k+1} \geqslant \ln\dfrac{k^2+2k}{k^2-1} - \ln\dfrac{(k+2)^2}{(k+1)^2} = \ln\dfrac{k^2+k}{k^2+k-2} > 0$，则由莱布尼兹判别法知$\sum\limits_{k=1}^{\infty}(-1)^k A_k$收敛，从而原级数收敛.

3. 讨论下列级数是否绝对收敛或条件收敛：

(1) $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n+x}$

(2) $\sum\limits_{n=1}^{\infty}\dfrac{\sin(2^n x)}{n!}$

(3) $\sum\limits_{n=1}^{\infty}\dfrac{\sin nx}{n}$, $(0<x<\pi)$

(4) $\sum\limits_{n=1}^{\infty}\dfrac{\cos nx}{n^p}$, $(0<x<\pi)$

解：

(1) $\sum\limits_{n=1}^{\infty}\left|\dfrac{(-1)^n}{n+x}\right| = \sum\limits_{n=1}^{\infty}\dfrac{1}{|n+x|}$

因$\lim\limits_{n\to\infty}\dfrac{\frac{1}{|n+x|}}{\frac{1}{n}} = \lim\limits_{n\to\infty}\dfrac{n}{|n+x|} = 1$，则由比较判别法，得$\sum\limits_{n=1}^{\infty}\dfrac{1}{|n+x|}$发散

当$x\geqslant 0$时，$\dfrac{1}{n+x}$单调减少，且$\lim\limits_{n\to\infty}\dfrac{1}{n+x} = 0$，则由莱布尼兹定理，得$\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n+x}$收敛

当$x<0$且不为负整数时，因x为定数，则当n充分大时，即存在$N\in Z^+$，当$n>N$时，有$n+x>0$，于是$\sum\limits_{n=N+1}^{\infty}\dfrac{(-1)^n}{n+x}$是交错级数，且由$\dfrac{1}{n+x}$单调减少及$\lim\limits_{n\to\infty}\dfrac{1}{n+x}=0$，则$\sum\limits_{n=N+1}^{\infty}\dfrac{(-1)^n}{n+x}$收敛，从而$\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n+x}$收敛

则当x不为负整数时，此级数为条件收敛.

(2) 因$\left|\dfrac{\sin(2^n x)}{n!}\right| \leqslant \dfrac{1}{n!}$，且$\lim\limits_{n\to\infty}\dfrac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim\limits_{n\to\infty}\dfrac{1}{n+1} = 0 < 1$，则由达朗贝尔判别法，得$\sum\limits_{n=1}^{\infty}\dfrac{1}{n!}$收敛

再据比较判别法，得$\sum\limits_{n=1}^{\infty}\left|\dfrac{\sin(2^n x)}{n!}\right|$收敛，从而$\sum\limits_{n=1}^{\infty}\dfrac{\sin(2^n x)}{n!}$绝对收敛.

(3) 因$\left|\sum\limits_{k=1}^{n}\sin kx\right| = \left|\dfrac{\cos\frac{x}{2} - \cos\frac{2n+1}{2}x}{2\sin\frac{x}{2}}\right| \leqslant \dfrac{1}{|\sin\frac{x}{2}|}$且数列$\left\{\dfrac{1}{n}\right\}$单调趋于0

则由狄立克莱判别法，得$\sum\limits_{n=1}^{\infty}\dfrac{\sin nx}{n}$收敛.

又$\left|\dfrac{\sin nx}{n}\right| \geqslant \dfrac{\sin^2 nx}{n} = \dfrac{1}{2n} - \dfrac{\cos 2nx}{2n}$且$\left|\sum\limits_{k=1}^{n}\cos 2kx\right| = \left|\dfrac{\sin x - \sin(2n+1)x}{2\sin x}\right| \leqslant \dfrac{1}{|\sin x|}$及数列$\left\{\dfrac{1}{2n}\right\}$单调趋于0

则由狄立克莱判别法，得$\sum\limits_{n=1}^{\infty}\dfrac{\cos 2nx}{n}$收敛.

又$\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$发散，则$\sum\limits_{n=1}^{\infty}\dfrac{1}{2n}$发散，于是$\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{2n} - \dfrac{\cos 2nx}{2n}\right)$发散，从而$\sum\limits_{n=1}^{\infty}\left|\dfrac{\sin nx}{n}\right|$发散

则级数$\sum\limits_{n=1}^{\infty}\dfrac{\sin nx}{n}\,(0<x<\pi)$条件收敛.

(4) (i) 当$p>1$时，因$\left|\dfrac{\cos nx}{n^p}\right| \leqslant \dfrac{1}{n^p}$且$\sum\limits_{n=1}^{\infty}\dfrac{1}{n^p}$当$p>1$时收敛，则级数$\sum\limits_{n=1}^{\infty}\dfrac{\cos nx}{n^p}\,(0<x<\pi)$绝对收敛.

(ii) 当$0<p\leqslant 1$时，因$\left|\sum\limits_{k=1}^{n}\cos kx\right| = \left|\dfrac{\sin\frac{x}{2} - \sin\frac{2n+1}{2}x}{2\sin\frac{x}{2}}\right| \leqslant \dfrac{1}{|\sin\frac{x}{2}|}$且数列$\left\{\dfrac{1}{n^p}\right\}$单调趋于0

则由狄立克莱判别法，得$\sum\limits_{n=1}^{\infty}\dfrac{\cos nx}{n^p}$收敛.

又$\left|\dfrac{\cos nx}{n^p}\right| \geqslant \dfrac{\cos^2 nx}{n^p} = \dfrac{1}{2n^p} + \dfrac{\cos 2nx}{2n^p}$且由刚才证明可得$\sum\limits_{n=1}^{\infty} \dfrac{\cos 2nx}{(2n)^p}$收敛.

则$\sum\limits_{n=1}^{\infty} \dfrac{\cos 2nx}{(2n)^p} \cdot 2^{p-1}$收敛，即$\sum\limits_{n=1}^{\infty} \dfrac{\cos 2nx}{2n^p}$收敛

又当$0 < p \leqslant 1$时，$\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$发散，则$\sum\limits_{n=1}^{\infty} \dfrac{1}{2n^p}$发散，于是$\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{2n^p} + \dfrac{\cos 2nx}{2n^p}\right)$发散

从而$\sum\limits_{n=1}^{\infty} \left|\dfrac{\cos nx}{n^p}\right|$发散

则级数$\sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n^p}$ $(0 < x < \pi)$ 当$0 < p \leqslant 1$时条件收敛.

(iii) 当$p \leqslant 0$时，因$\dfrac{\cos nx}{n^p} \nrightarrow 0$，则级数$\sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n^p}$ $(0 < x < \pi)$ 当$p \leqslant 0$时发散.

4. 若级数$\sum\limits_{n=1}^{\infty} a_n$ 收敛，并且$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$，能否断定$\sum\limits_{n=1}^{\infty} b_n$也收敛？

证明：

(1) 若级数$\sum\limits_{n=1}^{\infty} a_n$，$\sum\limits_{n=1}^{\infty} b_n$ 都是正项级数

由级数$\sum\limits_{n=1}^{\infty} a_n$ 收敛，$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$，则据正项级数比较判别法，得级数$\sum\limits_{n=1}^{\infty} b_n$ 收敛

(2) 若级数$\sum\limits_{n=1}^{\infty} a_n$，$\sum\limits_{n=1}^{\infty} b_n$ 不一定都是正项级数

由级数$\sum\limits_{n=1}^{\infty} a_n$ 收敛，不可断定$\sum\limits_{n=1}^{\infty} b_n$ 收敛

例：级数$\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$为莱布尼兹型级数，则其收敛且$\lim\limits_{n\to\infty} \dfrac{\frac{(-1)^n}{\sqrt{n}}}{\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}} = 1$

由于$\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$收敛，$\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$发散，则$\sum\limits_{n=1}^{\infty} \left(\dfrac{(-1)^n}{\sqrt{n}} + \dfrac{1}{n}\right)$发散.

5. 证明：若$\sum\limits_{n=1}^{\infty} \dfrac{a_n}{n^{x_0}}$收敛，那末当$x > x_0$时$\sum\limits_{n=1}^{\infty} \dfrac{a_n}{n^x}$也收敛.

证明：因$x > x_0$，则$\dfrac{\frac{1}{(n+1)^{x-x_0}}}{\frac{1}{n^{x-x_0}}} = \left(\dfrac{n}{n+1}\right)^{x-x_0} = \left(1 - \dfrac{1}{n+1}\right)^{x-x_0} < 1$，则$\dfrac{1}{(n+1)^{x-x_0}} < \dfrac{1}{n^{x-x_0}}$

且 $\dfrac{1}{n^{x-x_0}} \leqslant 1$

于是数列$\left\{\dfrac{1}{n^{x-x_0}}\right\}$单调有界，且 $\dfrac{1}{n^{x-x_0}} \leqslant 1$

又级数$\sum\limits_{n=1}^{\infty} \dfrac{a_n}{n^{x_0}}$收敛，则由阿贝尔判别法，得$\sum\limits_{n=1}^{\infty} \dfrac{a_n}{n^x}$收敛.

6. 设$\{na_n\}$收敛，$\sum\limits_{n=1}^{\infty} n(a_n - a_{n-1})$收敛，则$\sum\limits_{n=1}^{\infty} a_n$也收敛.

证明：因$\{na_n\}$收敛，设其极限为a

又$\sum\limits_{n=1}^{\infty} n(a_n - a_{n-1})$收敛，则其部分和数列$\left\{\sum\limits_{k=1}^{n} k(a_k - a_{k-1})\right\}$有极限，设其极限为$S$

又$\sum\limits_{k=1}^{n} k(a_k - a_{k-1}) = (a_1 - a_0) + 2(a_2 - a_1) + \cdots + n(a_n - a_{n-1}) = na_n - \sum\limits_{k=0}^{n-1} a_k$

即$\sum\limits_{k=0}^{n-1} a_k = na_n - \sum\limits_{k=1}^{n} k(a_k - a_{k-1})$，则$\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n-1} a_k = \lim\limits_{n\to\infty} na_n - \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} k(a_k - a_{k-1}) = a - S$

于是$\sum\limits_{n=0}^{\infty} a_n$收敛，从而$\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=0}^{\infty} a_n - a_0$收敛.

7. 若$\sum\limits_{v=1}^{\infty}(a_v - a_{v-1})$绝对收敛，$\sum\limits_{v=1}^{\infty} b_v$收敛，那末$\sum\limits_{v=1}^{\infty} a_v b_v$收敛.

证明：令$B_n^{n+m} = \sum\limits_{v=n+1}^{n+m} b_v$

由Abel 变换，得 $\sum\limits_{v=n+1}^{n+p} a_v b_v = a_{n+p} B_n^{n+p} + \sum\limits_{i=1}^{p-1} B_n^{n+i}(a_{n+i} - a_{n+i+1})$

故$\left|\sum\limits_{v=n+1}^{n+p} a_v b_v\right| \leqslant |a_{n+p}| \left|B_n^{n+p}\right| + \sum\limits_{i=1}^{p-1}\left|B_n^{n+i}\right| |a_{n+i} - a_{n+i+1}|$

令$H_n^p = \max\left\{\left|B_n^{n+1}\right|, \left|B_n^{n+2}\right|, \cdots, \left|B_n^{n+p}\right|\right\}$，则有$\left|\sum\limits_{v=n+1}^{n+p} a_v b_v\right| \leqslant H_n^p \left[|a_{n+p}| + \sum\limits_{i=1}^{p-1}|a_{n+i} - a_{n+i+1}|\right]$

因$\sum\limits_{v=1}^{\infty}|a_v - a_{v-1}|$收敛，故$\sum\limits_{v=1}^{\infty}(a_v - a_{v-1})$收敛且$\sum\limits_{v=1}^{\infty}(a_v - a_{v-1}) = -a_0 + a_n$，故$\lim\limits_{n\to\infty} a_n$存在

因而存在$M > 0$，使对一切n，有

$$\sum_{i=1}^{p-1}|a_{n+i} - a_{n+i+1}| + |a_{n+p}| < M \tag{4}$$

又$\sum\limits_{v=1}^{\infty} b_v$收敛，从而对$\forall\varepsilon > 0, \exists N \in Z^+$，当$n > N$时，对一切$p \in Z^+$，有

$$H_n^p < \frac{\varepsilon}{M} \tag{5}$$

由(**??**),(**??**)知，当$n > N$时，有$\left|\sum\limits_{v=n+1}^{n+p} a_v b_v\right| < \varepsilon$，这表明级数$\sum\limits_{v=1}^{\infty} a_v b_v$收敛

8. 利用柯西收敛原理证明交错级数的莱布尼兹定理.

证明：对任何自然数p，有

$|S_{n+p} - S_n| = \left|(-1)^{n+2}u_{n+1} + (-1)^{n+3}u_{n+2} + \cdots + (-1)^{n+p+1}u_{n+p}\right| = \left|(-1)^{n+2}(u_{n+1} - u_{n+2} + \cdots + (-1)^{p-1}u_{n+p})\right| = \left|u_{n+1} - u_{n+2} + \cdots + (-1)^{p-1}u_{n+p}\right|$

当p为偶数时，$(u_{n+1} - u_{n+2}) + \cdots + (u_{n+p-1} - u_{n+p}) \geqslant 0$

当p为奇数时，$(u_{n+1} - u_{n+2}) + \cdots + (u_{n+p-2} - u_{n+p-1}) + u_{n+p} \geqslant 0$

总之$\left|u_{n+1} - u_{n+2} + \cdots + (-1)^{p-1}u_{n+p}\right| = u_{n+1} - u_{n+2} + \cdots + (-1)^{p-1}u_{n+p}$

又当p为偶数时，$u_{n+1} - (u_{n+2} - u_{n+3}) - \cdots - (u_{n+p-2} - u_{n+p-1}) - u_{n+p} \leqslant u_{n+1}$

当p为奇数时，$u_{n+1} - (u_{n+2} - u_{n+3}) - \cdots - (u_{n+p-1} - u_{n+p}) \leqslant u_{n+1}$

总之$u_{n+1} - u_{n+2} + \cdots + (-1)^{p-1}u_{n+p} \leqslant u_{n+1}$

对任意$\varepsilon > 0$，因$\lim\limits_{n\to\infty} u_n = 0$，则$\lim\limits_{n\to\infty} u_{n+1} = 0$

于是必存在$N \in Z^+$，当$n > N$时，有$|u_{n+1} - 0| < \varepsilon$，则$u_{n+1} < \varepsilon$

由此当$n > N$时，对任何自然数p都有$|S_{n+p} - S_n| = u_{n+1} - u_{n+2} + \cdots + (-1)^p u_{n+p} \leqslant u_{n+1} < \varepsilon$

从而由柯西收敛原理，得$\sum\limits_{n=1}^{\infty}(-1)^{n+1}u_n$收敛.

§5. 绝对收敛级数和条件收敛级数的性质

1. 设$|x|<1,|y|<1$，证明$\sum\limits_{v=1}^{\infty}(x^{v-1}+x^{v-2}y+\cdots+y^{v-1})=\dfrac{1}{(1-x)(1-y)}$

证明：因$|x|<1,|y|<1$，则

$$\sum_{v=1}^{\infty}x^{v-1}=1+x+x^2+\cdots+x^v+\cdots=\frac{1}{1-x}\text{ 绝对收敛} \tag{6}$$

$$\sum_{v=1}^{\infty}y^{v-1}=1+y+y^2+\cdots+y^v+\cdots=\frac{1}{1-y}\text{ 绝对收敛} \tag{7}$$

(**??**)·(**??**)，得$\sum\limits_{v=1}^{\infty}x^{v-1}\sum\limits_{v=1}^{\infty}y^{v-1}=\dfrac{1}{(1-x)(1-y)}$

又$\sum\limits_{v=1}^{\infty}x^{v-1}\sum\limits_{v=1}^{\infty}y^{v-1}=(1+x+x^2+\cdots+x^v+\cdots)(1+y+y^2+\cdots+y^v+\cdots)=\sum\limits_{v=1}^{\infty}(x^{v-1}+x^{v-2}y+\cdots+y^{v-1})$，

则$\sum\limits_{v=1}^{\infty}(x^{v-1}+x^{v-2}y+\cdots+y^{v-1})=\dfrac{1}{(1-x)(1-y)}$.

2. 证明：$\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}\sum\limits_{n=0}^{\infty}\dfrac{y^n}{n!}=\sum\limits_{n=0}^{\infty}\dfrac{(x+y)^n}{n!}$

证明：因$\lim\limits_{n\to\infty}\dfrac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}}=\lim\limits_{n\to\infty}\dfrac{|x|}{n+1}=0<1$，则据达朗贝尔判别法的极限形式，得级数$\sum\limits_{n=0}^{\infty}\dfrac{|x|^n}{n!}$收敛

于是级数$\sum\limits_{n=0}^{\infty}\dfrac{|x|^n}{n!}$绝对收敛

同理，级数$\sum\limits_{n=0}^{\infty}\dfrac{|y|^n}{n!}$绝对收敛

可写成$\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}\sum\limits_{n=0}^{\infty}\dfrac{y^n}{n!}=\sum\limits_{n=0}^{\infty}C_n$

其中$C_n=\sum\limits_{i=0}^{n}\dfrac{x^i}{i!}\cdot\dfrac{y^{n-i}}{(n-i)!}=\dfrac{y^n}{n!}+\dfrac{x}{1!}\cdot\dfrac{y^{n-1}}{(n-1)!}+\cdots+\dfrac{x^n}{n!}=\dfrac{1}{n!}(C_n^0y^n+C_n^1xy^{n-1}+\cdots+C_n^nx^n)=\dfrac{(x+y)^n}{n!}$

则$\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}\sum\limits_{n=0}^{\infty}\dfrac{y^n}{n!}=\sum\limits_{n=0}^{\infty}\dfrac{(x+y)^n}{n!}$

3. 证明：可以作出条件收敛级数的更序级数，使其发散到$+\infty$.

证明：设$\sum\limits_{n=1}^{\infty}u_n$条件收敛

由定理1，得$\sum\limits_{n=1}^{\infty}v_n$和$\sum\limits_{n=1}^{\infty}w_n$都发散，且$\sum\limits_{n=1}^{\infty}v_n$发散到$+\infty$，$\sum\limits_{n=1}^{\infty}(-w_n)$发散到$-\infty$

选取发散到$+\infty$的数列$\{\beta_n\}$，即$\lim\limits_{n\to\infty}\beta_n=+\infty$

把$\sum\limits_{n=1}^{\infty}v_n$按顺序一项一项加起来

取m_1，使$v_1+v_2+\cdots+v_{m_1}>\beta_1+w_1$

然后取m_2，使$v_1+v_2+\cdots+v_{m_1}+v_{m_1+1}+\cdots+v_{m_2}>\beta_2+w_1+w_2$

一般地，可取充分大的$m_k>m_{k-1}$，使得$v_1+v_2+\cdots+v_{m_1}+\cdots+v_{m_2}+\cdots+v_{m_k}>\beta_k+w_1+w_2+\cdots+w_k$ $(k=3,4,\cdots)$

这样交错地放入一组正项和一个负项：

$$(v_1+\cdots+v_{m_1}-w_1)+(v_{m_1+1}+\cdots+v_{m_2}-w_2)+\cdots+(v_{m_{k-1}+1}+\cdots+v_{m_k}-w_k)+\cdots \tag{$*$}$$

此级数显然为原级数的更序级数

因$(*)$加括号后的级数$\sum\limits_{k=1}^{\infty}(v_{m_{k-1}+1}+\cdots+v_{m_k}-w_k)$的k次部分和

$$(v_1+\cdots+v_{m_1}-w_1)+(v_{m_1+1}+\cdots+v_{m_2}-w_2)+\cdots+(v_{m_{k-1}+1}+\cdots+v_{m_k}-w_k)>\beta_k$$

而$\lim\limits_{k\to\infty}\beta_k=+\infty$

则$\sum\limits_{k=1}^{\infty}(v_{m_{k-1}+1}+\cdots+v_{m_k}-w_k)$发散到$+\infty$

由发散级数可任意去括号，则可以作出条件收敛级数的更序级数，使其发散到$+\infty$.

§6. 无穷乘积

1. 讨论无穷乘积的收敛性：

(1) $\prod\limits_{n=3}^{\infty}\dfrac{n^2-4}{n^2-1}$

(2) $\prod\limits_{n=1}^{\infty}a^{\frac{(-1)^n}{n}}(a>0)$

(3) $\prod\limits_{n=0}^{\infty}\sqrt{\dfrac{n+1}{n+2}}$

解：

(1) 因$\dfrac{n^2-4}{n^2-1}=1-\dfrac{3}{n^2-1},n\geqslant 3$，且$-\dfrac{3}{n^2-1}<0$

又$\lim\limits_{n\to\infty}\dfrac{\frac{3}{n^2-1}}{\frac{1}{n^2}}=3$且$\sum\limits_{n=3}^{\infty}\dfrac{1}{n^2}$收敛，则$\sum\limits_{n=3}^{\infty}\dfrac{3}{n^2-1}$收敛，于是$\sum\limits_{n=3}^{\infty}\left(-\dfrac{3}{n^2-1}\right)$收敛

从而据定理2，得$\prod\limits_{n=3}^{\infty}\dfrac{n^2-4}{n^2-1}$收敛.

(2) $\sum\limits_{n=1}^{\infty}\ln a^{\frac{(-1)^n}{n}}=\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}\ln a=\ln a\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$

因级数$\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$为莱布尼兹型级数，则其收敛，于是级数$\sum\limits_{n=1}^{\infty}\ln a^{\frac{(-1)^n}{n}}$收敛，从而无穷乘积$\prod\limits_{n=1}^{\infty}a^{\frac{(-1)^n}{n}}$收敛.

(3) 由于部分乘积$P_n=\sqrt{\dfrac{1}{2}\cdot\dfrac{2}{3}\cdots\dfrac{n}{n+1}\cdot\dfrac{n+1}{n+2}}=\sqrt{\dfrac{1}{n+2}}\to 0(n\to\infty)$

故无穷乘积$\prod\limits_{n=0}^{\infty}\sqrt{\dfrac{n+1}{n+2}}$发散于0.

2. 证明：若$\sum\limits_{n=1}^{\infty}x_n^2$收敛，则$\prod\limits_{n=1}^{\infty}\cos x_n$收敛.

证明：因$\prod\limits_{n=1}^{\infty}\cos x_n=\prod\limits_{n=1}^{\infty}\left(1-2\sin^2\dfrac{x_n}{2}\right)$且$0\leqslant 2\sin^2\dfrac{x_n}{2}\leqslant 2\cdot\left(\dfrac{\sin x_n}{2}\right)^2=\dfrac{x_n^2}{2}$

又$\sum\limits_{n=1}^{\infty}x_n^2$收敛，则$\sum\limits_{n=1}^{\infty}2\sin^2\dfrac{x_n}{2}$收敛

于是据定理2，得$\prod\limits_{n=1}^{\infty}\cos x_n$收敛.

3. 证明：若$\sum\limits_{n=1}^{\infty}\alpha_n$绝对收敛，则$\prod\limits_{n=1}^{\infty}\tan\left(\dfrac{\pi}{4}+\alpha_n\right)$收敛$\left(其中|\alpha_n|<\dfrac{\pi}{4}\right)$.

证明：$\prod\limits_{n=1}^{\infty}\tan\left(\dfrac{\pi}{4}+\alpha_n\right)=\prod\limits_{n=1}^{\infty}\dfrac{1+\tan\alpha_n}{1-\tan\alpha_n}=\prod\limits_{n=1}^{\infty}\left(1+\dfrac{2\tan\alpha_n}{1-\tan\alpha_n}\right)$

因$\sum\limits_{n=1}^{\infty}\alpha_n$绝对收敛，则$\lim\limits_{n\to\infty}\alpha_n=0$，于是$\lim\limits_{n\to\infty}\dfrac{\left|\frac{2\tan\alpha_n}{1-\tan\alpha_n}\right|}{|\alpha_n|}=\lim\limits_{n\to\infty}\left|\dfrac{2}{1-\tan\alpha_n}\right|\left|\dfrac{\tan\alpha_n}{\alpha_n}\right|=2$

由$\sum\limits_{n=1}^{\infty}\alpha_n$绝对收敛，得$\sum\limits_{n=1}^{\infty}\left|\dfrac{2\tan\alpha_n}{1-\tan\alpha_n}\right|$收敛，于是$\sum\limits_{n=1}^{\infty}\dfrac{2\tan\alpha_n}{1-\tan\alpha_n}$绝对收敛

从而$\prod\limits_{n=1}^{\infty}\tan\left(\dfrac{\pi}{4}+\alpha_n\right)$绝对收敛.

第十章　广义积分

§1.　无穷限的广义积分

1. 求下列广义积分的值：

(1) $\displaystyle\int_2^{+\infty}\frac{1}{x^2-1}\,\mathrm{d}x$

(2) $\displaystyle\int_0^{+\infty}\frac{1}{(x^2+p)(x^2+q)}\,\mathrm{d}x\,,(p,q>0)$

(3) $\displaystyle\int_0^{+\infty}e^{-ax^2}x\,\mathrm{d}x\,(a>0)$

(4) $\displaystyle\int_0^{+\infty}e^{-ax}\sin bx\,\mathrm{d}x\,,(a>0)$

解:

(1) $\displaystyle\int_2^{+\infty}\frac{1}{x^2-1}\,\mathrm{d}x=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\Bigg|_2^{+\infty}=\frac{1}{2}\ln 3=\ln\sqrt{3}$

(2) $\displaystyle\int_0^{+\infty}\frac{1}{(x^2+p)(x^2+q)}\mathrm{d}x=\frac{1}{q-p}\left(\frac{1}{\sqrt{p}}\arctan\frac{x}{\sqrt{p}}-\frac{1}{\sqrt{q}}\arctan\frac{x}{\sqrt{q}}\right)\Bigg|_0^{+\infty}=\frac{\frac{\pi}{2}}{\sqrt{pq}(\sqrt{p}+\sqrt{q})}=\frac{\pi}{2(q\sqrt{p}+p\sqrt{q})}$

(3) $\displaystyle\int_0^{+\infty}e^{-ax^2}x\,\mathrm{d}x=-\frac{e^{-ax^2}}{2a}\Bigg|_0^{+\infty}=\frac{1}{2a}$

(4) $\displaystyle\int_0^{+\infty}e^{-ax}\sin bx\,\mathrm{d}x=\frac{-a\sin bx-b\cos bx}{a^2+b^2}e^{-ax}\Bigg|_0^{+\infty}=\frac{b}{a^2+b^2}$

2. 讨论下列积分的收敛性：

(1) $\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{\sqrt[3]{x^4+1}}$

(2) $\displaystyle\int_1^{+\infty}\frac{x\arctan x}{1+x^2}\,\mathrm{d}x$

(3) $\displaystyle\int_1^{+\infty}\sin\frac{1}{x^2}\,\mathrm{d}x$

(4) $\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{1+x|\sin x|}$

(5) $\displaystyle\int_0^{+\infty}\frac{x}{1+x^2\sin^2 x}\,\mathrm{d}x$

(6) $\displaystyle\int_0^{+\infty}\frac{x^m}{1+x^n}\,\mathrm{d}x\,,(n>0,m>0)$

解:

(1) $\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{\sqrt[3]{x^4+1}}=\int_0^1\frac{\mathrm{d}x}{\sqrt[3]{x^4+1}}+\int_1^{+\infty}\frac{\mathrm{d}x}{\sqrt[3]{x^4+1}}$

因$\displaystyle\int_0^1\frac{\mathrm{d}x}{\sqrt[3]{x^4+1}}$为正常积分，则其必收敛

对$\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{\sqrt[3]{x^4+1}}$，因$\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{\sqrt[3]{x^4}}$收敛，则$\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{\sqrt[3]{x^4+1}}$收敛，从而$\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{\sqrt[3]{x^4+1}}$收敛.

(2) 因$\displaystyle\lim_{x\to+\infty}\frac{\frac{x}{1+x^3}\arctan x}{\frac{1}{x^2}}=\lim_{x\to+\infty}\frac{x^3}{1+x^3}\arctan x=\frac{\pi}{2}$，且$\displaystyle\int_1^{+\infty}\frac{1}{x^2}$收敛

则由比较判别法的极限形式，得$\displaystyle\int_1^{+\infty}\frac{x\arctan x}{1+x^2}\,\mathrm{d}x$收敛.

(3) 因$\lim\limits_{x\to+\infty}\dfrac{\sin\frac{1}{x^2}}{\frac{1}{x^2}}=1$且$\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{x^2}$收敛，从而$\displaystyle\int_1^{+\infty}\sin\frac{1}{x^2}\,\mathrm{d}x$收敛.

(4) $\dfrac{1}{1+x|\sin x|}\geqslant\dfrac{1}{1+x}$

因$\lim\limits_{x\to+\infty}\dfrac{\frac{1}{1+x}}{\frac{1}{x}}=1$且$\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{x}$发散，从而由比较判别法的极限形式，得$\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{1+x}$发散

又$\displaystyle\int_0^1\frac{\mathrm{d}x}{1+x}$为正常积分则收敛，于是$\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{1+x}$发散

从而由比较判别法，得$\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{1+x|\sin x|}$发散.

(5) 因$x\in[0,+\infty)$时，有$\dfrac{x}{1+x^2\sin^2 x}\geqslant\dfrac{x}{1+x^2}$且$\displaystyle\int_0^{+\infty}\frac{x}{1+x^2}\,\mathrm{d}x=\frac{1}{2}\ln(1+x^2)\Big|_0^{+\infty}=+\infty$

则由比较判别法，得$\displaystyle\int_0^{+\infty}\frac{x}{1+x^2\sin^2 x}\,\mathrm{d}x$发散.

(6) $\displaystyle\int_0^{+\infty}\frac{x^m}{1+x^n}\,\mathrm{d}x=\int_0^1\frac{x^m}{1+x^n}\,\mathrm{d}x+\int_1^{+\infty}\frac{x^m}{1+x^n}\,\mathrm{d}x$且$\displaystyle\int_0^1\frac{x^m}{1+x^n}\,\mathrm{d}x$为常义积分

(i) 当$n-m>1$时，有$\dfrac{1}{x^{n-m}}\cdot\dfrac{1}{1+\frac{1}{x^n}}<\dfrac{1}{x^{n-m}}$且积分$\displaystyle\int_1^{+\infty}\frac{x^m}{1+x^n}\,\mathrm{d}x$收敛，故原积分收敛；

(ii) 当$n-m\leqslant 1$且$x\geqslant 1$时，有$\dfrac{x^m}{1+x^n}\geqslant\dfrac{1}{2x^{n-m}}$，且$\displaystyle\int_1^{+\infty}\frac{1}{x^{n-m}}\,\mathrm{d}x$发散，故原积分发散

则当$n-m>1$时，$\displaystyle\int_0^{+\infty}\frac{x^m}{1+x^n}\,\mathrm{d}x$收敛；当$n-m\leqslant 1$时，$\displaystyle\int_0^{+\infty}\frac{x^m}{1+x^n}\,\mathrm{d}x$发散.

3. 证明绝对收敛的广义积分必收敛，但反之不然.

证明：已知$\displaystyle\int_a^{+\infty}|f(x)|\,\mathrm{d}x$收敛，由柯西判别原理，得对$\forall\varepsilon>0,\exists A>0$，当$A''>A'>A$时，有$\left|\displaystyle\int_{A'}^{A''}|f(x)|\,\mathrm{d}x\right|<\varepsilon$，则$\displaystyle\int_{A'}^{A''}|f(x)|\,\mathrm{d}x<\varepsilon$，于是$\left|\displaystyle\int_{A'}^{A''}f(x)\,\mathrm{d}x\right|\leqslant\displaystyle\int_{A'}^{A''}|f(x)|\,\mathrm{d}x<\varepsilon$，

从而$\displaystyle\int_a^{+\infty}f(x)\,\mathrm{d}x$收敛.

收敛的广义积分未必绝对收敛.

例：$\displaystyle\int_1^{+\infty}\frac{\sin x}{x}\,\mathrm{d}x$收敛；而$\displaystyle\int_1^{+\infty}\left|\frac{\sin x}{x}\right|\mathrm{d}x$发散(见书上55页).

4. 证明对于无穷限积分，分部积分公式成立（当公式中各部分有意义时）

$$\int_a^{+\infty}f(x)g'(x)\,\mathrm{d}x=f(x)g(x)\Big|_a^{+\infty}-\int_a^{+\infty}g(x)f'(x)\,\mathrm{d}x$$

证明：对于任意$A>a$，成立$\displaystyle\int_a^A f(x)g'(x)\,\mathrm{d}x=f(x)g(x)\Big|_a^A-\int_a^A g(x)f'(x)\,\mathrm{d}x$

两边取极限，得$\displaystyle\lim_{A\to+\infty}\int_a^A f(x)g'(x)\,\mathrm{d}x=\lim_{A\to+\infty}\left(f(x)g(x)\Big|_a^A\right)-\lim_{A\to+\infty}\left(\int_a^A g(x)f'(x)\,\mathrm{d}x\right)$

则$\displaystyle\int_a^{+\infty}f(x)g'(x)\,\mathrm{d}x=f(x)g(x)\Big|_a^{+\infty}-\int_a^{+\infty}g(x)f'(x)\,\mathrm{d}x$

5. 证明：

(1) 设$f(x)$为$[0,+\infty)$上的一致连续函数，并且积分$\displaystyle\int_0^{+\infty}f(x)\,\mathrm{d}x$收敛，则$\lim\limits_{x\to+\infty}f(x)=0$；如果仅仅积分$\displaystyle\int_0^{+\infty}f(x)\,\mathrm{d}x$收敛，以及$f(x)$在$[0,+\infty)$连续，$f(x)\geqslant 0$，是否仍旧成立$\lim\limits_{x\to+\infty}f(x)=0$？

证明：用反证法.设$\lim\limits_{x\to+\infty}f(x)\neq 0$，则$\exists\varepsilon>0$，对任意大的$A>0$，都存在$x_A>A$，使得$|f(x_A)|\geqslant 2\varepsilon$.

取序列$A_n\to+\infty(n\to\infty)$，有序列$x_n\to+\infty$且$x_n>A_n(n=1,2,\cdots)$，使$|f(x_n)|\geqslant 2\varepsilon$

另一方面，由$f(x)$的一致收敛性，对上述$\varepsilon>0,\exists\delta>0$，使得当$|x'-x''|<\delta$时，有$|f(x')-f(x'')|<\varepsilon$

因此，对一切n，当$x\in\left(x_n-\dfrac{\delta}{2},x_n+\dfrac{\delta}{2}\right)$时，有$|f(x)-f(x_n)|<\varepsilon$，即$f(x_n)-\varepsilon<f(x)<f(x_n)+\varepsilon$

当$f(x_n)>0$时，$|f(x_n)|=f(x_n)\geqslant 2\varepsilon$，由左端不等式，得$f(x)>2\varepsilon-\varepsilon=\varepsilon$
当$f(x_n)<0$时，$|f(x_n)|=-f(x_n)\geqslant 2\varepsilon$，由右端不等式，得$f(x)<-2\varepsilon+\varepsilon=-\varepsilon$
从而，$\displaystyle\int_{x_n-\frac{\delta}{2}}^{x_n+\frac{\delta}{2}} f(x)\,\mathrm{d}x>\varepsilon\delta$（当$f(x_n)>0$时）或$\displaystyle\int_{x_n-\frac{\delta}{2}}^{x_n+\frac{\delta}{2}} f(x)\,\mathrm{d}x<-\varepsilon\delta$（当$f(x_n)<0$时）
此与$\displaystyle\int_0^{+\infty} f(x)\,\mathrm{d}x$收敛矛盾，则假设不成立，于是$\lim\limits_{x\to+\infty} f(x)=0$.

(2) 积分$\displaystyle\int_0^{+\infty} f(x)\,\mathrm{d}x$收敛，以及$f(x)$在$[0,+\infty)$连续，$f(x)\geqslant 0$，并不能保证$\lim\limits_{x\to+\infty} f(x)=0$.
例：$\displaystyle\int_0^{+\infty}\frac{x}{1+x^6\sin^2 x}\,\mathrm{d}x$.
它是绝对收敛的.
因$\displaystyle\int_0^{+\infty}\frac{x}{1+x^6\sin^2 x}\,\mathrm{d}x=\sum_{n=0}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{x}{1+x^6\sin^2 x}\,\mathrm{d}x=\int_0^{\pi}\frac{x}{1+x^6\sin^2 x}\,\mathrm{d}x+\sum_{n=1}^{\infty}\left(I_n^1+I_n^2\right)$
其中$\displaystyle I_n^1=\int_{n\pi}^{\left(n+\frac{1}{2}\right)\pi}\frac{x}{1+x^6\sin^2 x}\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{n\pi+z}{1+(n\pi+z)^6\sin^2 z}\,\mathrm{d}z$,
$\displaystyle I_n^2=\int_{\left(n+\frac{1}{2}\right)\pi}^{(n+1)\pi}\frac{x}{1+x^6\sin^2 x}\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{n\pi+\pi-z}{1+(n\pi+\pi-z)^6\sin^2 z}\,\mathrm{d}z$
注意到当$0<z<\dfrac{\pi}{2}$时，$\dfrac{2}{\pi}<\dfrac{\sin z}{z}\leqslant 1$，于是$(n\pi+z)^6\sin^2 z\geqslant(n\pi)^6\left(\dfrac{2z}{\pi}\right)^2=(2\pi^2n^3z)^2$,
$(n\pi+\pi-z)^6\sin^2 z\geqslant(2\pi^2n^3z)^2$
故有$\displaystyle I_n^1\leqslant\int_0^{\frac{\pi}{2}}\frac{(n+1)\pi}{1+(2\pi^2n^3z)^2}\,\mathrm{d}z=\frac{n+1}{2n^3\pi}\int_0^{(n\pi)^3}\frac{\mathrm{d}y}{1+y^2}\leqslant\frac{n+1}{4n^3}$
同理，有$I_n^2\leqslant\dfrac{n+1}{4n^3}$
因$\displaystyle\int_0^{\pi}\frac{x}{1+x^6\sin^2 x}\,\mathrm{d}x$为正常积分，则必收敛
又$\dfrac{n+1}{2n^3}<\dfrac{1}{n^2}$且$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$收敛，则$\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{2n^3}$收敛
于是$\displaystyle\int_0^{+\infty}\frac{x}{1+x^6\sin^2 x}\,\mathrm{d}x\leqslant\int_0^{\pi}\frac{x}{1+x^6\sin^2 x}\,\mathrm{d}x+\sum_{n=1}^{\infty}\frac{n+1}{2n^3}$绝对收敛
显然$f(x)=\dfrac{x}{1+x^6\sin^2 x}$在$[0,+\infty)$上非负连续
但若取$x_n=2n\pi(n=0,1,2,\cdots)$，有$f(x_n)=f(2n\pi)=2n\pi\to+\infty(n\to\infty)$
则$\lim\limits_{x\to\infty} f(x)\neq 0$.

6. 证明：若 $f(x),g(x)$ 在任何区间 $[a,A]$ 可积，又设 $f^2(x),g^2(x)$ 在 $[a,+\infty)$ 积分收敛，那末 $[f(x)+g(x)]^2$ 和 $|f(x)\cdot g(x)|$ 在 $[a,+\infty)$ 上皆可积.

证明：因 $f(x),g(x)$ 在任何区间 $[a,A]$ 可积，则$\displaystyle\int_a^A|f(x)\cdot g(x)|\,\mathrm{d}x$存在，$\displaystyle\int_a^A[f(x)+g(x)]^2\,\mathrm{d}x$存在
又$\displaystyle\int_a^{+\infty}f^2(x)\,\mathrm{d}x$和$\displaystyle\int_a^{+\infty}g^2(x)\,\mathrm{d}x$都收敛，则$\displaystyle\int_a^{+\infty}[f^2(x)+g^2(x)]\,\mathrm{d}x$收敛，
于是$\displaystyle\int_a^{+\infty}2[f^2(x)+g^2(x)]\,\mathrm{d}x$和$\displaystyle\int_a^{+\infty}\frac{1}{2}[f^2(x)+g^2(x)]\,\mathrm{d}x$都收敛
又$[|f(x)|-|g(x)|]^2=f^2(x)+g^2(x)-2|f(x)\cdot g(x)|\geqslant 0$即$|f(x)\cdot g(x)|\leqslant\dfrac{1}{2}[f^2(x)+g^2(x)]$
则由比较判别法，得$|f(x)\cdot g(x)|$在$[a,+\infty)$上可积
又$[f(x)+g(x)]^2=f^2(x)+g^2(x)+2f(x)\cdot g(x)\leqslant f^2(x)+g^2(x)+2|f(x)\cdot g(x)|\leqslant 2[f^2(x)+g^2(x)]$
则由比较判别法，得$[f(x)+g(x)]^2$在$[a,+\infty)$上可积.

7. 对无穷限广义积分，讨论平方可积和绝对可积的关系.考察例子：$\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{x^{3/2}}$和$\displaystyle\int_1^{+\infty}f(x)\,\mathrm{d}x$，其中
$f(x)=n^2\left(\text{当}\,n\leqslant x<n+\dfrac{1}{n^4}\right)$，$f(x)=0\left(\text{当}\,n+\dfrac{1}{n^4}\leqslant x<n+1\right)$.

证明：平方可积$\nRightarrow$绝对可积
例：$\displaystyle\int_1^{+\infty}\frac{1}{x^{3/2}}\,\mathrm{d}x$收敛，但$\displaystyle\int_1^{+\infty}\left|\frac{1}{x^{3/4}}\right|\,\mathrm{d}x=\int_1^{+\infty}\frac{\mathrm{d}x}{x^{3/4}}$发散；

$\int_1^{+\infty} \frac{1}{x^3}\,\mathrm{d}x$收敛，且$\int_1^{+\infty} \left|\frac{1}{x^{3/2}}\right|\,\mathrm{d}x = \int_1^{+\infty} \frac{1}{x^{3/2}}\,\mathrm{d}x$收敛

绝对可积$\nRightarrow$平方可积

例：$\int_1^{+\infty} f(x)\,\mathrm{d}x$，其中$f(x)=n^2\left(当 n \leqslant x < n+\frac{1}{n^4}\right)$，$f(x)=0\left(当 n+\frac{1}{n^4} \leqslant x < n+1\right)$

$\int_1^{+\infty} |f(x)|\,\mathrm{d}x = \int_1^{+\infty} f(x)\,\mathrm{d}x = \frac{1}{1^4}\cdot 1^2 + \frac{1}{2^4}\cdot 2^2 + \cdots + \frac{1}{n^4}\cdot n^2 + \cdots = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$收敛

$\int_1^{+\infty} f(x)\,\mathrm{d}x$，其中$f(x)=n^4\left(当 n \leqslant x < n+\frac{1}{n^4}\right)$，$f(x)=0\left(当 n+\frac{1}{n^4} \leqslant x < n+1\right)$

$\int_1^{+\infty} f^2(x)\,\mathrm{d}x = \frac{1}{1^4}\cdot 1^4 + \frac{1}{2^4}\cdot 2^4 + \cdots + \frac{1}{n^4}\cdot n^4 + \cdots = 1+1+\cdots+1+\cdots = \sum_{n=1}^{\infty} 1$发散；

$\int_1^{+\infty} \left|\frac{1}{x^{3/2}}\right|\,\mathrm{d}x = \int_1^{+\infty} \frac{1}{x^{3/2}}\,\mathrm{d}x$收敛，且$\int_1^{+\infty} \frac{1}{x^3}\,\mathrm{d}x$收敛

8. 讨论下列积分的绝对收敛性及条件收敛性：

(1) $\int_0^{+\infty} \frac{\sqrt{x}\cos x}{x+100}\,\mathrm{d}x$

(2) $\int_1^{+\infty} \frac{\cos x}{x^\lambda}\,\mathrm{d}x\,, \int_1^{+\infty} \frac{\sin x}{x^\lambda}\,\mathrm{d}x$

(3) $\int_a^{+\infty} \frac{P_m(x)}{Q_n(x)}\sin x\,\mathrm{d}x\,, P_m(x), Q_n(x)$各为$m, n$次多项式且当$x \geqslant a$时，$Q_n(x) \neq 0$

(4) $\int_2^{+\infty} \frac{\ln\ln x}{\ln x}\sin x\,\mathrm{d}x$

解:

(1) 对$A>0$，由于$\left|\int_0^A \cos\,\mathrm{d}x\right| = |\sin A - \sin 0| \leqslant 1$

又$\lim\limits_{x\to+\infty} \frac{\sqrt{x}}{x+100} = 0, \left(\frac{\sqrt{x}}{x+100}\right)' = \frac{100-x}{2\sqrt{x}(x+100)^2}$，当$x>100$时，$\left(\frac{\sqrt{x}}{x+100}\right)' < 0$，则$\frac{\sqrt{x}}{x+100}$单调减少

于是由狄立克莱判别法，得$\int_0^{+\infty} \frac{\sqrt{x}\cos x}{x+100}\,\mathrm{d}x$ 收敛.

但它不为绝对收敛.

由于$\frac{\sqrt{x}|\cos x|}{x+100} \geqslant \frac{\sqrt{x}\cos^2 x}{x+100} = \frac{1}{2}\left(\frac{\sqrt{x}}{x+100} + \frac{\sqrt{x}\cos 2x}{x+100}\right)$

因$\lim\limits_{x\to+\infty}\left(x^{\frac{1}{2}}\cdot\frac{\sqrt{x}}{x+100}\right) = 1$，则由柯西判别法的极限形式，得$\int_1^{+\infty} \frac{\sqrt{x}}{x+100}\,\mathrm{d}x$发散

又$\int_0^1 \frac{\sqrt{x}}{x+100}\,\mathrm{d}x$为正常积分，则$\int_0^{+\infty} \frac{\sqrt{x}}{x+100}\,\mathrm{d}x = \int_0^1 \frac{\sqrt{x}}{x+100}\,\mathrm{d}x + \int_1^{+\infty} \frac{\sqrt{x}}{x+100}\,\mathrm{d}x$ 发散

依前半段的证明，可知$\int_0^{+\infty} \frac{\sqrt{x}\cos 2x}{x+100}\,\mathrm{d}x$收敛，从而积分$\int_0^{+\infty} \frac{\sqrt{x}\cos^2 x}{x+100}\,\mathrm{d}x$发散

则$\int_0^{+\infty} \left|\frac{\sqrt{x}\cos x}{x+100}\right|\,\mathrm{d}x = \int_0^{+\infty} \frac{\sqrt{x}|\cos x|}{x+100}\,\mathrm{d}x$发散，从而积分$\int_0^{+\infty} \frac{\sqrt{x}\cos x}{x+100}\,\mathrm{d}x$条件收敛.

(2) (i) 当$\lambda > 1$时，因$\left|\frac{\cos x}{x^\lambda}\right| = \frac{|\cos x|}{x^\lambda} \leqslant \frac{1}{x^\lambda}$且当$\lambda>1$时，$\int_1^{+\infty} \frac{\mathrm{d}x}{x^\lambda}$收敛，从而$\int_1^{+\infty} \frac{\cos x}{x^\lambda}\,\mathrm{d}x$ 绝对收敛

同理$\int_1^{+\infty} \frac{\sin x}{x^\lambda}\,\mathrm{d}x$ 绝对收敛

(ii) 当$0<\lambda\leqslant 1$时

因$\left|\int_1^A \cos x\,\mathrm{d}x\right| = |\sin A - \sin 1| \leqslant 2$且$\frac{1}{x^\lambda}$当$0<\lambda\leqslant 1$时单调减少，当$x\to+\infty$时趋于0，

则由狄立克莱判别法，得$\int_1^{+\infty}\dfrac{\cos x}{x^\lambda}\,\mathrm{d}x$收敛

但$\dfrac{|\cos x|}{x^\lambda}\geqslant\dfrac{\cos^2 x}{x^\lambda}=\dfrac{1}{2}\left(\dfrac{1}{x^\lambda}+\dfrac{\cos 2x}{x^\lambda}\right)$，由前面证明，可知$\int_1^{+\infty}\dfrac{\cos 2x}{x^\lambda}\,\mathrm{d}x$收敛

又$\int_1^{+\infty}\dfrac{\mathrm{d}x}{x^\lambda}(0<\lambda\leqslant 1)$发散，则$\int_1^{+\infty}\dfrac{|\cos x|}{x^\lambda}\,\mathrm{d}x$发散，从而$\int_1^{+\infty}\dfrac{\cos x}{x^\lambda}\,\mathrm{d}x$条件收敛

同理，$\int_1^{+\infty}\dfrac{\sin x}{x^\lambda}\,\mathrm{d}x$ 条件收敛

(iii) 当$\lambda\leqslant 0$时

因$n\to+\infty, 2n\pi\to+\infty$，于是对任意$A>0$，至少可以找到$(2n+1)\pi>2n\pi>A$

取$\varepsilon_0=2$，当$(2n+1)\pi>2n\pi>A$时，$\left|\int_{2n\pi}^{(2n+1)\pi}\dfrac{\sin x}{x^\lambda}\,\mathrm{d}x\right|=\int_{2n\pi}^{(2n+1)\pi}\dfrac{\sin x}{x^\lambda}\,\mathrm{d}x\geqslant\int_{2n\pi}^{(2n+1)\pi}\sin x\,\mathrm{d}x=2=\varepsilon_0$

则当$\lambda\leqslant 0$时，$\int_1^{+\infty}\dfrac{\sin x}{x^\lambda}\,\mathrm{d}x$ 发散

同理，$\int_1^{+\infty}\dfrac{\cos x}{x^\lambda}\,\mathrm{d}x$发散.

综合知，$\lambda>1$时，$\int_1^{+\infty}\dfrac{\cos x}{x^\lambda}\,\mathrm{d}x$，$\int_1^{+\infty}\dfrac{\sin x}{x^\lambda}\,\mathrm{d}x$绝对收敛；

$0<\lambda\leqslant 1$时，$\int_1^{+\infty}\dfrac{\cos x}{x^\lambda}\,\mathrm{d}x$，$\int_1^{+\infty}\dfrac{\sin x}{x^\lambda}\,\mathrm{d}x$条件收敛；

$\lambda<0$时，$\int_1^{+\infty}\dfrac{\cos x}{x^\lambda}\,\mathrm{d}x$，$\int_1^{+\infty}\dfrac{\sin x}{x^\lambda}\,\mathrm{d}x$发散.

(3) (i) 设$m<n$.此时，真分式$\dfrac{P_m(x)}{Q_n(x)}$当x足够大时，随$x\to+\infty$而单调下降趋于0

又$\left|\int_a^A\sin x\,\mathrm{d}x\right|\leqslant 2$(对$\forall A>a$)，则据狄立克莱判别法，原积分收敛

(ii) 设$Q_n(x)\equiv 1$.此时多项式为$P_m(x)=a_mx^m+\cdots+a_0$，不妨设$a_m>0$

由于$\lim\limits_{x\to+\infty}\dfrac{P_m(x)}{x^m}=a_m>0$，故存在$b\pi+\pi>0$，使当$x>b\pi+\pi$时，$P_m(x)=\dfrac{a_m}{2}x^m$

于是有$\int_a^{+\infty}P_m(x)\sin x\,\mathrm{d}x=\int_a^{b\pi+\pi}P_m(x)\sin x\,\mathrm{d}x+\sum\limits_{n=b+1}^{\infty}I_n$，其中$I_n=\int_{n\pi}^{(n+1)\pi}P_m(x)\sin x\,\mathrm{d}x$

此时有$|I_n|=\left|\int_{n\pi}^{(n+1)\pi}P_m(x)\sin x\,\mathrm{d}x\right|=\left|\int_0^{\pi}P_m(n\pi+z)(-1)^n\sin z\,\mathrm{d}z\right|\geqslant\dfrac{a_m}{2}(n\pi)^m\int_0^{\pi}\sin z\,\mathrm{d}z=a_m(n\pi)^m$，则$I_n\to\infty(n\to\infty)$

又$\int_a^{b\pi+\pi}P_m(x)\sin x\,\mathrm{d}x$为正常积分，则必收敛，于是$\int_a^{+\infty}P_m(x)\sin x\,\mathrm{d}x$发散

(iii) 当$m\geqslant n$时，$\dfrac{P_m(x)}{Q_n(x)}=R(x)+S(x)$，其中$R(x)$为真分式，$S(x)$为整式

由(ii)知，$\int_a^{+\infty}S(x)\sin x\,\mathrm{d}x$发散；由(i)知，$\int_a^{+\infty}R(x)\sin x\,\mathrm{d}x$收敛，故$\int_a^{+\infty}\dfrac{P_m(x)}{Q_n(x)}\sin x\,\mathrm{d}x$发散

(iv) 设$Q_n(x)=b_nx^n+\cdots+b_1x+b_0$

由于$\lim\limits_{x\to+\infty}\dfrac{\left|\frac{P_n(x)}{Q_m(x)}\sin x\right|}{\left|\frac{a_m}{b_n}x^{m-n}\sin x\right|}=1$，则由8(2)知，当$\lambda=n-m>1$时，积分绝对收敛

综合知：$m\geqslant n$时，积分发散；$m=n-1$时，积分条件收敛；$m<n-1$时，积分绝对收敛.

(4) 对$A>2$，$\left|\int_2^A\sin x\,\mathrm{d}x\right|\leqslant 2$，$\lim\limits_{x\to+\infty}\dfrac{\ln\ln x}{\ln x}=\lim\limits_{x\to+\infty}\dfrac{\frac{1}{\ln x}\cdot\frac{1}{x}}{\frac{1}{x}}=\lim\limits_{x\to+\infty}\dfrac{1}{\ln x}=0$，

$\left(\dfrac{\ln\ln x}{\ln x}\right)'=\dfrac{1-\ln\ln x}{x(\ln x)^2}$，当$x>e^e$时，$\left(\dfrac{\ln\ln x}{\ln x}\right)'<0$，此时此函数单调减趋于0

则由狄立克莱判别法，得$\int_{e^e}^{+\infty}\dfrac{\ln\ln x}{\ln x}\sin x\,\mathrm{d}x$收敛

又$\int_2^{e^e}\dfrac{\ln\ln x}{\ln x}\sin x\,\mathrm{d}x$为正常积分，则必收敛，于是$\int_2^{+\infty}\dfrac{\ln\ln x}{\ln x}\sin x\,\mathrm{d}x$收敛

又$\displaystyle\int_2^{+\infty}\left|\frac{\ln\ln x}{\ln x}\sin x\right|\mathrm{d}x=\int_2^{n_0\pi}\left|\frac{\ln\ln x}{\ln x}\sin x\right|\mathrm{d}x+\sum_{n=n_0}^{\infty}I_n$，其中$n_0>\dfrac{e^e}{\pi}$为正整数

$\displaystyle I_n=\int_{n\pi}^{(n+1)\pi}\frac{\ln\ln x}{\ln x}|\sin x|\,\mathrm{d}x=\int_0^{\pi}\frac{\ln\ln(n\pi+z)}{\ln(n\pi+z)}\sin z\,\mathrm{d}z\geqslant\frac{\ln\ln(n+1)\pi}{\ln(n+1)\pi}\int_0^{\pi}\sin z\,\mathrm{d}z=2\frac{\ln\ln(n+1)\pi}{\ln(n+1)\pi}$

因$\displaystyle\int_{e^e+\pi}^{+\infty}\frac{\ln\ln x}{\ln x}\,\mathrm{d}x>\int_{e^e+\pi}^{+\infty}\frac{\ln\ln x}{x}\,\mathrm{d}x=\ln x(\ln\ln x-1)\Big|_{e^e+\pi}^{+\infty}=+\infty$，则由柯西判别法，得$\displaystyle\sum_{n=n_0}^{\infty}\frac{\ln\ln(n+1)\pi}{\ln(n+1)\pi}$发散，于是$\displaystyle\sum_{n=n_0}^{+\infty}I_n$发散，从而原积分条件收敛.

§2. 无界函数的广义积分

1. 下列积分是否收敛？如果收敛，求其值.

(1) $\int_0^{\frac{1}{2}} \cot x \, \mathrm{d}x$

(2) $\int_0^1 \ln x \, \mathrm{d}x$

解：

(1) 因$\lim\limits_{x\to+0} \cot x = \infty$，则$x=0$为$\cot x$的奇点

又$\int_{0+\eta}^{\frac{1}{2}} \cot x \, \mathrm{d}x = \ln|\sin x|\Big|_{\eta}^{\frac{1}{2}} = \ln\left|\sin\frac{1}{2}\right| - \ln|\sin\eta| \to +\infty(\eta\to+0)$，则积分$\int_0^{\frac{1}{2}} \cot x \, \mathrm{d}x$发散.

(2) 因$\lim\limits_{x\to+0} \ln x = \infty$，则$x=0$为$\ln x$的奇点

又$\int_{0+\eta}^{1} \ln x \, \mathrm{d}x = x(\ln x - 1)\Big|_{\eta}^{1} = -\eta\ln\eta - 1 + \eta \to -1(\eta\to+0)$，则积分$\int_0^1 \ln x \, \mathrm{d}x$收敛于$-1$.

2. 讨论下列积分的收敛性：

(1) $\int_0^1 \frac{\sin x}{x^{\frac{3}{2}}} \, \mathrm{d}x$

(2) $\int_0^1 \frac{\mathrm{d}x}{\sqrt[3]{x^2(1-x)}}$

(3) $\int_0^1 \frac{\ln x}{1-x^2} \, \mathrm{d}x$

(4) $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sin^2 x\cdot\cos^2 x}$

(5) $\int_0^1 |\ln x|^p \, \mathrm{d}x$

(6) $\int_0^{\frac{\pi}{2}} \frac{1-\cos x}{x^m} \, \mathrm{d}x$

(7) $\int_0^1 x^{a-1}(1-x)^{b-1} \, \mathrm{d}x$

(8) $\int_0^1 x^{a-1}(1-x)^{b-1} \ln x \, \mathrm{d}x$

解：

(1) $x=0$为$\dfrac{\sin x}{x^{\frac{3}{2}}}$ 的奇点

因$\lim\limits_{x\to+0} x^{\frac{1}{2}} \cdot \dfrac{\sin x}{x^{\frac{3}{2}}} = \lim\limits_{x\to+0} \dfrac{\sin x}{x} = 1$，则据柯西判别法，得$\int_0^1 \dfrac{\sin x}{x^{\frac{3}{2}}} \, \mathrm{d}x$绝对收敛.

(2) $x=0$，$x=1$均为被积函数的奇点，$\int_0^1 \dfrac{\mathrm{d}x}{\sqrt[3]{x^2(1-x)}} = \int_0^{\frac{1}{2}} \dfrac{\mathrm{d}x}{\sqrt[3]{x^2(1-x)}} + \int_{\frac{1}{2}}^1 \dfrac{\mathrm{d}x}{\sqrt[3]{x^2(1-x)}}$

因$\lim\limits_{x\to+0} x^{\frac{2}{3}} \cdot \dfrac{1}{\sqrt[3]{x^2(1-x)}} = \lim\limits_{x\to+0} \dfrac{1}{\sqrt[3]{1-x}} = 1$，则据柯西判别法，得$\int_0^{\frac{1}{2}} \dfrac{\mathrm{d}x}{\sqrt[3]{x^2(1-x)}}$绝对收敛；

又$\lim\limits_{x\to1-0} (1-x)^{\frac{1}{3}} \dfrac{\mathrm{d}x}{\sqrt[3]{x^2(1-x)}} = \lim\limits_{x\to1-0} \dfrac{1}{\sqrt[3]{x^2}} = 1$，则据柯西判别法，得$\int_{\frac{1}{2}}^1 \dfrac{\mathrm{d}x}{\sqrt[3]{x^2(1-x)}}$绝对收敛

从而$\int_0^1 \dfrac{\mathrm{d}x}{\sqrt[3]{x^2(1-x)}}$绝对收敛.

(3) 因$\lim\limits_{x\to1}\dfrac{\ln x}{1-x^2}=\lim\limits_{x\to1}\dfrac{\frac{1}{x}}{-2x}=-\lim\limits_{x\to1}\dfrac{1}{2x^2}=-\dfrac{1}{2}$，则$x=1$不是奇点，于是此积分只有一个奇点0

又$\lim\limits_{x\to+0}x^{\frac{1}{2}}\cdot\dfrac{\ln x}{1-x^2}=\lim\limits_{x\to+0}x^{\frac{1}{2}}\ln x=\lim\limits_{x\to+0}\dfrac{\ln x}{x^{-\frac{1}{2}}}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}}=-2\lim\limits_{x\to+0}x^{\frac{1}{2}}=0$

则由柯西判别法，得$\displaystyle\int_0^1\frac{\ln x}{1-x^2}\,\mathrm{d}x$收敛.

(4) $x=0$，$x=\dfrac{\pi}{2}$均为被积函数的奇点，则$\displaystyle\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{\sin^2x\cdot\cos^2x}=\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}x}{\sin^2x\cdot\cos^2x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\mathrm{d}x}{\sin^2x\cdot\cos^2x}$

因$\lim\limits_{x\to+0}x^2\cdot\dfrac{1}{\sin^2x\cdot\cos^2x}=1$，且$\dfrac{1}{\sin^2x\cdot\cos^2x}\geqslant0$，则据柯西判别法，得$\displaystyle\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}x}{\sin^2x\cdot\cos^2x}$发散至$+\infty$

又$\dfrac{1}{\sin^2x\cdot\cos^2x}\geqslant0$，则$\displaystyle\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{\sin^2x\cdot\cos^2x}$发散.

(5) $\displaystyle\int_0^1|\ln x|^p\,\mathrm{d}x=\int_0^{\frac{1}{2}}|\ln x|^p\,\mathrm{d}x+\int_{\frac{1}{2}}^1|\ln x|^p\,\mathrm{d}x$

对$\displaystyle\int_0^{\frac{1}{2}}|\ln x|^p\,\mathrm{d}x$，当$p>0$时，0为奇点

因$\lim\limits_{x\to+0}x^{\frac{1}{2}}|\ln x|^p=\lim\limits_{x\to+0}\dfrac{|\ln x|^p}{x^{-\frac{1}{2}}}=\lim\limits_{x\to+0}\dfrac{(-\ln x)^p}{x^{-\frac{1}{2}}}=(-1)^p\lim\limits_{x\to+0}\dfrac{p\ln^{p-1}x\cdot\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}}=(-1)^{p+1}2p\lim\limits_{x\to+0}\dfrac{\ln^{p-1}x}{x^{-\frac{1}{2}}}$
$=0$(见书上册231页1.(16))

则由柯西判别法，得$\displaystyle\int_0^{\frac{1}{2}}|\ln x|^p\,\mathrm{d}x$当$p>0$时收敛

当$p\leqslant0$时，$\displaystyle\int_0^{\frac{1}{2}}|\ln x|^p\,\mathrm{d}x$为正常积分，则$\displaystyle\int_0^{\frac{1}{2}}|\ln x|^p\,\mathrm{d}x$，于是$p$为任何值时，$\displaystyle\int_0^{\frac{1}{2}}|\ln x|^p\,\mathrm{d}x$均收敛.

对$\displaystyle\int_{\frac{1}{2}}^1|\ln x|^p\,\mathrm{d}x$，$p<0$时，1为奇点

$$因\lim_{x\to1-0}(1-x)^{-p}|\ln x|^p=\lim_{x\to1-0}\frac{|\ln x|^p}{(1-x)^p}=\lim_{x\to1-0}\left(\frac{\ln\dfrac{1}{x}}{1-x}\right)^p=\left(\lim_{x\to1-0}\frac{1}{x}\right)^p=1$$

则据柯西判别法，得当$-p<1$即$0>p>-1$时，$\displaystyle\int_{\frac{1}{2}}^1|\ln x|^p\,\mathrm{d}x$收敛；

当$-p\geqslant1$即$p\leqslant-1$时，$\displaystyle\int_{\frac{1}{2}}^1|\ln x|^p\,\mathrm{d}x$发散

当$p\geqslant0$时，$\displaystyle\int_{\frac{1}{2}}^1|\ln x|^p\,\mathrm{d}x$为正常积分，故收敛

于是当$p>-1$时，$\displaystyle\int_{\frac{1}{2}}^1|\ln x|^p\,\mathrm{d}x$收敛；当$p\leqslant-1$时，$\displaystyle\int_{\frac{1}{2}}^1|\ln x|^p\,\mathrm{d}x$发散

综合知，当$p>-1$时，$\displaystyle\int_0^1|\ln x|^p\,\mathrm{d}x$收敛；当$p\leqslant-1$时，$\displaystyle\int_0^1|\ln x|^p\,\mathrm{d}x$发散.

(6) $$因\lim_{x\to+0}\frac{1-\cos x}{x^m}=\begin{cases}0,\quad m\leqslant0\\ \lim\limits_{x\to+0}\dfrac{\sin x}{mx^{m-1}}=\begin{cases}0,\quad 0<m<1\\ \lim\limits_{x\to+0}\dfrac{\cos x}{m(m-1)x^{m-2}}=\begin{cases}0,& 1<m<2\\ \dfrac{1}{2},& m=2\\ \infty,& m>2\end{cases}\end{cases}\end{cases}$$

$$则\lim_{x\to+0}\frac{1-\cos x}{x^m}=\begin{cases}0,& m<2\\ \dfrac{1}{2},& m=2\\ \infty,& m>2\end{cases}$$

从而当$m\leqslant2$时，$\displaystyle\int_0^{\frac{\pi}{2}}\frac{1-\cos x}{x^m}\,\mathrm{d}x$为正常积分，故收敛

当$m>2$时，$x=0$为$\displaystyle\int_0^{\frac{\pi}{2}}\frac{1-\cos x}{x^m}\,\mathrm{d}x$的奇点

又$\lim\limits_{x\to+0}x^{m-2}\dfrac{1-\cos x}{x^m}=\lim\limits_{x\to+0}\dfrac{1-\cos x}{x^2}=\dfrac{1}{2}$

则当$0 < m-2 < 1$即$2 < m < 3$时，积分收敛；当$m-2 \geqslant 1$即$m \geqslant 3$时，积分发散

从而当$m < 3$时，$\displaystyle\int_0^{\frac{\pi}{2}} \frac{1-\cos x}{x^m}\,\mathrm{d}x$收敛；当$m \geqslant 3$时，$\displaystyle\int_0^{\frac{\pi}{2}} \frac{1-\cos x}{x^m}\,\mathrm{d}x$发散.

(7) 当$a \geqslant 1$且$b \geqslant 1$时，$\displaystyle\int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$为正常积分，故收敛

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x = \int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x + \int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$$

对积分$\displaystyle\int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x = \int_0^{\frac{1}{2}} \frac{(1-x)^{b-1}}{x^{1-a}}\,\mathrm{d}x$

因$\displaystyle\lim_{x\to+0} \frac{(1-x)^{b-1}}{x^{1-a}} = \begin{cases} 0, & a>1 \\ 1, & a=1 \\ \infty, & a<1 \end{cases}$ 且$\displaystyle\lim_{x\to+0} x^{1-a}\frac{(1-x)^{b-1}}{x^{1-a}} = \lim_{x\to+0}(1-x)^{b-1} = 1$

则由柯西判别法的极限形式，得当$1-a<1$即$a>0$时积分收敛；当$1-a \geqslant 1$即$a \leqslant 0$时，积分发散；

对积分$\displaystyle\int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x = \int_{\frac{1}{2}}^1 \frac{x^{a-1}}{(1-x)^{1-b}}\,\mathrm{d}x$

因$\displaystyle\lim_{x\to1-0} \frac{x^{a-1}}{(1-x)^{1-b}} = \begin{cases} 0, & b>1 \\ 1, & b=1 \\ \infty, & b<1 \end{cases}$ 且$\displaystyle\lim_{x\to1-0}(1-x)^{1-b}\frac{x^{a-1}}{(1-x)^{1-b}} = \lim_{x\to1-0} x^{a-1} = 1$

则由柯西判别法的极限形式，得当$1-b<1$即$b>0$时积分收敛；当$1-b \geqslant 1$即$b \leqslant 0$时，积分发散；

综上所述，当$a>0$且$b>0$时，$\displaystyle\int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$收敛，其余情形积分均发散.

(8) $\displaystyle\int_0^1 x^{a-1}(1-x)^{b-1}\ln x\,\mathrm{d}x = \int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}\ln x\,\mathrm{d}x + \int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{b-1}\ln x\,\mathrm{d}x$

对积分$\displaystyle\int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}\ln x\,\mathrm{d}x = \int_0^{\frac{1}{2}} \frac{(1-x)^{b-1}\ln x}{x^{1-a}}\,\mathrm{d}x$

因$\displaystyle\lim_{x\to+0} \frac{(1-x)^{b-1}\ln x}{x^{1-a}} = \begin{cases} 0, & a>1 \\ \infty, & a \leqslant 1 \end{cases}$

且对$\forall c>0$，$\displaystyle\lim_{x\to+0} x^{1-a+c}\frac{(1-x)^{b-1}}{x^{1-a}}|\ln x| = \lim_{x\to+0}\frac{-\ln x}{x^{-c}} = \lim_{x\to+0}\frac{-\frac{1}{x}}{-cx^{-c-1}} = \lim_{x\to+0}\frac{x^c}{c} = 0$

则由柯西判别法的极限形式，得当$1-a+c<1$即$a>c>0$时收敛

又$\displaystyle\lim_{x\to+0} x^{1-a}\frac{(1-x)^{b-1}}{x^{1-a}}|\ln x| = -\lim_{x\to+0}(1-x)^{b-1}\ln x = \infty$

则由柯西判别法的极限形式，得当$1-a \geqslant 1$即$a \leqslant 0$时发散

对积分$\displaystyle\int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{b-1}\ln x\,\mathrm{d}x = \int_{\frac{1}{2}}^1 \frac{x^{a-1}\ln x}{(1-x)^{1-b}}\,\mathrm{d}x$

因$\displaystyle\lim_{x\to1-0} \frac{x^{a-1}\ln x}{(1-x)^{1-b}} = \begin{cases} 0, & b>0 \\ -1, & b=0 \\ \infty, & b<0 \end{cases}$

且$\displaystyle\lim_{x\to1-0}(1-x)^{-b}\frac{x^{a-1}}{(1-x)^{1-b}}|\ln x| = \lim_{x\to1-0}\frac{-\ln x}{1-x} = \lim_{x\to1-0}\frac{1}{x} = 1$

则由柯西判别法的极限形式，得当$-b<1$即$b>-1$时收敛；当$-b \geqslant 1$即$b \leqslant -1$时发散

综上所述，得当$a>0$且$b>-1$时，积分$\displaystyle\int_0^1 x^{a-1}(1-x)^{b-1}\ln x\,\mathrm{d}x$收敛，其余情形积分均发散.

3. 证明无界函数广义积分的柯西判别法及其极限形式.

证明：

(1) 柯西判别法：

(i) 由$|f(x)| \leqslant \dfrac{C}{(x-a)^p}(C>0), p<1$，知$\displaystyle\lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b |f(x)|\,\mathrm{d}x \leqslant \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b \frac{C}{(x-a)^p}\,\mathrm{d}x =$

$\displaystyle\lim_{\varepsilon\to+0}\frac{C}{1-p}(1-a)^{1-p}\bigg|_{a+\varepsilon}^b = \lim_{\varepsilon\to+0}\left[\frac{C}{1-p}(b-a)^{1-p} - \frac{C}{1-p}\varepsilon^{1-p}\right] = \frac{C}{1-p}(b-a)^{1-p}$

即$\displaystyle\lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b |f(x)|\,\mathrm{d}x$存在，故$\displaystyle\int_a^b f(x)\,\mathrm{d}x$绝对收敛

(ii) 因有$\displaystyle\int_{a+\varepsilon}^b |f(x)|\,\mathrm{d}x \geqslant \int_{a+\varepsilon}^b \frac{C}{(x-a)^p}\,\mathrm{d}x = \left[\frac{C}{1-p}(b-a)^{1-p} - \frac{C}{1-p}\varepsilon^{1-p}\right] \to \infty$(当$p>1, C>$

0且$\varepsilon \to 0$时)

又当$p=1$时，$\int_a^b \frac{C}{x-a}\,\mathrm{d}x$发散，从而$\int_a^b |f(x)|\,\mathrm{d}x$发散.

(2) 柯西判别法的极限形式：

(i) 设$\lim\limits_{x\to a}(x-a)^p|f(x)|=k(0<k<\infty)$

则对$\forall k>\varepsilon>0$，存在$\delta>0$，使当$a<x<a+\delta$时，有$0<k-\varepsilon<(x-a)^p|f(x)|<k+\varepsilon$

即有$\frac{k-\varepsilon}{(x-a)^p}<|f(x)|<\frac{k+\varepsilon}{(x-a)^p}$

于是$\int_a^b \frac{\mathrm{d}x}{(x-a)^p}$与$\int_a^b |f(x)\,\mathrm{d}x$同时收敛或发散（归结为柯西判别法）

从而当$p<1$时，$\int_a^b f(x)\,\mathrm{d}x$绝对收敛；$p\geqslant 1$时，$f(x)$有定号，则$\int_a^b f(x)\,\mathrm{d}x$发散

(ii) $k=0$时，取$\varepsilon_0=1$，则$\exists\delta>0$，使当$a<x<a+\delta$时，

$|(x-a)^p f(x)|=(x-a)^p|f(x)|<1$即$|f(x)|<\frac{1}{(x-a)^p}$，

则由柯西判别法，得$p<1$时，$\int_a^b f(x)\,\mathrm{d}x$绝对收敛

(iii) $k=\infty$时，取$G=1$，则$\exists\delta>0$，使当$a<x<a+\delta$时，有$|(x-a)^p f(x)|=(x-a)^p|f(x)|>1$

即$|f(x)|>\frac{1}{(x-a)^p}$

则由柯西判别法，得当$p\geqslant 1$时，$\int_a^b |f(x)|\,\mathrm{d}x$发散；又$f(x)$有定号，从而$\int_a^b f(x)\,\mathrm{d}x$发散.

综上，得若$0\leqslant k<+\infty, p<1$，那末$\int_a^b f(x)\,\mathrm{d}x$绝对收敛；若$0<k\leqslant+\infty, p\geqslant 1$，那末$\int_a^b f(x)\,\mathrm{d}x$发散.

4. 讨论下列积分的收敛性：

(1) $\int_0^{+\infty}\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$

(2) $\int_0^{+\infty}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$

(3) $\int_0^{+\infty}\frac{\mathrm{d}x}{x^p+x^q}$

(4) $\int_0^{+\infty}\frac{\arctan x}{x^\alpha}\,\mathrm{d}x$

(5) $\int_1^{+\infty}\frac{\mathrm{d}x}{x^p\ln^q x}$

(6) $\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{|x-a_1|^{p_1}|x-a_2|^{p_2}\cdots|x-a_n|^{p_n}}$

解：

(1) $x=0,1,2$均为被积函数的奇点

$$\int_0^{+\infty}\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}=\left(\int_0^{\frac{1}{2}}+\int_{\frac{1}{2}}^1+\int_1^{\frac{3}{2}}+\int_{\frac{3}{2}}^2+\int_2^3+\int_3^{+\infty}\right)\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$$

对积分$\int_0^{\frac{1}{2}}\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$

因$\lim\limits_{x\to+0}x^{\frac{2}{3}}\left|\frac{1}{\sqrt[3]{(x-1)^2x(x-2)}}\right|=\lim\limits_{x\to+0}\left|\frac{x}{(x-1)^2(x-2)}\right|^{\frac{1}{3}}=0$

则由柯西判别法的极限形式，得积分$\int_0^{\frac{1}{2}}\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$绝对收敛

对积分$\int_{\frac{1}{2}}^1\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$

因$\lim\limits_{x\to1-0}(1-x)^{\frac{5}{6}}\left|\dfrac{1}{\sqrt[3]{(x-1)^2x(x-2)}}\right|=\lim\limits_{x\to1-0}\left|\dfrac{(1-x)^{\frac{1}{6}}}{[x(x-2)]^{\frac{1}{3}}}\right|=0$

则由柯西判别法的极限形式，得积分$\displaystyle\int_{\frac{1}{2}}^{1}\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$绝对收敛

同此证法，得积分$\displaystyle\int_{1}^{\frac{3}{2}}\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$绝对收敛

对积分$\displaystyle\int_{\frac{3}{2}}^{2}\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$

因$\lim\limits_{x\to2-0}(2-x)^{\frac{2}{3}}\left|\dfrac{1}{\sqrt[3]{(x-1)^2x(x-2)}}\right|=\lim\limits_{x\to2-0}\left|\dfrac{2-x}{x(x-1)^2}\right|^{\frac{1}{3}}=0$

则由柯西判别法的极限形式，得积分$\displaystyle\int_{\frac{3}{2}}^{2}\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$绝对收敛

同此证法，得$\displaystyle\int_{2}^{3}\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$绝对收敛

因$x\geqslant3$时，$\dfrac{1}{\sqrt[3]{(x-1)^2x(x-2)}}\leqslant\dfrac{1}{(x-2)^{\frac{4}{3}}}$且$\displaystyle\int_{3}^{+\infty}\frac{\mathrm{d}x}{(x-2)^{\frac{4}{3}}}$绝对收敛

则由比较判别法，得$\displaystyle\int_{3}^{+\infty}\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$绝对收敛

综上可知，$\displaystyle\int_{0}^{+\infty}\frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2x(x-2)}}$绝对收敛

(2) $\displaystyle\int_{0}^{+\infty}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x=\int_{0}^{1}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x+\int_{1}^{+\infty}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$

对$\displaystyle\int_{0}^{1}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$

因$\lim\limits_{x\to+0}\dfrac{\ln(1+x)}{x^\alpha}=\begin{cases}0,\ \alpha\leqslant0\\ \lim\limits_{x\to+0}\dfrac{\frac{1}{1+x}}{\alpha x^{\alpha-1}}=\begin{cases}0, & 0<\alpha<1\\ 1, & \alpha=1\\ \infty, & \alpha>1\end{cases}\end{cases}$，则当$\alpha>1$时，0为奇点

又$\lim\limits_{x\to+0}x^{\alpha-1}\left|\dfrac{\ln(1+x)}{x^\alpha}\right|=\lim\limits_{x\to+0}\left|\dfrac{\ln(1+x)}{x}\right|=1$

则当$\alpha-1<1$即$1<\alpha<2$时，$\displaystyle\int_{0}^{1}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$绝对收敛；当$\alpha\geqslant2$时，$\displaystyle\int_{0}^{1}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$发散；

当$\alpha\leqslant1$时，$\displaystyle\int_{0}^{1}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$为正常积分，故必收敛

从而当$\alpha<2$时，$\displaystyle\int_{0}^{1}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$绝对收敛；当$\alpha\geqslant2$时，$\displaystyle\int_{0}^{1}-\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$发散

对$\displaystyle\int_{1}^{+\infty}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$

取$\lambda>1$，当$\alpha-\lambda>0$时，因$\lim\limits_{x\to+\infty}x^\lambda\dfrac{\ln(1+x)}{x^\alpha}=\lim\limits_{x\to+\infty}\dfrac{\frac{1}{1+x}}{(\alpha-\lambda)x^{\alpha-\lambda-1}}=0$

则当$\alpha-\lambda>0$即$\alpha>\lambda>1$时，积分$\displaystyle\int_{1}^{+\infty}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$绝对收敛；

又$\lim\limits_{x\to+\infty}x^\alpha\dfrac{\ln(1+x)}{x^\alpha}=+\infty$，则当$\alpha\leqslant1$时，积分$\displaystyle\int_{1}^{+\infty}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$发散

从而当$1<\alpha<2$时，$\displaystyle\int_{0}^{+\infty}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$绝对收敛；其它情形，$\displaystyle\int_{0}^{+\infty}\frac{\ln(1+x)}{x^\alpha}\,\mathrm{d}x$都发散.

(3) $\displaystyle\int_{0}^{+\infty}\frac{\mathrm{d}x}{x^p+x^q}=\int_{0}^{1}\frac{\mathrm{d}x}{x^p+x^q}+\int_{1}^{+\infty}\frac{\mathrm{d}x}{x^p+x^q}$

对积分$\displaystyle\int_{0}^{1}\frac{\mathrm{d}x}{x^p+x^q}$，设$\min(p,q)=p$

若$p\leqslant0$，则$\displaystyle\int_{0}^{1}\frac{\mathrm{d}x}{x^p+x^q}$为正常积分，故$\displaystyle\int_{0}^{1}\frac{\mathrm{d}x}{x^p+x^q}$收敛

若$p>0$，由于$\lim\limits_{x\to+0} x^p\dfrac{1}{x^p+x^q}=\lim\limits_{x\to+0}\dfrac{1}{1+x^{q-p}}=\begin{cases}\dfrac{1}{2}, & p=q\\ 1, & p\neq q\end{cases}$

故积分$\displaystyle\int_0^1\frac{\mathrm{d}x}{x^p+x^q}$仅当$p<1$即$\min(p,q)<1$时收敛

对积分$\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{x^p+x^q}$，设$\max(p,q)=q$

由于$\lim\limits_{x\to+\infty} x^q\dfrac{1}{x^p+x^q}=\lim\limits_{x\to+\infty}\dfrac{1}{x^{-(q-p)}+1}=\begin{cases}\dfrac{1}{2}, & p=q\\ 1, & p\neq q\end{cases}$

故积分$\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{x^p+x^q}$仅当$q>1$即$\max(p,q)>1$时收敛

则积分$\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{x^p+x^q}$当$\min(p,q)<1$且$\max(p,q)>1$时收敛.

(4) 当$0<\alpha\leqslant 1$时，$\lim\limits_{x\to+0}\dfrac{\arctan x}{x^\alpha}=\lim\limits_{x\to+0}\dfrac{\frac{1}{1+x^2}}{\alpha x^{\alpha-1}}=\lim\limits_{x\to+0}\dfrac{x^{1-\alpha}}{\alpha(1+x^2)}=0$

当$\alpha\leqslant 0$，$\lim\limits_{x\to+0}\dfrac{\arctan x}{x^\alpha}=\lim\limits_{x\to+0}x^{-\alpha}\arctan x=0$

则$\alpha\leqslant 1$时，0不为奇点

又$\lim\limits_{x\to+\infty}x^\alpha\dfrac{\arctan x}{x^\alpha}=\lim\limits_{x\to+\infty}\arctan x=\dfrac{\pi}{2}$

则$\alpha\leqslant 1$时，由柯西判别法的极限形式，得$\displaystyle\int_0^{+\infty}\frac{\arctan x}{x^\alpha}\mathrm{d}x$发散

当$\alpha>1$时，因$\lim\limits_{x\to+0}\dfrac{\arctan x}{x^\alpha}=\lim\limits_{x\to+0}\dfrac{\frac{1}{1+x^2}}{\alpha x^{\alpha-1}}=\lim\limits_{x\to+0}\dfrac{1}{\alpha x^{\alpha-1}(1+x^2)}=+\infty$，则0为奇点

则$\displaystyle\int_0^{+\infty}\frac{\arctan x}{x^\alpha}\,\mathrm{d}x=\int_0^1\frac{\arctan x}{x^\alpha}\,\mathrm{d}x+\int_1^{+\infty}\frac{\arctan x}{x^\alpha}\,\mathrm{d}x$

对$\displaystyle\int_0^1\frac{\arctan x}{x^\alpha}\,\mathrm{d}x$，$\alpha>1$，0为奇点

因$\lim\limits_{x\to+0}x^{\alpha-1}\dfrac{\arctan x}{x^\alpha}=\lim\limits_{x\to+0}\dfrac{\arctan x}{x}=1$

则由柯西判别法的极限形式，得$\alpha-1<1$即$\alpha<2$时积分收敛；当$\alpha\geqslant 2$时，积分发散

对$\displaystyle\int_1^{+\infty}\frac{\arctan x}{x^\alpha}\,\mathrm{d}x$

因$\dfrac{\frac{\pi}{4}}{x^\alpha}\leqslant\dfrac{\arctan x}{x^\alpha}\leqslant\dfrac{\frac{\pi}{2}}{x^\alpha}$且$\displaystyle\int_1^{+\infty}\frac{\frac{\pi}{2}}{x^\alpha}\,\mathrm{d}x$当$\alpha>1$时积分收敛；$\displaystyle\int_1^{+\infty}\frac{\frac{\pi}{4}}{x^\alpha}\,\mathrm{d}x$当$\alpha\leqslant 1$时积分发散

则由比较判别法，得当$\alpha>1$时，$\displaystyle\int_1^{+\infty}\frac{\arctan x}{x^\alpha}\,\mathrm{d}x$收敛；当$\alpha\leqslant 1$时，$\displaystyle\int_1^{+\infty}\frac{\arctan x}{x^\alpha}\,\mathrm{d}x$发散

总之，当$1<\alpha<2$时，$\displaystyle\int_0^{+\infty}\frac{\arctan x}{x^\alpha}\,\mathrm{d}x$收敛；其余情形此积分均发散.

(5) $\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{x^p\ln^q x}=\int_1^2\frac{\mathrm{d}x}{x^p\ln^q x}+\int_2^{+\infty}\frac{\mathrm{d}x}{x^p\ln^q x}$

考虑$\displaystyle\int_1^2\frac{\mathrm{d}x}{x^p\ln^q x}$，对任意的$p$

由于$\lim\limits_{x\to1+0}\left[(x-1)^q\dfrac{1}{x^p\ln^q x}\right]=\lim\limits_{x\to1+0}\dfrac{1}{x^p}\cdot\dfrac{(x-1)^q}{\ln^q x}=\lim\limits_{x\to1+0}\dfrac{(x-1)^q}{\ln^q x}=\left(\lim\limits_{x\to1+0}\dfrac{x-1}{\ln x}\right)^q$

$=\left(\lim\limits_{x\to1+0}\dfrac{1}{\frac{1}{x}}\right)^q=1$

则积分$\displaystyle\int_1^2\frac{\mathrm{d}x}{x^p\ln^q x}$仅当$q<1$且$p$为任意值时收敛

考虑$\displaystyle\int_2^{+\infty}\frac{\mathrm{d}x}{x^p\ln^q x}$，若$p>1$，取$\alpha>0$充分小，使$p-\alpha>1$，则对任意$q$，

由于$\lim\limits_{x\to+\infty}\left(x^{p-\alpha}\dfrac{1}{x^p\ln^q x}\right)=\lim\limits_{x\to+\infty}\dfrac{1}{x^\alpha\ln^q x}=0$

于是积分$\displaystyle\int_2^{+\infty}\frac{\mathrm{d}x}{x^p\ln^q x}$当$p>1$且$q$为任意值时收敛；

若$p \leqslant 1, q < 1$，由于$\int_2^{+\infty} \frac{\mathrm{d}x}{x^p \ln^q x} \geqslant \int_2^{+\infty} \frac{\mathrm{d}x}{x \ln^q x} = \frac{1}{1-q} (\ln x)^{1-q} \bigg|_2^{+\infty} = +\infty$

则此时积分$\int_2^{+\infty} \frac{\mathrm{d}x}{x^p \ln^q x}$发散

从而积分$\int_1^{+\infty} \frac{\mathrm{d}x}{x^p \ln^q x}$当$p > 1$且$q < 1$时收敛.

(6) 首先，被积函数关于$\frac{1}{x}$是$\sum_{i=1}^{n} p_i$级无穷小(当$x \to \pm\infty$时)

其次(不妨设为$i \neq j$时，$a_i \neq a_j$)

因$\lim_{x \to a_i} \left[|x-a_i|^{p_i} \frac{1}{|x-a_1|^{p_1}|x-a_2|^{p_2} \cdots |x-a_n|^{p_n}} \right] = c_i$，$0 < c_i < +\infty (i = 1, 2, \cdots, n)$

故积分$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{|x-a_1|^{p_1}|x-a_2|^{p_2} \cdots |x-a_n|^{p_n}}$仅当$\sum_{i=1}^{n} p_i > 1$且$p_i < 1 (i = 1, 2, \cdots, n)$时收敛.

5. 设$f(x)$当$x \to +0$时单调趋向于$+\infty$，试证明：若$\int_0^1 f(x) \,\mathrm{d}x$收敛，必须$\lim_{x \to 0} x f(x) \,\mathrm{d}x = 0$.

证明： 由题设知0是$f(x)$的奇点，即$\int_0^1 f(x) \,\mathrm{d}x$是无界函数的广义积分，且当$x$充分靠近0时，$f(x) \geqslant 0$，在$[0, x]$上单调减

又$\int_0^1 f(x) \,\mathrm{d}x$收敛，则由柯西收敛原理，对$\forall \varepsilon > 0, \exists \delta > 0$，当$0 < \frac{x}{2} < x < \delta$时，

有$\left| \int_{\frac{x}{2}}^{x} f(x) \,\mathrm{d}x \right| = \int_{\frac{x}{2}}^{x} f(x) \,\mathrm{d}x < \frac{\varepsilon}{2}$

由第一积分中值定理，得$\int_{\frac{x}{2}}^{x} f(x) \,\mathrm{d}x = f(\xi) \left(x - \frac{x}{2} \right) = \frac{x}{2} f(\xi) > \frac{x}{2} f(x) \left(\frac{x}{2} < \xi < x \right)$

于是 $\frac{x}{2} f(x) < \int_{\frac{x}{2}}^{x} f(x) \,\mathrm{d}x < \frac{\varepsilon}{2}$ 即$0 \leqslant x f(x) < \varepsilon$，从而$\lim_{x \to 0} x f(x) \,\mathrm{d}x = 0$.

6. 讨论下列积分的绝对收敛和条件收敛性：

(1) $\int_0^{+\infty} \frac{x^p \sin x}{1+x^q} \,\mathrm{d}x (q \geqslant 0)$

(2) $\int_0^{+\infty} \frac{e^{\sin x} \sin 2x}{x^\lambda} \,\mathrm{d}x (\lambda > 0)$

(3) $\int_0^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} \,\mathrm{d}x$

解：

(1) $\int_0^{+\infty} \frac{x^p \sin x}{1+x^q} \,\mathrm{d}x = \int_0^1 \frac{x^p \sin x}{1+x^q} \,\mathrm{d}x + \int_1^{+\infty} \frac{x^p \sin x}{1+x^q} \,\mathrm{d}x$

对$\int_0^1 \frac{x^p \sin x}{1+x^q} \,\mathrm{d}x$

因$\lim_{x \to +0} x^{-(p+1)} \frac{x^p \sin x}{1+x^q} = \lim_{x \to +0} \frac{\frac{\sin x}{x}}{1+x^q} = 1$

则当$-(p+1) < 1$即$p > -2$时，$\int_0^1 \frac{x^p \sin x}{1+x^q} \,\mathrm{d}x$绝对收敛；当$p \leqslant -2$时，积分$\int_0^1 \frac{x^p \sin x}{1+x^q} \,\mathrm{d}x$发散

对$\int_1^{+\infty} \left| \frac{x^p \sin x}{1+x^q} \right| \,\mathrm{d}x$

因$\left| \frac{x^q \sin x}{1+x^q} \right| \leqslant \frac{x^p}{1+x^q}$而$\lim_{x \to +\infty} x^{q-p} \frac{x^p}{1+x^q} = \lim_{x \to +\infty} \frac{x^q}{1+x^q} = 1$

则当$q - p > 1$即$q > p + 1$时$\int_1^{+\infty} \frac{x^p}{1+x^q} \,\mathrm{d}x$ 收敛，于是由比较判别法，得$\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} \,\mathrm{d}x$绝对收敛

又$\frac{2x^q}{1+x^q} \geqslant 1, \frac{2x^p}{1+x^q} \geqslant \frac{1}{x^{q-p}}, \frac{x^p |\sin x|}{1+x^q} \geqslant \frac{|\sin x|}{2x^{q-p}}$且由56页8.(2)知，$\int_1^{+\infty} \frac{\sin x}{x^{q-p}} \,\mathrm{d}x$当$q \leqslant p+1$时非绝对收敛

总之，当$p>-2,q>p+1$时，$\displaystyle\int_0^{+\infty}\frac{x^p\sin x}{1+x^q}\,\mathrm{d}x$绝对收敛

考虑$\displaystyle\int_1^{+\infty}\frac{x^p\sin x}{1+x^q}\,\mathrm{d}x$

当$q>p$时，$\displaystyle\left|\int_1^A\sin x\,\mathrm{d}x\right|\leqslant 2$，$\displaystyle\frac{x^p}{1+x^q}$单调减趋于$0(x\to+\infty)$

则由狄立克莱判别法，得$\displaystyle\int_1^{+\infty}\frac{x^p\sin x}{1+x^q}\,\mathrm{d}x$收敛

当$q\leqslant p$时，当$q=p$时，$\displaystyle\frac{x^p}{1+x^q}\to 1(x\to+\infty)$；当$q<p$时，$\displaystyle\frac{x^p}{1+x^q}\to+\infty(x\to+\infty)$

则对充分大的x，恒有$\displaystyle\frac{x^p}{1+x^q}\geqslant\frac{1}{3}$

于是对$\forall A>1$，必$\exists N\in Z^+$，使得$2N\pi+\dfrac{\pi}{4}>A$且当$x\geqslant 2N\pi+\dfrac{\pi}{4}$时，恒有$\displaystyle\frac{x^p}{1+x^q}\geqslant\frac{1}{3}$

从而对$A'=2N\pi+\dfrac{\pi}{4},A''=2N\pi+\dfrac{\pi}{2}$，有$\displaystyle\left|\int_{A'}^{A''}\frac{x^p}{1+x^q}\sin x\,\mathrm{d}x\right|\geqslant\frac{1}{3}\left|\int_{A'}^{A''}\sin x\,\mathrm{d}x\right|=\frac{\sqrt{2}}{6}$

则由柯西原理，得$\displaystyle\int_1^{+\infty}\frac{x^p\sin x}{1+x^q}\,\mathrm{d}x$发散

综上所述，当$q>p+1>-1$时，$\displaystyle\int_0^{+\infty}\frac{x^p\sin x}{1+x^q}\,\mathrm{d}x$绝对收敛；当$p+1\geqslant q>p>-2$时，$\displaystyle\int_0^{+\infty}\frac{x^p\sin x}{1+x^q}\,\mathrm{d}x$条件收敛.

(2) 因$$\lim_{x\to+0}\frac{e^{\sin x}\sin 2x}{x^\lambda}=\lim_{x\to+0}\frac{\sin 2x}{x^\lambda}=\begin{cases}0, & \lambda\leqslant 0\\ \displaystyle\lim_{x\to+0}\frac{2\cos 2x}{\lambda x^{\lambda-1}}=\begin{cases}0, & 0<\lambda<1\\ 2, & \lambda=1\\ \infty, & \lambda>1\end{cases}\end{cases}=\begin{cases}0, & \lambda<1\\ 2, & \lambda=1\\ \infty, & \lambda>1\end{cases}$$

则当$\lambda>1$时，0为奇点

$$\int_0^{+\infty}\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x=\int_0^1\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x+\int_1^{+\infty}\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x$$

对$\displaystyle\int_0^1\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x$，被积函数为正，0为奇点

因$\displaystyle\lim_{x\to+0}x^{\lambda-1}\frac{e^{\sin x}\sin 2x}{x^\lambda}=\lim_{x\to+0}\frac{e^{\sin x}\sin 2x}{x}=2$

则由柯西判别法的极限形式，得当$\lambda-1<1$即$\lambda<2$时，$\displaystyle\int_0^1\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x$绝对收敛；

当$\lambda\geqslant 2$时，$\displaystyle\int_0^1\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x$发散

对$\displaystyle\int_1^{+\infty}\left|\frac{e^{\sin x}\sin 2x}{x^\lambda}\right|\,\mathrm{d}x$

当$\lambda>1$时，$\displaystyle\left|\frac{e^{\sin x}\sin 2x}{x^\lambda}\right|\leqslant\frac{e}{x^\lambda}$，则$\displaystyle\int_1^{+\infty}\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x$绝对收敛

于是$\displaystyle\int_0^{+\infty}\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x$当$1<\lambda<2$时绝对收敛；

当$\lambda\leqslant 1$时，$\displaystyle\frac{e^{\sin x}|\sin 2x|}{x^\lambda}>\frac{e^{-1}\sin^2 2x}{x^\lambda}=e^{-1}\left(\frac{1-\cos 4x}{2x^\lambda}\right)=\frac{1}{2e}\left(\frac{1}{x^\lambda}-\frac{\cos 4x}{x^\lambda}\right)$

因$\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{x^\lambda}$发散,$\displaystyle\int_1^{+\infty}\frac{\cos 4x}{x^\lambda}\,\mathrm{d}x$收敛，则$\displaystyle\int_1^{+\infty}\left|\frac{e^{\sin x}\sin 2x}{x^\lambda}\right|\,\mathrm{d}x$发散

对$\displaystyle\int_1^{+\infty}\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x$

因$\displaystyle\left|\int_1^A e^{\sin x}\sin 2x\,\mathrm{d}x\right|=2\left|\int_1^A e^{\sin x}\sin x\,\mathrm{d}\sin x\right|\leqslant 4e$

又$\dfrac{1}{x^\lambda}$单调减趋于0，则据狄立克莱判别法，得对$\lambda>0$有$\displaystyle\int_1^{+\infty}\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x$收敛

于是$\displaystyle\int_0^{+\infty}\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x$当$0<\lambda\leqslant 1$时条件收敛

从而当$1<\lambda<2$时$\int_0^{+\infty}\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x$绝对收敛；当$0<\lambda\leqslant 1$时$\int_0^{+\infty}\frac{e^{\sin x}\sin 2x}{x^\lambda}\,\mathrm{d}x$条件收敛.

(3) $\int_0^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^n}\,\mathrm{d}x=\int_0^{1}\frac{\sin\left(x+\frac{1}{x}\right)}{x^n}\,\mathrm{d}x+\int_0^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^n}\,\mathrm{d}x=I_1+I_2$

对I_1，令$x=\frac{1}{t}$, $\mathrm{d}x=-\frac{\mathrm{d}t}{t^2}$，则$I_1=\int_0^{1}\frac{\sin\left(x+\frac{1}{x}\right)}{x^n}\,\mathrm{d}x=\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^{2-n}}\,\mathrm{d}x$

研究I_2

因$\left|\frac{\sin\left(x+\frac{1}{x}\right)}{x^n}\right|\leqslant\frac{1}{x^n}$，则当$n>1$时，$I_2$绝对收敛

因$\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^n}\,\mathrm{d}x=\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)}{x^n\left(1-\frac{1}{x^2}\right)}\,\mathrm{d}x$

$\left|\int_1^{A}\sin\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\mathrm{d}x\right|=\left|\cos\left(x+\frac{1}{x}\right)\Big|_1^A\right|\leqslant 2$

且$\left[x^n\left(1-\frac{1}{x^2}\right)\right]'=nx^{n-1}-(n-2)x^{n-3}=x^{n-3}[nx^2-(n-2)]$

则当$n\in(0,1]$时，$x^n\left(1-\frac{1}{x^2}\right)$单调增即$\frac{1}{x^n\left(1-\frac{1}{x^2}\right)}$单调减趋于$0(x\to+\infty)$

于是由狄立克莱判别法，得当$0<n\leqslant 1$时，I_2收敛

又$\left|\frac{\sin\left(x+\frac{1}{x}\right)}{x^n}\right|\geqslant\frac{\sin^2\left(x+\frac{1}{x}\right)}{x^n}=\frac{1}{2x^n}-\frac{\cos 2\left(x+\frac{1}{x}\right)}{2x^n}$

当$0<n\leqslant 1$时，$\int_1^{+\infty}\frac{\mathrm{d}x}{2x^n}$发散，$\int_1^{+\infty}\frac{\cos 2\left(x+\frac{1}{x}\right)}{2x^n}\,\mathrm{d}x$收敛

则当$0<n\leqslant 1$时，$\int_1^{+\infty}\frac{\sin^2\left(x+\frac{1}{x}\right)}{x^n}\,\mathrm{d}x$发散

于是由比较判别法，知$\int_1^{+\infty}\left|\frac{\sin\left(x+\frac{1}{x}\right)}{x^n}\right|\,\mathrm{d}x$当$0<n\leqslant 1$时发散

从而当$0<n\leqslant 1$时，$\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^n}\,\mathrm{d}x$条件收敛

当$n\leqslant 0$时，对$\forall A>1$，必$\exists k\in Z^+$，使得$2k\pi+\frac{\pi}{6}>A$且当$x\geqslant 2k\pi+\frac{\pi}{6}$时，

恒有$x^{-n}\sin\left(x+\frac{1}{x}\right)\geqslant\sin\left(2k\pi+\frac{\pi}{6}\right)=\frac{1}{2}$

于是，对$A'=2k\pi+\frac{\pi}{6}, A''=2k\pi+\frac{\pi}{3}$，有$\left|\int_{A'}^{A''}\frac{\sin\left(x+\frac{1}{x}\right)}{x^n}\,\mathrm{d}x\right|\geqslant\int_{A'}^{A''}\sin\left(2k\pi+\frac{\pi}{6}\right)\mathrm{d}x=\frac{\pi}{12}$

则由柯西收敛原理，得当$n\leqslant 0$时，I_2发散

对I_1，由I_2的结论，得当$2-n>1$即$n<1$时绝对收敛；当$1\geqslant 2-n>0$即$1\leqslant n<2$时条件收敛；当$2-n\leqslant 0$即$n\leqslant 2$时发散

总之，$\int_0^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^n}\,\mathrm{d}x$当$0<n<2$时条件收敛.

7. 设$f(x)$单调下降，$\lim\limits_{x\to+\infty}f(x)=0$，如果导数$f'(x)$在$[0,+\infty)$上连续，那末积分$\int_0^{+\infty}f'(x)\sin^2 x\,\mathrm{d}x$收敛.

证明：因$(\sin^2 x)'=\sin 2x$，导数$f'(x)$在$[0,+\infty)$上连续，$\lim\limits_{x\to+\infty}f(x)\sin^2 x=0$

则由分部积分公式，得$\int_0^{+\infty}f'(x)\sin^2 x\,\mathrm{d}x=f(x)\sin^2 x\Big|_0^{+\infty}-\int_0^{+\infty}f(x)\sin 2x\,\mathrm{d}x=-\int_0^{+\infty}f(x)\sin 2x\,\mathrm{d}x$

对于$\int_0^{+\infty}f(x)\sin 2x\,\mathrm{d}x$，由已知$f(x)$单调下降，$\lim\limits_{x\to+\infty}f(x)=0$及$\left|\int_0^{A}\sin 2x\,\mathrm{d}x\right|=\frac{1}{2}|\cos 2A-1|\leqslant 1$

则由狄立克莱判别法，得$\int_0^{+\infty}f(x)\sin 2x\,\mathrm{d}x$收敛，从而积分$\int_0^{+\infty}f'(x)\sin^2 x\,\mathrm{d}x$收敛.

8. 在无界函数的广义积分(积分限为有限)中，证明平方可积一定绝对可积，但反之不然.

证明：由已知$f^2(x)$可积，则$\frac{f^2(x)}{2}$也可积

因$(|f(x)|-1)^2=f^2(x)-2|f(x)|+1\geqslant 0$，则$|f(x)|\leqslant\frac{f^2(x)+1}{2}$

于是由比较判别法，得$|f(x)|$可积 即平方可积定绝对可积.
反之不然.
例：由57页例1，得$\displaystyle\int_1^2\left|\frac{1}{(x-1)^{\frac{1}{2}}}\right|\mathrm{d}x$收敛即$\displaystyle\int_1^2\frac{\mathrm{d}x}{(x-1)^{\frac{1}{2}}}$绝对收敛
但$\displaystyle\int_1^2\frac{\mathrm{d}x}{x-1}$发散，即$\dfrac{1}{x-1}$在$[1,2]$上不可积.

9. 计算下列积分的柯西主值：

(1) $\displaystyle\int_0^3\frac{\mathrm{d}x}{1-x}$

(2) $\displaystyle\int_{-\infty}^{+\infty}\sin x\,\mathrm{d}x$

解：

(1) $\displaystyle \text{P.V.}\int_0^3\frac{\mathrm{d}x}{1-x}=\lim_{\eta\to0}\left[\int_0^{1-\eta}\frac{\mathrm{d}x}{1-x}+\int_{1+\eta}^3\frac{\mathrm{d}x}{1-x}\right]=\lim_{\eta\to0}\left[-\ln(1-x)\Big|_0^{1-\eta}-\ln(x-1)\Big|_{1+\eta}^3\right]=-\ln2$

(2) $\displaystyle \text{P.V.}\int_{-\infty}^{+\infty}\sin x\,\mathrm{d}x=\lim_{A\to+\infty}\left(\int_{-A}^{A}\sin x\,\mathrm{d}x\right)=\lim_{A\to+\infty}\left(-\cos x\Big|_{-A}^{A}\right)=\lim_{A\to+\infty}(\cos(-A)-\cos A)=0$

10. 证明广义积分及柯西主值之间的关系：

(1) 若$\displaystyle\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x$收敛，其值为$A$，则柯西主值$\displaystyle\text{P.V.}\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x$存在，且等于$A$，但反之不然；

(2) 若$f(x)\geqslant0$，$\displaystyle\text{P.V.}\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x$存在，其值为$A$，则$\displaystyle\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x$收敛，且收敛于$A$.

证明：

(1) 由$\displaystyle\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x$收敛，知$\displaystyle\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x=\int_{-\infty}^{0}f(x)\,\mathrm{d}x+\int_0^{+\infty}f(x)\,\mathrm{d}x$收敛，
则有$\displaystyle\lim_{B\to-\infty}\int_B^0f(x)\,\mathrm{d}x+\lim_{A\to+\infty}\int_0^Af(x)\,\mathrm{d}x$存在，特别取$B=-A$，有$\displaystyle\lim_{A\to+\infty}\int_{-A}^{A}f(x)\,\mathrm{d}x$存在，且等于$A$
这表明$\displaystyle\text{P.V.}\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x$存在，且等于$A$
但反之不然.
例如：$\displaystyle\text{P.V.}\int_{-\infty}^{+\infty}\sin x\,\mathrm{d}x=\lim_{A\to+\infty}\int_{-A}^{A}\sin x\,\mathrm{d}x=0$，但$\displaystyle\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x=\int_{-\infty}^{+\infty}\sin x\,\mathrm{d}x$不收敛.

(2) 用反证法.
若不然，则由于$f(x)\geqslant0$，得$\displaystyle\int_{-\infty}^{a}f(x)\,\mathrm{d}x$和$\displaystyle\int_a^{+\infty}f(x)\,\mathrm{d}x$中至少有一为$+\infty$
于是$\displaystyle\int_{-A}^{a}f(x)\,\mathrm{d}x$和$\displaystyle\int_a^{A}f(x)\,\mathrm{d}x$中当$A\to+\infty$时至少有一趋于$+\infty$，而另一个大于等于0，从而它们的和趋于$+\infty$，这与已知$\displaystyle\text{P.V.}\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x$存在矛盾，则$\displaystyle\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x$收敛.
又由$\displaystyle\text{P.V.}\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x=A$，则据极限唯一性，得$\displaystyle\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x=A$.

第二部分　函数项级数

第十一章　函数项级数、幂级数

§1.　函数项级数的一致收敛

1. 讨论下列函数序列在所示区域内的一致收敛性:

(1) $f_n(x)=\sqrt{x^2+\dfrac{1}{n^2}}$，　$-\infty<x<\infty$

(2) $f_n(x)=x^2-x^{2n}$,　　$0\leqslant x\leqslant 1$

(3) $f_n(x)=\sin\dfrac{x}{n}$

(i) $-l<x<l$

(ii) $-\infty<x<\infty$

(4) $f_n(x)=x^n(1-x)$，　$0\leqslant x\leqslant 1$

(5) $f_n(x)=\dfrac{nx}{1+nx}$，　　$0\leqslant x\leqslant 1$

(6) $f_n(x)=\dfrac{x}{n}\ln\dfrac{x}{n}$，　　$0<x<1$

解:

(1) 当$-\infty<x<+\infty$时，$f(x)=\lim\limits_{n\to\infty}f_n(x)=\lim\limits_{n\to\infty}\sqrt{x^2+\dfrac{1}{n^2}}=|x|$

则$||f_n-f||=\sup\limits_{x\in(-\infty,+\infty)}|f_n(x)-f(x)|=\sup\limits_{x\in(-\infty,+\infty)}\left|\sqrt{x^2+\dfrac{1}{n^2}}-|x|\right|=\dfrac{1}{n}\to 0(n\to\infty)$

于是由定义2，得$f_n(x)$在$(-\infty,+\infty)$内一致收敛于$|x|$.

(2) 当$x=1$时，$f_n(1)=0,f(x)=0$；当$0\leqslant x<1$时，$f(x)=\lim\limits_{n\to\infty}f_n(x)=0$，则$f(x)=0(0\leqslant x\leqslant 1)$

$||f_n-f||=\sup\limits_{x\in[0,1]}|f_n(x)-f(x)|=\sup\limits_{x\in[0,1]}|x^n-x^{2n}|=\max\limits_{x\in[0,1]}|x^n-x^{2n}|$

令$(x^n-x^{2n})'=nx^{n-1}(1-2x^n)=0$，则得$x=0,x=\sqrt[n]{\dfrac{1}{2}}$

又$f_n(0)=0,f_n\left(\sqrt[n]{\dfrac{1}{2}}\right)=\dfrac{1}{4},f_n(1)=0$，则$||f_n-f||=\dfrac{1}{4}\neq 0$，于是由定义2，得此函数序列在所示区域内不一致收敛.

(3) (i) 当$-l<x<l$时，$f(x)=\lim\limits_{n\to\infty}f_n(x)=0$

$||f_n-f||=\sup\limits_{x\in(-l,l)}|f_n(x)-f(x)|=\sup\limits_{x\in(-l,l)}\left|\sin\dfrac{x}{n}\right|\leqslant\dfrac{l}{n}\to 0(n\to\infty)$

于是据定义2，得$f_n(x)$在$(-l,l)$上一致收敛于0.

(ii) 当$-\infty<x<+\infty$时，$f(x)=\lim\limits_{n\to\infty}f_n(x)=0$

取ε_0使$0<\varepsilon_0<1$，不论n多大，只要取$x=\dfrac{n}{2}\pi$，就有$\left|f\left(\dfrac{n}{2}\pi\right)-f\left(\dfrac{n}{2}\pi\right)\right|=1>\varepsilon_0$

则$f_n(x)$在$(-\infty,+\infty)$上不一致收敛.

(4) 当$0\leqslant x<1$时，$f(x)=\lim\limits_{n\to\infty}f_n(x)=0$；当$x=1$时，$f_n(1)=0,f(1)=0$，则$f(x)=\lim\limits_{n\to\infty}f_n(x)=0$

$||f_n-f||=\sup\limits_{x\in[0,1]}|f_n(x)-f(x)|=\sup\limits_{x\in[0,1]}|x^n(1-x)|=\max\limits_{x\in[0,1]}(x^n-x^{n+1})$

令$(x^n-x^{n+1})'=x^{n-1}[n-(n+1)x]=0$。则得$x=0,x=\dfrac{n}{n+1}$

又$f_n(0)=f_n(1)=0,f_n\left(\dfrac{n}{n+1}\right)=\left(\dfrac{n}{n+1}\right)^n\left(1-\dfrac{n}{n+1}\right)>0$

则$\max\limits_{x\in[0,1]}(x^n-x^{n+1})=\left(\dfrac{n}{n+1}\right)^n\left(1-\dfrac{n}{n+1}\right)\to 0(n\to\infty)$即$||f_n-f||\to 0(n\to\infty)$

于是由定义2，得此函数序列在所示区域内一致收敛于0.

(5) $f(x)=\lim_{n\to\infty}f_n(x)=\begin{cases}1, & 0<x\leqslant 1\\ 0, & x=0\end{cases}$

于是$f(x)$在$[0,1]$上不连续，而$f_n(x)$在$[0,1]$上连续，则$f_n(x)=\dfrac{nx}{1+nx}$在$[0,1]$上不一致收敛.

(6) 因$\lim_{t\to+0}t\ln t=0$，则$f(x)=\lim_{n\to\infty}f_n(x)=0$

$|f_n(x)-f(x)|=\left|\dfrac{x}{n}\ln\dfrac{x}{n}\right|$

对$\forall\varepsilon>0$，因$\lim_{t\to+0}t\ln t=0$，则存在$\delta(\varepsilon)>0$，当$0<t<\delta$时，有$|t\ln t-0|<\varepsilon$

取$N=\left[\dfrac{1}{\delta}\right]$，当$n>N$时，$\dfrac{1}{n}<\delta$

从而对一切$0<x<1$，有$0<\dfrac{x}{n}<\delta$，故$|f_n(x)-f(x)|\leqslant\left|\dfrac{x}{n}\ln\dfrac{x}{n}\right|<\varepsilon$

从而由定义1，得此函数在$(0,1)$内一致收敛于0.

2. 讨论下列级数的一致收敛性：

(1) $\sum\limits_{n=0}^{\infty}(1-x)x^n$，　$0\leqslant x\leqslant 1$

(2) $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^2}{(1+x^2)^n}$，　$-\infty<x<+\infty$

(3) $\sum\limits_{n=1}^{\infty}\dfrac{\sin nx}{\sqrt[3]{n^4+x^4}}$，　$-\infty<x<+\infty$

(4) $\sum\limits_{n=1}^{\infty}\dfrac{x}{1+n^4x^2}$，　$-\infty<x<+\infty$

(5) $\sum\limits_{n=1}^{\infty}\dfrac{\sin nx\sin x}{\sqrt{n+x}}$，　$0\leqslant x\leqslant 2\pi$

(6) $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n(1-e^{-nx})}{n^2+x^2}$，　$0\leqslant x<+\infty$

(7) $\sum\limits_{n=1}^{\infty}2^n\sin\dfrac{1}{3^nx}$，　$0<x<+\infty$

解:

(1) 因部分和$S_n(x)=\sum\limits_{k=0}^{n}(1-x)x^k=1-x^{n+1}$，则$S(x)=\lim_{n\to\infty}S_n(x)=\begin{cases}1, & 0\leqslant x<1\\ 0, & x=1\end{cases}$

于是$S(x)$在$[0,1]$上不连续，而$S_n(x)$在$[0,1]$上连续，则$\sum\limits_{n=0}^{\infty}(1-x)x^n$在$[0,1]$上不一致收敛.

(2) 因此级数为交错级数，且$\dfrac{x^2}{(1+x^2)^{n+1}}\leqslant\dfrac{x^2}{(1+x^2)^n}$，则余式的绝对值不会超过它的首项的绝对值，

即$|r_n(x)|\leqslant\dfrac{x^2}{(1+x^2)^n}=\dfrac{x^2}{1+nx^2+\cdots+x^{2n}}<\dfrac{1}{n}$ $(\forall x\in(-\infty,+\infty))$

从而对$\forall\varepsilon>0,\exists N=\left[\dfrac{1}{\varepsilon}\right]$，当$n>N$时，有$|r_n(x)|<\varepsilon$，则此级数在$(-\infty,+\infty)$上一致收敛.

(3) 当$-\infty<x<+\infty$时，$\left|\dfrac{\sin nx}{\sqrt[3]{n^4+x^4}}\right|\leqslant\dfrac{1}{n^{\frac{4}{3}}}$恒成立，且级数$\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{\frac{4}{3}}}$收敛

则由魏氏判别法，得级数$\sum\limits_{n=1}^{\infty}\dfrac{\sin nx}{\sqrt[3]{n^4+x^4}}$在$(-\infty,+\infty)$上一致收敛.

(4) 因$0\leqslant(1-n^2|x|)^2=1-2n^2|x|+n^4x^2$，则$2n^2|x|\leqslant 1+n^4x^2$即$\dfrac{2n^2|x|}{1+n^4x^2}\leqslant 1$ $(x\in(-\infty,+\infty))$

从而$\left|\dfrac{x}{1+n^4x^2}\right|=\dfrac{2n^2|x|}{2n^2(1+n^4x^2)}\leqslant\dfrac{1}{2n^2}$

又级数$\sum_{n=1}^{\infty}\frac{1}{2n^2}$收敛，则据魏氏判别法，得级数$\sum_{n=1}^{\infty}\frac{x}{1+n^4x^2}$在$(-\infty,+\infty)$上一致收敛.

(5) 当$x=0,2\pi$时，$\sum_{k=1}^{n}\sin kx\sin x=0$

当$x\neq 0,2\pi$时，$\left|\sum_{k=1}^{n}\sin kx\sin x\right|=|\sin x|\left|\sum_{k=1}^{n}\sin kx\right|\leqslant|\sin x|\frac{1}{\left|\sin\frac{x}{2}\right|}=2\left|\cos\frac{x}{2}\right|\leqslant 2$

则当$0\leqslant x\leqslant 2\pi$时，$\left|\sum_{k=1}^{n}\sin kx\sin x\right|\leqslant 2$

又$\frac{1}{\sqrt{n+x}}$对$x\in[0,2\pi]$关于n单调递减且由$\frac{1}{\sqrt{n+x}}\leqslant\frac{1}{\sqrt{n}}$得当$n\to\infty$时，$\frac{1}{\sqrt{n+x}}$关于$x$在$[0,2\pi]$上一致地趋于0(由定义2)

则据狄立克莱判别法，得级数$\sum_{n=1}^{\infty}\frac{\sin nx\sin x}{\sqrt{n+x}}$在$[0,2\pi]$上一致收敛.

(6) 由于对$\forall x\in[0,+\infty)$，有$0\leqslant 1-e^{-nx}<1$，则$\left|\frac{(-1)^n(1-e^{-nx})}{n^2+x^2}\right|\leqslant\frac{1}{n^2}$

又级数$\sum_{n=1}^{\infty}\frac{1}{n^1}$收敛，则据魏氏判别法，得级数$\sum_{n=1}^{\infty}\frac{(-1)^n(1-e^{-nx})}{n^2+x^2}$在$[0,+\infty)$上一致收敛.

(7) 记$u_n(x)=2^n\sin\frac{1}{3^nx}$

当$0<x<+\infty$时，由于$|u_n(x)|\leqslant\frac{1}{x}\left(\frac{2}{3}\right)^n$且$\sum_{n=1}^{\infty}\frac{1}{x}\left(\frac{2}{3}\right)^n$收敛，故原级数绝对收敛，从而收敛，但它在$(0,+\infty)$内并不一致收敛.

如若不然，设它一致收敛，则对任给$\varepsilon>0$，取$\varepsilon=1$，必存在$N=N(\varepsilon)\in Z^+$(它与x无关)，使当$n>N$时，对于$(0,+\infty)$内的一切x，均有$|u_{n+1}(x)+u_{n+2}(x)+\cdots+u_{n+p}(x)|<\varepsilon$，其中$p$为任意正整数

今取$p=1,n=N$，则对一切$x\in(0,+\infty)$，应有$|u_{N+1}(x)|<\varepsilon=1$

又取$x_0=\frac{2}{3^{N+1}\pi}\in(0,+\infty)$，也应有$|u_{N+1}(x_0)|<1$

但事实上却有$u_{N+1}(x_0)=2^{N+1}\sin\frac{1}{3^{N+1}x_0}=2^{N+1}>1$这与$|u_{N+1}(x_0)|<1$矛盾

则假设不成立，即级数$\sum_{n=1}^{\infty}2^n\sin\frac{1}{3^nx}$在$(0,+\infty)$上收敛但非一致收敛.

3. 证明一致收敛定义1和定义2的等价性.

证明：定义1$\Longrightarrow$定义2

已知对任给的$\varepsilon>0$，存在只依赖于ε的正整数$N(\varepsilon)$，使$n>N(\varepsilon)$时，有$|S_n(x)-S(x)|<\frac{\varepsilon}{2}$对一切$x\in X$都成立

于是$||S_n-S||=\sup_{x\in X}|S_n(x)-S(x)|\leqslant\frac{\varepsilon}{2}<\varepsilon$，从而$\lim_{n\to\infty}||S_n-S||=0$.

定义2$\Longrightarrow$定义1

已知$\lim_{n\to\infty}||S_n-S||=0$，即对$\forall\varepsilon>0,\exists N(\varepsilon)\in Z^+$，使当$n>N$时，对一切$x\in X$，都有

$\big|\,||S_n-S||-0\,\big|=\sup_{x\in X}|S_n(x)-S(x)|<\varepsilon$

而$|S_n(x)-S(x)|\leqslant\big|\,||S_n-S||-0\,\big|<\varepsilon$对一切$x\in X$都成立.

(完全类似地可证明函数项级数$\sum_{n=1}^{\infty}u_n$定义1$\Longleftrightarrow$定义2).

4. 试证级数$\sum_{n=1}^{\infty}\frac{\ln(1+nx)}{nx^n}$在任何区间$[1+\alpha,\infty),\alpha>0$为一致收敛.

证明：因当$h>0$时，$\ln(1+h)<h$，则$\left|\frac{\ln(1+nx)}{nx^n}\right|=\frac{\ln(1+nx)}{nx^n}<\frac{nx}{nx^n}=\frac{1}{x^{n-1}}\leqslant\frac{1}{(1+\alpha)^{n-1}}$ $(1+\alpha\leqslant x<+\infty)$

又$\sum_{n=1}^{\infty}\frac{1}{(1+\alpha)^{n-1}}$收敛，则据M判别法，得原级数在$[1+\alpha,+\infty)(\alpha>0)$上一致收敛.

5. 若$\sum\limits_{n=1}^{\infty}u_n(x)$的一般项$|u_n(x)|\leqslant c_n(x)$，并且$\sum\limits_{n=1}^{\infty}c_n(x)$在$X$上一致收敛，则$\sum\limits_{n=1}^{\infty}u_n(x)$在$X$上亦一致收敛且绝对收敛.

证明：因$\sum\limits_{n=1}^{\infty}c_n(x)$在$X$上一致收敛

则由一致收敛的柯西充要条件，得对$\forall\varepsilon>0,\exists N=N(\varepsilon)\in Z^{+}$，使当$n>N$时，对一切$x\in X$和任意的正整数$p$，有$|c_{n+1}(x)+c_{n+2}(x)+\cdots+c_{n+p}(x)|<\varepsilon$

又$\sum\limits_{n=1}^{\infty}u_n(x)$的一般项$|u_n(x)|\leqslant c_n(x)$

则对上述$\varepsilon>0$，正整数$N=N(\varepsilon)$，使当$n>N$时，对一切$x\in X$和上述正整数p，有
$|u_{n+1}(x)+u_{n+2}(x)+\cdots+u_{n+p}(x)|\leqslant\big||u_{n+1}(x)|+|u_{n+2}(x)|+\cdots+|u_{n+p}(x)|\big|\leqslant|c_{n+1}(x)+c_{n+2}(x)+\cdots+c_{n+p}(x)|<\varepsilon$

由一致收敛的柯西充要条件，得$\sum\limits_{n=1}^{\infty}u_n(x)$在$X$上一致收敛且绝对收敛.

6. 设$f_0(x)$在$[0,a]$上连续，又$f_n(x)=\int_0^x f_{n-1}(t)\,\mathrm{d}t$，证明$\{f_n(x)\}$在$[0,a]$上一致收敛于零.

证明：因$f_0(x)$在$[0,a]$上连续，则其有界，即存在$M>0$，有$|f_0(x)|\leqslant M$

又$f_n(x)=\int_0^x f_{n-1}(t)\,\mathrm{d}t$，则

$$|f_1(x)=\left|\int_0^x f_0(t)\,\mathrm{d}t\right|\leqslant\int_0^x|f_0(t)|\,\mathrm{d}t\leqslant\int_0^x M\,\mathrm{d}t=Mx\leqslant Ma$$

$$|f_2(x)=\left|\int_0^x f_1(t)\,\mathrm{d}t\right|\leqslant\int_0^x|f_1(t)|\,\mathrm{d}t\leqslant\int_0^x Mt\,\mathrm{d}t=\frac{Mx^2}{2}\leqslant\frac{Ma^2}{2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|f_n(x)=\left|\int_0^x f_{n-1}(t)\,\mathrm{d}t\right|\leqslant\int_0^x|f_{n-1}(t)|\,\mathrm{d}t\leqslant\int_0^x M\frac{t^{n-1}}{(n-1)!}\,\mathrm{d}t=M\frac{x^n}{n!}\leqslant M\frac{a^n}{n!}$$

因$\lim\limits_{n\to\infty}\dfrac{a^n}{n!}=0$即对$\forall\varepsilon>0,\exists N\in Z^{+}$，当$n>N$时，有$\left|\dfrac{a^n}{n!}-0\right|=\dfrac{a^n}{n!}<\varepsilon$

于是$|f_n(x)-0|<M\varepsilon$对一切$x\in[0,a]$均成立

从而由定义1，得$\{f_n(x)\}$在$[0,a]$上一致收敛于零.

7. 证明级数$\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{1}{n+x^2}$关于$x$在$(-\infty,+\infty)$上为一致收敛，但对任何$x$并非绝对收敛，而级数$\sum\limits_{n=1}^{\infty}\dfrac{x^2}{(1+x^2)^n}$虽在$x\in(-\infty,+\infty)$上绝对收敛，但并不一致收敛.

证明：因$\left|\sum\limits_{k=1}^{n}(-1)^{k-1}\right|\leqslant1$即$\sum\limits_{k=1}^{n}(-1)^{k-1}$在$(-\infty,+\infty)$上一致有界

又$\dfrac{1}{n+1+x^2}<\dfrac{1}{n+x^2}$，则函数列$\left\{\dfrac{1}{n+x^2}\right\}$对于$x\in(-\infty,+\infty)$单调减

又对$\forall\varepsilon>0$，取$N=\left[\dfrac{1}{\varepsilon}\right]$，则当$n>N$时，对一切$x\in(-\infty,+\infty)$，都有$\left|\dfrac{1}{n+x^2}\ -0\right|=\dfrac{1}{n+x^2}\leqslant\dfrac{1}{n}<\varepsilon$

则$\left\{\dfrac{1}{n+x^2}\right\}$关于$x\in(-\infty,+\infty)$一致收敛于0，于是由狄立克莱判别法，得$\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{1}{n+x^2}$在$(-\infty,+\infty)$内一致收敛.

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n+x^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n+x^2}$$

因$\lim\limits_{n\to\infty}\dfrac{\frac{1}{n+x^2}}{\frac{1}{n}}=1$且$\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$发散，则由比较判别法的极限形式，得$\sum\limits_{n=1}^{\infty}\dfrac{1}{n+x^2}$发散，于是对任何$x$级数非绝对收敛.

$$\sum_{n=1}^{\infty}\left|\frac{x^2}{(1+x^2)^n}\right|=\sum_{n=1}^{\infty}\frac{x^2}{(1+x^2)^n}$$

对固定的$x\in(-\infty,+\infty)$，因$\overline{\lim\limits_{n\to\infty}}\sqrt[n]{\dfrac{x^2}{(1+x^2)^n}}=\begin{cases}\dfrac{1}{1+x^2}, & x\neq0\\ 0, & x=0\end{cases}$

由柯西判别法，得$\sum\limits_{n=1}^{\infty}\dfrac{x^2}{(1+x^2)^n}$在$(-\infty,+\infty)$收敛，于是绝对收敛.

当$x \neq 0$时，$S_n(x)=\sum\limits_{k=1}^{n}\frac{x^2}{(1+x^2)^k}=1-\frac{1}{(1+x^2)^n}$，$S(x)=\lim\limits_{n\to\infty}S_n(x)=1$

当$x=0$时，$S_n(0)=0, S(0)=0$，则$S(x)=\begin{cases}1, & x\neq 0\\ 0, & x=0\end{cases}$

因$S_n(x)$在$(-\infty,+\infty)$上连续，而$S(x)$在$(-\infty,+\infty)$上不连续，则$\sum\limits_{n=1}^{\infty}\frac{x^2}{(1+x^2)^n}$在$(-\infty,+\infty)$内不一致收敛.

8. 证明：

(1) 如果$\sum|f_n(x)|$在$[a,b]$上一致收敛，那末$\sum\limits_1^{\infty}f_n(x)$在$[a,b]$上也一致收敛；

(2) 如果$\sum f_n(x)$在$[a,b]$上一致收敛，但$\sum|f_n(x)|$未必一致收敛，以$\sum\limits_1^{\infty}(-1)^n(x^n-x^{n+1}), 0\leqslant x\leqslant 1$为例来说明.

证明:

(1) 由柯西准则及题设，得
对$\forall\varepsilon>0, \exists N=N(\varepsilon)\in Z^+$，使当$n>N$时，对一切$x\in[a,b]$和任意$p\in Z^=$，有
$|f_{n+1}(x)|+|f_{n+2}(x)|+\cdots+|f_{n+p}(x)|<\varepsilon$
从而$|f_{n+1}(x)+f_{n+2}(x)+\cdots+f_{n+p}(x)|\leqslant|f_{n+1}(x)|+|f_{n+2}(x)|+\cdots+|f_{n+p}(x)|<\varepsilon$
则据一致收敛的柯西准则，得$\sum\limits_1^{\infty}f_n(x)$在$[a,b]$上一致收敛.

(2) 例：$\sum\limits_1^{\infty}(-1)^n(x^n-x^{n+1})$在$[0,1]$上一致收敛
因$x^n-x^{n+1}=0$(当$x=0,1$时)；当$0<x<1$时，$x^n-x^{n+1}=x^n(1-x)$，则x^n-x^{n+1}在$[0,1]$上关于n单调减少
由1.(4)，得$x^n-x^{n+1}=x^n(1-x)$在$[0,1]$上一致收敛于0，则由狄立克莱判别法，得
$\sum\limits_1^{\infty}(-1)^n(x^n-x^{n+1})$ 在$[0,1]$上一致收敛
但$\sum\limits_1^{\infty}\left|(-1)^n(x^n-x^{n+1})\right|=\sum\limits_1^{\infty}(x^n-x^{n+1})$在$[0,1]$上非一致收敛(由2.(1)得).

9. 设每一项$\varphi_n(x)$都是$[a,b]$上的单调函数，如果$\sum\varphi_n(x)$在$[a,b]$的端点为绝对收敛，那末这级数在$[a,b]$上一致收敛.

证明: 因$\varphi_n(x)$在$[a,b]$上单调，故有$|\varphi_n(x)|\leqslant|\varphi_n(a)|+|\varphi_n(b)|$ $(\forall x\in[a,b])$
由于$\sum|\varphi_n(a)|$和$\sum|\varphi_n(b)|$收敛，则$\sum(|\varphi_n(a)|+|\varphi_n(b)|)$收敛
则据M判别法，得级数$\sum\varphi_n(x)$在$[a,b]$上一致收敛.

10. 下列函数列是否一致收敛？

(1) $f_n(x)=(\sin x)^n, \qquad 0\leqslant x\leqslant\pi$

(2) $f_n(x)=(\sin x)^{\frac{1}{n}}$

(i) $0\leqslant x\leqslant\pi$

(ii) $\delta\leqslant x\leqslant\pi-\delta$

(3) $f_n(x)=\frac{x^n}{1+x^n}$

(i) $0\leqslant x\leqslant 1-\varepsilon$

(ii) $1-\varepsilon<x<1+\varepsilon \quad (\varepsilon>0)$

(iii) $1+\varepsilon\leqslant x<\infty$

解:

(1) $f(x)=\lim\limits_{n\to\infty}f_n(x)=\begin{cases}0, & 0\leqslant x\leqslant\pi\text{且}x\neq\frac{\pi}{2}\\ 1, & x=\frac{\pi}{2}\end{cases}$
因$f(x)$在$[0,\pi]$上不连续，但$f_n(x)$在$[0,\pi]$上连续，则$f_n(x)=(\sin x)^n$在$[0,\pi]$上不一致收敛.

(2) (i) $f(x)=\lim\limits_{n\to\infty}f_n(x)=\begin{cases}0, & x=0,\pi\\ 1, & 0<x<\pi\end{cases}$

因$f(x)$在$[0,\pi]$上不连续，但$f_n(x)$在$[0,\pi]$上连续，则$f_n(x)=(\sin x)^{\frac{1}{n}}$在$[0,\pi]$上不一致收敛.

(ii) 因$f(x)=\lim\limits_{n\to\infty}f_n(x)=1, |f_n(x)-f(x)|=1-(\sin x)^{\frac{1}{n}}\leqslant 1-(\sin\delta)^{\frac{1}{n}}$

即$||f_n-f||=\sup\limits_{x\in[\delta,\pi-\delta]}|f_n(x)-f(x)|=1-(\sin\delta)^{\frac{1}{n}}\to 0(n\to\infty)$

则由定义2，得$f_n(x)=(\sin x)^{\frac{1}{n}}$在$[\delta,\pi-\delta]$上一致收敛于1.

(3) (i) 当$0\leqslant x\leqslant 1-\varepsilon$时，$f(x)=\lim\limits_{n\to\infty}f_n(x)=0$，则$|f_n(x)-f(x)|=\dfrac{x^n}{1+x^n}\leqslant x^n\leqslant(1-\varepsilon)^n$

于是$||f_n-f||=\sup\limits_{x\in[0,1-\varepsilon]}|f_n(x)-f(x)|=(1-\varepsilon)^n\to 0\ (n\to\infty)$

则由定义2，得$f_n(x)=\dfrac{x^n}{1+x^n}$在$[0,1-\varepsilon]$上一致收敛于0.

(ii) $f(x)=\lim\limits_{n\to\infty}f_n(x)=\begin{cases}0, & 1-\varepsilon<x<1\\ \dfrac{1}{2}, & x=1\\ 1, & 1<x<1+\varepsilon\end{cases}$

因$f(x)$在$(1-\varepsilon,1+\varepsilon)$上不连续，而$f_n(x)$在$(1-\varepsilon,1+\varepsilon)$上连续，则$f_n(x)=\dfrac{x^n}{1+x^n}$在$(1-\varepsilon,1+\varepsilon)$上不一致收敛.

(iii) 当$1+\varepsilon\leqslant x<+\infty$时，$f(x)=\lim\limits_{n\to\infty}f_n(x)=1$，则$|f_n(x)-f(x)|=\dfrac{1}{1+x^n}\leqslant\dfrac{1}{1+(1+\varepsilon)^n}$

于是$||f_n-f||=\sup\limits_{x\in[1+\varepsilon,+\infty)}|f_n(x)-f(x)|=\dfrac{1}{1+(1+\varepsilon)^n}\to 0\ (n\to\infty)$

从而由定义2，得$f_n(x)=\dfrac{x^n}{1+x^n}$在$[1+\varepsilon,+\infty)$上一致收敛于1.

11. 证明$\sum\limits_{1}^{\infty}ne^{-nx}$在$(0,+\infty)$内连续.

证明：任取$x_0\in(0,+\infty)$，则存在$\alpha,\beta>0$，使$\alpha<x_0<\beta$，在$[\alpha,\beta]$上$0<ne^{-nx}\leqslant ne^{-n\alpha}$

因$\alpha>0$，则$e^\alpha>1$，于是$\lim\limits_{n\to\infty}\dfrac{(n+1)e^{-(n+1)\alpha}}{ne^{-n\alpha}}=\dfrac{1}{e^\alpha}<1$，则由达朗贝尔判别法的极限形式，得级数$\sum\limits_{1}^{\infty}ne^{-n\alpha}$收敛，从而据M判别法，得$\sum\limits_{1}^{\infty}ne^{-nx}$在$[\alpha,\beta]$上一致收敛.

又ne^{-nx}在$[\alpha,\beta]$上连续，从而$\sum\limits_{1}^{\infty}ne^{-nx}$在$[\alpha,\beta]$上连续

因$x_0\in[\alpha,\beta]$，则$\sum\limits_{1}^{\infty}ne^{-nx}$在$x_0$点连续

由于x_0是$(0,+\infty)$的任意点，故$\sum\limits_{1}^{\infty}ne^{-nx}$在$(0,+\infty)$内连续.

12. 证明函数$f(x)=\sum\limits_{1}^{\infty}\dfrac{\sin nx}{n^3}$在$(-\infty,+\infty)$内连续，并有连续导函数.

证明：因$\left|\dfrac{\sin nx}{n^3}\right|\leqslant\dfrac{1}{n^3}$且$\sum\limits_{1}^{\infty}\dfrac{1}{n^3}$收敛，则据M判别法，得$f(x)=\sum\limits_{1}^{\infty}\dfrac{\sin nx}{n^3}$在$(-\infty,+\infty)$一致收敛

又$\dfrac{\sin nx}{n^3}$在$(-\infty,+\infty)$内连续，则$f(x)=\sum\limits_{1}^{\infty}\dfrac{\sin nx}{n^3}$在$(-\infty,+\infty)$内连续

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin nx}{n^3}\right)=\frac{\cos nx}{n^2}$$

因$\left|\dfrac{\cos nx}{n^2}\right|\leqslant\dfrac{1}{n^2}$且$\sum\limits_{1}^{\infty}\dfrac{1}{n^2}$收敛，则据M判别法，得$\sum\limits_{1}^{\infty}\dfrac{\cos nx}{n^2}$在$(-\infty,+\infty)$一致收敛

于是$f'(x)=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sum\limits_{1}^{\infty}\dfrac{\sin nx}{n^3}\right)=\sum\limits_{1}^{\infty}\dfrac{\cos nx}{n^2}$

又$\dfrac{\cos nx}{n^2}$在$(-\infty,+\infty)$内连续，则$\sum\limits_{1}^{\infty}\dfrac{\cos nx}{n^2}$在$(-\infty,+\infty)$内连续

即$f'(x)$在$(-\infty,+\infty)$内连续且$f'(x)=\sum\limits_{1}^{\infty}\dfrac{\cos nx}{n^2}$.

13. 证明函数$\zeta(x)=\sum\limits_{1}^{\infty}\dfrac{1}{n^x}$在$(1,+\infty)$连续，并有连续各阶导函数.

证明: 各项求导数所得级数为$-\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^x}$.下证它在$1<a\leqslant x<+\infty$上一致连续(a为大于1的任何数)

当$a\leqslant x<+\infty$时，有$0<\dfrac{\ln n}{n^x}\leqslant\dfrac{\ln n}{n^a}$

由于$\lim\limits_{n\to\infty}\dfrac{\frac{\ln n}{n^a}}{\frac{1}{n^{(a+1)/2}}}=0$且$\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{(a+1)/2}}$收敛

则级数$\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^a}$收敛，于是由M判别法，得级数$\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^x}$在$a\leqslant x<+\infty$上一致收敛

注意到每项$\dfrac{\ln n}{n^x}$都是x的连续函数，则级数$\sum\limits_{n=1}^{\infty}\dfrac{1}{n^x}$在$a\leqslant x<+\infty$上可逐项求导数，得$\zeta'(x)=-\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^x}$

且$\zeta'(x)$在$a\leqslant x<+\infty$上连续

由$a>1$的任意性，得$\zeta'(x)=-\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^x}$对一切$1<x<+\infty$成立且$\zeta'(x)$在$1<x<+\infty$上连续，当然$\zeta(x)$更在$1<x<+\infty$上连续

利用数学归纳法，并注意到对任何正整数k，级数$\sum\limits_{n=1}^{\infty}\dfrac{(\ln n)^k}{n^a}$ $(a>1)$都收敛，仿照上述，可证：对任何正整数k，$\zeta^{(k)}(x)$在$1<x<+\infty$上都存在且连续，且可由原级数逐项求导数k次，得

$\zeta^{(k)}(x)=(-1)^k\sum\limits_{n=1}^{\infty}\dfrac{(\ln n)^k}{n^x}$ $(1<x<+\infty)$.

14. 试证级数$\sum\limits_{1}^{\infty}\dfrac{\sin(2^n\pi x)}{2^n}$在整个实数轴上一致收敛，但在任何区间内不能逐项求微商.

证明: 因$\left|\dfrac{\sin(2^n\pi x)}{2^n}\right|\leqslant\dfrac{1}{2^n}$对$\forall x\in(-\infty<+\infty)$皆成立且级数$\sum\limits_{1}^{\infty}\dfrac{1}{2^n}$收敛，则据M判别法，得$\sum\limits_{1}^{\infty}\dfrac{\sin(2^n\pi x)}{2^n}$在整个实数轴上一致收敛

$\left(\dfrac{\sin(2^n\pi x)}{2^n}\right)'=\pi\cos(2^n\pi x)$

下证$\sum\limits_{1}^{\infty}\pi\cos(2^n\pi x)$在任何区间内都有不连续点

任取$x\in(-\infty,+\infty)$，总存在$k\in Z$，使$x=k+y$，其中$0\leqslant y<1$

将其代入，得$\sum\limits_{1}^{\infty}\cos(2^n\pi x)=\sum\limits_{1}^{\infty}\cos(2^n\pi y)$，特别的，取$y=2^{-m}h$，其中$m\in Z^+,h=0,1,2,\cdots,2^m-1$

当$n>m$时，$\cos(2^n\pi y)=1$，此时级数一般项不趋于0，则$\sum\limits_{1}^{\infty}\cos(2^n\pi x)=\sum\limits_{1}^{\infty}\cos(2^n\pi y)$发散，于是$\sum\limits_{1}^{\infty}\pi\cos(2^n\pi x)$发散

又在任何区间内都存在$x=k+2^{-m}h(h=0,1,2,\cdots,2^m-1)$这样的点，$k$为$x$的最小整数部分

则级数$\sum\limits_{1}^{\infty}\dfrac{\sin(2^n\pi x)}{2^n}$在任何区间内不能逐项求微商.

15. 先证

$$\frac{1-r^2}{1-2r\cos x+r^2}=1+2\sum_{n=1}^{\infty}r^n\cos nx$$

当$|r|<1$时成立，从而证明：

$$\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos x+r^2}\ \mathrm{d}x=2\pi\ (|r|<1)$$

.

证明: $|r^n\cos nx|\leqslant|r|^n$对$\forall x\in(-\infty,+\infty)$都成立

因$|r|<1$，则$\sum\limits_{n=1}^{\infty}|r|^n$收敛，于是由M判别法，得$\sum\limits_{n=1}^{\infty}r^n\cos nx$在$(-\infty,+\infty)$内一致收敛

从而设$f(x)=1+2\sum_{n=1}^{\infty}r^n\cos nx$

因$1-2r\cos x+r^2\neq 0$，上式两端同乘以$1-2r\cos x+r^2$，则得

$$(1-2r\cos x+r^2)f(x)=(1-2r\cos x+r^2)\left(1+2\sum_{n=1}^{\infty}r^n\cos nx\right)$$

$$=\left[1-2r\cos x+r^2+2\sum_{n=1}^{\infty}r^n\cos nx-2\sum_{n=1}^{\infty}r^{n+1}(2\cos nx\cos x)+2\sum_{n=1}^{\infty}r^{n+2}\cos nx\right]$$

$$=\left[1-2r\cos x+r^2+2\sum_{n=1}^{\infty}r^n\cos nx-2\sum_{n=1}^{\infty}r^{n+1}\cos(n+1)x-2\sum_{n=1}^{\infty}r^{n+1}\cos(n-1)x+2\sum_{n=1}^{\infty}r^{n+2}\cos nx\right]$$

$$=\left[1-r^2+2\sum_{n=1}^{\infty}r^n\cos nx-2\left(\sum_{n=1}^{\infty}r^{n+1}\cos(n+1)x+r\cos x\right)-2\left(\sum_{n=1}^{\infty}r^{n+1}\cos(n-1)x-r^2\right)+2\sum_{n=1}^{\infty}r^{n+2}\cos nx\right]$$

$$=1-r^2$$

于是$f(x)=\dfrac{1-r^2}{1-2r\cos x+r^2}$ 即$\dfrac{1-r^2}{1-2r\cos x+r^2}=1+2\sum_{n=1}^{\infty}r^n\cos nx$

由于上式右端级数在$[-\pi,\pi]$上一致收敛，且$r^n\cos nx$在$[-\pi,\pi]$上连续，则上式级数可以逐项积分，得

$$\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos x+r^2}\,\mathrm{d}x=\int_{-\pi}^{\pi}\left(1+2\sum_{n=1}^{\infty}r^n\cos nx\right)\mathrm{d}x=2\pi+2\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}r^n\cos nx\,\mathrm{d}x=2\pi.$$

16. 用有限覆盖定理证明狄尼定理.

证明：因$\{S_n(x)\}$在$[a,b]$上收敛于$S(x)$，故对$\forall\varepsilon>0,\forall x\in[a,b],\exists N(\varepsilon,x)\in Z^+$，使得当$n\geqslant N(\varepsilon,x)$时，都应有$|S_n(x)-S(x)|<\varepsilon$，特别有$|S_{N(\varepsilon,x)}-S(x)|<\varepsilon$

由$S_{N(\varepsilon,x)}(x)-S(x)$在$x$点连续，得存在$x$点的开邻域$O_x$，使得$|S_{N(\varepsilon,x)}(y)-S(y)|,\forall y\in O_x$

于是$\{O_x\big|x\in[a,b]\}$构成$[a,b]$的开覆盖(对端点a,b可作连续延拓)

据有限覆盖定理，从中选出有限个开邻域$O_{x_1},\cdots,O_{x_m}$同样覆盖$[a,b]$且满足$|S_{N(\varepsilon,x_i)}(y)-S(y)|<\varepsilon,\forall y\in O_{x_i},i=1,2,\cdots,m$

取$N=\max\limits_{i\leqslant i\leqslant m}N(\varepsilon,x_i)$，则当$n>N$时，对$\forall x\in[a,b]$，由$\{S_n(x)\}$单调性和$\bigcup\limits_{i=1}^{m}O_{x_i}\supset[a,b]$，必存在某个$O_{x_i}$，使$x\in O_{x_i}$，且有$|S_n(x)-S(x)|\leqslant|S_N(x)-S(x)|\leqslant|S_N(\varepsilon,x_i)(x)-S(x)|<\varepsilon$

即$\{S_n(x)\}$在$[a,b]$上一致收敛于$S(x)$.

17. 若$S_n(x)$在c点左连续$(n=1,2,3,\cdots)$，但$\{S_n(c)\}$发散，则在任何开区间$(c-\delta,c)$内$(\delta>0)$，$\{S_n(x)\}$必不一致收敛.

证明：用反证法.

假设存在$\delta_0>0$，使得$\{S_n(x)\}$在$(c-\delta_0,c)$内一致收敛

由一致收敛的柯西原理，得对$\forall\varepsilon>0,\exists N(\varepsilon)\in Z^+$，使得当$n>N(\varepsilon)$时，对$\forall x\in(c-\delta_0,c)$和$\forall p\in Z^+$，都应有$|S_{n+p}(x)-S_n(x)|<\dfrac{\varepsilon}{2}$ $(*)$成立

因每一个$S_n(x)$在c点左连续，则$S_{n+p}(x)-S_n(x)$也在c点左连续

于是$\lim\limits_{x\to c-0}[S_{n+p}(x)-S_n(x)]=S_{n+p}(c)-S_n(c)$

在$(*)$式两端令$x\to c-0$，得$|S_{n+p}(c)-S_n(c)|\leqslant\dfrac{\varepsilon}{2}<\varepsilon$

由数列的柯西收敛原理，得$\{S_n(c)\}$收敛，与已知$\{S_n(c)\}$发散矛盾

故假设不正确，则在任何开区间$(c-\delta,c)$内$(\delta>0)$，$\{S_n(x)\}$必不一致收敛.

§2. 幂级数

1. 求下列各幂级数的收敛区间：

(1) $\sum\limits_{n=1}^{\infty}\dfrac{(2x)^n}{n!}$

(2) $\sum\limits_{n=1}^{\infty}\dfrac{\ln(n+1)}{n+1}\,x^{n+1}$

(3) $\sum\limits_{n=1}^{\infty}\left[\left(\dfrac{n+1}{n}\right)^n x\right]^n$

(4) $\sum\limits_{n=1}^{\infty}\dfrac{x^{n^2}}{2^n}$

(5) $\sum\limits_{n=1}^{\infty}\dfrac{[3+(-1)^n]^n}{n}\,x^n$

(6) $\sum\limits_{n=1}^{\infty}\dfrac{3^n+(-2)^n}{n}\,(x+1)^n$

解：

(1) $a_n=\dfrac{2^n}{n!}$

因$R=\lim\limits_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right|=+\infty$，则其收敛域为$(-\infty,+\infty)$.

(2) $\sum\limits_{n=1}^{\infty}\dfrac{\ln(n+1)}{n+1}\,x^{n+1}=\sum\limits_{n=2}^{\infty}\dfrac{\ln n}{n}\,x^n$ ， $a_n=\dfrac{\ln n}{n}$

由于$\lim\limits_{y\to+\infty}\dfrac{(y+1)\ln y}{y\ln(y+1)}=\lim\limits_{y\to+\infty}\dfrac{y+1}{y}\lim\limits_{y\to+\infty}\dfrac{\ln y}{\ln(y+1)}=1$，则$R=\lim\limits_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right|=1$，于是其收敛区间为$(-1,1)$

当$x=-1$时，原级数为$\sum\limits_{n=2}^{\infty}(-1)^n\dfrac{\ln n}{n}\,x^n$

因$\left(\dfrac{\ln x}{x}\right)'=\dfrac{1-\ln x}{x^2}$且当$x\geqslant 3$时，$\left(\dfrac{\ln x}{x}\right)'<0$，则$\left\{\dfrac{\ln n}{n}\right\}$单调减少

又$\lim\limits_{n\to\infty}\dfrac{\ln n}{n}=0$，则级数$\sum\limits_{n=2}^{\infty}(-1)^n\dfrac{\ln n}{n}\,x^n$ 为莱布尼兹级数，于是级数$\sum\limits_{n=2}^{\infty}(-1)^n\dfrac{\ln n}{n}\,x^n$ 收敛

当$x=1$时，原级数为$\sum\limits_{n=2}^{\infty}\dfrac{\ln n}{n}\,x^n$

因$\lim\limits_{n\to\infty}\dfrac{\frac{\ln n}{n}}{n}=+\infty$，则据正项级数的比较判别法及级数$\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$发散，得级数$\sum\limits_{n=2}^{\infty}\dfrac{\ln n}{n}\,x^n$ 发散

则此级数的收敛域为$[-1,1)$.

(3) 因$\sum\limits_{n=1}^{\infty}\left[\left(\dfrac{n+1}{n}\right)^n x\right]^n=\sum\limits_{n=1}^{\infty}\left(1+\dfrac{1}{n}\right)^{n^2}x^n$，则$a_n=\left(1+\dfrac{1}{n}\right)^{n^2}$

又$\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=e$，则其收敛半径为$R=\dfrac{1}{e}$，收敛区间为$\left(-\dfrac{1}{e},\dfrac{1}{e}\right)$.

当$x=\pm\dfrac{1}{e}$时，原级数为$\sum\limits_{n=1}^{\infty}(\pm1)^n\left(1+\dfrac{1}{n}\right)^{n^2}\left(\dfrac{1}{e}\right)^n$，则$u_n=(\pm1)^n\left(1+\dfrac{1}{n}\right)^{n^2}\left(\dfrac{1}{e}\right)^n$

由洛必达法则，得$\lim\limits_{n\to\infty}|u_n|=e^{-\frac{1}{2}}\neq 0$

则级数$\sum\limits_{n=1}^{\infty}(\pm1)^n\left(1+\dfrac{1}{n}\right)^{n^2}\left(\dfrac{1}{e}\right)^n$发散，于是原级数的收敛域为$\left(-\dfrac{1}{e},\dfrac{1}{e}\right)$.

(4) $a_n = \dfrac{1}{2^n}$

由 $\overline{\lim\limits_{n\to\infty}} \sqrt[n^2]{|a_n|x^{n^2}} = |x| \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{1}{2}} = |x| < 1$，得其收敛半径为$R=1$，收敛区间为$(-1,1)$

当$|x|=1$即$x=\pm 1$时，原级数变为$\sum\limits_{n=1}^{\infty} \dfrac{(\pm 1)^{n^2}}{2^n}$

由于级数$\sum\limits_{n=1}^{\infty} \left|\dfrac{(\pm 1)^{n^2}}{2^n}\right| = \sum\limits_{n=1}^{\infty} \dfrac{1}{2^n}$ 收敛，则级数$\sum\limits_{n=1}^{\infty} \dfrac{(\pm 1)^{n^2}}{2^n}$绝对收敛则收敛

从而幂级数$\sum\limits_{n=1}^{\infty} \dfrac{x^{n^2}}{2^n}$ 的收敛域为$[-1,1]$.

(5) $a_n = \dfrac{[3+(-1)^n]^n}{n}$

因 $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\dfrac{[3+(-1)^n]^n}{n}} = 4$，则级数收敛半径为$R = \dfrac{1}{4}$，收敛区间为$\left(-\dfrac{1}{4}, \dfrac{1}{4}\right)$

当$x = \dfrac{1}{4}$ 时，原级数变为$\sum\limits_{n=1}^{\infty} \dfrac{[3+(-1)^n]^n}{n\cdot 4^n} = \sum\limits_{k=1}^{\infty} \dfrac{1}{2k} + \sum\limits_{k=1}^{\infty} \dfrac{1}{(2k+1)2^{2k+1}}$

对级数$\sum\limits_{k=1}^{\infty} \dfrac{1}{(2k+1)2^{2k+1}}$

因$\lim\limits_{k\to\infty} \dfrac{\frac{1}{(2k+3)2^{2k+3}}}{\dfrac{1}{(2k+1)2^{2k+1}}} = \dfrac{1}{4} < 1$ ，则据达朗贝尔判别法，得级数$\sum\limits_{k=1}^{\infty} \dfrac{1}{(2k+1)2^{2k+1}}$ 收敛

又级数$\sum\limits_{k=1}^{\infty} \dfrac{1}{2k}$ 发散，则级数$\sum\limits_{n=1}^{\infty} \dfrac{[3+(-1)^n]^n}{n\cdot 4^n}$ 发散

同法可得，当$x = -\dfrac{1}{4}$ 时，级数$\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{[3+(-1)^n]^n}{n\cdot 4^n}$ 发散

则级数$\sum\limits_{n=1}^{\infty} \dfrac{[3+(-1)^n]^n}{n} x^n$的收敛域为$\left(-\dfrac{1}{4}, \dfrac{1}{4}\right)$.

(6) $a_n = \dfrac{3^n + (-2)^n}{n}$

因$\lim\limits_{n\to\infty} \left|\dfrac{a_n}{a_{n+1}}\right| = \dfrac{1}{3}$ ，则级数的收敛半径为$R = \dfrac{1}{3}$ ，收敛区间为$\left(-\dfrac{4}{3}, -\dfrac{2}{3}\right)$

当$x = -\dfrac{4}{3}$ 时，原级数变为$\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{3^n + (-2)^n}{n} \left(\dfrac{1}{3}\right)^n = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n} + \sum\limits_{n=1}^{\infty} \dfrac{\left(\frac{2}{3}\right)^n}{n}$

对级数$\sum\limits_{n=1}^{\infty} \dfrac{\left(\frac{2}{3}\right)^n}{n}$

因$\lim\limits_{n\to\infty} \dfrac{\left(\frac{2}{3}\right)^{n+1}/(n+1)}{\left(\frac{2}{3}\right)^n/n} = \dfrac{2}{3} < 1$ ，则据达朗贝尔判别法，得$\sum\limits_{n=1}^{\infty} \dfrac{\left(\frac{2}{3}\right)^n}{n}$ 收敛

又级数$\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ 收敛，则当$x = -\dfrac{4}{3}$ 时，原级数收敛；

同法可得，当$x = -\dfrac{2}{3}$ 时，原级数发散

则级数$\sum\limits_{n=1}^{\infty} \dfrac{3^n + (-2)^n}{n} (x+1)^n$的收敛域为$\left[-\dfrac{4}{3}, -\dfrac{2}{3}\right)$.

2. 求级数的收敛半径：

(1) $\sum\limits_{n=1}^{\infty} \left(1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}\right) x^n$

(2) $\sum \dfrac{(2n)!}{(n!)^2} x^n$

解：

(1) $a_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$

因$1=\sqrt[n]{n\cdot\frac{1}{n}}\leqslant\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}\leqslant\sqrt[n]{n\cdot 1}\to 1(n\to\infty)$

则$\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=1$，于是其收敛半径为$R=1$.

(2) $a_n=\frac{(2n)!}{(n!)^2}$

因$\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim\limits_{n\to\infty}\frac{(n+1)^2}{2(n+1)(2n+1)}=\frac{1}{4}$，于是其收敛半径为$R=\frac{1}{4}$.

3. $\sum\limits_{n=1}^{\infty}\frac{3^n+(-2)^n}{n}(x+1)^n$ 设幂级数$\sum a_nx^n$的收敛半径为R，$\sum b_nx^n$的收敛半径为Q，讨论下列级数的收敛半径：

(1) $\sum a_nx^{2n}$

(2) $\sum(a_n+b_n)x^n$

(3) $\sum a_nb_nx^n$

解：

(1) $\overline{\lim\limits_{n\to\infty}}\sqrt[2n]{|a_n|}=\overline{\lim\limits_{n\to\infty}}\left(\sqrt[n]{|a_n|}\right)^{\frac{1}{2}}=\sqrt{\frac{1}{R}}=\frac{1}{\sqrt{R}}$，则其收敛半径为$R_1=\sqrt{R}$.

(2) 设$A_n=a_n+b_n$

则有$\sqrt[n]{|A_n|}=\sqrt[n]{|a_n+b_n|}\leqslant\sqrt[n]{|a_n|+|b_n|}\leqslant\sqrt[n]{2\max(|a_n|,|b_n|)}=\sqrt[n]{2}\cdot\sqrt[n]{\max(|a_n,b_n|)}=\sqrt[n]{2}\max(\sqrt[n]{|a_n|},\sqrt[n]{|b_n|})$

因$\lim\limits_{n\to\infty}\sqrt[n]{2}=1$

则$\frac{1}{R_2}=\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|A_n|}\leqslant\overline{\lim\limits_{n\to\infty}}\{\sqrt[n]{2}\max(\sqrt[n]{|a_n|},\sqrt[n]{|b_n|})\}=\overline{\lim\limits_{n\to\infty}}\{\max(\sqrt[n]{|a_n|},\sqrt[n]{|b_n|})\}=\max\left(\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|},\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}\right)=\max\left(\frac{1}{R},\frac{1}{Q}\right)$

从而，得$R_2\geqslant\frac{1}{\max\left(\frac{1}{R},\frac{1}{Q}\right)}=\min(R,Q)$.

(3) 设$B_n=a_nb_n$

则有$\sqrt[n]{|B_n|}=\sqrt[n]{|a_nb_n|}=\sqrt[n]{|a_n|}\cdot\sqrt[n]{|b_n|}$

于是$\frac{1}{R_3}=\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|B_n|}=\overline{\lim\limits_{n\to\infty}}\{\sqrt[n]{|a_n|}\cdot\sqrt[n]{|b_n|}\leqslant\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}\cdot\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|b_n|}\}=\frac{1}{R}\cdot\frac{1}{Q}=\frac{1}{RQ}$

从而$R_3\geqslant RQ$.

4. 设对充分大的n，$|a_n|\leqslant|b_n|$，那末级数$\sum a_nx^n$的收敛半径不小于$\sum b_nx^n$的收敛半径.

证明：因对充分大的n，$|a_n|\leqslant|b_n|$，则$\sqrt[n]{|a_n|}\leqslant\sqrt[n]{|b_n|}$，于是$\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}\leqslant\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|b_n|}$

设级数$\sum a_nx^n$的收敛半径为R，级数$\sum b_nx^n$的收敛半径为Q

则当$0<\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}\leqslant\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|b_n|}<\infty$时，由$R=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}},Q=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|b_n|}}$，得$R\geqslant Q$；

当$0=\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}\leqslant\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|b_n|}$，则$R=\infty,Q\leqslant\infty$，于是$R\geqslant Q$；

当$\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}\leqslant\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|b_n|}=\infty$时，则$R\geqslant 0,Q=0$，于是$R\geqslant Q$

综上知，级数$\sum a_nx^n$的收敛半径不小于$\sum b_nx^n$的收敛半径.

5. 证明幂级数的性质1和性质2.

证明：性质1.

设x为(x_0-R,x_0+R)内任一点，总可以选取$0<r<R$，使得$|x-x_0|\leqslant r$

由阿贝尔第二定理，得$\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$在$[x_0-r,x_0+r]$上一致收敛

又$a_n(x-x_0)^n(n=0,1,2,\cdots)$在$[x_0-r,x_0+r]$连续，则由函数项级数的和的连续性知$S(x)$在$[x_0-r,x_0+r]$连续，当然在$x$这一点连续

而x为(x_0-R,x_0+R)上任一点，则$S(x)$在(x_0-R,x_0+R)连续

又若$\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$在$x_0+R$收敛，则由阿贝尔第二定理，得$\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$在$[a,x_0+R]$(取$a\in(x_0-R,x_0+R)$)上一致收敛

由于$a_n(x-x_0)^n(n=0,1,2,\cdots)$在$[a,x_0+R]$连续，则由函数项级数的和的连续性定理，得

$S(x)$在$[a,x_0+R]$连续，当然也在x_0+R连续，于是$S(x)$在$(x_0-R,x_0+R]$上连续

同理若$\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$在$x_0-R$收敛，则$S(x)$在$[x_0-R,x_0+R)$上连续.

性质2.

(1) 设x为(x_0-R,x_0+R)内任一点，由阿贝尔第二定理，得$\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$在$[x_0,x]$上一致收敛(若$x<x_0$，则取$[x,x_0]$即可)

又$a_n(x-x_0)^n(n=0,1,2,\cdots)$在$[x_0,x]$连续

则由函数项级数逐项求积分定理，得

$$\int_{x_0}^{x}S(x)\,\mathrm{d}x=\int_{x_0}^{x}\left(\sum_{n=0}^{\infty}a_n(x-x_0)^n\right)\mathrm{d}x=\sum_{n=0}^{\infty}\int_{x_0}^{x}[a_n(x-x_0)^n]\,\mathrm{d}x=\sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-x_0)^{n+1}$$

(2) 由第5页习题3(2)知，若$\{x_n\}$收敛，且$\lim\limits_{n\to\infty}x_n=0$，则对任何$\{y_n\}$，有$\overline{\lim\limits_{n\to\infty}}(x_n\cdot y_n)=\lim\limits_{n\to\infty}x_n\cdot\overline{\lim\limits_{n\to\infty}}y_n$

则$\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|na_n|}=\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}$

这说明：$\sum\limits_{n=1}^{\infty}na_n(x-x_0)^{n-1}$与$\sum\limits_{n=1}^{\infty}a_n(x-x_0)^n$有相同的收敛半径$R$

设x是(x_0-R,x_0+R)内任一点，总可选取一点$0<r<R$，使得$|x-x_0|\leqslant r$

由阿贝尔第二定理，得$\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$在$[x_0-r,x_0+r]$上一致收敛，因而收敛

又$\sum\limits_{n=1}^{\infty}na_n(x-x_0)^{n-1}$的收敛半径为$R$，则由阿贝尔第二定理，得$\sum\limits_{n=1}^{\infty}na_n(x-x_0)^{n-1}$在$[x_0-r,x_0+r]$上一致收敛

又$na_n(x-x_0)^{n-1}(n=1,2,\cdots)$在$[x_0-r,x_0+r]$连续，则由函数项级数逐项微分定理，得

在$[x_0-r,x_0+r]$当然也就在x点，有$\dfrac{\mathrm{d}}{\mathrm{d}x}S(x)=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n\right)=\sum\limits_{n=1}^{\infty}na_n(x-x_0)^{n-1}$

再由x在(x_0-R,x_0+R)的任意性，得在(x_0-R,x_0+R)上式也成立

(3) 设$\sum\limits_{n=0}^{\infty}\dfrac{a_n}{n+1}(x-x_0)^{n+1}$收敛半径为$R'$

由(1)，得当$\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$在$(x_0-R,x_0+R)$收敛(收敛到$S(x)$)时，有

$\sum\limits_{n=0}^{\infty}\dfrac{a_n}{n+1}(x-x_0)^{n+1}$在$(x_0-R,x_0+R)$上收敛$\left(\text{收敛到}\int_{x_0}^{x}S(x)\,\mathrm{d}x\right)$，那末$R\leqslant R'$

另一方面，由(2)，当$\sum\limits_{n=0}^{\infty}\dfrac{a_n}{n+1}(x-x_0)^{n+1}$在$(x_0-R',x_0+R')$上收敛$\left(\text{收敛到}\int_{x_0}^{x}S(x)\,\mathrm{d}x\right)$时，有

$\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$在$(x_0-R',x_0+R')$收敛(收敛到$S(x)$)，那末$R'\leqslant R$

于是$R=R'$

6. 设$\sum\limits_{0}^{\infty}a_n$收敛于$A$，$\sum\limits_{0}^{\infty}b_n$收敛于$B$，如果它们的柯西乘积

$$\sum_{0}^{\infty}c_n=\sum_{0}^{\infty}(a_0b_n+a_1b_{n-1}+\cdots+a_nb_0)$$

收敛，则一定收敛于AB.

证明：作$A(x)=\sum\limits_{0}^{\infty}a_nx^n,B(x)=\sum\limits_{0}^{\infty}b_nx^n,C(x)=\sum\limits_{0}^{\infty}c_nx^n$

当$x=1$时，$A=A(1)=\sum\limits_{0}^{\infty}a_n,B=B(1)=\sum\limits_{0}^{\infty}b_n,C=C(1)=\sum\limits_{0}^{\infty}c_n$

即幂级数$\sum\limits_{0}^{\infty} a_n x^n, \sum\limits_{0}^{\infty} b_n x^n, \sum\limits_{0}^{\infty} c_n x^n$在$x=1$收敛

由Abel第一定理，得上述的幂级数在$|x|<1$内绝对收敛

由柯西定理，得级数$\sum\limits_{0}^{\infty} c_n x^n$收敛于$\left(\sum\limits_{0}^{\infty} a_n x^n\right)\left(\sum\limits_{0}^{\infty} b_n x^n\right)$即$C(x)=A(x)B(x)$

因$\sum\limits_{0}^{\infty} a_n x^n, \sum\limits_{0}^{\infty} b_n x^n, \sum\limits_{0}^{\infty} c_n x^n$在$x=1$收敛

由幂级数类似性质1，则$A(x), B(x), C(x)$在$x=1$左连续

$C(1)=\lim\limits_{x\to 1-0} C(x)=\lim\limits_{x\to 1-0} A(x)B(x)=A(1)B(1)$

则$C=AB$，于是$\sum\limits_{0}^{\infty} c_n=AB.$

7. 设$f(x)=\sum\limits_{0}^{\infty} a_n x^n$当$|x|<r$时收敛，那末当$\sum\limits_{0}^{\infty} \dfrac{a_n}{n+1} r^{n+1}$收敛时成立

$$\int_0^r f(x)\,\mathrm{d}x=\sum_{0}^{\infty}\frac{a_n}{n+1} r^{n+1}$$

不论$\sum\limits_{0}^{\infty} a_n x^n$当$x=r$时是否收敛.

证明：因$f(x)=\sum\limits_{0}^{\infty} a_n x^n$当$|x|<r$时收敛，则其收敛半径为$R$，且$r\leqslant R$，从而$f(x)$在$(-r,r)$内收敛.

则据性质2，当$x\in(-r,r)$时，有$\displaystyle\int_0^{\theta} f(x)\,\mathrm{d}x=\int_0^{\theta}\left[\sum_{0}^{\infty} a_n x^n\right]\mathrm{d}x=\sum_{0}^{\infty}\frac{a_n}{n+1}\theta^{n+1}, \theta\in(0,r)$

即$\displaystyle\int_0^{\theta} f(x)\,\mathrm{d}x=\sum_{0}^{\infty}\frac{a_n}{n+1}\theta^{n+1}\ \theta\in(0,r)$

因$\sum\limits_{0}^{\infty}\dfrac{a_n}{n+1} r^{n+1}$收敛，则$\sum\limits_{0}^{\infty}\dfrac{a_n}{n+1}\theta^{n+1}$在$\theta=r$收敛，于是其和$S(\theta)$在$r$点左连续

$\displaystyle S(r)=\sum_{0}^{\infty}\frac{a_n}{n+1} r^{n+1}=\lim_{\theta\to r-0} S(\theta)=\lim_{\theta\to r-0}\int_0^{\theta} f(x)\,\mathrm{d}x=\int_0^r f(x)\,\mathrm{d}x$

从而不论$\sum\limits_{0}^{\infty} a_n x^n$当$x=r$时是否收敛，均有$\displaystyle\int_0^r f(x)\,\mathrm{d}x=\sum_{0}^{\infty}\frac{a_n}{n+1} r^{n+1}$

8. 利用上题证明$\displaystyle\int_0^1 \frac{\ln(1-t)}{t}\,\mathrm{d}t=-\sum_{n=1}^{\infty}\frac{1}{n^2}$.

证明：因$\ln(1+x)=\sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{x^n}{n}\ (-1<x<1)$，则$\dfrac{\ln(1-x)}{x}=-\sum\limits_{n=1}^{\infty}\dfrac{x^{n-1}}{n}\ (-1<x<1$且$x\neq 0)$

即$f(x)=\begin{cases}\dfrac{\ln(1-x)}{x}, & x\in(-1,0)\bigcup(0,1)\\ -1, & x=0\end{cases}$，$f(x)=-\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n+1}\ (-1<x<1)$

因$\displaystyle\sum_{n=0}^{\infty}\frac{-\frac{1}{n+1}}{n+1}=-\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$收敛，则由上题结论，得$\displaystyle\int_0^1\frac{\ln(1-x)}{x}\,\mathrm{d}x=-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}=-\sum_{n=1}^{\infty}\frac{1}{n^2}$

9. 求$f(x)=\sum\limits_{n=1}^{\infty}\dfrac{\sin(2^n\cdot x)}{n!}$的麦克劳林级数，说明它的麦克劳林级数并不表示这个函数.

证明：因$\left|\dfrac{\sin(2^n\cdot x)}{n!}\right|\leqslant\dfrac{1}{n!}\,(x\in(-\infty,+\infty))$，且$\sum\limits_{n=0}^{\infty}\dfrac{1}{n!}$收敛，则由M判别法，得$\sum\limits_{n=1}^{\infty}\dfrac{\sin(2^n\cdot x)}{n!}$在$(-\infty,+\infty)$内一致收敛，从而收敛

$f(0)=0, \displaystyle\sum_{n=1}^{\infty}\left[\frac{\sin(2^n\cdot x)}{n!}\right]'=\sum_{n=1}^{\infty}\frac{2^n\cos(2^n\cdot x)}{n!}$

又$\left|\dfrac{2^n\cos(2^n\cdot x)}{n!}\right|\leqslant\dfrac{2^n}{n!}\,(x\in(-\infty,+\infty))$，且$\sum\limits_{n=0}^{\infty}\dfrac{2^n}{n!}$收敛，则由M判别法，得$\sum\limits_{n=1}^{\infty}\dfrac{2^n\cos(2^n\cdot x)}{n!}$在$(-\infty,+\infty)$内一致收敛

又$\dfrac{2^n\cos(2^n\cdot x)}{n!}(n=0,1,2,\cdots)$ 在$(-\infty,+\infty)$内连续，则由逐项求导定理，得在$(-\infty,+\infty)$上

$f'(x)=\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\sum\limits_{n=1}^{\infty}\dfrac{\sin(2^n\cdot x)}{n!}\right]=\sum\limits_{n=1}^{\infty}\dfrac{2^n\cos(2^n\cdot x)}{n!}$

于是$f'(0)=\sum\limits_{n=1}^{\infty}\dfrac{2^n}{n!}=e^2-1$

如此下去，用数学归纳法，得

$f^{(m)}(x)=\sum\limits_{n=1}^{\infty}\dfrac{2^{mn}\sin\left(\frac{m\pi}{2}+2^n\pi\right)}{n!}$，

$$f^{(m)}(0)=\begin{cases}0, & m=2k\\ \sum\limits_{n=1}^{\infty}(-1)^k\dfrac{2^{(2k+1)n}}{n!}=(-1)^k(e^{2^{2k+1}}-1), & m=2k+1\end{cases}$$

则$f(x)$的麦克劳林级数为$\sum\limits_{k=0}^{\infty}(-1)^k(e^{2^{2k+1}}-1)\dfrac{x^{2k+1}}{(2k+1)!}$ 其收敛半径为$R=\lim\limits_{k\to\infty}\dfrac{(e^{2^{2k+1}}-1)/(2k+1)!}{(e^{2^{2k+3}}-1)/(2k+3)!}$

因$0\leqslant\dfrac{(e^{2^{2k+1}}-1)/(2k+1)!}{(e^{2^{2k+3}}-1)/(2k+3)!}=(2k+2)(2k+3)\dfrac{e^{2^{2k+1}}-1}{e^{2^{2k+3}}-1}\leqslant(2k+2)(2k+3)\dfrac{e^{2^{2k+1}}}{e^{2^{2k+3}}}=\dfrac{(2k+2)(2k+3)}{e^{6\cdot 2^{2k}}}$，

且$\lim\limits_{x\to\infty}\dfrac{(2x+2)(2x+3)}{e^{6\cdot2^{2x}}}=0$，则$\lim\limits_{k\to\infty}\dfrac{(2k+2)(2k+3)}{e^{6\cdot2^{2k}}}=0$

于是$R=0$，即其麦克劳林级数仅在$x=0$收敛

但由前面可知其在$(-\infty,+\infty)$内均收敛，则它的麦克劳林级数并不表示此函数.

10. 证明：

(1) $\sum\limits_{n=0}^{\infty}\dfrac{x^{4n}}{(4n)!}$ 满足$y^{(IV)}=y$；

(2) $\sum\limits_{n=0}^{\infty}\dfrac{x^n}{(n!)^2}$ 满足$xy''+y'-y=0$.

证明：

(1) $a_n=\sqrt[n]{\dfrac{1}{(4n)!}}$，$R=\lim\limits_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right|=+\infty$

则知对任一x，幂级数都收敛，即其收敛域为$(-\infty,+\infty)$

在收敛域内逐项微分之，得

$y'=\sum\limits_{n=1}^{\infty}\dfrac{x^{4n-1}}{(4n-1)!}$，$y''=\sum\limits_{n=1}^{\infty}\dfrac{x^{4n-2}}{(4n-2)!}$，$y'''=\sum\limits_{n=1}^{\infty}\dfrac{x^{4n-3}}{(4n-3)!}$，$y^{(4)}=\sum\limits_{n=1}^{\infty}\dfrac{x^{4n-4}}{(4n-4)!}=\sum\limits_{n=0}^{\infty}\dfrac{x^{4n}}{(4n)!}=y$

即$y^{(IV)}=y$.

(2) $a_n=\dfrac{1}{(n!)^2}$，则$R=\lim\limits_{x\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right|=+\infty$

则知对任一x，幂级数都收敛，即其收敛域为$(-\infty,+\infty)$

在收敛域内逐项微分之，得$y'=\sum\limits_{n=1}^{\infty}\dfrac{x^{n-1}}{(n-1)!n!}=\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!(n+1)!}$，$y''=\sum\limits_{n=1}^{\infty}\dfrac{x^{n-1}}{(n-1)!(n+1)!}$

于是$xy''+y'=\sum\limits_{n=1}^{\infty}\dfrac{x^n}{(n-1)!(n+1)!}+\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!(n+1)!}=1+\sum\limits_{n=1}^{\infty}\left[\dfrac{1}{(n-1)!(n+1)!}+\dfrac{1}{n!(n+1)!}\right]x^n=$

$1+\sum\limits_{n=1}^{\infty}\dfrac{n+1}{n!(n+1)!}x^n=1+\sum\limits_{n=1}^{\infty}\dfrac{x^n}{(n!)^2}=\sum\limits_{n=0}^{\infty}\dfrac{x^n}{(n!)^2}=y$

即$xy''+y'-y=0$.

11. 展开：

(1) $f(x)=\dfrac{1}{a-x}$ $(a\neq0)$成为x的幂级数，并确定收敛范围；

(2) $f(x)=\ln x$为$(x-2)$的幂级数.

解：

(1) 因$f(x)=\dfrac{1}{a-x}=\dfrac{1}{a}\left(\dfrac{1}{1-\frac{x}{a}}\right)$，且$\sum\limits_{n=0}^{\infty}\left(\dfrac{x}{a}\right)^{n}=\dfrac{1}{1-\frac{x}{a}}$，此时$\left|\dfrac{x}{a}\right|<1$

则$f(x)=\dfrac{1}{a}\left[\sum\limits_{n=0}^{\infty}\left(\dfrac{x}{a}\right)^{n}\right]=\sum\limits_{n=0}^{\infty}\dfrac{x^{n}}{a^{n+1}}$，$|x|<|a|$

(2) $f(x)=\ln x=\ln[2+(x-2)]=\ln 2+\ln\left(1+\dfrac{x-2}{2}\right)$

因$\ln\left(1+\dfrac{x-2}{2}\right)=\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\left(\dfrac{x-2}{2}\right)^{n}=\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{2^{n}\cdot n}(x-2)^{n}$，$0<x\leqslant 4$

则$f(x)=\ln 2+\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{2^{n}\cdot n}(x-2)^{n}$，收敛域为$(0,4]$.

12. 利用已知展开式展开下列函数为幂级数，并确定收敛范围：

(1) $\dfrac{e^{x}-e^{-x}}{2}$

(2) $\sin^{2}x=\dfrac{1-\cos 2x}{2}$

解：

(1) 因$e^{x}=\sum\limits_{n=0}^{\infty}\dfrac{x^{n}}{n!}\ (-\infty<x<+\infty), e^{-x}=\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n}x^{n}}{n!}\ (-\infty<x<+\infty)$

则$f(x)=\dfrac{e^{x}-e^{-x}}{2}=\dfrac{1}{2}\left[\sum\limits_{n=0}^{\infty}\dfrac{x^{n}}{n!}-\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n}x^{n}}{n!}\right]$

当$n=2k$时，$f(x)=0$；当$n=2k+1$时，$f(x)=\sum\limits_{k=0}^{\infty}\dfrac{x^{2k+1}}{(2k+1)!}$

综上可知，$f(x)=\sum\limits_{n=0}^{\infty}\dfrac{x^{2n+1}}{(2n+1)!}$，收敛域为$(-\infty,+\infty)$.

(2) 因$\cos 2x=\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n}(2x)^{2n}}{(2n)!}\ (-\infty<x<+\infty)$

则$\sin^{2}x=\dfrac{1-\cos 2x}{2}=\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}2^{2n-1}}{(2n)!}x^{2n}$，收敛域为$(-\infty,+\infty)$.

13. 展开$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{e^{x}-1}{x}\right)$为$x$的幂级数，并推出$1=\sum\limits_{n=1}^{\infty}\dfrac{n}{(n+1)!}$.

解：因$e^{x}=\sum\limits_{n=0}^{\infty}\dfrac{x^{n}}{n!}\ (-\infty<x<+\infty)$，则$\dfrac{e^{x}-1}{x}=\sum\limits_{n=0}^{\infty}\dfrac{x^{n}}{(n+1)!}\ (x\neq 0)$

令$f(x)=\begin{cases}\dfrac{e^{x}-1}{x}, & x\neq 0\\ 1, & x=0\end{cases}$，则$\sum\limits_{n=0}^{\infty}\dfrac{x^{n}}{(n+1)!}$为$f(x)$的幂级数，其收敛范围为$(-\infty,+\infty)$

由幂级数的逐项求导定理，得$\sum\limits_{n=0}^{\infty}\dfrac{x^{n}}{(n+1)!}$在$(-\infty,+\infty)$内逐项求导

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x)=\begin{cases}\sum\limits_{n=1}^{\infty}\dfrac{n}{(n+1)!}x^{n-1}, & x\neq 0\\ \dfrac{1}{2}, & x=0\end{cases}$$

于是$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{e^{x}-1}{x}\right)=\sum\limits_{n=0}^{\infty}\dfrac{n}{(n+1)!}x^{n-1}$

因$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{e^{x}-1}{x}\right)\Bigg|_{x=1}=\dfrac{(x-1)e^{x}+1}{x^{2}}\Bigg|_{x=1}=1$，则$\sum\limits_{n=0}^{\infty}\dfrac{n}{(n+1)!}x^{n-1}\Bigg|_{x=1}=\sum\limits_{n=0}^{\infty}\dfrac{n}{(n+1)!}=1$

14. 求下列函数的幂级数展开式，并推出收敛半径：

(1) $\displaystyle\int_{0}^{x}\frac{\sin t}{t}\,\mathrm{d}t$

(2) $\int_0^x \cos t^2 \,\mathrm{d}t$

解:

(1) 因$\sin t = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n t^{2n+1}}{(2n+1)!}$，则$\dfrac{\sin t}{t} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n t^{2n}}{(2n+1)!}\ (t \neq 0)$

令$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$，则$\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n t^{2n}}{(2n+1)!}$为$f(t)$的幂级数，收敛域为$(-\infty, +\infty)$

由幂级数逐项积分定理，得$\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n t^{2n}}{(2n+1)!}$在$(-\infty, +\infty)$内逐项积分

$$\int_0^x f(t)\,\mathrm{d}t = \begin{cases} \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

则$\int_0^x \dfrac{\sin t}{t}\ \mathrm{d}t = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!}$，其收敛半径为$R = +\infty$.

(2) 因$\cos t^2 = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (t^2)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n t^{4n}}{(2n)!}$，其收敛域为$(-\infty, +\infty)$，收敛半径为$R = \infty$

由幂级数的逐项积分定理，得$\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n t^{4n}}{(2n)!}$在$(-\infty, +\infty)$内逐项积分

$\int_0^x \cos t^2 \,\mathrm{d}t = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}$，其收敛半径为$R = \infty$.

15. 求下列级数的和：

(1) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n!}$

(2) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^{n+1}}{n(n+1)}$

(3) $\sum\limits_{n=1}^{\infty} n^2 x^{n-1}$

(4) $\sum\limits_{n=0}^{\infty} \dfrac{(2n+1)x^{2n}}{n!}$

解:

(1) 因$e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}\ (-\infty < x < +\infty)$，则$\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} - 1 = e^x - 1(-\infty < x < \infty)$

(2) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^{n+1}}{n(n+1)} = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}\ x^{n+1} + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n+1}\ x^{n+1}$

因$\ln(1+x) = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}\ x^n\ \ (-1 < x \leqslant 1)$

则$\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^{n+1}}{n(n+1)} = x\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}\ x^n + \sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n+1}}{n}\ x^n = x\ln(1+x) + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}\ x^n - x = (1+x)\ln(1+x) - x(-1 < x \leqslant 1)$

$x = -1$时，$f(x) = \sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+1)} = \lim\limits_{N\to\infty} \sum\limits_{n=1}^{N} \left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) = 1$

(3) $\sum\limits_{n=1}^{\infty} n^2 x^{n-1} = \sum\limits_{n=0}^{\infty} (n+1)^2 x^n, a_n = (n+1)^2$

则$\lim\limits_{n\to\infty} \left|\dfrac{a_n}{a_{n+1}}\right| = 1$，于是其收敛半径为$R = 1$

当$|x| = 1$时，由于$(n+1)^2 \to +\infty$，则级数发散，于是级数的收敛域为$(-1, 1)$

当$x\in(-1,1)$时，令$f(x)=\sum_{n=1}^{\infty}n^2x^{n-1},|x|<1$

由性质2，得$\sum_{n=1}^{\infty}n^2x^{n-1}$在$(-1,1)$可逐项积分，$\int_0^x f(x)\,\mathrm{d}x=\sum_{n=1}^{\infty}nx^n$，且其收敛半径不变，仍为1.

又由性质2，得$\sum_{n=1}^{\infty}nx^n$在$(-1,1)$上可逐项积分

$\int_0^x\left(\int_0^x f(x)\,\mathrm{d}x\right)\,\mathrm{d}x=\sum_{n=1}^{\infty}\int_0^x nx^n\,\mathrm{d}x=\sum_{n=1}^{\infty}\frac{n}{n+1}\,x^{n+1}=x^2\sum_{n=0}^{\infty}x^n+\sum_{n=1}^{\infty}\left(-\frac{x^n}{n}\right)+x=$
$\frac{x^2}{1-x}+\ln(1-x)+x,|x|<1$

则$\int_0^x f(x)\,\mathrm{d}x=\left(\frac{x^2}{1-x}+\ln(1-x)+x\right)'=\frac{x}{(1-x)^2}$

于是$f(x)=\left(\frac{x}{(1-x)^2}\right)'=\frac{1+x}{(1-x)^3}\,,|x|<1$

(4) $\sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{n!}=1+\sum_{n=1}^{\infty}\frac{(2n+1)x^{2n}}{n!}=1+\sum_{n=1}^{\infty}\frac{2x^2}{(n-1)!}\,x^{2(n-1)}+\sum_{n=1}^{\infty}\frac{x^{2n}}{n!}$

因$e^{x^2}=1+\sum_{n=1}^{\infty}\frac{x^{2n}}{n!}=\sum_{n=1}^{\infty}\frac{x^{2(n-1)}}{(n-1)!}$

则$\sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{n!}=(2x^2+1)e^{x^2},(-\infty<x<+\infty)$

§3. 逼近定理

1. 在闭区间$[-1,1]$上用伯恩斯坦多项式$B_4(x)$逼近函数$f(x)=\dfrac{x+|x|}{2}$，作出函数$y=\dfrac{x+|x|}{2}$和$y=B_4(x)$的图形.

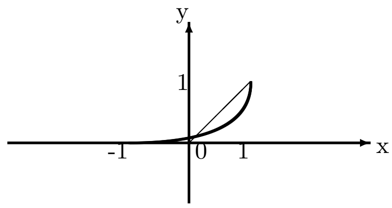
解: 令$x=-1+2y$，则当$0\leqslant y\leqslant 1$时，$-1\leqslant x\leqslant 1$，此时$y=\dfrac{x+1}{2}$,$1-y=\dfrac{1-x}{2}$,$f(x)=f(-1+2y)$

则$f(x)$在$[-1.1]$上用伯恩斯坦多项式为$B_n(x)=\displaystyle\sum_{k=0}^{n}f\left(-1+2\cdot\frac{k}{n}\right)C_n^k\frac{(x+1)^k(1-x)^{n-k}}{2^n}$

则$B_4(x)=\displaystyle\sum_{k=0}^{4}f\left(-1+\frac{k}{2}\right)C_4^k\frac{(x+1)^k(1-x)^{4-k}}{2^4}$

又$f(x)$当$-1\leqslant x\leqslant 0$时，$f(x)=0$，

则$B_4(x)=f\left(\dfrac{1}{2}\right)C_4^3\dfrac{(x+1)^3(1-x)}{2^4}+f(1)C_4^4\dfrac{(x+1)^k}{2^4}=\dfrac{1}{8}(1-x)(x+1)^3+\dfrac{1}{16}(1+x)^4.$



2. 设$f(x)$是$[a,b]$上的连续函数，证明存在有理系数的多项式$P(x)$，使得$\displaystyle\max_{x\in[a,b]}|f(x)-P(x)|<\varepsilon$.其中$\varepsilon$是预先给定的任意正数.

证明: 因$f(x)$是$[a,b]$上的连续函数

则由逼近定理，得对任意给定的$\varepsilon>0$，定存在多项式$Q(x)$，使得$||f(x)-Q(x)||=\displaystyle\max_{x\in[a,b]}|f(x)-Q(x)|<\frac{\varepsilon}{2}$

其中$Q(x)=a_0+a_1x+\cdots+a_nx^n(a_0,a_1,\cdots,a_n$均为实数)

设$C=\max(|a|,|b|)$，由实数的稠密性，得必存在有理数b_i，使得$|b_i-a_i|<\dfrac{\varepsilon}{4(n+1)^2C^i}(i=0,1,\cdots,n)$

并设$P(x)=b_0+b_1x+\cdots+b_nx^n$

则$|P(x)-Q(x)|=\left|\displaystyle\sum_{i=0}^{n}(b_i-a_i)x^i\right|\leqslant\sum_{i=1}^{n}|b_i-a_i||x|^i<\sum_{i=0}^{n}\frac{\varepsilon}{4(n+1)^2C^i}C^i<\frac{\varepsilon}{2}$

于是$||P(x)-Q(x)||=\displaystyle\max_{x\in[a,b]}|P(x)-Q(x)|<\frac{\varepsilon}{2}$

从而$||f(x)-P(x)||\leqslant||f(x)-Q(x)||+||Q(x)-P(x)||<\varepsilon$

即存在有理系数的多项式$P(x)$，使得$\displaystyle\max_{x\in[a,b]}|f(x)-P(x)|<\varepsilon$

第十二章　富里埃级数和富里埃变换

§1.　富里埃级数

1. 证明：

(1) $1, \cos x, \cos 2x, \cdots, \cos nx, \cdots$

(2) $\sin x, \sin 2x, \sin 3x, \cdots, \sin nx, \cdots$

是$[0,\pi]$上的正交系；但$1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots, \cos nx, \sin nx, \cdots$不是$[0,\pi]$上的正交系.

证明：

(1) 因$\int_0^\pi 1\cdot\cos kx\,\mathrm{d}x = 0\ (k=1,2,\cdots)$, $\int_0^\pi \cos kx\cdot\cos lx\,\mathrm{d}x = \begin{cases} 0, & k\neq l, k, l = 1,2,\cdots \\ \dfrac{\pi}{2}, & k=l=1,2,\cdots \end{cases}$

$\int_0^\pi 1^2\,\mathrm{d}x = \pi$

则$1, \cos x, \cos 2x, \cdots, \cos nx, \cdots$是$[0,\pi]$上的正交系

(2) 因$\int_0^\pi \sin kx\sin lx\,\mathrm{d}x = \begin{cases} 0, & k\neq l, k, l = 1,2,\cdots \\ \dfrac{\pi}{2}, & k=l=1,2,\cdots \end{cases}$

则$\sin x, \sin 2x, \sin 3x, \cdots, \sin nx, \cdots$是$[0,\pi]$上的正交系

又$\int_0^\pi 1\cdot\sin x\,\mathrm{d}x = 2\neq 0$，则$1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots, \cos nx, \sin nx, \cdots$不是$[0,\pi]$上的正交系.

2. 证明：$\sin x, \sin 3x, \cdots, \sin(2n+1)x, \cdots$是$\left[0, \dfrac{\pi}{2}\right]$上的正交系，写出它的标准正交系$\left(\text{即不仅正交，而且每个函数的平方在}\left[0,\dfrac{\pi}{2}\right]\text{上的积分为}1\right)$，并导出$\sin\dfrac{\pi x}{2l}, \sin\dfrac{3\pi x}{2l}, \cdots, \sin\dfrac{(2n+1)\pi x}{2l}, \cdots$是$[0,l]$上的正交系.

证明：因$\int_0^{\frac{\pi}{2}} [\sin(2k+1)x\sin(2l+1)x]\,\mathrm{d}x = \begin{cases} 0, & k\neq l, k, l = 1,2,\cdots \\ \dfrac{\pi}{4}, & k=l=1,2,\cdots \end{cases}$

则$\sin x, \sin 3x, \cdots, \sin(2n+1)x, \cdots$是$\left[0,\dfrac{\pi}{2}\right]$上的正交系

又由$\int_0^{\frac{\pi}{2}} \left[\dfrac{\sin(2k+1)x}{a}\right]^2\mathrm{d}x = \dfrac{\pi}{4a^2} = 1(k=1,2,\cdots)$，得$a = \dfrac{\sqrt{\pi}}{2}$

则在$\left[0,\dfrac{\pi}{2}\right]$上它的标准正交系为$\dfrac{2\sin x}{\sqrt{\pi}}, \dfrac{2\sin 3x}{\sqrt{\pi}}, \cdots, \dfrac{2\sin(2n+1)x}{\sqrt{\pi}}, \cdots$

又$\int_0^l \sin\dfrac{(2k+1)\pi x}{2l}\sin\dfrac{\sin(2m+1)\pi x}{2l}\,\mathrm{d}x = \begin{cases} 0, & k\neq m, k, m = 1,2,\cdots \\ \dfrac{l}{2}\neq 0, & k=m=1,2,\cdots \end{cases}$

则$\sin\dfrac{\pi x}{2l}, \sin\dfrac{3\pi x}{2l}, \cdots, \sin\dfrac{(2n+1)\pi x}{2l}, \cdots$是$[0,l]$上的正交系.

3. 设$f(t)$是周期为T的方波，它在$\left[-\dfrac{T}{2}, \dfrac{T}{2}\right]$上的函数表示式为

$$f(t) = \begin{cases} E, & \text{当}0\leqslant t<\dfrac{T}{2}\text{时} \\ 0, & \text{当}-\dfrac{T}{2}\leqslant t<0\text{时} \end{cases}$$

将这个方波展开成富里埃级数.

解：因$\omega = \dfrac{T}{2}, f(t) = \begin{cases} E, & \text{当}0\leqslant t<\dfrac{T}{2}\text{时} \\ 0, & \text{当}-\dfrac{T}{2}\leqslant t<0\text{时} \end{cases}$

则$a_k=\dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)\cos k\omega t\,\mathrm{d}t=\dfrac{2}{T}\displaystyle\int_{0}^{\frac{T}{2}}E\cos\dfrac{2k\pi}{T}\,t\,\mathrm{d}t=0$

$a_0=\dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}\mathrm{d}t=\dfrac{2}{T}\displaystyle\int_{0}^{\frac{T}{2}}E\,\mathrm{d}t=E$

$b_k=\dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)\sin k\omega t\,\mathrm{d}t=\dfrac{T}{2}\displaystyle\int_{0}^{\frac{T}{2}}E\sin\dfrac{2k\pi}{T}\,t\,\mathrm{d}t=\begin{cases}0, & k\text{为偶}\\ \dfrac{2E}{k\pi}, & k\text{为奇}\end{cases}$

则$f(x)\sim\dfrac{E}{2}+\dfrac{2E}{\pi}\displaystyle\sum_{k=1}^{\infty}\dfrac{1}{2k-1}\sin\dfrac{(4k-2)\pi x}{T}=\begin{cases}E, & 0<x<\dfrac{T}{2}\\ 0, & -\dfrac{T}{2}<x<0\\ \dfrac{E}{2}, & x=0,\pm\dfrac{T}{2}\end{cases}$

4. 设$f(t)$是周期为T的半波整流波，它在$\left[-\dfrac{T}{2},\dfrac{T}{2}\right)$上的函数表示式为

$$f(t)=\begin{cases}U_m\sin\omega t, & \text{当}0\leqslant t<\dfrac{T}{2}\text{时}\\ 0, & \text{当}-\dfrac{T}{2}\leqslant t<0\text{时}\end{cases}$$

把这半波整流波展开成富里埃级数.

解：因$\omega'=\dfrac{2\pi}{T}$，$f(t)=\begin{cases}U_m\sin\omega t, & \text{当}0\leqslant t<\dfrac{T}{2}\text{时}\\ 0, & \text{当}-\dfrac{T}{2}\leqslant t<0\text{时}\end{cases}$

则$a_0=\dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)\,\mathrm{d}t=\dfrac{2U_m}{\omega T}\left(1-\cos\dfrac{T\omega}{2}\right)$

$a_k=\dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)\cos k\omega' t\,\mathrm{d}t=\dfrac{U_m}{\omega T+2k\pi}\left(1-\cos\dfrac{T\omega+2k\pi}{2}\right)+\dfrac{U_m}{\omega T-2k\pi}\left(1-\cos\dfrac{T\omega-2k\pi}{2}\right)$

$b_k=\dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)\sin k\omega' t\,\mathrm{d}t=\dfrac{U_m}{\omega T-2k\pi}\sin\dfrac{T\omega-2k\pi}{2}-\dfrac{U_m}{\omega T+2k\pi}\sin\dfrac{T\omega+2k\pi}{2}$

则$f(t)\sim\dfrac{2U_m}{\omega T}\left(1-\cos\dfrac{T\omega}{2}\right)+\displaystyle\sum_{k=1}^{\infty}\left(a_k\cos\dfrac{2k\pi}{T}\,t+b_k\sin\dfrac{2k\pi}{T}\,t\right)=\begin{cases}U_m\sin\omega t, & 0\leqslant t<\dfrac{T}{2}\\ 0, & -\dfrac{T}{2}<t<0\\ \dfrac{U_m}{2}\sin\dfrac{T\omega}{2}, & t=\pm\dfrac{T}{2}\end{cases}$

5. 设$f(t)$以2π为周期，在$[-\pi,\pi)$内

$$f(t)=\begin{cases}t, & \text{当}-\pi\leqslant t<0\text{时}\\ 0, & \text{当}0\leqslant t<\pi\text{时}\end{cases}$$

把$f(t)$展开成富里埃级数.

解：因$a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(t)\,\mathrm{d}t=-\dfrac{\pi}{2}$

$a_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(t)\cos kt\,\mathrm{d}t=\dfrac{1}{k^2\pi}[1-(-1)^k]$

$b_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(t)\sin kt\,\mathrm{d}t=\dfrac{(-1)^{k+1}}{k}$

则$f(t)\sim-\dfrac{\pi}{4}+\displaystyle\sum_{k=1}^{\infty}\dfrac{(-1)^{k+1}}{k}\sin kt+\dfrac{1}{\pi}\displaystyle\sum_{k=1}^{\infty}\dfrac{1-(-1)^k}{k^2}\cos kt=$

$-\dfrac{\pi}{4}+\displaystyle\sum_{k=1}^{\infty}\dfrac{(-1)^{k+1}}{k}\sin kt+\dfrac{2}{\pi}\displaystyle\sum_{k=0}^{\infty}\dfrac{1}{(2k+1)^2}\cos(2k+1)t=\begin{cases}t, & -\pi<t<0\\ 0, & 0\leqslant t<\pi\\ -\dfrac{\pi}{2}, & t=\pm\pi\end{cases}$

6. 设$f(t)$是周期为2π、高为h的锯齿形波，它在$[0,2\pi)$上的函数表示式为$f(t)=\dfrac{h}{2\pi}\,t$，将这个锯齿形波展开成富里埃级数.

解：因$a_0=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(t)\,\mathrm{d}t=h$

$a_k=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(t)\cos kt\,\mathrm{d}t=0$

$b_k=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(t)\sin kt\,\mathrm{d}t=-\dfrac{h}{k\pi}$

则$f(t)\sim\dfrac{h}{2}+\displaystyle\sum_{k=1}^{\infty}(a_k\cos kt+b_k\sin kt)=\dfrac{h}{2}-\dfrac{h}{\pi}\sum_{k=1}^{\infty}\dfrac{\sin kt}{k}=\begin{cases}\dfrac{h}{2\pi}\,t, & 0<t<2\pi\\ \dfrac{h}{2}, & t=0,2\pi\end{cases}$

7. 将宽度为τ、高为h、周期为T的矩形波展开成余弦级数.

解：在一个周期$\left[-\dfrac{T}{2},\dfrac{T}{2}\right]$内矩形波函数表达式为$f(t)=\begin{cases}0, & -\dfrac{T}{2}\leqslant t<-\dfrac{\tau}{2}\\ h, & -\dfrac{\tau}{2}\leqslant t\leqslant\dfrac{\tau}{2}\\ 0, & \dfrac{\tau}{2}<t\leqslant\dfrac{T}{2}\end{cases}$

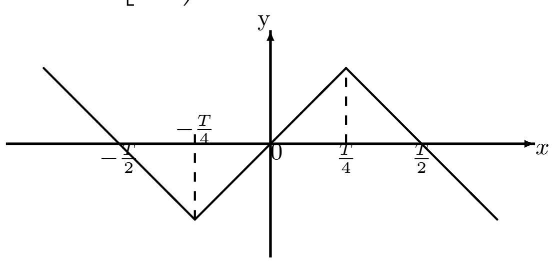
则$a_0=\dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,\mathrm{d}t=\dfrac{2h}{T}\tau$

$a_k=\dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\dfrac{2k\pi}{T}t\,\mathrm{d}t=\dfrac{2h}{k\pi}\sin\dfrac{k\tau}{T}\pi$

$b_k=\dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\dfrac{2k\pi}{T}t\,\mathrm{d}t=0$

于是$f(t)\sim\dfrac{h}{T}\tau+\displaystyle\sum_{k=1}^{\infty}\dfrac{2h}{k\pi}\sin\dfrac{k\pi}{T}\tau\cos\dfrac{2k\pi}{T}t$

8. 写出如图12-5所示的周期为T的三角波在$\left[0,\dfrac{T}{2}\right)$内的函数表示式，并将它展开成正弦级数.



解：如图所示的周期为T的三角波在$\left[0,\dfrac{T}{2}\right)$的函数表达式为$f(t)=\begin{cases}\dfrac{4E}{T}t, & 0\leqslant t<\dfrac{T}{4}\\ \dfrac{4E}{T}\left(\dfrac{T}{2}-t\right), & \dfrac{T}{4}\leqslant t<\dfrac{T}{2}\end{cases}$

先把$f(t)$延拓成$\left[-\dfrac{T}{2},\dfrac{T}{2}\right]$上的函数，再据题意，还必须把它延拓成奇函数，于是$a_0=a_k=0$

$b_k=\dfrac{4}{T}\displaystyle\int_0^{\frac{T}{2}} f(t)\sin\dfrac{2k\pi}{T}t\,\mathrm{d}t=\dfrac{8E}{k^2\pi^2}\sin\dfrac{k}{2}\pi=\begin{cases}0, & k\text{为偶}\\ \dfrac{(-1)^{\frac{k-1}{2}}\cdot 8E}{k^2\pi^2}, & k\text{为奇}\end{cases}$

则$f(t)\sim\dfrac{8E}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)^2}\sin\dfrac{2(2n-1)\pi}{T}t$

9. 在区间$(0,2\pi)$中展开$f(x)=\dfrac{\pi-x}{2}$成富里埃级数.

解：因$a_0=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\dfrac{\pi-x}{2}\,\mathrm{d}x=0$

$a_k=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\dfrac{\pi-x}{2}\cos kx\,\mathrm{d}x=0$

$b_k=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\dfrac{\pi-x}{2}\sin kx\,\mathrm{d}x=\dfrac{1}{k}$

则$f(x)\sim\sum_{k=1}^{\infty}\frac{1}{k}\,\sin kx=\frac{\pi-x}{2}\ (0<x<2\pi)$

10. 在区间$(-\pi,\pi)$中展开$f(x)=\pi^2-x^2$成富里埃级数.

解：因在$(-\pi,\pi)$上，$f(x)=\pi^2-x^2$为偶函数，则$b_k=0$

又$a_0=\frac{2}{\pi}\int_0^{\pi}(\pi^2-x^2)\,\mathrm{d}x=\frac{4}{3}\,\pi^2$

$a_k=\frac{2}{\pi}\int_0^{\pi}(\pi^2-x^2)\cos kx\,\mathrm{d}x=(-1)^{k+1}\frac{4}{k^2}$

则$f(x)\sim\frac{2}{3}\,\pi^2+4\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\,\cos kx=\pi^2-x^2\ (-\pi<x<\pi)$

11. 将$f(x)=\operatorname{sgn}(\cos x)$展开成富里埃级数.

解：因$f(x+2\pi)=\operatorname{sgn}[\cos(x+2\pi)]=\operatorname{sgn}(\cos x)=f(x)$，则$f(x)$是以$2\pi$为周期的周期函数

由$f(-x)=f(x)$，则$f(x)$为偶函数，于是$b_k=0$

又$a_0=\frac{2}{\pi}\int_0^{\pi}\operatorname{sgn}(\cos x)\,\mathrm{d}x=\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}(-1)\,\mathrm{d}x\right]=0$

$$a_k=\int_0^{\pi}\operatorname{sgn}(\cos x)\cos kx\,\mathrm{d}x=\frac{4}{k\pi}\,\sin\frac{k\pi}{2}=\begin{cases}0, & k=2n\\ (-1)^n\dfrac{4}{(2n+1)\pi}\,, & k=2n+1\end{cases}\quad(n=0,1,2,\cdots)$$

则$f(x)$在$(-\infty,+\infty)$上可展为$f(x)\sim\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\,\cos(2n+1)x=\operatorname{sgn}(\cos x)$

12. 应当如何把区间$\left(0,\frac{\pi}{2}\right)$内的可积函数$f(x)$延拓后，使它展开成的富里埃级数的形状如下：

$$f(x)\sim\sum_{n=1}^{\infty}a_n\cos(2n-1)x\ (-\pi<x<\pi)$$

解：因展开式中无正弦项，则$f(x)$延拓后应为偶函数

设$f(x)$延拓到$\left(\frac{\pi}{2}\,,\pi\right)$内的部分为$\varphi(x)$

因展开式中偶数项的系数$a_{2n}=0$即$a_{2n}=\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}f(x)\cos 2nx\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}\varphi(x)\cos 2nx\,\mathrm{d}x\right]=0$

则$\int_0^{\frac{\pi}{2}}f(x)\cos 2nx\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}\varphi(x)\cos 2nx\,\mathrm{d}x=0$

在左端前一积分中作变量代换，令$x=\pi-t$

则$-\int_{\pi}^{\frac{\pi}{2}}f(\pi-t)\cos 2n(\pi-t)\,\mathrm{d}t+\int_{\frac{\pi}{2}}^{\pi}\varphi(x)\cos 2nx\,\mathrm{d}x=0$即$\int_{\frac{\pi}{2}}^{\pi}[f(\pi-x)+\varphi(x)]\cos 2nx\,\mathrm{d}x=0$

要使上式成立，则必须当$x\in\left(\frac{\pi}{2}\,,\pi\right)$时。有$f(\pi-x)+\varphi(x)=0$即$\varphi(x)=-f(\pi-x)$

于是就求出了延拓后的函数在$\left(\frac{\pi}{2}\,,\pi\right)$内的表达式为$-f(\pi-x)$

又延拓后的函数为偶函数，则它在$\left(-\frac{\pi}{2}\,,0\right)$的表达式为$f(-x)$，在$\left(-\pi,-\frac{\pi}{2}\right)$的表达式为$-f(\pi+x)$

$$\text{不妨设延拓后的函数为}\psi(x)\text{，则}\psi(x)=\begin{cases}-f(\pi+x), & -\pi<x<-\dfrac{\pi}{2}\\ f(-x), & -\dfrac{\pi}{2}<x<0\\ f(x), & 0<x<\dfrac{\pi}{2}\\ -f(\pi-x), & \dfrac{\pi}{2}<x<\pi\end{cases}$$

13. 同上一题，但展开的富里埃级数形状为：

$$f(x)\sim\sum_{n=1}^{\infty}b_n\sin(2n-1)x\ (-\pi<x<\pi)$$

解：因展开式中无余弦项，则$f(x)$延拓后应为奇函数

设$f(x)$延拓到$\left(\dfrac{\pi}{2},\pi\right)$内的部分为$\varphi(x)$

因展开式中偶数项的系数$b_{2n}=0$即$b_{2n}=\dfrac{2}{\pi}\left[\displaystyle\int_0^{\frac{\pi}{2}} f(x)\sin 2nx\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}\varphi(x)\sin 2nx\,\mathrm{d}x\right]=0$

则$\displaystyle\int_0^{\frac{\pi}{2}} f(x)\sin 2nx\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}\varphi(x)\sin 2nx\,\mathrm{d}x=0$

在左端前一积分中作变量代换，令$x=\pi-t$

则$-\displaystyle\int_{\pi}^{\frac{\pi}{2}} f(\pi-t)\sin 2n(\pi-t)\,\mathrm{d}t+\int_{\frac{\pi}{2}}^{\pi}\varphi(x)\sin 2nx\,\mathrm{d}x=0$即$\displaystyle\int_{\frac{\pi}{2}}^{\pi}[-f(\pi-x)+\varphi(x)]\sin 2nx\,\mathrm{d}x=0$

要使上式成立，则必须当$x\in\left(\dfrac{\pi}{2},\pi\right)$时。有$-f(\pi-x)+\varphi(x)=0$即$\varphi(x)=f(\pi-x)$

于是就求出了延拓后的函数在$\left(\dfrac{\pi}{2},\pi\right)$内的表达式为$f(\pi-x)$

又延拓后的函数为奇函数，则它在$\left(-\dfrac{\pi}{2},0\right)$的表达式为$-f(-x)$，在$\left(-\pi,-\dfrac{\pi}{2}\right)$的表达式为$-f(\pi+x)$

不妨设延拓后的函数为$\psi(x)$，则$\psi(x)=\begin{cases}-f(\pi+x), & -\pi<x<-\dfrac{\pi}{2}\\ -f(-x), & -\dfrac{\pi}{2}<x<0\\ f(x), & 0<x<\dfrac{\pi}{2}\\ f(\pi-x), & \dfrac{\pi}{2}<x<\pi\end{cases}$

14. 设$f(x)$可积、绝对可积，证明：

(1) 如果函数$f(x)$在$[-\pi,\pi]$上满足$f(x+\pi)=f(x)$，那末$a_{2m-1}=b_{2m-1}=0$

(2) 如果函数$f(x)$在$[-\pi,\pi]$上满足$f(x+\pi)=-f(x)$，那末$a_{2m}=b_{2m}=0$

证明：

(1) 因$f(x)$可积、绝对可积且函数$f(x)$在$[-\pi,\pi]$上满足$f(x+\pi)=f(x)$
则$f(x)$在$[-\pi,\pi]$上可积、绝对可积且以π为周期
于是$a_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos kx\,\mathrm{d}x=\dfrac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos kx\,\mathrm{d}x+\int_0^{\pi} f(x)\cos kx\,\mathrm{d}x\right]$
对右端第二式作变量代换：$t=x-\pi$，则其变为$\dfrac{1}{\pi}\displaystyle\int_0^{\pi} f(x)\cos kx\,\mathrm{d}x=\dfrac{1}{\pi}\int_{-\pi}^{0} f(t)\cos k(t+\pi)\,\mathrm{d}t$
于是$a_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{0}[1+(-1)^k]f(x)\cos kx\,\mathrm{d}x$
从而，得$a_{2m-1}=0(m=1,2,\cdots)$
同理，得$b_{2m-1}=0(m=1,2,\cdots)$

(2) 因$f(x)$可积、绝对可积且函数$f(x)$在$[-\pi,\pi]$上满足$f(x+\pi)=-f(x)$，则$f(x+2\pi)=f(x)$
于是$f(x)$在$[-\pi,\pi]$上可积、绝对可积且以2π为周期
于是$a_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos kx\,\mathrm{d}x=\dfrac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos kx\,\mathrm{d}x+\int_0^{\pi} f(x)\cos kx\,\mathrm{d}x\right]$
对右端第二式作变量代换：$t=x-\pi$，则其变为$\dfrac{1}{\pi}\displaystyle\int_0^{\pi} f(x)\cos kx\,\mathrm{d}x=-\dfrac{1}{\pi}\int_{-\pi}^{0} f(t)\cos k(t+\pi)\,\mathrm{d}t$
于是$a_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{0}[1+(-1)^{k+1}]f(x)\cos kx\,\mathrm{d}x$
从而，得$a_{2m}=0(m=1,2,\cdots)$
同理，得$b_{2m}=0(m=1,2,\cdots)$

15. 周期为2π的可积和绝对可积函数$f(x)$的富里埃系数为a_n,b_n，计算：

(1) 函数$f(x+k)$ (k为常数)的富里埃系数$\overline{a}_n,\overline{b}_n$；

(2) $F(x)=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t)f(x-t)\,\mathrm{d}t$的富里埃系数$A_n,B_n$，设有关的积分顺序可交换.

解:

(1) 由已知，得$a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\,\mathrm{d}x, a_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,\mathrm{d}x, b_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,\mathrm{d}x$

则作代换$x+k=y$且$f(x)$是以2π为周期的函数，有

$\overline{a}_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x+k)\,\mathrm{d}x=\dfrac{1}{\pi}\int_{-\pi+k}^{\pi+k}f(y)\,\mathrm{d}y=a_0$

$\overline{a}_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x+k)\cos nx\,\mathrm{d}x=\dfrac{1}{\pi}\int_{-\pi+k}^{\pi+k}f(y)\cos n(y-k)\,\mathrm{d}y=a_n\cos nk+b_n\sin nk$

即$\overline{a}_n=a_n\cos nk+b_n\sin nk\ (n=0,1,2,\cdots)$

同理，可求得$\overline{b}_n=b_n\cos nk-a_n\sin nk$

(2) 因$f(x)$是周期为2π的可积和绝对可积函数

则$F(x+2\pi)=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(t)f(x+2\pi-t)\,\mathrm{d}t=F(x)$，于是$F(x)$仍是以$2\pi$为周期的函数

又$a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\,\mathrm{d}x, a_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,\mathrm{d}x, b_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,\mathrm{d}x$

则$A_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}F(x)\,\mathrm{d}x=\dfrac{1}{\pi^2}\int_{-\pi}^{\pi}f(t)\,\mathrm{d}t\int_{-\pi}^{\pi}f(x-t)\,\mathrm{d}x$

对$\displaystyle\int_{-\pi}^{\pi}f(x-t)\,\mathrm{d}x$作代换$x-t=y$且$f(x)$是以$2\pi$为周期的函数，有

$A_0=\dfrac{1}{\pi^2}\displaystyle\int_{-\pi}^{\pi}f(t)\,\mathrm{d}t\int_{-\pi-t}^{\pi-t}f(y)\,\mathrm{d}y=\dfrac{1}{\pi^2}\left[\int_{-\pi}^{\pi}f(t)\,\mathrm{d}t\right]^2=a_0^2$

同理，可求得$A_n=a_n^2-b_n^2$

$B_n=2a_nb_n$

16. 如果$\varphi(-x)=\psi(x)$，问$\varphi(x)$与$\psi(x)$的富里埃系数之间有什么关系？

解: 函数$\varphi(x)$与$\psi(x)$的富里埃系数分为$a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\varphi(x)\cos nx\,\mathrm{d}x, b_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x)\sin nx\,\mathrm{d}x$

$\alpha_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\psi(x)\cos nx\,\mathrm{d}x, \beta_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}\psi(x)\sin nx\,\mathrm{d}x$

对$a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\varphi(x)\cos nx\,\mathrm{d}x$右端作变量代换$y=-x$，并将$\varphi(-x)=\psi(x)$代入，得

$a_n=\dfrac{1}{\pi}\displaystyle\int_{\pi}^{-\pi}\varphi(-y)\cos n(-y)\,\mathrm{d}(-y)=\dfrac{1}{\pi}\int_{-\pi}^{\pi}\psi(x)\cos nx\,\mathrm{d}x=\alpha_n\ (n=0,1,2,\cdots)$

同理，得$b_n=-\beta_n\ (n=1,2,\cdots)$

17. 如果$\varphi(-x)=-\psi(x)$，问$\varphi(x)$与$\psi(x)$的富里埃系数之间有什么关系？

解: 函数$\varphi(x)$与$\psi(x)$的富里埃系数分为$a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\varphi(x)\cos nx\,\mathrm{d}x, b_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x)\sin nx\,\mathrm{d}x$

$\alpha_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\psi(x)\cos nx\,\mathrm{d}x, \beta_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}\psi(x)\sin nx\,\mathrm{d}x$

对$a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\varphi(x)\cos nx\,\mathrm{d}x$右端作变量代换$y=-x$，并将$\varphi(-x)=-\psi(x)$代入，得

$a_n=\dfrac{1}{\pi}\displaystyle\int_{\pi}^{-\pi}\varphi(-y)\cos n(-y)\,\mathrm{d}(-y)=-\dfrac{1}{\pi}\int_{-\pi}^{\pi}\psi(x)\cos nx\,\mathrm{d}x=-\alpha_n\ (n=0,1,2,\cdots)$

同理，得$b_n=\beta_n\ (n=1,2,\cdots)$

18. 设$f(t)$在$(-\pi,\pi)$上分段连续，当$t=0$连续且有单侧导数，证明当$p\to\infty$时

$\displaystyle\int_{-\pi}^{\pi}f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t\to\frac{1}{2}\int_0^{\pi}[f(t)-f(-t)]\cot\frac{t}{2}\,\mathrm{d}t$

证明: $\displaystyle\int_{-\pi}^{\pi}f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t=\int_{-\pi}^{0}f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t+\int_0^{\pi}f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

在右端前一积分中令$t=-x$，则$\displaystyle\int_{-\pi}^{0}f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t=-\int_0^{\pi}f(-t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

代回原式，得$\displaystyle\int_{-\pi}^{\pi}f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t=-\int_0^{\pi}f(-t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t+\int_0^{\pi}f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t=$
$\displaystyle\frac{1}{2}\int_0^{\pi}[f(t)-f(-t)]\cot\frac{t}{2}\,\mathrm{d}t-\frac{1}{2}\int_0^{\pi}\frac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt\,\mathrm{d}t$

下证 $\lim\limits_{p\to\infty}\int_0^\pi \dfrac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt\,\mathrm{d}t=0$

因 $\int_0^\pi \dfrac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt\,\mathrm{d}t=\left[\int_0^\delta+\int_\delta^\pi\right]\dfrac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt\,\mathrm{d}t$(其中$0<\delta<\pi$)

对于 $\int_\delta^\pi \dfrac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt\,\mathrm{d}t$

因$f(t)$在$(-\pi,\pi)$上分段连续，$\dfrac{1}{2\sin\frac{t}{2}}$在$(\delta,\pi)$上连续，则$\dfrac{f(t)-f(-t)}{2\sin\frac{t}{2}}$在$(\delta,\pi)$上分段连续因而可积

则由黎曼引理，得 $\lim\limits_{p\to\infty}\int_\delta^\pi \dfrac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt\,\mathrm{d}t=0$

对于 $\int_0^\delta \dfrac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt\,\mathrm{d}t$

$$\lim_{p\to\infty}\int_0^\delta \frac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt\,\mathrm{d}t=\lim_{p\to\infty}\int_0^\delta[f(t)-f(-t)]\left(\frac{1}{2\sin\frac{t}{2}}-\frac{1}{t}\right)\cos pt\,\mathrm{d}t+\lim_{p\to\infty}\int_0^\delta \frac{f(t)-f(-t)}{t}\cos pt\,\mathrm{d}t$$

因$\lim\limits_{t\to0}\left(\dfrac{1}{2\sin\frac{t}{2}}-\dfrac{1}{t}\right)=0$，补充定义，$t=0$时，函数$\dfrac{1}{2\sin\frac{t}{2}}-\dfrac{1}{t}$的值为0，则$\dfrac{1}{2\sin\frac{t}{2}}-\dfrac{1}{t}$是$[0,\delta]$上的连续函数

又$f(t)$为$(-\pi,\pi)$上的分段连续函数，则$[f(t)-f(-t)]\left(\dfrac{1}{\sin\frac{t}{2}}-\dfrac{1}{t}\right)$在$[0,\delta]$上分段连续，因而可积，则由黎曼引理，得 $\lim\limits_{p\to\infty}\int_0^\delta \dfrac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt\,\mathrm{d}t=0$

因$f'(+0),f'(-0)$存在，则 $\lim\limits_{t\to+0}\dfrac{f(t)-f(-t)}{t}=f'(+0)+f'(-0)$存在

补充定义，$t=0$时，函数$\dfrac{f(t)-f(-t)}{t}$值为$f'(+0)+f'(-0)$，则$\dfrac{f(t)-f(-t)}{t}$是$[0,\delta]$上的分段函数，因而可积，于是由黎曼引理，得 $\lim\limits_{p\to\infty}\int_0^\delta \dfrac{f(t)-f(-t)}{t}\cos pt\,\mathrm{d}t=0$

综上可得，当$p\to\infty$时，$\int_0^\pi \dfrac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt\,\mathrm{d}t\to\dfrac{1}{2}\int_0^\pi[f(t)-f(-t)]\cot\dfrac{t}{2}\,\mathrm{d}t$

19. 设$T_n(x)=\dfrac{1}{2}+\sum\limits_{v=1}^n\cos vx, T_0(x)=\dfrac{1}{2}, \sigma_n(x)=\dfrac{T_0(x)+\cdots+T_n(x)}{n+1}$

证明

(1) $\sigma_n(x)=\dfrac{1}{2(n+1)}\left(\dfrac{\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}\right)^2$

(2) $\int_{-\pi}^\pi\sigma_n(x)\,\mathrm{d}x=\pi$

证明：

(1) 因$2\sin\dfrac{x}{2}\left(\dfrac{1}{2}+\sum\limits_{v=1}^n\cos vx\right)=\sin\dfrac{2n+1}{2}x$，则$T_n(x)=\dfrac{\sin\frac{2n+1}{2}x}{2\sin\frac{x}{2}}$

于是
$$\sigma_n(x)=\frac{T_0(x)+\cdots+T_n(x)}{n+1}=\frac{\frac{1}{2}+\sum\limits_{k=1}^n T_k(x)}{n+1}=$$
$$\frac{1}{n+1}\left(\frac{1}{2}+\sum_{k=1}^n\frac{\sin\frac{2n+1}{2}x}{2\sin\frac{x}{2}}\right)=\frac{1}{2(n+1)\sin^2\frac{x}{2}}\left(\sin^2\frac{x}{2}+\sum_{k=1}^n\sin\frac{x}{2}\sin\frac{2k+1}{2}x\right)=\frac{1}{2(n+1)\sin^2\frac{x}{2}}\left[\frac{1}{2}-\frac{1}{2}\cos(n+1)x\right]=$$
$$\frac{1}{2(n+1)}\left(\frac{\sin\frac{2n+1}{2}x}{2\sin\frac{x}{2}}\right)^2$$

(2)
$$\int_{-\pi}^\pi\sigma_n(x)\,\mathrm{d}x=\int_{-\pi}^\pi\frac{\frac{1}{2}+\sum\limits_{k=1}^n T_k(x)}{n+1}\,\mathrm{d}x=\frac{1}{n+1}\int_{-\pi}^\pi\left[\frac{1}{2}+\sum_{k=1}^n\left(\frac{1}{2}+\sum_{v=1}^n\cos vx\right)\right]\mathrm{d}x=$$
$$\frac{1}{n+1}\left[\pi+\sum_{k=1}^n\left(\pi+\sum_{v=1}^n\int_{-\pi}^\pi\cos vx\,\mathrm{d}x\right)\right]=\pi.$$

20. 设$\varphi(x)$在$[a,b]$上为单调增加函数，证明

(1) 如果$a=0,b<0$，有$\frac{1}{\pi}\int_a^b \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z \to -\frac{1}{2}\varphi(-0)\ (p\to\infty)$

(2) 如果$a<0,b>0$，有$\frac{1}{\pi}\int_a^b \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z \to \frac{\varphi(+0)+\varphi(-0)}{2}\ (p\to\infty)$

证明：

(1) 因$\varphi(x)$在$[a,b]$上为单调增加函数，则$\varphi(-t)$在$[-b,-a]$上为单调减少函数
当$a=0,b<0$时，$\varphi(-t)$在$[0,-b](-b>0)$上为单调增加函数
对$\int_a^b \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z$作变量代换$z=-t$，则$\int_a^b \varphi(z)\frac{\sin pz}{z}=-\int_0^{-b}\varphi(-t)\frac{\sin pt}{t}\,\mathrm{d}t$
则由狄立克莱引理，得$\lim\limits_{p\to\infty}\int_0^{-b}\varphi(-t)\frac{\sin pt}{t}\,\mathrm{d}t=\frac{\pi}{2}\varphi(-0)$即$\lim\limits_{p\to\infty}\int_a^b \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z=-\frac{\pi}{2}\varphi(-0)$
于是$\frac{1}{\pi}\int_a^b \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z\to -\frac{1}{2}\varphi(-0)\ (p\to\infty)$

(2) 因$a<0,b>0$，$\varphi(x)$在$[a,b]$上为单调增加函数，$\int_a^b \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z=\int_a^0 \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z+\int_0^b \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z$
据(1)，得$\lim\limits_{p\to\infty}\int_0^a \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z=-\frac{\pi}{2}\varphi(-0)$，则$\lim\limits_{p\to\infty}\int_a^0 \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z=\frac{\pi}{2}\varphi(-0)$
又由狄立克莱引理，得$\lim\limits_{p\to\infty}\int_0^b \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}t=\frac{\pi}{2}\varphi(+0)$
则$\lim\limits_{p\to\infty}\int_a^b \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z=\frac{\pi}{2}[\varphi(+0)+\varphi(-0)]$
于是$\frac{1}{\pi}\int_a^b \varphi(z)\frac{\sin pz}{z}\,\mathrm{d}z\to\frac{\varphi(-0)+\varphi(+0)}{2}\ (p\to\infty)$

§2. 富里埃变换

1. 设$f(x)$在$(-\infty,+\infty)$内绝对可积，证明$\widehat{f}(\omega)$在$(-\infty,+\infty)$内连续.

证明：对$\forall\omega\in(-\infty,+\infty)$，总有$A',A''$，使得$\omega\in[A',A'']$

由于$\left|\widehat{f}(\omega)\right|=\left|\int_{-\infty}^{+\infty}f(x)e^{-i\omega x}\,\mathrm{d}x\right|\leqslant\int_{-\infty}^{+\infty}|f(x)|\,\mathrm{d}x$

后者收敛且不含参量ω，这表明积分$\widehat{f}(\omega)=\int_{-\infty}^{+\infty}f(x)e^{-i\omega x}\,\mathrm{d}x$在$[A',A'']$上一致收敛

据一致收敛积分的连续性，得$\widehat{f}(\omega)$在$[A',A'']$上连续，从而在点ω处连续

由ω的任意性，得$\widehat{f}(\omega)$在$(-\infty,+\infty)$内连续.

2. 设$f(x)$在$(-\infty,+\infty)$内绝对可积，证明$\lim\limits_{\omega\to\infty}\widehat{f}(\omega)=0$.

证明：由$f(x)$在$(-\infty,+\infty)$内绝对可积，得对于任给的$\varepsilon>0$，存在$A>0$，使有$\int_{A}^{+\infty}|f(x)|\,\mathrm{d}x<\dfrac{\varepsilon}{3}$

$\left|\int_{A}^{+\infty}f(x)\sin\omega x\,\mathrm{d}x\right|\leqslant\int_{A}^{+\infty}|f(x)|\,\mathrm{d}x<\dfrac{\varepsilon}{3}$

设$f(x)$在$[0,A]$内无瑕点，则在$[0,A]$中插入分点$0=t_0<t_1<\cdots<t_m=A$，并设$f(x)$在$[t_{k-1},t_k]$上的下确界为m_k，于是

$$\int_{0}^{A}f(x)\sin\omega x\,\mathrm{d}x=\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}f(x)\sin\omega x\,\mathrm{d}x=\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}[f(x)-m_k]\sin\omega x\,\mathrm{d}x+\sum_{k=1}^{m}m_k\int_{t_{k-1}}^{t_k}\sin\omega x\,\mathrm{d}x$$

从而$\left|\int_{0}^{A}f(x)\sin\omega x\,\mathrm{d}x\right|\leqslant\sum_{k=1}^{m}\omega_k\Delta t_k+\sum_{k=1}^{m}|m_k|\dfrac{|\cos nt_{k-1}-\cos nt_k|}{n}\leqslant\sum_{k=1}^{m}\omega_k\Delta t_k+\dfrac{2}{\omega}\sum_{k=1}^{m}|m_k|$

其中ω_k为$f(x)$在区间$[t_{k-1},t_k]$上的振幅，$\Delta t_k=t_k-t_{k-1}$

由于$f(x)$在$[0,A]$上可积，故可取某一分法，使有$\left|\sum_{k=1}^{m}\omega_k\Delta t_k\right|<\dfrac{\varepsilon}{3}$

对于这样固定的分法，$\sum_{k=1}^{m}|m_k|$为一定值，因而存在$\delta>0$，使当$\omega>\delta$时，恒有$\dfrac{2}{\omega}\sum_{k=1}^{m}|m_k|<\dfrac{\varepsilon}{3}$

于是对上述所选取的δ，当$\omega>\delta$时

$\left|\int_{0}^{+\infty}f(x)\sin\omega x\,\mathrm{d}x\right|\leqslant\left|\int_{0}^{A}f(x)\sin\omega x\,\mathrm{d}x\right|+\left|\int_{A}^{+\infty}f(x)\sin\omega x\,\mathrm{d}x\right|\leqslant\varepsilon$即$\lim\limits_{\omega\to\infty}\int_{0}^{+\infty}f(x)\sin\omega x\,\mathrm{d}x=0$

其次，设$f(x)$在区间$[0,A]$中有瑕点，为简便起见，不妨设只有一个瑕点且为0

于是对任给的$\varepsilon>0$，存在$\eta>0$，使有$\int_{0}^{\eta}|f(x)|\,\mathrm{d}x<\dfrac{\varepsilon}{3}$

又$f(x)$在$[\eta,A]$上无瑕点，故应用上述结果可得存在δ，使当$\omega>\delta$时，恒有$\left|\int_{\eta}^{A}f(x)\sin\omega x\,\mathrm{d}x\right|<\dfrac{\varepsilon}{3}$

于是当$\omega>\delta$时，有$\left|\int_{0}^{+\infty}f(x)\sin\omega x\,\mathrm{d}x\right|\leqslant\int_{0}^{\eta}|f(x)|\,\mathrm{d}x+\left|\int_{\eta}^{A}f(x)\sin\omega x\,\mathrm{d}x\right|+\int_{A}^{+\infty}|f(x)|\,\mathrm{d}x<\varepsilon$

即$\lim\limits_{\omega\to\infty}\int_{0}^{+\infty}f(x)\sin\omega x\,\mathrm{d}x=0$

同法，得当$f(x)$在$(-\infty,+\infty)$内绝对可积时，均有$\lim\limits_{\omega\to\infty}\int_{-\infty}^{+\infty}f(x)\sin\omega x\,\mathrm{d}x=0$

同法可证得当$f(x)$在$(-\infty,+\infty)$内绝对可积时，$\lim\limits_{\omega\to\infty}\int_{-\infty}^{+\infty}f(x)\cos\omega x\,\mathrm{d}x=0$

于是$\lim\limits_{\omega\to\infty}\widehat{f}(x)=0$.

3. 求下列函数的富里埃变换：

(1) $f(x)=\begin{cases}E\sin\omega_0 x, & |x|<\dfrac{\pi}{\omega_0}\\ 0, & |x|\geqslant\dfrac{\pi}{\omega_0}\end{cases}$

(2) $f(x)=\begin{cases}0, & -\infty<x\leqslant-\dfrac{\pi}{2}\\ \dfrac{2h}{\tau}x+h, & -\dfrac{\tau}{2}<x<0\\ -\dfrac{2h}{\tau}x+h, & 0\leqslant x<\dfrac{\tau}{2}\\ 0, & \dfrac{\tau}{2}\leqslant x<+\infty\end{cases}$

解：

(1) $\widehat{f}(\omega)=\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,\mathrm{d}x=\int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}}E\sin\omega_0xe^{-i\omega x}\,\mathrm{d}x=E\int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}}\sin\omega_0x(\cos\omega x-i\sin\omega x)\,\mathrm{d}x=$
$2Ei\int_0^{\frac{\pi}{\omega_0}}\sin\omega_0x\sin\omega x\,\mathrm{d}x=iE\int_0^{\frac{\pi}{\omega_0}}[\cos(\omega_0+\omega)x-\cos(\omega-\omega_0)x]\,\mathrm{d}x=iE\left(\left.\frac{\sin(\omega_0+\omega)x}{\omega_0+\omega}\right|_0^{\frac{\pi}{\omega_0}}-\left.\frac{\sin(\omega-\omega_0)x}{\omega-\omega_0}\right|_0^{\frac{\pi}{\omega_0}}\right)=$
$\frac{2E\omega_0i}{\omega^2-\omega_0^2}\,\sin\frac{\omega}{\omega_0}\,\pi(\omega\neq\pm\omega_0)$

因$\widehat{f}(\omega)$为$(-\infty,+\infty)$内的连续函数，则$\widehat{f}(\mp\omega_0)=\lim\limits_{\omega\to\mp\omega_0}\widehat{f}(\omega)=\pm\frac{iE\pi}{\omega_0}$.

(2) $\widehat{f}(\omega)=\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,\mathrm{d}x=\int_{-\frac{\tau}{2}}^{0}\left(\frac{2h}{\tau}x+h\right)e^{-i\omega x}\,\mathrm{d}x+\int_0^{\frac{\tau}{2}}\left(-\frac{2h}{\tau}x+h\right)e^{-i\omega x}\,\mathrm{d}x=$
$\frac{2h}{\tau}\left[\int_{-\frac{\tau}{2}}^{0}xe^{-i\omega x}\,\mathrm{d}x-\int_0^{\frac{\tau}{2}}xe^{-i\omega x}\,\mathrm{d}x\right]+\frac{2h}{\omega}\,\sin\frac{\omega\tau}{2}=\frac{4h}{\tau\omega^2}-\frac{4h}{\tau\omega^2}\,\cos\frac{\omega\tau}{2}\,(\omega\neq0)$

因$\widehat{f}(\omega)$为$(-\infty,+\infty)$内的连续函数，则$\widehat{f}(0)=\lim\limits_{\omega\to0}\widehat{f}(\omega)=\frac{h\tau}{2}$.

第四篇　多变量微积分学

第一部分　多元函数的极限论

第十三章　多元函数的极限与连续

§1.　平面点集

1. 证明$(x_n,y_n)\to(x_0,y_0)$的充要条件是：$x_n\to x_0, y_n\to y_0(n\to\infty)$

证明：$\Rightarrow$

因$\lim\limits_{n\to\infty}M_n=M_0$，则对$\forall\varepsilon>0,\exists N\in Z^+$，当$n>N$时，有$r(M_n,M_0)<\varepsilon$

即$\sqrt{(x_n-x_0)^2+(y_n-y_0)^2}<\varepsilon$

于是一定有$|x_n-x_0|\leqslant r(M_n,M_0)<\varepsilon, |y_n-y_0|\leqslant r(M_n,M_0)<\varepsilon$即$x_n\to x_0, y_n\to y_0(n\to\infty)$

$\Leftarrow$

因$(|x_n-x_0|+|y_n-y_0|)^2\geqslant|x_n-x_0|^2+|y_n-y_0|^2$即$0\leqslant\sqrt{|x_n-x_0|^2+|y_n-y_0|^2}\leqslant|x_n-x_0|+|y_n-y_0|$

又$x_n\to x_0, y_n\to y_0(n\to\infty)$，则$\sqrt{|x_n-x_0|^2+|y_n-y_0|^2}\to 0(n\to\infty)$即$(x_n,y_n)\to(x_0,y_0)(n\to\infty)$

2. 证明：若平面上的点列$\{M_n\}$收敛，则它只有一个极限.

证明：设$\lim\limits_{n\to\infty}M_n=M_0$，假设又有$\lim\limits_{n\to\infty}M_n={M_0}'$

由定义，对$\forall\varepsilon>0,\exists N\in Z^+$，当$n>N$时，有$r(M_n,M_0)<\dfrac{\varepsilon}{2}, r(M_n,{M_0}')<\dfrac{\varepsilon}{2}$

由三角不等式，有$r(M_0,{M_0}')\leqslant r(M_n,M_0)+r(M_n,{M_0}')<\varepsilon$

又$M_0,{M_0}'$为固定的两点，由ε的任意性，得$r(M_0,{M_0}')=0$即$M_0={M_0}'$.

3. 证明：若$M_n\to M_0(n\to\infty)$，那么它的任何一个子列$M_{n_k}\to M_0$.

证明：因$M_n\to M_0(n\to\infty)$，则对$\forall\varepsilon>0,\exists N\in Z^+$，当$n>N$时，有$r(M_n,M_0)<\varepsilon$

今取$K=N$，则对一切$k>K$，有$n_k>n_K=n_N\geqslant N$，自然有$r(M_{n_k},M_0)<\varepsilon$即$M_{n_k}\to M_0(k\to\infty)$.

4. 求下列点集E的内点，外点，边界点：

(1) E由满足$y<x^2$的点所组成；

(2) E由满足$1\leqslant x^2+\dfrac{y^2}{4}<4$的点所组成；

(3) E由满足$0<x^2+y^2<1$的点所组成；

(4) E由所有这样的点(x,y)所组成，其中x和y都是有理数.

解：

(1) 凡满足$y<x^2$的点(x,y)是E的内点；凡满足$y>x^2$的点(x,y)是E的外点；凡满足$y=x^2$的点(x,y)是E的边界点.

(2) 凡满足$1<x^2+\dfrac{y^2}{4}<4$的点(x,y)是E的内点；凡满足$x^2+\dfrac{y^2}{4}<1$或$x^2+\dfrac{y^2}{4}>4$的点(x,y)是E的外点；凡满足$x^2+\dfrac{y^2}{4}=1$或$x^2+\dfrac{y^2}{4}=4$的点(x,y)是E的边界点.

(3) 凡满足$0<x^2+y^2<1$的点(x,y)是E的内点；凡满足$x^2+y^2>1$的点(x,y)是E的外点；原点θ及满足$x^2+y^2=1$的点(x,y)是E的边界点.

(4) 由有理数及无理数的稠密性，得平面上所有点(x,y)都是E的边界点.

5. 证明：若M_0是平面点集E的聚点，则在E中存在点列$M_n\to M_0(n\to\infty)$.

证明：已知M_0是平面点集E的聚点，取$\delta_n=\dfrac{1}{n}$，在$O(M_0,\delta_1)$中定存在E的点$M_1\neq M_0$；在$O(M_0,\delta_2)$中定存在E的点$M_2, M_2\neq M_i(i\neq 0,1)$

如此进行下去，得到点列$\{M_n\}(M_n\neq M_i)(i=0,1,\cdots,n-1)$且$r(M_0,M_n)<\dfrac{1}{n}$

于是当$n\to\infty$时，$r(M_0,M_n)\to 0$即$M_n\to M_0(n\to\infty)$.

6. 证明平面点列的收敛原理.

证明：$\Rightarrow$

设$M_n\to M_0(n\to\infty)$，则对$\forall\varepsilon>0,\exists N\in Z^+$，当$n,m>N$时，有$r(M_n,M_0)<\dfrac{\varepsilon}{2}, r(M_m,M_0)<\dfrac{\varepsilon}{2}$

由距离的三角不等式，得$r(M_m,M_n)\leqslant r(M_n,M_0)+r(M_m,M_0)<\varepsilon$

$\Leftarrow$

设点列$\{M_n\}$满足对$\forall\varepsilon>0,\exists N\in Z^+$，当$n,m>N$时，有$r(M_n,M_m)<\varepsilon$

将$\{M_n\}$分别投影到两根坐标轴上，得数列$\{x_n\},\{y_n\}$

因$|x_m-x_n|<r(M_m,M_n)<\varepsilon,|y_m-y_n|<r(M_m,M_n)<\varepsilon$

由R^1上的柯西收敛原理，得$\{x_n\},\{y_n\}$都收敛

设$x_n\to x_0,y_n\to y_0(n\to\infty)$，则$\lim\limits_{n\to\infty}M_n=M_0(M_0(x_0,y_0))$即$\{M_n\}$收敛.

7. 用平面上的有限覆盖定理证明魏尔斯特拉斯定理.

证明:

(1) 若$\{M_n(x_n,y_n)\}$是有界有限点集，定理成立;

(2) 若$\{M_n(x_n,y_n)\}$是有界无穷点集，据5，只需证$E=\{M_n(x_n,y_n)\big|n=1,2,\cdots\}$中至少有一个聚点.

反证.设E没有聚点.

由于$a\leqslant x_n\leqslant b,c\leqslant y_n\leqslant d(n=1,2,\cdots)$，而矩形域$R=\{(x,y)\big|a\leqslant x\leqslant b,c\leqslant d\}$是有界闭区域且$E\subset R$

对$\forall M(x,y)\in R$，都不是E的聚点，因而存在δ_M，使得$O(M,\delta_M)$至多有E中有限个点，

$\{O(M,\delta_M)\big|M\in R\}$覆盖$R$

据有限覆盖定理，存在有限个开集$O(M_1,\delta_{M_1}),\cdots,O(M_k,\delta_{M_k})$同样覆盖$R$，其中每个$O(M_i,\delta_{M_i})(i=1,2,\cdots,k)$中至多有有限个$E$中的点

于是$\bigcup\limits_{i=1}^{k}O(M_i,\delta_{M_i})$至多含$E$中有限个点

但由于$\bigcup\limits_{i=1}^{k}O(M_i,\delta_{M_i})\supset R\supset E$，于是矛盾.

§2. 多元函数的极限和连续性

1. 确定并绘出下列函数之定义域：

(1) $u=\sqrt{x}-\sqrt{1-y}$

(2) $u=\sqrt{x-y+1}$

(3) $u=\ln(-x-y)$

(4) $u\sqrt{\sin(x^2+y^2)}$

(5) $u\sqrt{R^2-x^2-y^2-z^2}+\sqrt{x^2+y^2+z^2-r^2}$

解：

(1) 定义域为$x\geqslant 0$且$y\leqslant 1$

(2) 定义域为满足不等式$y\leqslant x+1$的点集

(3) 定义域为半平面$x+y<0$

(4) 定义域为满足不等式$2k\pi\leqslant x^2+y^2\leqslant(2k+1)\pi(k=0,1,2,\cdots)$的点集

(5) 定义域为满足不等式$r^2\leqslant x^2+y^2+z^2\leqslant R^2$的点集

2. 求下列极限：

(1) $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^2+y^2}{|x|+|y|}$

(2) $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

(3) $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{1+x^2+y^2}{x^2+y^2}$

(4) $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\sin(x^3+y^3)}{x^2+y^2}$

(5) $\lim\limits_{\substack{x\to +\infty\\ y\to +\infty}}(x^2+y^2)e^{-(x+y)}$

(6) $\lim\limits_{\substack{x\to 1\\ y\to 0}}\dfrac{\ln(x+e^y)}{\sqrt{x^2+y^2}}$

解：

(1) 因$0\leqslant\dfrac{x^2+y^2}{|x|+|y|}\leqslant\dfrac{(|x|+|y|)^2}{|x|+|y|}=|x|+|y|$且$\lim\limits_{\substack{x\to 0\\ y\to 0}}(|x|+|y|)=0$，则$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^2+y^2}{|x|+|y|}=0$

(2) 因$\lim\limits_{t\to+0}\dfrac{t}{\sqrt{t+1}-1}=\lim\limits_{t\to+0}(\sqrt{t+1}+1)=2$，则$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}=2$

(3) 因$\lim\limits_{t\to+0}\dfrac{1+t}{t}=+\infty$，则$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{1+x^2+y^2}{x^2+y^2}=+\infty$

(4) 因$0\leqslant\left|\dfrac{\sin(x^3+y^3)}{x^2+y^2}\right|\leqslant\dfrac{|x^3+y^3|}{x^2+y^2}\leqslant\dfrac{|x|^3+|y|^3}{x^2+y^2}=\dfrac{|x|^3}{x^2+y^2}+\dfrac{|y|^3}{x^2+y^2}\leqslant|x|+|y|$且$\lim\limits_{\substack{x\to 0\\ y\to 0}}(|x|+|y|)=0$

则$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\sin(x^3+y^3)}{x^2+y^2}=0$

(5) 因$\lim\limits_{t\to+\infty}\dfrac{t}{e^t}=0,\lim\limits_{t\to+\infty}\dfrac{t^2}{e^t}=0$

则$\lim\limits_{\substack{x\to +\infty\\ y\to +\infty}}(x^2+y^2)e^{-(x+y)}=\lim\limits_{\substack{x\to +\infty\\ y\to +\infty}}\left[\dfrac{(x+y)^2}{e^{-(x+y)}}-2\dfrac{x}{e^x}\cdot\dfrac{y}{e^y}\right]=0$

(6) $\lim\limits_{\substack{x\to 1\\ y\to 0}}\dfrac{\ln(x+e^y)}{\sqrt{x^2+y^2}}=\ln 2$

3. 试证若 $\lim\limits_{\substack{y\to a\\x\to b}} f(x,y)=A$存在，而当$x$取任何与$a$邻近之值时，极限$\lim\limits_{y\to b} f(x,y)=\varphi(x)$存在，则二次极限存在，且等于$A$：

$$\lim_{x\to a}\lim_{y\to b} f(x,y)=\lim_{\substack{y\to a\\x\to b}} f(x,y)=A$$

.

证明：因二重极限存在，则对$\forall\varepsilon>0,\exists\delta>0$，当$|x-a|<\delta,|y-b|<\delta$且$(x-a)^2+(y-b)^2\neq 0$时，恒有$|f(x,y)-A|<\varepsilon$

现在$0<|x-a|<\delta$中固定x，而在上式中令$y\to b$，即得$|\varphi(x)-A|\leqslant\varepsilon$，这就证明了$\lim\limits_{x\to a}\varphi(x)=A$

于是$\lim\limits_{x\to a}\lim\limits_{y\to b} f(x,y)=\lim\limits_{x\to a}\varphi(x)=A=\lim\limits_{\substack{y\to a\\x\to b}} f(x,y)$

4. (1) 试举出两个二次极限不相等的例子；

   (2) 试举出只有一个二次极限存在的例子；

   (3) 试举出二重极限存在，但二次极限不全存在的例子.

解：

(1) 例：$f(x,y)=\begin{cases}\dfrac{x-y}{x+y}, & x+y\neq 0\\ 0, & x+y=0\end{cases}$ 在点$(0,0)$的二次极限

$\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)=\lim\limits_{x\to 0}\dfrac{x}{x}=1,\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)=\lim\limits_{y\to 0}\dfrac{-y}{y}=-1$

则$\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)\neq\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$.

(2) 例：$f(x,y)=\dfrac{x\sin\frac{1}{x}+y}{x+y}$ 在点$(0,0)$的二次极限

$\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)=\lim\limits_{y\to 0}\dfrac{y}{y}=1$

但$\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)=\lim\limits_{x\to 0}\dfrac{x\sin\frac{1}{x}}{x}=\lim\limits_{x\to 0}\sin\frac{1}{x}$ 不存在.

(3) 例：$f(x,y)=\begin{cases}x\sin\dfrac{1}{y}, & y\neq 0\\ 0, & y=0\end{cases}$ 在点$(0,0)$的二次极限和二重极限

因$0\leqslant|f(x,y)|=\left|x\sin\dfrac{1}{y}\right|\leqslant|x|$，则$\lim\limits_{\substack{x\to 0\\y\to 0}} f(x,y)=0$即其二重极限存在

$\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)=0$，而当$y\to 0$时，$x\sin\dfrac{1}{y}$ 极限不存在，即$\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$不存在.

5. 讨论下列函数在点$(0,0)$的二次极限和二重极限：

   (1) $f(x,y)=\dfrac{x^2y^2}{x^2y^2+(x-y)^2}$

   (2) $f(x,y)=(x+y)\cdot\sin\dfrac{1}{x}\cdot\sin\dfrac{1}{y}$

解：

(1) $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)=0,\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)=0$

若按$y=kx\to 0$的方向取极限，则有$\lim\limits_{\substack{y=kx\\x\to 0}} f(x,y)=\lim\limits_{x\to 0}\dfrac{x^2k^2}{x^2k^2+(1-k)^2}$

特别的，分别取$k\neq 1$及$k=1$，便得到不同的极限0及1，因此$\lim\limits_{\substack{x\to 0\\y\to 0}} f(x,y)$不存在.

(2) 因$0\leqslant|f(x,y)|\leqslant|x+y|\leqslant|x|+|y|$，则$\lim\limits_{\substack{x\to 0\\y\to 0}} f(x,y)=0$即其二重极限存在

又$\lim\limits_{y\to 0}(x+y)\cdot\sin\dfrac{1}{x}\cdot\sin\dfrac{1}{y}$ 不存在$\left(当x\neq\dfrac{1}{k\pi}\right)(k=\pm 1,\pm 2,\cdots)$，$\lim\limits_{x\to 0}(x+y)\cdot\sin\dfrac{1}{x}\cdot\sin\dfrac{1}{y}$ 不存在$\left(当y\neq\dfrac{1}{k\pi}\right)(k=\pm 1,\pm 2,\cdots)$

即$\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$及$\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$都不存在.

6. 讨论下列函数的连续范围：

(1) $u=\dfrac{1}{\sqrt{x^2+y^2}}$

(2) $u=\ln(1-x^2-y^2)$

(3) $u=\dfrac{1}{\sin x\sin y}$

(4) $u=\ln\dfrac{1}{(x-1)^a+(y-b)^2+(z-c)^2}$

解：

(1) 函数$u=\dfrac{1}{\sqrt{x^2+y^2}}$在点$(0,0)$无定义，故原点$(0,0)$为此函数的不连续点，除此点外均连续；

(2) 单位圆内的点，即满足$x^2+y^2<1$的各点为函数$u=\ln(1-x^2-y^2)$的连续点；

(3) 连续范围为$x\neq m\pi, y\neq n\pi(m,n=0,\pm1,\pm2,\cdots)$.

(4) 除点$(a.b.c)$外均连续.

7. 证明函数

$$f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

分别对于每一变量x和y是连续的，但非关于二变量的连续函数.

证明：先固定$y=a\neq 0$，则得x的函数$g(x)=f(x,a)=\dfrac{2ax}{x^2+a^2}\,(-\infty<x<+\infty)$
它是处处有定义的有理函数
又当$y=0$时，$f(x,0)\equiv 0$，它显然是连续的
于是当变数y固定时，函数$f(x,y)$对于变数x是连续的
同理可证，当变数x固定时，函数$f(x,y)$对于变数y是连续的
作为二元函数，$f(x,y)$虽在除点$(0,0)$外的各点均连续，但在点$(0,0)$不连续
当动点$P(x,y)$沿射线$y=mx$趋于原点时，有$\lim\limits_{\substack{y=mx\\x\to 0}}f(x,y)=\lim\limits_{x\to 0}\dfrac{2mx^2}{(1+m^2)x^2}=\dfrac{2m}{1+m^2}$
取不同的m，则极限值不同，说明其二重极限不存在，于是$\lim\limits_{\substack{x\to 0\\y\to 0}}f(x,y)\neq f(0,0)$
则其关于二变量的函数在$(0,0)$点不连续，从而其非关于二变量的连续函数.

8. 证明函数

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

在$(0,0)$点沿每一条射线$x=t\cos\theta, y=t\sin\theta(0\leqslant t+\infty)$连续，但它在$(0,0)$点不连续.

证明：当$\sin\theta=0$时，$\cos\theta=1$或-1，于是当$t\neq 0$时，$f(t\cos\theta,t\sin\theta)=0$，而$f(0,0)=0$
则有$\lim\limits_{t\to 0}f(t\cos\theta,t\sin\theta)=f(0,0)$
当$\sin\theta\neq 0$，有$\lim\limits_{t\to 0}f(t\cos\theta,t\sin\theta)=0$，故有$\lim\limits_{t\to 0}f(t\cos\theta,t\sin\theta)=f(0,0)$
其次，设动点$P(x,y)$沿抛物线$y=x^2$趋于原点，得$\lim\limits_{\substack{y=x^2\\x\to 0}}f(x,y)=\dfrac{1}{2}\neq f(0,0)$，则函数$f(x,y)$在点$(0,0)$不连续.

9. 若$f(x,y)$在某一区域G内对变量x为连续，对变量y满足李普希兹条件，即对任何

$$(x,y')\in G,(x,y'')\in G$$

有$|f(x,y')-f(x,y'')|\leqslant L|y'-y''|$
其中L为常数，则此函数在G内连续.

证明：因$f(x,y)$在区域G内对变量x为连续，则对G内任一点(x_0,y_0)，对$\forall\varepsilon<0,\exists\delta_1>0$，当$|x-x_0|<\delta_1$时，有$|f(x,y_0)-f(x_0,y_0)|<\dfrac{\varepsilon}{2}$
又因$f(x,y)$在G内对y满足李普希兹条件，则对任何$(x,y)\in G,(x,y_0)\in G$，有$|f(x,y)-f(x,y_0)|\leqslant L|y-y_0|$

令$L|y-y_0|<\dfrac{\varepsilon}{2}$，则$|y-y_0|<\dfrac{\varepsilon}{2L}$

取$\delta=\min\left(\delta_1,\dfrac{\varepsilon}{2L}\right)$，当$|x-x_0|<\delta,|y-y_0|<\delta$时，定有

$$|f(x,y)-f(x_0,y_0)|\leqslant|f(x,y)-f(x,y_0)|+|f(x,y_0)-f(x_0,y_0)|<\varepsilon$$

即此函数在G内连续.

第十四章　偏导数和全微分

§1.　偏导数和全微分的概念

1. 求下列函数的偏导数：

(1) $z = x^2\ln(x^2+y^2)$

(2) $u = e^{xy}$

(3) $z = xy + \dfrac{x}{y}$

(4) $u = \arctan\dfrac{y}{x}$

(5) $u = x^2+y^2+z^2+2xy+2yz+2zx$

(6) $u = e^{\varphi-\theta}\cos(\theta+\varphi)$

解：

(1) $z_x = 2x\left[\ln(x^2+y^2)+\dfrac{x^2}{x^2+y^2}\right], z_y = \dfrac{2x^2y}{x^2+y^2}.$

(2) $u_x = ye^{xy}, u_y = xe^{xy}.$

(3) $z_x = y+\dfrac{1}{y}, z_y = \dfrac{x(y^2-1)}{y^2}.$

(4) $u_x = -\dfrac{y}{x^2+y^2}, u_y = \dfrac{x}{x^2+y^2}.$

(5) $u_x = 2(x+y+z), u_y = 2(x+y+z), u_z = 2(x+y+z).$

(6) $u_\varphi = e^{\varphi-\theta}[\cos(\theta+\varphi)-\sin(\theta+\varphi)], u_\theta = -e^{\varphi-\theta}[\sin(\theta+\varphi)+\cos(\theta+\varphi)].$

2. 设$f(x,y) = x^2y^2-2y$，求$f_x(x,y), f_y(x,y), f_x(2,3), f_y(0,0), f_y(x,y)\Big|_{\substack{x=y\\y=x}}$.

解：$f_x(x,y) = 2xy^2, f_y(x,y) = 2x^2y-2, f_x(2,3) = 36, f_y(0,0) = -2, f_y(x,y)\Big|_{\substack{x=y\\y=x}} = 2xy^2-2$

3. 设$z = \ln(\sqrt{x}+\sqrt{y})$，证明$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y} = \dfrac{1}{2}$.

证明：因$z = \ln(\sqrt{x}+\sqrt{y})$，则$\dfrac{\partial z}{\partial x} = \dfrac{1}{2\sqrt{x}(\sqrt{x}+\sqrt{y})}, \dfrac{\partial z}{\partial y} = \dfrac{1}{2\sqrt{y}(\sqrt{x}+\sqrt{y})}$

于是$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y} = \dfrac{1}{2}$.

4. 求下列函数在给定点(x_0,y_0)的全微分：

(1) $u = x^4+y^4-4x^2y^2,\ (0,0),(1,1)$

(2) $u = \dfrac{x}{\sqrt{x^2+y^2}}, (1,0),(0,1)$

(3) $u = x\sin(x+y),\ (0,0),\left(\dfrac{\pi}{4},\dfrac{\pi}{4}\right)$

(4) $u = \ln(x+y^2), (0,1),(1,1)$

解：

(1) 因$\mathrm{d}u = 4x(x^2-2y^2)\,\mathrm{d}x+4y(y^2-2x^2)\,\mathrm{d}y$，则
在$(0,0)$点$\mathrm{d}u = 0$；在$(1,1)$点$\mathrm{d}u = -4\,\mathrm{d}x-4\,\mathrm{d}y$.

(2) 因$\mathrm{d}u = \dfrac{y^2}{(x^2+y^2)^{\frac{3}{2}}}\,\mathrm{d}x-\dfrac{xy}{(x^2+y^2)^{\frac{3}{2}}}\,\mathrm{d}y$，则
在$(1,0)$点$\mathrm{d}u = 0$；在$(0,1)$点$\mathrm{d}u = \mathrm{d}x$.

(3) 因$\mathrm{d}u = [\sin(x+y)+x\cos(x+y)]\,\mathrm{d}x+x\cos(x+y)\,\mathrm{d}y$，则
在$(0,0)$点$\mathrm{d}u = 0$；在$\left(\dfrac{\pi}{4},\dfrac{\pi}{4}\right)$点$\mathrm{d}u = \mathrm{d}x$.

(4) 因 $\mathrm{d}u = \dfrac{\mathrm{d}x}{x+y^2} + \dfrac{2y}{x+y^2}\,\mathrm{d}y$，则

在$(0,1)$点 $\mathrm{d}u = \mathrm{d}x + 2\,\mathrm{d}y$；在$(1,1)$点 $\mathrm{d}u = \dfrac{\mathrm{d}x}{2} + \mathrm{d}y$.

5. 求下列函数的全微分：

(1) $u = \sin(x^2+y^2)$

(2) $u = x^m \cdot y^n$

(3) $u = e^{xy}$

(4) $u = x^y$

(5) $u = \sqrt{x^2+y^2+z^2}$

(6) $u = \ln(x^2+y^2+z^2)$

解：

(1) $\mathrm{d}u = 2\cos(x^2+y^2)(x\,\mathrm{d}x + y\,\mathrm{d}y)$

(2) $\mathrm{d}u = x^{m-1}y^{n-1}(my\,\mathrm{d}x + nx\,\mathrm{d}y)$

(3) $\mathrm{d}u = e^{xy}(y\,\mathrm{d}x + x\,\mathrm{d}y)$

(4) $\mathrm{d}u = x^{y-1}(y\,\mathrm{d}x + x\ln x\,\mathrm{d}y)$

(5) $\mathrm{d}u = \dfrac{x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z}{\sqrt{x^2+y^2+z^2}}$

(6) $\mathrm{d}u = \dfrac{2(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)}{x^2+y^2+z^2}$

6. 证明：$f(x,y) = \sqrt{|xy|}$在$(0,0)$连续，$f_x(0,0), f_y(0,0)$存在，但在$(0,0)$点不可微.

证明：由 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \sqrt{|xy|} = 0$，得 $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0 = f(0,0)$，则$f(x,y)$在$(0,0)$点连续

因$f_x(0,0) = \lim\limits_{\Delta x\to 0} \dfrac{f(\Delta x,0)-f(0,0)}{\Delta x} = 0, f_y(0,0) = \lim\limits_{\Delta y\to 0} \dfrac{f(0,\Delta y)-f(0,0)}{\Delta y} = 0$

则$f_x(0,0), f_y(0,0)$存在

但$f(x,y) = \sqrt{|xy|}$在$(0,0)$点不可微.若可微，则有$\Delta f = f_x(0,0)\Delta x + f_y(0,0)\Delta y + o(\rho)$即$\Delta f = o(\rho)$

考虑点$P(x,y)$沿$y = x$趋于0时，有$\dfrac{\Delta f}{\rho} = \dfrac{\sqrt{|\Delta x\Delta y|}}{\sqrt{\Delta x^2+\Delta y^2}} = \dfrac{1}{\sqrt{2}} \not\to 0(\rho\to 0)$矛盾，于是假设不成立，则$f(x,y)$在$(0,0)$点不可微.

7. 证明：$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$ 在$(0,0)$点的邻域中连续，$f_x(x,y), f_y(x,y)$有界，但在$(0,0)$点不可微.

证明：由于$\dfrac{xy}{\sqrt{x^2+y^2}}$是二元初等函数，在其定义域内必连续，则$f(x,y)$在$x^2+y^2\neq 0$连续

又$0 < \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| = \dfrac{|xy|}{\sqrt{x^2+y^2}} \leqslant \dfrac{\sqrt{x^2+y^2}}{2}$，$f(0,0) = 0$，则 $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = f(0,0)$，于是$f(x,y)$在$(0,0)$点连续，从而$f(x,y)$在$(0,0)$点的任何邻域内连续

因$f_x(0,0) = \lim\limits_{\Delta x\to 0} \dfrac{f(\Delta x,0)-f(0,0)}{\Delta x} = 0, f_y(0,0) = \lim\limits_{\Delta y\to 0} \dfrac{f(0,\Delta y)-f(0,0)}{\Delta y} = 0$

当$x^2+y^2\neq 0$时，$f_x(x,y) = \dfrac{y^3}{(x^2+y^2)^{\frac{3}{2}}}$，$|f_x(x,y)| = \left|\dfrac{y^3}{(x^2+y^2)^{\frac{3}{2}}}\right| \leqslant 1$，则$f_x(x,y)$有界

同理可得$f_y(x,y)$有界

但$f(x,y)$在$(0,0)$点不可微.若可微，则有$\Delta f = f_x(0,0)\Delta x + f_y(0,0)\Delta y + o(\rho)$即$\Delta f = o(\rho)$

考虑点$P(x,y)$沿$y = x$趋于0时，有$\dfrac{\Delta f}{\rho} = \dfrac{\frac{\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}}{\sqrt{\Delta x^2+\Delta y^2}} = \dfrac{1}{2} \not\to 0(\rho\to 0)$矛盾，于是假设不成立，则$f(x,y)$在$(0,0)$点不可微.

8. 设$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

证明$f_x(x,y),f_y(x,y)$存在但不连续，在$(0,0)$点的任何邻域中无界，但在$(0,0)$点可微.

证明：当$x^2+y^2\neq 0$时，$f_x(x,y)=2x\sin\dfrac{1}{x^2+y^2}-\dfrac{2x}{x^2+y^2}\cos\dfrac{1}{x^2+y^2}$

$f_x(0,0)=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$，则$f_x(0,0)$存在

考察在点$\left(\dfrac{1}{\sqrt{2n\pi}},0\right)$的偏导数

$f_x\left(\dfrac{1}{\sqrt{2n\pi}},0\right)=-2\sqrt{2n\pi}\to-\infty(n\to\infty)$

这说明$f_x(x,y)$在$(0,0)$点的任何邻域内无界，则其在$(0,0)$点不连续，于是$f_x(x,y)$不连续

同理可得$f_y(x,y)$存在但不连续且$f_y(0,0)=0$，在$(0,0)$点的任何邻域中无界

又$\dfrac{\Delta f-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\rho}=\sqrt{\Delta x^2+\Delta y^2}\sin\dfrac{1}{\Delta x^2+\Delta y^2}\to 0(\rho\to 0)$

则$f(x,y)$在$(0,0)$点可微.

9. 求下列函数的高阶偏导数：

(1) $u=x\sin(x+y)+y\cos(x+y)$，所有二阶偏导数

(2) $u=\dfrac{1}{2}\ln(x^2+y^2)$，所有二阶偏导数

(3) $u=x\ln(xy)$，$\dfrac{\partial^3 u}{\partial x^2\partial y}$

(4) $u=\ln(ax+by+cz)$，$\dfrac{\partial^4 u}{\partial x^4},\dfrac{\partial^4 u}{\partial x^2\partial y^2}$

(5) $u=(x-x_0)^p\cdot(y-y_0)^q$，$\dfrac{\partial^{p+q}u}{\partial x^p\partial y^q}$

(6) $u=x\cdot y\cdot ze^{x+y+z}$，$\dfrac{\partial^{p+q+r}u}{\partial x^p\cdot\partial y^q\cdot\partial z^r}$

解：

(1) 因$u_x=(1-y)\sin(x+y)+x\cos(x+y),u_y=-y\sin(x+y)+(x+1)\cos(x+y)$

则$u_{x^2}=(2-y)\cos(x+y)-x\sin(x+y),u_{xy}=u_{yx}=(1-y)\cos(x+y)-(x+1)\sin(x+y)$,

$u_{y^2}=-y\cos(x+y)-(x+2)\sin(x+y)$

(2) 因$u_x=\dfrac{x}{x^2+y^2},u_y=\dfrac{y}{x^2+y^2}$

则$u_{x^2}=\dfrac{y^2-x^2}{(x^2+y^2)^2},u_{xy}=u_{yx}=-\dfrac{2xy}{(x^2+y^2)^2},u_{y^2}=\dfrac{x^2-y^2}{(x^2+y^2)^2}$

(3) 因$\dfrac{\partial u}{\partial y}=\dfrac{x}{y}$，则$\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{1}{y}$，于是$\dfrac{\partial^3 u}{\partial x^2\partial y}=0$

(4) $\dfrac{\partial u}{\partial x}=\dfrac{a}{ax+by+cz},\dfrac{\partial u}{\partial y}=\dfrac{b}{ax+by+cz}$

$\dfrac{\partial^2 u}{\partial x^2}=-\dfrac{a^2}{(ax+by+cz)^2},\dfrac{\partial^2 u}{\partial y^2}=-\dfrac{b^2}{(ax+by+cz)^2}$

$\dfrac{\partial^3 u}{\partial x^3}=\dfrac{2a^3}{(ax+by+cz)^3},\dfrac{\partial^3 u}{\partial x\partial y^2}=\dfrac{2ab^2}{(ax+by+cz)^3}$

$\dfrac{\partial^4 u}{\partial x^4}=-\dfrac{6a^4}{(ax+by+cz)^4},\dfrac{\partial^4 u}{\partial x^2\partial y^2}=-\dfrac{6a^2b^2}{(ax+by+cz)^4}$

(5) 因$\dfrac{\partial^q u}{\partial y^q}=q!(x-x_0)^p$，则$\dfrac{\partial^{p+q}u}{\partial x^p\partial y^q}=p\,!q!(p,q$均为自然数$)$

(6) $\dfrac{\partial^{p+q+r}u}{\partial x^p\cdot\partial y^q\cdot\partial z^r}=\dfrac{\partial^p}{\partial x^p}(xe^x)\dfrac{\partial^q}{\partial y^q}(ye^y)\dfrac{\partial^r}{\partial z^r}(ze^z)=e^{x+y+z}(x+p)(y+q)(z+r)$

10. 设

(1) $u=x^2-2xy-3y^2$

(2) $u=x^{y^2}$

(3) $u=\arccos\sqrt{\dfrac{x}{y}}$

验证成立等式$\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$

证明：

(1) 因$u_x=2x-2y,u_y=-2x-6y$，则$\dfrac{\partial^2 u}{\partial x\partial y}=-2,\dfrac{\partial^2 u}{\partial y\partial x}=-2$，于是$\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$

(2) 因$u_x=y^2x^{y^2-1},u_y=2yx^{y^2}\ln x$，则$\dfrac{\partial^2 u}{\partial x\partial y}=2yx^{y^2-1}(1+y^2\ln x),\dfrac{\partial^2 u}{\partial y\partial x}=2yx^{y^2-1}(1+y^2\ln x)$

于是$\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$

(3) 当$0<x\leqslant y$时，$u=\arccos\sqrt{\dfrac{x}{y}}=\arccos\dfrac{\sqrt{x}}{\sqrt{y}}$

则$u_x=-\dfrac{1}{2\sqrt{x(y-x)}},u_y=\dfrac{\sqrt{x}}{2y\sqrt{y-x}}$

于是$\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{1}{4\sqrt{x}(y-x)^{\frac{3}{2}}},\dfrac{\partial^2 u}{\partial y\partial x}=\dfrac{1}{4\sqrt{x}(y-x)^{\frac{3}{2}}}$

从而$\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$

同理可证，当$y\leqslant x<0$时，$\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$也成立

综上，得$\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$.

§2. 求复合函数偏导数的链式法则

1. 求下列函数的偏导数：

(1) $u = f(x, y)$，其中$x = r\cos\theta, y = r\sin\theta$，求$\dfrac{\partial u}{\partial r}, \dfrac{\partial^2 u}{\partial r^2}$；

(2) $u = f(x, y)$，其中$x = a\xi, y = b\eta$，求$\dfrac{\partial u}{\partial \xi}, \dfrac{\partial^2 u}{\partial \xi^2}, \dfrac{\partial^2 u}{\partial \xi\partial\eta}, \dfrac{\partial u}{\partial \eta}, \dfrac{\partial^2 u}{\partial \eta^2}$

(3) $u = f(x^2+y^2+z^2)$，求$\dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2}{\partial x\partial y}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial z}$

(4) $u = f\left(x, \dfrac{x}{y}\right)$，求$\dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial u}{\partial y}$.

解：

(1) $\dfrac{\partial u}{\partial r} = f_x\cos\theta + f_y\sin\theta$

$\dfrac{\partial^2 u}{\partial r^2} = f_{x^2}\cos^2\theta + f_{xy}\sin 2\theta + f_{y^2}\sin^2\theta$

(2) $\dfrac{\partial u}{\partial \xi} = af_x, \dfrac{\partial^2 u}{\partial \xi^2} = a^2 f_{x^2}, \dfrac{\partial^2 u}{\partial\xi\partial\eta} = abf_{xy}, \dfrac{\partial u}{\partial\eta} = bf_y, \dfrac{\partial^2 u}{\partial\eta^2} = b^2 f_{y^2}$

(3) $\dfrac{\partial u}{\partial x} = 2xf'(x^2+y^2+z^2), \dfrac{\partial^2 u}{\partial x^2} = 2f'(x^2+y^2+z^2) + 4x^2 f''(x^2+y^2+z^2), \dfrac{\partial^2}{\partial x\partial y} = 4xyf''(x^2+y^2+z^2),$

$\dfrac{\partial u}{\partial y} = 2yf'(x^2+y^2+z^2), \dfrac{\partial u}{\partial z} = 2zf'(x^2+y^2+z^2)$

(4) $\dfrac{\partial u}{\partial x} = f_1 + \dfrac{1}{y}f_2, \dfrac{\partial^2 u}{\partial x^2} = f_{11} + \dfrac{2}{y}f_{12} + \dfrac{1}{y^2}f_{22}, \dfrac{\partial u}{\partial y} = -\dfrac{x}{y^2}f_2$

2. 设$\Phi = \Phi(x, y, z), x = u+v, y = u-v, z = uv$，求$\Phi_u, \Phi_v$.

解：$\Phi_u = \Phi_x + \Phi_y + v\Phi_z, \Phi_v = \Phi_x - \Phi_y + u\Phi_z$

3. 求下列函数的全微分(设其可微)：

(1) $u = f(x+y)$

(2) $u = f(x+y, x-y)$

(3) $u = f(ax^2+by^2+cz^2)$

解：

(1) $\mathrm{d}u = f'(x+y)(\,\mathrm{d}x + \mathrm{d}y)$

(2) $\mathrm{d}u = (f_1+f_2)\,\mathrm{d}x + (f_1-f_2)\,\mathrm{d}y$

(3) $\mathrm{d}u = 2f'(ax^2+by^2+cz^2)(ax\,\mathrm{d}x + by\,\mathrm{d}y + cz\,\mathrm{d}z)$

4. 验证下列各式：

(1) 设$z = \varphi(x^2+y^2)$，则$y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0$；

(2) 设$u = y\varphi(x^2-y^2)$，则$y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = \dfrac{xu}{y}$；

(3) 设$u = x\varphi(x+y) + y\psi(x+y)$，则$\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

证明：

(1) 因$\dfrac{\partial z}{\partial x} = 2x\varphi'(x^2+y^2), \dfrac{\partial z}{\partial y} = 2y\varphi'(x^2+y^2)$

则$y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0$.

(2) 因$\dfrac{\partial u}{\partial x} = 2xy\varphi'(x^2-y^2), \dfrac{\partial u}{\partial y} = \varphi(x^2-y^2) - 2y^2\varphi'(x^2-y^2)$

则$y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = x\varphi(x^2-y^2) = \dfrac{xu}{y}$.

(3) 因$\dfrac{\partial u}{\partial x}=\varphi(x+y)+x\varphi'(x+y)+y\psi'(x+y), \dfrac{\partial u}{\partial y}=x\varphi'(x+y)+\psi(x+y)+y\psi'(x+y)$

则$\dfrac{\partial^2 u}{\partial x^2}=2\varphi'(x+y)+x\varphi''(x+y)+y\psi''(x+y), \dfrac{\partial^2 u}{\partial x\partial y}=\varphi'(x+y)+\psi'(x+y)+x\varphi''(x+y)+y\psi''(x+y)$

$\dfrac{\partial^2 u}{\partial y^2}=2\psi'(x+y)+x\varphi''(x+y)+y\psi''(x+y)$

于是$\dfrac{\partial^2 u}{\partial x^2}-2\dfrac{\partial^2 u}{\partial x\partial y}+\dfrac{\partial^2 u}{\partial y^2}=0.$

5. 求$\Delta u=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}, u=f(x+y+z, x^2+y^2+z^2).$

解：因$\dfrac{\partial u}{\partial x}=f_1+2xf_2$，则$\dfrac{\partial^2 u}{\partial x^2}=f_{11}+4xf_{12}+4x^2f_{22}+2f_2$

据对称性，得$\dfrac{\partial^2 u}{\partial y^2}=f_{11}+4yf_{12}+4y^2f_{22}+2f_2, \dfrac{\partial^2 u}{\partial z^2}=f_{11}+4zf_{12}+4z^2f_{22}+2f_2$

于是$\Delta u=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=3f_{11}+4(x+y+z)f_{12}+4(x^2+y^2+z^2)f_{22}+6f_2.$

6. 若$u=f(r), r=\sqrt{x^2+y^2}$，其中$f(r)$二次可微，试证明

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\mathrm{d}^2 u}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}$$

证明：因$\dfrac{\partial u}{\partial x}=\dfrac{x}{\sqrt{x^2+y^2}}f'(r)$，则$\dfrac{\partial^2 u}{\partial x^2}=\dfrac{y^2}{(x^2+y^2)^{\frac{3}{2}}}f'(r)+\dfrac{x^2}{x^2+y^2}f''(r)$

据对称性，得$\dfrac{\partial^2 u}{\partial y^2}=\dfrac{x^2}{(x^2+y^2)^{\frac{3}{2}}}f'(r)+\dfrac{y^2}{x^2+y^2}f''(r)$

于是$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\mathrm{d}^2 u}{\mathrm{d}r^2}+\dfrac{1}{r}\dfrac{\mathrm{d}u}{\mathrm{d}r}$

7. 若u, v为x, y的函数，$x=r\cos\theta, y=r\sin\theta$，试由

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

证明等式$\dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial \theta}, \dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

证明：因u, v为x, y的函数，$x=r\cos\theta, y=r\sin\theta$

则$\dfrac{\partial u}{\partial r}=\cos\theta\dfrac{\partial u}{\partial x}+\sin\theta\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial r}=\cos\theta\dfrac{\partial v}{\partial x}+\sin\theta\dfrac{\partial v}{\partial y}$,

$\dfrac{\partial u}{\partial \theta}=-r\sin\theta\dfrac{\partial u}{\partial x}+r\cos\theta\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial \theta}=-r\sin\theta\dfrac{\partial v}{\partial x}+r\cos\theta\dfrac{\partial v}{\partial y}$

又$\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$，则$\dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial \theta}, \dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

8. 设$f(tx, ty)=t^nf(x, y)$，则有

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$

具有这样性质的函数，称为n次齐次函数.利用这结果，对$z=\sqrt{x^2+y^2}$，求$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}$.

证明：因$f(tx, ty)=t^nf(x, y)$，则两端对t求偏导，得$f_1(tx, ty)x+f_2(tx, ty)y=nt^{n-1}f(x, y)$

令$t=1$，则$f_1(x, y)x+f_2(x, y)y=nf(x, y)$即$x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=nf$

因$z(x, y)=\sqrt{x^2+y^2}$，则$z(tx, ty)=t\sqrt{x^2+y^2}\ (t\geqslant 0)$

于是$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z=\sqrt{x^2+y^2}$.

9. 设φ与ψ是任意的二阶可导函数，证明：

$$z=x\varphi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)$$

满足$x^2\dfrac{\partial^2 z}{\partial x^2}+2xy\dfrac{\partial^2 z}{\partial x\partial y}+y^2\dfrac{\partial^2 z}{\partial y^2}=0$

证明：因$\dfrac{\partial z}{\partial x}=\varphi\left(\dfrac{y}{x}\right)-\dfrac{y}{x}\varphi'\left(\dfrac{y}{x}\right)-\dfrac{y}{x^2}\psi'\left(\dfrac{y}{x}\right),\dfrac{\partial z}{\partial y}=\varphi'\left(\dfrac{y}{x}\right)+\dfrac{1}{x}\psi'\left(\dfrac{y}{x}\right)$

则$\dfrac{\partial^2 z}{\partial x^2}=\dfrac{y^2}{x^3}\varphi''+\dfrac{2y}{x^3}\psi'+\dfrac{y^2}{x^4}\psi'',\dfrac{\partial^2 z}{\partial x\partial y}=-\dfrac{y}{x^2}\varphi''-\dfrac{1}{x^2}\psi'-\dfrac{y}{x^3}\psi'',\dfrac{\partial^2 z}{\partial y^2}=\dfrac{1}{x}\varphi''+\dfrac{1}{x^2}\psi''$

于是$x^2\dfrac{\partial^2 z}{\partial x^2}+2xy\dfrac{\partial^2 z}{\partial x\partial y}+y^2\dfrac{\partial^2 z}{\partial y^2}=0$

10. 设$u=\varphi(x+at)+\psi(x-at)$，其中$\varphi,\psi$是任意的二次可微函数，求证

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}.$$

证明：因$u=\varphi(x+at)+\psi(x-at)$，$\varphi,\psi$是任意的二次可微函数

则$\dfrac{\partial u}{\partial t}=a(\varphi'-\psi'),\dfrac{\partial u}{\partial x}=\varphi'+\psi'$，于是$\dfrac{\partial^2 u}{\partial t^2}=a^2(\varphi''+\psi''),\dfrac{\partial^2 u}{\partial x^2}=\varphi''+\psi''$

从而$\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$.

§3. 由方程(组)所确定的函数的求导法

1. 求由下列方程所确定的函数$z=f(x,y)$的一阶和二阶的偏导数:

(1) $x+y+z=e^z$

(2) $xyz=x+y+z$

解:

(1) 两边关于x求导，得$1+z_x=z_xe^z$，则$z_x=\dfrac{1}{e^z-1}$，于是$z_{x^2}=\dfrac{e^z}{(1-e^z)^3}$

同法可得，$z_y=\dfrac{1}{e^z-1}$, $z_{y^2}=\dfrac{e^z}{(1-e^z)^3}$, $z_{xy}=z_{yx}=\dfrac{e^z}{(1-e^z)^3}$

(2) 两边关于x求导，得$yz+xyz_x=1+z_x$ $(*)$，则$z_x=\dfrac{yz-1}{1-xy}$

将$(*)$式两边关于x求导，得$2yz_x+xyz_{x^2}=z_{x^2}$，则$z_{x^2}=\dfrac{2yz_x}{1-xy}=\dfrac{2y(yz-1)}{(xy-1)^2}$

同法可得，$z_y=\dfrac{xz-1}{1-xy}$, $z_{y^2}=\dfrac{2x(xz-1)}{(xy-1)^2}$, $z_{xy}=z_{yx}=\dfrac{2z}{(xy-1)^2}$

2. 求由下列方程所确定的函数的全微分或偏导数:

(1) $f(x+y,y+z,z+x)=0$，求$\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$;

(2) $z=f(xz,z-y)$，求$\mathrm{d}z$;

(3) $F(x-y,y-z,z-x)=0$，求$\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$;

(4) $F(x,x+y,x+y+z)=0$，求$\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$, $\dfrac{\partial^2 z}{\partial x^2}$.

解:

(1) 两边关于x求导，且$z=z(x,y)$，得$f_1+f_2z_x+f_3(z_x+1)=0$，则$z_x=-\dfrac{f_1+f_3}{f_2+f_3}$

同法可得，$z_y=-\dfrac{f_1+f_2}{f_2+f_3}$

(2) 两端微分，得$\mathrm{d}z=(x\,\mathrm{d}z+z\,\mathrm{d}x)f_1+(\mathrm{d}z-\mathrm{d}y)f_2$，则$\mathrm{d}z=\dfrac{zf_1\,\mathrm{d}x-f_2\,\mathrm{d}y}{1-xf_1-f_2}$

(3) 两边关于x求导，且$z=z(x,y)$，得$F_1-F_2z_x+F_3(z_x-1)=0$，则$z_x=\dfrac{F_1-F_3}{F_2-F_3}$

同法可得，$z_y=\dfrac{F_2-F_1}{F_2-F_3}$

(4) 两边关于x求导，且$z=z(x,y)$，得$F_1+F_2+F_3(1+z_x)=0$ $(*)$，则$z_x=-\dfrac{F_1+F_2+F_3}{F_3}$

在$(*)$式两边再关于x求导，得

$F_{11}+F_{12}+F_{13}(1+z_x)+F_{21}+F_{22}+F_{23}(1+z_x)+z_{x^2}F_3+(1+z_x)[F_{13}+F_{23}+F_{33}(1+z_x)]=0$

则$z_{x^2}=-\dfrac{1}{F_3^3}\left[F_3^2(F_{11}+2F_{12}+F_{22})-2F_3(F_1+F_2)(F_{13}+F_{23})+F_{33}(F_1+F_2)^2\right]$

同法可得，$z_y=-\dfrac{F_2+F_3}{F_3}$

3. 设由方程$z=x+y\cdot\varphi(z)$确定函数$z=z(x,y)$，设$1-y\varphi'(z)\neq 0$，证明

$$\frac{\partial z}{\partial y}=\varphi(z)\cdot\frac{\partial z}{\partial x}$$

证明: 方程两端微分，且$z=z(x,y)$，得$\mathrm{d}z=\mathrm{d}x+\varphi(z)\,\mathrm{d}y+y\varphi'(z)\,\mathrm{d}z$

又$1-y\varphi'(z)\neq 0$，则$\mathrm{d}z=\dfrac{\mathrm{d}x+\varphi(z)\,\mathrm{d}y}{1-y\varphi'(z)}$

于是$\dfrac{\partial z}{\partial y}=\dfrac{\varphi(z)}{1-y\varphi'(z)}$, $\dfrac{\partial z}{\partial x}=\dfrac{1}{1-y\varphi'(z)}$，从而$\dfrac{\partial z}{\partial y}=\varphi(z)\cdot\dfrac{\partial z}{\partial x}$

4. 证明由方程$ax+by+cz=\Phi(x^2+y^2+z^2)$所定义的函数$z=z(x,y)$满足方程$(cy-bz)\dfrac{\partial z}{\partial x}+(az-cx)\dfrac{\partial z}{\partial y}=bx-ay$，其中$\Phi(u)$是$u$的可微函数，$a,b,c$为常数.

证明：方程两端微分，且$z=z(x,y)$，$\Phi(u)$是u的可微函数

则得$a\,\mathrm{d}x+b\,\mathrm{d}y+c\,\mathrm{d}z=2(x\,\mathrm{d}x+y\,\mathrm{d}y+z\,\mathrm{d}z)\Phi'$

于是$\dfrac{\partial z}{\partial x}=\dfrac{2x\Phi'-a}{c-2z\Phi'},\dfrac{\partial z}{\partial y}=\dfrac{2y\Phi'-b}{c-2z\Phi'}$

从而$(cy-bz)\dfrac{\partial z}{\partial x}+(az-cx)\dfrac{\partial z}{\partial y}=bx-ay$

5. 设φ为任意的可微函数，证明由方程$\varphi(cx-az,cy-bz)=0$所定义的函数$z=z(x,y)$满足$a\dfrac{\partial z}{\partial x}+b\dfrac{\partial z}{\partial y}=c$.

证明：对方程两端分别关于x,y求导，且$z=z(x,y)$，得

$c\varphi_1-a\varphi_1z_x-b\varphi_2z_x=0,-a\varphi_1z_y+c\varphi_2-b\varphi_2z_y=0$

于是$\dfrac{\partial z}{\partial x}=\dfrac{c\varphi_1}{a\varphi_1+b\varphi_2},\dfrac{\partial z}{\partial y}=\dfrac{c\varphi_2}{a\varphi_1+b\varphi_2}$

从而$a\dfrac{\partial z}{\partial x}+b\dfrac{\partial z}{\partial y}=c$.

6. 证明由方程$F(x+zy^{-1},y+zx^{-1})=0$所确定的函数$z=z(x,y)$满足$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z-xy$.

证明：对方程两端分别关于x,y求导，且$z=z(x,y)$，得

$F_1\left(1+\dfrac{z_x}{y}\right)+F_2\left(\dfrac{z_x}{x}-\dfrac{z}{x^2}\right)=0,F_1\left(\dfrac{z_y}{y}-\dfrac{z}{y^2}\right)+F_2\left(1+\dfrac{z_y}{x}\right)=0$

于是$\dfrac{\partial z}{\partial x}=\dfrac{yzF_2-x^2yF_1}{x(xF_1+yF_2)},\dfrac{\partial z}{\partial y}=\dfrac{xzF_1-xy^2F_2}{y(xF_1+yF_2)}$

从而$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z-xy$.

7. 求下列方程组所确定的函数的导数或偏导数或全微分：

(1) $\begin{cases}x+y+z=0,\\ x\cdot y\cdot z=1,\end{cases}$ 求$\dfrac{\mathrm{d}y}{\mathrm{d}x},\dfrac{\mathrm{d}z}{\mathrm{d}x},\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$；

(2) $\begin{cases}x+y=u+v,\\ \dfrac{x}{y}=\dfrac{\sin u}{\sin v},\end{cases}$ 求$\mathrm{d}u,\mathrm{d}v$；

(3) $\begin{cases}xu+yv=0,\\ yu+xv=1,\end{cases}$ 求$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial x},\dfrac{\partial v}{\partial y},\dfrac{\partial^2u}{\partial x\partial y}$；

(4) $\begin{cases}x=\cos\theta\cos\varphi,\\ y=\cos\theta\sin\varphi,\\ z=\sin\theta,\end{cases}$ 求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$；

(5) $\begin{cases}u=f(u,x,v+y),\\ v=g(u-x,u^2\cdot y),\end{cases}$ 求$\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$.

解：

(1) 对x求导，得$\begin{cases}1+\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{\mathrm{d}z}{\mathrm{d}x}=0\\ yz+xz\dfrac{\mathrm{d}y}{\mathrm{d}x}+xy\dfrac{\mathrm{d}z}{\mathrm{d}x}=0\end{cases}\quad(*)$

联立求解，得$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y(z-x)}{x(y-z)},\dfrac{\mathrm{d}z}{\mathrm{d}x}=\dfrac{z(x-y)}{x(y-z)}$

$(*)$式再对x求导，得$\begin{cases}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}^2z}{\mathrm{d}x^2}=0\\ z\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\dfrac{\mathrm{d}z}{\mathrm{d}x}+z\dfrac{\mathrm{d}y}{\mathrm{d}x}+x\dfrac{\mathrm{d}y}{\mathrm{d}x}\cdot\dfrac{\mathrm{d}z}{\mathrm{d}x}+xz\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+y\dfrac{\mathrm{d}z}{\mathrm{d}x}+x\dfrac{\mathrm{d}y}{\mathrm{d}x}\cdot\dfrac{\mathrm{d}z}{\mathrm{d}x}+xy\dfrac{\mathrm{d}^2z}{\mathrm{d}x^2}=0\end{cases}$

联立，得$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{2z\frac{\mathrm{d}y}{\mathrm{d}x}+2y\frac{\mathrm{d}z}{\mathrm{d}x}+2x\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}z}{\mathrm{d}x}}{x(y-z)}$

将$\dfrac{\mathrm{d}y}{\mathrm{d}x},\dfrac{\mathrm{d}z}{\mathrm{d}x}$代入，得$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{yz[(x-y)^2+(x-z)^2+(y-z)^2]}{x^2(z-y)^3}$

(2) 将原式改写为$\begin{cases} u+v=x+y \\ y\sin u=x\sin v \end{cases}$ 微分，得$\begin{cases} \mathrm{d}u+\mathrm{d}v=\mathrm{d}x+\mathrm{d}y \\ \sin u\,\mathrm{d}y+y\cos u\,\mathrm{d}u=\sin v\,\mathrm{d}x+x\cos v\,\mathrm{d}v \end{cases}$

则$\mathrm{d}u=\dfrac{1}{x\cos v+y\cos u}[(\sin v+x\cos v)\,\mathrm{d}x-(\sin u-x\cos v)\,\mathrm{d}y]$

$\mathrm{d}v=\dfrac{1}{x\cos v+y\cos u}[-(\sin v-y\cos u)\,\mathrm{d}x+(\sin u+y\cos u)\,\mathrm{d}y]$

(3) 微分，得$\begin{cases} x\,\mathrm{d}u+y\,\mathrm{d}v=-u\,\mathrm{d}x-v\,\mathrm{d}y \\ y\,\mathrm{d}u+x\,\mathrm{d}v=-v\,\mathrm{d}x-u\,\mathrm{d}y \end{cases}$

于是$\mathrm{d}u=\dfrac{1}{x^2-y^2}[(yv-xu)\,\mathrm{d}x+(yu-xv)\,\mathrm{d}y]$, $\mathrm{d}v=\dfrac{1}{x^2-y^2}[(yu-xv)\,\mathrm{d}x+(yv-xu)\,\mathrm{d}y]$

则$\dfrac{\partial u}{\partial x}=\dfrac{yv-xu}{x^2-y^2}$, $\dfrac{\partial u}{\partial y}=\dfrac{yu-xv}{x^2-y^2}$, $\dfrac{\partial v}{\partial x}=\dfrac{yu-xv}{x^2-y^2}$, $\dfrac{\partial u}{\partial x}=\dfrac{yv-xu}{x^2-y^2}$

于是$\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{(yu_x-v-xv_x)(x^2-y^2)-2x(yu-xv)}{(x^2-y^2)^2}$

将$\dfrac{\partial u}{\partial x}$, $\dfrac{\partial v}{\partial x}$代入，得$\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{2(x^2v+y^2v-2xyu)}{(x^2-y^2)^2}$

(4) 由x,y对x求偏导数，得$\begin{cases} 1=-\sin\theta\cdot\cos\varphi\dfrac{\partial\theta}{\partial x}-\cos\theta\cdot\sin\varphi\dfrac{\partial\varphi}{\partial x} \\ 0=-\sin\theta\cdot\sin\varphi\dfrac{\partial\theta}{\partial x}+\cos\theta\cdot\cos\varphi\dfrac{\partial\varphi}{\partial x} \end{cases}$

则$\dfrac{\partial\theta}{\partial x}=-\dfrac{\cos\varphi}{\sin\theta}$, $\dfrac{\partial\varphi}{\partial x}=-\dfrac{\sin\varphi}{\cos\theta}$，于是$\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial\theta}\cdot\dfrac{\partial\theta}{\partial x}=-\cot\theta\cos\varphi=-\dfrac{x}{z}$

同理可得，$\dfrac{\partial z}{\partial y}=-\dfrac{y}{z}$

(5) 对x求偏导，得$\begin{cases} \dfrac{\partial u}{\partial x}=f_1\dfrac{\partial u}{\partial x}+f_2+f_3\dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial x}=g_1\left(\dfrac{\partial u}{\partial x}-1\right)+2vyg_2\dfrac{\partial v}{\partial x} \end{cases}$

则$\dfrac{\partial u}{\partial x}=\dfrac{f_2(1-2vyg_2)-g_1f_3}{(f_1-1)(2vyg_2-1)-g_1f_3}$, $\dfrac{\partial v}{\partial x}=\dfrac{g_1(f_1+f_2-1)}{(f_1-1)(2vyg_2-1)-g_1f_3}$.

8. 方程$x=u+v, y=u^2+v^2, z=u^3+v^3$定义$z$为$x,y$的函数，求$\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$.

解：因$x^2-y=2uv$，则$z=(u+v)(u^2-uv+v^2)=\dfrac{x}{2}(3y-x^2)$

于是$\dfrac{\partial z}{\partial x}=\dfrac{3}{2}(y-x^2)$, $\dfrac{\partial z}{\partial y}=\dfrac{3}{2}x$.

9. 设$x=r\cos\theta, y=r\sin\theta$，变换方程

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}=y+kx(x^2+y^2) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=-x+ky(x^2+y^2) \end{cases}$$

为极坐标方程.

解：由方程知，x,y是t的函数，从极坐标变换知r,θ也是t的函数，$x=r\cos\theta, y=r\sin\theta$

两端对t求导，得$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}=\cos\theta\dfrac{\mathrm{d}r}{\mathrm{d}t}-r\sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=\sin\theta\dfrac{\mathrm{d}r}{\mathrm{d}t}+r\cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \end{cases}$

将$x, y, \dfrac{\mathrm{d}x}{\mathrm{d}t}$, $\dfrac{\mathrm{d}y}{\mathrm{d}t}$代入原方程组，得$\begin{cases} \cos\theta\dfrac{\mathrm{d}r}{\mathrm{d}t}-r\sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=r\sin\theta+kr\cos\theta\cdot r^2 \\ \sin\theta\dfrac{\mathrm{d}r}{\mathrm{d}t}+r\cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=-r\cos\theta+kr\sin\theta\cdot r^2 \end{cases}$

于是$\dfrac{\mathrm{d}r}{\mathrm{d}t}=kr^3$, $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=-1$.

10. 设$x=e^u\cos\theta, y=e^u\sin\theta$，变换方程$\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$.

解：因$x=e^u\cos\theta, y=e^u\sin\theta$，则$u=\ln(x^2+y^2), \theta=\arctan\dfrac{y}{x}$

则$\dfrac{\partial u}{\partial x}=\dfrac{x}{x^2+y^2}$, $\dfrac{\partial u}{\partial y}=\dfrac{y}{x^2+y^2}$; $\dfrac{\partial \theta}{\partial x}=-\dfrac{y}{x^2+y^2}$, $\dfrac{\partial \theta}{\partial y}=\dfrac{x}{x^2+y^2}$，于是$\dfrac{\partial u}{\partial x}=\dfrac{\partial \theta}{\partial y}$, $\dfrac{\partial u}{\partial y}=-\dfrac{\partial \theta}{\partial x}$

又$\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial \theta}\cdot\dfrac{\partial \theta}{\partial x}$, $\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial \theta}\cdot\dfrac{\partial \theta}{\partial y}$

则$\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial^2 z}{\partial u^2}\left(\dfrac{\partial u}{\partial x}\right)^2+2\dfrac{\partial^2 z}{\partial\theta\partial u}\cdot\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial \theta}{\partial x}+\dfrac{\partial^2 z}{\partial \theta^2}\left(\dfrac{\partial \theta}{\partial x}\right)^2+\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial z}{\partial \theta}\cdot\dfrac{\partial^2 \theta}{\partial x^2}$

$\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial^2 z}{\partial u^2}\left(\dfrac{\partial u}{\partial y}\right)^2+2\dfrac{\partial^2 z}{\partial\theta\partial u}\cdot\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial \theta}{\partial y}+\dfrac{\partial^2 z}{\partial \theta^2}\left(\dfrac{\partial \theta}{\partial y}\right)^2+\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial z}{\partial \theta}\cdot\dfrac{\partial^2 \theta}{\partial y^2}$

又$\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial \theta}{\partial y}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial \theta}{\partial x}\right)=\dfrac{\partial}{\partial y}\left(-\dfrac{\partial u}{\partial y}\right)=-\dfrac{\partial^2 u}{\partial y^2}$

同法可得，$\dfrac{\partial^2 \theta}{\partial x^2}=-\dfrac{\partial^2 \theta}{\partial y^2}$

则$\dfrac{\partial^2 \theta}{\partial x^2}+\dfrac{\partial^2 \theta}{\partial y^2}=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

又$\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2=\left(\dfrac{\partial \theta}{\partial x}\right)^2+\left(\dfrac{\partial \theta}{\partial y}\right)^2$, $\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial \theta}{\partial x}=-\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial \theta}{\partial y}$

则$\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=e^{-2u}\left(\dfrac{\partial^2 z}{\partial u^2}+\dfrac{\partial^2 z}{\partial \theta^2}\right)=0$

即$\dfrac{\partial^2 z}{\partial u^2}+\dfrac{\partial^2 z}{\partial \theta^2}=0.$

11. 设$x=r\cos\theta, y=r\sin\theta$，则$f(x,y)=\Phi(r,\theta)$，用$\Phi$关于$r,\theta$的偏导数来表示$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}$.

解：将$f(x,y)=\Phi(r,\theta)$关于r,θ求偏导，得

$\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial r}+\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial r}=\dfrac{\partial \Phi}{\partial r}$ 即$\dfrac{\partial f}{\partial x}\cos\theta+\dfrac{\partial f}{\partial y}\sin\theta=\dfrac{\partial \Phi}{\partial r}$

$\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial \theta}+\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial \theta}=\dfrac{\partial \Phi}{\partial \theta}$ 即$-\dfrac{\partial f}{\partial x}r\sin\theta+\dfrac{\partial f}{\partial y}r\cos\theta=\dfrac{\partial \Phi}{\partial \theta}$

则$\dfrac{\partial^2 \Phi}{\partial r^2}=\dfrac{\partial^2 f}{\partial x^2}\cos^2\theta+\dfrac{\partial^2 f}{\partial x\partial y}\sin 2\theta+\dfrac{\partial^2 f}{\partial y^2}\sin^2\theta$

$\dfrac{\partial^2 \Phi}{\partial \theta^2}=r^2\left(\dfrac{\partial^2 f}{\partial x^2}\sin^2\theta-\dfrac{\partial^2 f}{\partial x\partial y}\sin 2\theta+\dfrac{\partial^2 f}{\partial y^2}\cos^2\theta\right)-\dfrac{\partial f}{\partial x}r\cos\theta-\dfrac{\partial f}{\partial y}r\sin\theta$

于是$\dfrac{\partial^2 \Phi}{\partial r^2}+\dfrac{1}{r^2}\dfrac{\partial^2 \Phi}{\partial \theta^2}=\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}-\dfrac{1}{r}\dfrac{\partial \Phi}{\partial r}$

从而$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=\dfrac{\partial^2 \Phi}{\partial r^2}+\dfrac{1}{r^2}\dfrac{\partial^2 \Phi}{\partial \theta^2}+\dfrac{1}{r}\dfrac{\partial \Phi}{\partial r}$

12. 设$x=e^{\xi}, y=e^{\eta}$，变换方程$ax^2\dfrac{\partial^2 z}{\partial x^2}+2bxy\dfrac{\partial^2 z}{\partial x\partial y}+cy^2\dfrac{\partial^2 z}{\partial y^2}=0$($a,b,c$为常数).

解：因$x=e^{\xi}, y=e^{\eta}$，则$\xi=\ln x, \eta=\ln y$，于是$\dfrac{\mathrm{d}\xi}{\mathrm{d}x}=\dfrac{1}{x}$, $\dfrac{\mathrm{d}\eta}{\mathrm{d}y}=\dfrac{1}{y}$

则$\dfrac{\partial z}{\partial x}=\dfrac{1}{x}\dfrac{\partial z}{\partial \xi}$, $\dfrac{\partial z}{\partial y}=\dfrac{1}{y}\dfrac{\partial z}{\partial \eta}$

于是$\dfrac{\partial^2 z}{\partial x^2}=\dfrac{1}{x^2}\left(\dfrac{\partial^2 z}{\partial \xi^2}-\dfrac{\partial z}{\partial \xi}\right)$, $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{1}{y^2}\left(\dfrac{\partial^2 z}{\partial \eta^2}-\dfrac{\partial z}{\partial \eta}\right)$, $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{1}{xy}\dfrac{\partial^2 z}{\partial \xi\partial \eta}$

代入原方程，化简整理，得$a\left(\dfrac{\partial^2 z}{\partial \xi^2}-\dfrac{\partial z}{\partial \xi}\right)+2b\dfrac{\partial^2 z}{\partial \xi\partial \eta}+c\left(\dfrac{\partial^2 z}{\partial \eta^2}-\dfrac{\partial z}{\partial \eta}\right)=0.$

13. 设$\xi=x, \eta=x^2+y^2$，变换方程$y\dfrac{\partial z}{\partial x}-x\dfrac{\partial z}{\partial y}=0$.

解：由方程知z是x,y的函数，而ξ,η又是x,y的函数，从而z可看成是通过中间变量ξ,η关于x,y的复合函数

于是$\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial \xi}+2x\dfrac{\partial z}{\partial \eta}$, $\dfrac{\partial z}{\partial y}=2y\dfrac{\partial z}{\partial \eta}$

因而$y\dfrac{\partial z}{\partial x}-x\dfrac{\partial z}{\partial y}=y\dfrac{\partial z}{\partial \xi}$

因$y \neq 0$，则由$y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0$，得$\dfrac{\partial z}{\partial \xi} = 0.$

14. 设$\xi = x, \eta = y - x, \zeta = z - x$，变换方程$\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0.$

解：由方程知u是x, y, z的函数，而ξ, η, ζ又是x, y, z的函数，从而u可看成是通过中间变量ξ, η, ζ关于$x, y.z$的复合函数

于是$\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \xi} - \dfrac{\partial u}{\partial \eta} - \dfrac{\partial u}{\partial \zeta}, \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial \eta}, \dfrac{\partial u}{\partial z} = \dfrac{\partial u}{\partial \zeta}$

则由$\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$，得$\dfrac{\partial z}{\partial \xi} = 0$

15. 设线性变换$\xi = x + \lambda_1 y, \eta = x + \lambda_2 y$，现在要把方程$A\dfrac{\partial^2 u}{\partial x^2} + 2B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} = 0$($A, B, C$为常数，且$AC - B^2 < 0$)变换为$\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0$，证明$\lambda_1, \lambda_2$为方程$C\lambda^2 + 2B\lambda + A = 0$的两个相异实根.

证明：由方程知u是x, y的函数，因而可以把u视为以ξ, η为中间变量的关于x, y的复合函数，于是

$\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \xi} + \dfrac{\partial u}{\partial \eta}, \dfrac{\partial u}{\partial y} = \lambda_1 \dfrac{\partial u}{\partial \xi} + \lambda_2 \dfrac{\partial u}{\partial \eta}$

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial \xi^2} + 2\dfrac{\partial^2 u}{\partial \xi \partial \eta} + \dfrac{\partial^2 u}{\partial \eta^2}, \dfrac{\partial^2 u}{\partial y^2} = \lambda_1^2 \dfrac{\partial^2 u}{\partial \xi^2} + 2\lambda_1\lambda_2 \dfrac{\partial^2 u}{\partial \xi \partial \eta} + \lambda_2^2 \dfrac{\partial^2 u}{\partial \eta^2},$

$\dfrac{\partial^2 u}{\partial x \partial y} = \lambda_1 \dfrac{\partial^2 u}{\partial \xi^2} + (\lambda_1 + \lambda_2)\dfrac{\partial^2 u}{\partial \xi \partial \eta} + \lambda_2 \dfrac{\partial^2 u}{\partial \eta^2}$

因而$A\dfrac{\partial^2 u}{\partial x^2} + 2B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial \xi^2}(A + 2B\lambda_1 + C\lambda_1^2) + 2\dfrac{\partial^2 u}{\partial \xi \partial \eta}[A + B(\lambda_1 + \lambda_2) + C\lambda_1\lambda_2] + \dfrac{\partial^2 u}{\partial \eta^2}(A + 2B\lambda_2 + C\lambda_2^2) = 0$

要使$A\dfrac{\partial^2 u}{\partial x^2} + 2B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} = 0$变换为$\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0$，必须$\begin{cases} A + 2B\lambda_1 + C\lambda_1^2 = 0 \\ A + 2B\lambda_2 + C\lambda_2^2 = 0 \\ A + B(\lambda_1 + \lambda_2) + C\lambda_1\lambda_2 \neq 0 \end{cases}$

由前两个方程，得λ_1, λ_2是方程$C\lambda^2 + 2B\lambda + A = 0$的根

而由第三个方程，得$\lambda_1 \neq \lambda_2$，则λ_1, λ_2是$C\lambda^2 + 2B\lambda + A = 0$的两个相异实根

又因$\lambda_1 + \lambda_2 = -\dfrac{2B}{C}, \lambda_1\lambda_1 = \dfrac{A}{C}$，则$A + B(\lambda_1 + \lambda_2) + C\lambda_1\lambda_2 = \dfrac{2}{C}(AC - B^2) \neq 0$

于是方程$A\dfrac{\partial^2 u}{\partial x^2} + 2B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} = 0$在线性变换$\xi = x + \lambda_1 y, \eta = x + \lambda_2 y$下确实变换为$\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0$，且$\lambda_1, \lambda_2$为方程$C\lambda^2 + 2B\lambda + A = 0$的两个相异实根.

16. 证明拉普拉斯方程$\Delta w \equiv \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0$在变化$x = \varphi(u, v), y = \psi(u, v)$ $\left(\text{它们满足}\dfrac{\partial \varphi}{\partial u} = \dfrac{\partial \psi}{\partial v}, \dfrac{\partial \varphi}{\partial v} = -\dfrac{\partial \psi}{\partial u}\right)$下形状保持不变.

证明：从方程知ω是x, y的函数，x, y是u, v的函数，则w是以x, y为中间变量的u, v的函数

于是$\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x} \cdot \dfrac{\partial \varphi}{\partial u} + \dfrac{\partial w}{\partial y} \cdot \dfrac{\partial \psi}{\partial u}, \dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x} \cdot \dfrac{\partial \varphi}{\partial v} + \dfrac{\partial w}{\partial y} \cdot \dfrac{\partial \psi}{\partial v}$

$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial \varphi}{\partial u}\right)^2 + 2\frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial \varphi}{\partial u} \cdot \frac{\partial \psi}{\partial u} + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial \psi}{\partial u}\right)^2 + \frac{\partial w}{\partial x} \cdot \frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial w}{\partial y} \cdot \frac{\partial^2 \psi}{\partial u^2}$$

$$\frac{\partial^2 w}{\partial v^2} = \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial \varphi}{\partial v}\right)^2 + 2\frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial \varphi}{\partial v} \cdot \frac{\partial \psi}{\partial v} + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial \psi}{\partial v}\right)^2 + \frac{\partial w}{\partial x} \cdot \frac{\partial^2 \varphi}{\partial v^2} + \frac{\partial w}{\partial y} \cdot \frac{\partial^2 \psi}{\partial v^2}$$

注意非退化条件$\dfrac{\partial \varphi}{\partial u} = \dfrac{\partial \psi}{\partial v}, \dfrac{\partial \varphi}{\partial v} = -\dfrac{\partial \psi}{\partial u}$，则$\dfrac{\partial^2 \varphi}{\partial u^2} = \dfrac{\partial^2 \psi}{\partial u \partial v}, \dfrac{\partial^2 \varphi}{\partial v^2} = -\dfrac{\partial^2 \psi}{\partial v \partial u}, \dfrac{\partial^2 \psi}{\partial u^2} = -\dfrac{\partial^2 \varphi}{\partial u \partial v}, \dfrac{\partial^2 \psi}{\partial v^2} = \dfrac{\partial^2 \varphi}{\partial v \partial u}$

将$\dfrac{\partial^2 w}{\partial u^2}, \dfrac{\partial^2 w}{\partial v^2}$相加，并将上述各式代入，得

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\left[\left(\frac{\partial \varphi}{\partial u}\right)^2 + \left(\frac{\partial \varphi}{\partial v}\right)^2\right]$$

因$\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0$，则$\dfrac{\partial^2 w}{\partial u^2} + \dfrac{\partial^2 w}{\partial v^2} = 0$

这表明拉普拉斯方程$\Delta w \equiv \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0$在变化$x = \varphi(u, v), y = \psi(u, v)$下形状保持不变.

17. 设$\xi = x - at, \eta = x + at$，变换方程$\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}$.

解：由方程知u是t,x的函数，ξ,η也是t,x的函数，故可将u视为以ξ,η为中间变量的关于t,x的函数

则$\dfrac{\partial u}{\partial t}=-a\dfrac{\partial u}{\partial \xi}+a\dfrac{\partial u}{\partial \eta},\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial \xi}+\dfrac{\partial u}{\partial \eta}$

$\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial \xi^2}-2a^2\dfrac{\partial^2 u}{\partial \xi\partial\eta}+a^2\dfrac{\partial^2 u}{\partial \eta^2},\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial \xi^2}+\dfrac{\partial^2 u}{\partial \xi\partial\eta}+\dfrac{\partial^2 u}{\partial \eta^2}$

于是由$\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$，得$4a^2\dfrac{\partial^2 u}{\partial \xi\partial\eta}=0$

又$a\neq 0$，则$\dfrac{\partial^2 u}{\partial \xi\partial\eta}=0.$

18. 作自变数和因变数的变换，取u,v为新的自变数，$w=w(u,v)$为新的因变数：

(1) 设$u=x^2+y^2,v=\dfrac{1}{x}+\dfrac{1}{y},w=\ln z-(x+y)$，变换方程

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=(y-x)\cdot z$$

(2) 设$u=x+y,v=\dfrac{y}{x},w=\dfrac{z}{x}$，变换方程

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0$$

(3) 设$x=u,y=\dfrac{u}{1+uv},z=\dfrac{u}{1+u\cdot w}$，变换方程

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=z^2$$

(4) 设$u=\dfrac{x}{y},v=x,w=xz-y$，变换方程

$$y\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial y}=\frac{2}{x}$$

解：

(1) 由已知，得$\mathrm{d}u=2x\,\mathrm{d}x+2y\,\mathrm{d}y,\mathrm{d}v=-\dfrac{1}{x^2}\,\mathrm{d}x-\dfrac{1}{y^2}\,\mathrm{d}y,\mathrm{d}w=\dfrac{1}{z}\,\mathrm{d}z-\mathrm{d}x-\mathrm{d}y$

另一方面，$\mathrm{d}w=\dfrac{\partial w}{\partial u}\,\mathrm{d}u+\dfrac{\partial w}{\partial v}\,\mathrm{d}v$

则$\dfrac{1}{z}\,\mathrm{d}z-\mathrm{d}x-\mathrm{d}y=\dfrac{\partial w}{\partial u}(2x\,\mathrm{d}x+2y\,\mathrm{d}y)+\dfrac{\partial w}{\partial v}\left(-\dfrac{1}{x^2}\,\mathrm{d}x-\dfrac{1}{y^2}\,\mathrm{d}y\right)$

整理，得$\mathrm{d}z=\left(2xz\dfrac{\partial w}{\partial u}-\dfrac{z}{x^2}\cdot\dfrac{\partial w}{\partial v}+z\right)\mathrm{d}x+\left(2yz\dfrac{\partial w}{\partial u}-\dfrac{z}{y^2}\cdot\dfrac{\partial w}{\partial v}+z\right)\mathrm{d}y$

将上式所确定的$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$代入原方程，得$z\left(\dfrac{x}{y^2}-\dfrac{y}{x^2}\right)\dfrac{\partial w}{\partial v}=0$

又$z\left(\dfrac{x}{y^2}-\dfrac{y}{x^2}\right)\not\equiv 0$，则$\dfrac{\partial w}{\partial v}=0.$

(2) 由已知，得$\mathrm{d}u=\mathrm{d}x+\mathrm{d}y,\mathrm{d}v=-\dfrac{y}{x^2}\,\mathrm{d}x+\dfrac{1}{x}\,\mathrm{d}y,\mathrm{d}w=-\dfrac{z}{x^2}\,\mathrm{d}x+\dfrac{1}{x}\,\mathrm{d}z$

另一方面，$\mathrm{d}w=\dfrac{\partial w}{\partial u}\,\mathrm{d}u+\dfrac{\partial w}{\partial v}\,\mathrm{d}v$

则$-\dfrac{z}{x^2}\,\mathrm{d}x+\dfrac{1}{x}\,\mathrm{d}z=\dfrac{\partial w}{\partial u}(\mathrm{d}x+\mathrm{d}y)+\dfrac{\partial w}{\partial v}\left(-\dfrac{y}{x^2}\,\mathrm{d}x+\dfrac{1}{x}\,\mathrm{d}y\right)$

整理，得$\mathrm{d}z=\left(x\dfrac{\partial w}{\partial u}-\dfrac{y}{x}\cdot\dfrac{\partial w}{\partial v}+\dfrac{z}{x}\right)\mathrm{d}x+\left(x\dfrac{\partial w}{\partial u}+\dfrac{\partial w}{\partial v}\right)\mathrm{d}y$

则$\dfrac{\partial z}{\partial x}=x\dfrac{\partial w}{\partial u}-\dfrac{y}{x}\cdot\dfrac{\partial w}{\partial v}+\dfrac{z}{x},\dfrac{\partial z}{\partial y}=x\dfrac{\partial w}{\partial u}+\dfrac{\partial w}{\partial v}$

令$R=\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}=w-(1+v)\dfrac{\partial w}{\partial v}$

则$\dfrac{\partial^2 z}{\partial x^2}-2\dfrac{\partial^2 z}{\partial x\partial y}+\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)-\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial R}{\partial x}-\dfrac{\partial R}{\partial y}=\dfrac{\partial R}{\partial u}\left(\dfrac{\partial u}{\partial x}-\dfrac{\partial u}{\partial y}\right)+$
$\dfrac{\partial R}{\partial v}\left(\dfrac{\partial v}{\partial x}-\dfrac{\partial v}{\partial y}\right)=\dfrac{\partial}{\partial v}\left[w-(1+v)\dfrac{\partial w}{\partial v}\right]\left(-\dfrac{y}{x^2}-\dfrac{1}{x}\right)=\dfrac{(1+v)^2}{x}\dfrac{\partial^2 w}{\partial v^2}=0$

因$\dfrac{(1+v)^2}{x}\not\equiv 0$，则原方程变为$\dfrac{\partial^2 w}{\partial v^2}=0.$

(3) 因$x=u,y=\dfrac{u}{1+uv},z=\dfrac{u}{1+u\cdot w}$，则$u=x,v=\dfrac{1}{y}-\dfrac{1}{x},w=\dfrac{1}{z}-\dfrac{1}{x}$

于是$\mathrm{d}u=\mathrm{d}x$, $\mathrm{d}v=\dfrac{1}{x^2}\,\mathrm{d}x-\dfrac{1}{y^2}\,\mathrm{d}y$, $\mathrm{d}w=\dfrac{1}{x^2}\,\mathrm{d}x-\dfrac{1}{z^2}\mathrm{d}z$

另一方面，$\mathrm{d}w=\dfrac{\partial w}{\partial u}\,\mathrm{d}u+\dfrac{\partial w}{\partial v}\,\mathrm{d}v$

则$\dfrac{1}{x^2}\,\mathrm{d}x-\dfrac{1}{z^2}\mathrm{d}z=\dfrac{\partial w}{\partial u}\,\mathrm{d}x+\dfrac{\partial w}{\partial v}\left(\dfrac{1}{x^2}\,\mathrm{d}x-\dfrac{1}{y^2}\,\mathrm{d}y\right)$

整理，得$\mathrm{d}z=z^2\left(\dfrac{1}{x^2}-\dfrac{\partial w}{\partial u}-\dfrac{1}{x^2}\cdot\dfrac{\partial w}{\partial v}\right)\mathrm{d}x+\dfrac{z^2}{y^2}\cdot\dfrac{\partial w}{\partial v}\,\mathrm{d}y$

将上式所确定的$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$代入原方程，得$x^2z^2\dfrac{\partial w}{\partial u}=0$

又$xz\not\equiv 0$，则$\dfrac{\partial w}{\partial u}=0.$

(4) 因$u=\dfrac{x}{y},v=x,w=xz-y$，则$\dfrac{\partial w}{\partial y}=x\dfrac{\partial z}{\partial y}-1,\dfrac{\partial w}{\partial y}=\dfrac{\partial w}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial w}{\partial v}\cdot\dfrac{\partial v}{\partial y},\dfrac{\partial u}{\partial y}=-\dfrac{x}{y^2},\dfrac{\partial v}{\partial y}=0$

于是$\dfrac{\partial w}{\partial y}=-\dfrac{x}{y^2}\dfrac{\partial w}{\partial u}$，则$\dfrac{\partial z}{\partial y}=\dfrac{1}{x}-\dfrac{1}{y^2}\dfrac{\partial w}{\partial u}$

于是$y\dfrac{\partial^2 z}{\partial y^2}+2\dfrac{\partial z}{\partial y}=\dfrac{2}{x}+\dfrac{x}{y^3}\dfrac{\partial^2 w}{\partial u^2}=\dfrac{2}{x}$

又$\dfrac{x}{y^3}\not\equiv 0$，则原方程变换为$\dfrac{\partial^2 w}{\partial u^2}=0.$

§4. 空间曲线的切线与法平面

1. 求下列曲线在所示点处的切线与法平面：

(1) $x = a\sin^2 t, y = b\sin t\cdot\cos t, z = c\cos^2 t$，在$t = \dfrac{\pi}{4}$的点处；

(2) $x^2 + y^2 + z^2 = 6, x + y + z = 0$，在点$(1, -2, 1)$.

解：

(1) $x_0 = \dfrac{a}{2}, y_0 = \dfrac{b}{2}, z_0 = \dfrac{c}{2}, x'(t_0) = a, y'(t_0) = 0, z'(t_0) = -c$

则曲线在$t = \dfrac{\pi}{4}$的点处的切线方程为$\begin{cases} \dfrac{x - \frac{a}{2}}{a} = \dfrac{z - \frac{c}{2}}{-c} \\ y = \dfrac{b}{2} \end{cases}$ 即$\begin{cases} \dfrac{x}{a} + \dfrac{z}{c} = 1 \\ y = \dfrac{b}{2} \end{cases}$

法平面方程为$a\left(x - \dfrac{a}{2}\right) - c\left(z - \dfrac{c}{2}\right) = 0$即$ax - cz = \dfrac{1}{2}(a^2 - c^2)$.

(2) 因$\left.\dfrac{D(F_1, F_2)}{D(y, z)}\right|_{(1,-2,1)} = \left.\begin{vmatrix} 2y & 2z \\ 1 & 1 \end{vmatrix}\right|_{(1,-2,1)} = -6$, $\left.\dfrac{D(F_1, F_2)}{D(z, x)}\right|_{(1,-2,1)} = \left.\begin{vmatrix} 2z & 2x \\ 1 & 1 \end{vmatrix}\right|_{(1,-2,1)} = 0$,

$\left.\dfrac{D(F_1, F_2)}{D(x, y)}\right|_{(1,-2,1)} = \left.\begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix}\right|_{(1,-2,1)} = 6$

则曲线在点$(1, -2, 1)$的切线方程为$\begin{cases} x + z - 2 = 0 \\ y = -2 \end{cases}$

法平面方程为$x - z = 0$.

2. 在曲线$x = t, y = t^2, z = t^3$上求出一点，使此点的切线平行于平面$x + 2y + z = 4$.

解：设所求点为(t_0, t_0^2, t_0^3)，则$x'(t_0) = 1, y'(t_0) = 2t_0, z'(t_0) = 3t_0^2$

于是曲线的切线方向矢量为$\mathbf{v} = \{1, 2t_0, 3t_0^2\}$

又平面法矢量$\mathbf{n} = \{1, 2, 1\}$，则据题意，应有$\mathbf{v}\cdot\mathbf{n} = 1 + 4t_0 + 3t_0^2 = 0$，于是$t_0 = -1, t_0 = -\dfrac{1}{3}$

则所求点为$(-1, 1, -1), \left(-\dfrac{1}{3}, \dfrac{1}{9}, -\dfrac{1}{27}\right)$.

3. 证明曲线$x = ae^t\cos t, y = ae^t\sin t, z = ae^t$与锥面$x^2 + y^2 = z^2$的母线相交成同一角.

证明：将x, y, z代入$x^2 + y^2 = z^2$，得$a^2e^{2t}\cos^2 t + a^2e^{2t}\sin^2 t = a^2e^{2t} = z^2$，则曲线应在曲面上

圆锥$x^2 + y^2 = z^2$的顶点在原点，过圆锥上任一点$P(x_0, y_0, z_0)$的母线也过原点

则母线的方向矢量为$\mathbf{v}_1 = \{x_0, y_0, z_0\}$

又曲线在点P的切向量为$\mathbf{v}_2 = \{ae^{t_0}(\cos t_0 - \sin t_0), ae^{t_0}(\sin t_0 + \cos t_0), ae^{t_0}\} = \{x_0 - y_0, x_0 + y_0, z_0\}$

$x_0^2 + y_0^2 = z_0^2$

则$\cos(\widehat{\mathbf{v}_1, \mathbf{v}_2}) = \dfrac{\mathbf{v}_1\cdot\mathbf{v}_2}{|\mathbf{v}_1||\mathbf{v}_2|} = \dfrac{2}{\sqrt{6}} = \dfrac{\sqrt{6}}{3}$，这与曲线上点$(x, y, z)$的位置没有关系

因而曲线与锥面的母线相交成同一角.

4. 求下列各曲线在所示点的切线的方向余弦：

(1) $x = t^2, y = t^3, z = t^4$，在$t = 1$的点上；

(2) $xyz = 1, y^2 = x$，在点$(1, 1, 1)$.

解：

(1) 因$x'(t_0) = 2, y'(t_0) = 3, z'(t_0) = 4$，则切向量为$\{2, 3, 4\}$

于是方向余弦为：$\cos\alpha = \pm\dfrac{2}{29}\sqrt{29}, \cos\beta = \pm\dfrac{3}{29}\sqrt{29}, \cos\gamma = \pm\dfrac{4}{29}\sqrt{29}$.

(2) 因$\left.\dfrac{D(F_1, F_2)}{D(y, z)}\right|_{(1,1,1)} = \left.\begin{vmatrix} xz & xy \\ -2y & 0 \end{vmatrix}\right|_{(1,1,1)} = 2$, $\left.\dfrac{D(F_1, F_2)}{D(z, x)}\right|_{(1,1,1)} = \left.\begin{vmatrix} xy & yz \\ 0 & 1 \end{vmatrix}\right|_{(1,1,1)} = 1$,

$\left.\dfrac{D(F_1, F_2)}{D(x, y)}\right|_{(1,1,1)} = \left.\begin{vmatrix} yz & xz \\ 1 & -2y \end{vmatrix}\right|_{(1,1,1)} = -3$，则切向量为$\{2, 1, -3\}$

于是方向余弦为：$\cos\alpha = \pm\dfrac{\sqrt{14}}{7}, \cos\beta = \pm\dfrac{\sqrt{14}}{14}, \cos\gamma = \pm\dfrac{3}{14}\sqrt{14}$.

§5. 曲面的切平面与法线

1. 求下列曲面在所示点的切平面及法线方程：

(1) $x = a\sin\varphi\cos\theta, y = a\sin\varphi\sin\theta, z = a\cos\varphi$，在$(\theta_0, \varphi_0)$；

(2) $e^{\frac{x}{z}} + e^{\frac{y}{z}} = 4$，在点$(\ln 2, \ln 2, 1)$；

(3) $z = 2x^2 + 4y^2$，在点$(2, 1, 12)$；

(4) $ax^2 + by^2 + cz^2 + d = 0$，在点$(x_0, y_0, z_0)$.

解:

(1) 因$\left.\dfrac{D(y,z)}{D(\theta,\varphi)}\right|_{(\theta_0,\varphi_0)} = \left.\begin{vmatrix} a\sin\varphi\cos\theta & a\cos\varphi\sin\theta \\ 0 & -a\sin\varphi \end{vmatrix}\right|_{(\theta_0,\varphi_0)} = -a\sin^2\varphi_0\cos\theta_0$,

$\left.\dfrac{D(z,x)}{D(\theta,\varphi)}\right|_{(\theta_0,\varphi_0)} = -a^2\sin^2\varphi_0\sin\theta_0, \left.\dfrac{D(x,y)}{D(\theta,\varphi)}\right|_{(\theta_0,\varphi_0)} = -a^2\sin\varphi_0\cos\varphi_0$

则切平面方程为$\sin\varphi_0\cos\theta_0 x + \sin\varphi_0\sin\theta_0 y + \cos\varphi_0 z = a$

法线方程为$\dfrac{x - a\sin\varphi_0\cos\theta_0}{\sin\varphi_0\cos\theta_0} = \dfrac{y - a\sin\varphi_0\sin\theta_0}{\sin\varphi_0\sin\theta_0} = \dfrac{z - a\cos\varphi_0}{\cos\varphi_0}$.

(2) 因在$(\ln 2, \ln 2, 1)$点$f_x = 2, f_y = 2, f_z = -\ln 16$

则切平面方程为$x + y - 2\ln 2 \cdot z = 0$；法线方程为$\dfrac{x - \ln 2}{1} = \dfrac{y - \ln 2}{1} = \dfrac{z - 1}{-2\ln 2}$.

(3) 因$z_x(2,1) = 8, z_y(2,1) = 8$

则切平面方程为$8x + 8y - z = 12$；法线方程为$\dfrac{x-2}{8} = \dfrac{y-1}{8} = \dfrac{z-12}{-1}$.

(4) 因在(x_0, y_0, z_0)点$f_x = 2ax_0, f_y = 2by_0, f_z = 2cz_0$

则切平面方程为$ax_0x + by_0y + cz_0z + d = 0$；法线方程为$\dfrac{x - x_0}{ax_0} = \dfrac{y - y_0}{by_0} = \dfrac{z - z_0}{cz_0}$.

2. 在曲面$z = xy$上求一点，使这点的法线垂直于平面$x + 3y + z + 9 = 0$，并写出此法线方程.

解: 过曲面上任一点$M_0(x_0, y_0, z_0)$的$\mathbf{n}_1 = \{y_0, x_0, -1\}$，法线的切向量为$\mathbf{n}_2 = \{1, 3, 1\}$

要使法线垂直于上述平面，则$\mathbf{n}_1 \parallel \mathbf{n}_2$即$\dfrac{-y}{1} = \dfrac{-x}{3} = \dfrac{1}{1}$

于是所求点为$(-3, -1, 3)$，则法线方程为$\dfrac{x+3}{1} = \dfrac{y+1}{3} = \dfrac{z-3}{1}$.

3. 证明曲面$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}, (a > 0)$上任何一点的切平面在各坐标轴上的截距之和等于$a$.

证明: 在曲面上任取一点$P_0(x_0, y_0, z_0)$

则曲面在该点的切平面方程为$\dfrac{1}{2\sqrt{x_0}}(x - x_0) + \dfrac{1}{2\sqrt{y_0}}(y - y_0) + \dfrac{1}{2\sqrt{z_0}}(z - z_0) = 0$

即$\sqrt{y_0z_0}(x - x_0) + \sqrt{x_0z_0}(y - y_0) + \sqrt{x_0y_0}(z - z_0) = 0$

于是切平面在坐标轴上的截距分为$\sqrt{ax_0}, \sqrt{ay_0}, \sqrt{az_0}$，其和为$\sqrt{a}(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}) = a$.

4. 求两曲面$x^2 + y^2 = a^2, bz = xy$的交角.

解: 设两曲面任一交点$M_0(x_0, y_0, z_0)$

此两曲面在M_0点的法向量为$\mathbf{n}_1 = \{2x_0, 2y_0, 0\}, \mathbf{n}_2 = \{y_0, x_0, -b\}$

于是交角φ满足$\cos\varphi = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \dfrac{2bz_0}{|a|\sqrt{a^2 + b^2}}$.

§6. 方向导数和梯度

1. 求$u=x^2-xy+y^2$在$(1,1)$处沿方向$\mathbf{l}=(\cos\alpha,\sin\alpha)$的方向导数.并进一步求：

   (1) 在哪个方向上其导数有最大值；

   (2) 在哪个方向上其导数有最小值；

   (3) 在哪个方向上其导数为0；

   (4) 求u的梯度.

解：因$u_x=2x-y, u_y=-x+2y$，则$u_x(1,1)=1, u_y(1,1)=1$

又$\dfrac{\partial u}{\partial l}=u_x(1,1)\cos\alpha+u_y(1,1)\sin\alpha$，则$\dfrac{\partial u}{\partial l}=\cos\alpha+\sin\alpha=\sqrt{2}\sin\left(\alpha+\dfrac{\pi}{4}\right)$

于是

(1) 当$\alpha=\dfrac{\pi}{4}$时，在方向$\mathbf{l}=\left(\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)$上其导数有最大值$\sqrt{2}$；

(2) 当$\alpha=-\dfrac{3}{4}\pi$时，在方向$\mathbf{l}=\left(-\dfrac{\sqrt{2}}{2},-\dfrac{\sqrt{2}}{2}\right)$上其导数有最小值$-\sqrt{2}$；

(3) 当$\alpha=-\dfrac{\pi}{4},\dfrac{3}{4}\pi$时，在方向$\mathbf{l}=\left(\dfrac{\sqrt{2}}{2},-\dfrac{\sqrt{2}}{2}\right)$或$\mathbf{l}=\left(-\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)$上其导数为0；

(4) $\mathrm{grad}u=u_x(1,1)\mathbf{i}+u_y(1,1)\mathbf{j}=\mathbf{i}+\mathbf{j}$.

2. 求$u=xyz$在点$M(1,1,1)$，沿$\mathbf{l}=(2,-1,3)$的方向导数及梯度.

解：因$u_x=yz, u_y=xz, u_z=xy$，则在$(1,1,1)$点$u_x=u_y=u_z=1$

又向量$\mathbf{l}$的方向余弦$\cos\alpha=\dfrac{2}{\sqrt{14}},\sin\beta=-\dfrac{1}{\sqrt{14}},\cos\gamma=\dfrac{3}{\sqrt{14}}$

则$\dfrac{\partial u}{\partial l}=u_x(1,1,1)\cos\alpha+u_y(1,1,1)\cos\beta+u_z(1,1,1)\cos\gamma=\dfrac{2}{7}\sqrt{14}$；$\mathrm{grad}u=\mathbf{i}+\mathbf{j}+\mathbf{k}$.

3. 求数量函数$u=x^2+2y^2+3z^2+xy+3x-2y-6z$在$O(0,0,0)$及$A(1,1,1)$的梯度及其大小.

解：因$u_x=2x+y+3, u_y=4y+x-2, u_z=6z-6$

则在$O(0,0,0)$点：$u_x=3, u_y=-2, u_z=-6$，于是$\mathrm{grad}u=3\mathbf{i}-2\mathbf{j}+6\mathbf{k}, |\mathrm{grad}u|=7$

在$A(1,1,1)$点：$u_x=6, u_y=3, u_z=0$，于是$\mathrm{grad}u=6\mathbf{i}+3\mathbf{j}, |\mathrm{grad}u|=3\sqrt{5}$.

4. 证明：

   (1) $\mathrm{grad}(\alpha u+\beta v)=\alpha\mathrm{grad}\,u+\beta\mathrm{grad}\,v$，其中$\alpha,\beta$都是常数；

   (2) $\mathrm{grad}(uv)=u\mathrm{grad}\,v+v\mathrm{grad}\,u$；

   (3) $\mathrm{grad}\,F(u)=F'(u)\mathrm{grad}\,u$

证明：以二元函数为例来证.令$u=u(x,y), v=v(x,y)$

(1) 因$\dfrac{\partial(\alpha u+\beta v)}{\partial x}=\alpha\dfrac{\partial u}{\partial x}+\beta\dfrac{\partial v}{\partial x},\dfrac{\partial(\alpha u+\beta v)}{\partial y}=\alpha\dfrac{\partial u}{\partial y}+\beta\dfrac{\partial v}{\partial y}$

则$\mathrm{grad}(\alpha u+\beta v)=\left(\dfrac{\partial(\alpha u+\beta v)}{\partial x},\dfrac{\partial(\alpha u+\beta v)}{\partial y}\right)=\alpha\left(\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial y}\right)+\beta\left(\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial y}\right)=\alpha\mathrm{grad}u+\beta\mathrm{grad}v$.

(2) 因$\dfrac{\partial(uv)}{\partial x}=v\dfrac{\partial u}{\partial x}+u\dfrac{\partial v}{\partial x},\dfrac{\partial(uv)}{\partial y}=v\dfrac{\partial u}{\partial y}+u\dfrac{\partial v}{\partial y}$

则$\mathrm{grad}(uv)=\left(\dfrac{\partial(uv)}{\partial x},\dfrac{\partial(uv)}{\partial y}\right)=v\left(\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}\right)+u\left(\dfrac{\partial v}{\partial x},\dfrac{\partial v}{\partial y}\right)=u\mathrm{grad}v+v\mathrm{grad}u$.

(3) $\mathrm{grad}F(u)=\left(\dfrac{\partial F}{\partial x},\dfrac{\partial F}{\partial y}\right)=\left(F'(u)\dfrac{\partial u}{\partial x},F'(u)\dfrac{\partial u}{\partial y}\right)=F'(u)\left(\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}\right)=F'(u)\mathrm{grad}u$

多元函数可仿二元函数证之.

5. 证明$\mathrm{grad}\dfrac{1}{r}=-\dfrac{\mathbf{r}}{r^3}$，$r=\sqrt{x^2+y^2+z^2},\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

证明：因$\dfrac{\partial r}{\partial x}=\dfrac{x}{r},\dfrac{\partial r}{\partial y}=\dfrac{y}{r},\dfrac{\partial r}{\partial z}=\dfrac{z}{r}$

则$\mathrm{grad}\dfrac{1}{r}=\dfrac{\mathrm{d}}{\mathrm{d}r}\left(\dfrac{1}{r}\right)\mathrm{grad}r=-\dfrac{1}{r^2}\left(\dfrac{\partial r}{\partial x}\mathbf{i}+\dfrac{\partial r}{\partial y}\mathbf{j}+\dfrac{\partial r}{\partial z}\mathbf{k}\right)=-\dfrac{1}{r^2}\cdot\dfrac{1}{r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=-\dfrac{\mathbf{r}}{r^3}$.

6. 设数量函数$u=\ln\frac{1}{r}$, $r=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$，在空间中哪些点上成立$|\mathrm{grad}u|=1$？

解：因$\frac{\partial r}{\partial x}=\frac{x-a}{r}$, $\frac{\partial r}{\partial y}=\frac{y-b}{r}$, $\frac{\partial r}{\partial z}=\frac{z-c}{r}$

则$\mathrm{grad}u=\frac{\mathrm{d}}{\mathrm{d}r}\left(\ln\frac{1}{r}\right)\mathrm{grad}r=-\frac{1}{r}\left(\frac{x-a}{r}\mathbf{i}+\frac{y-b}{r}\mathbf{j}+\frac{z-c}{r}\mathbf{k}\right)=-\frac{1}{r^2}\left[(x-a)\mathbf{i}+(y-b)\mathbf{j}+(z-c)\mathbf{k}\right]$

于是$|\mathrm{grad}u|=\frac{1}{r}=1$，则$r=1$即$(x-a)^2+(y-b)^2+(z-c)^2=1$

这表明在以(a,b,c)为球心，半径为1的球面上成立$|\mathrm{grad}u|=1$.

§7. 泰勒公式

1. 写出点$(1,-2)$附近函数$f(x,y)=2x^2-xy-y^2-6x-3y+5$的泰勒公式.
   解：因$f_x=4x-y-6, f_y=-x-2y-3, f_{x^2}=4, f_{xy}=-1, f_{y^2}=-2$，所有三阶偏导均为0
   则在点$(1,-2)$，$f=5, f_x=0, f_y=0, f_{x^2}=4, f_{xy}=-1, f_{y^2}=-2$
   于是$f(x,y)=5+2(x-1)^2-(x-1)(y+2)-(y+2)^2$.
2. 按x及y的乘幂展开函数$f(x,y)=e^x\ln(1+y)$到三项为止.
   解：因$f_x=e^x\ln(1+y), f_y=\dfrac{e^x}{1+y}, f_{x^2}=e^x\ln(1+y), f_{y^2}=-\dfrac{e^x}{(1+y)^2}, f_{xy}=\dfrac{e^x}{1+y}$
   $f_{x^3}=e^x\ln(1+y), f_{y^3}=\dfrac{2e^x}{(1+y)^3}, f_{xy^2}=-\dfrac{e^x}{(1+y)^2}, f_{yx^2}=\dfrac{e^x}{1+y}$
   则在点$(0,0)$处，$f=0, f_x=f_{x^2}=f_{x^3}=0, f_y=1, f_{xy}=1, f_{y^2}=-1, f_{xy^2}=-1, f_{yx^2}=1, f_{y^3}=2$
   于是$f(x,y)=y+xy-\dfrac{1}{2}y^2+\dfrac{1}{2}x^2y-\dfrac{1}{2}xy^2+\dfrac{1}{3}y^3$。

第十五章　　极值和条件极值

§1.　　极值和最小二乘法

1. 求下列函数的极值：

(1) $z = x^2 - (y-1)^2$

(2) $z = (x-y+1)^2$

(3) $z = 3axy - x^3 - y^3 \ (a > 0)$

(4) $z = \sin x + \cos y + \cos(x-y) \ \left(0 \leqslant \begin{matrix} x \\ y \end{matrix} \leqslant \dfrac{\pi}{2}\right)$

(5) $z = xy \cdot \sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}} \ (a, b > 0)$

解：

(1) 令$z_x = 2x = 0, z_y = -2(y-1) = 0$
则得$x = 0, y = 1$，于是点$(0,1)$为可能极值点
又$z_{x^2} = 2, z_{xy} = 0, z_{y^2} = -2$，则$A = 2, B = 0, C = -2$，于是$H = -4 < 0$，从而此函数无极值.

(2) 令$z_x = 2(x-y+1) = 0, z_y = -2(x-y+1) = 0$
则当点分布在$x-y+1 = 0$上时，函数可能有极值
又$A = 2, B = -2, C = 2$，则$H = 0$，故需进一步判断
因对直线$x-y+1 = 0$上的点均有$z = 0$，且$z \geqslant 0$恒成立
则函数z在直线$x-y+1 = 0$上各点取得极小值$z = 0$.

(3) 令$z_x = 3ay - 3x^2 = 0, z_y = 3ax - 3y^2 = 0$
则得$\begin{cases} x_1 = 0 \\ y_1 = 0 \end{cases} \begin{cases} x_2 = a \\ y_2 = a \end{cases}$ 即在点$(0,0), (a,a)$处可能有极值
又$z_{x^2} = -6x, z_{xy} = 3a, z_{y^2} = -6y$
则在点$(0,0)$，$A = 0, B = 3a, C = 0$，于是$H = -9a^2 < 0$，此时函数无极值;
在点(a,a)，$A = -6a < 0, B = 3a, C = -6a$，于是$H = 27a^2 > 0$，此时函数有极大值$z = a^3$.

(4) 令$z_x = \cos x - \sin(x-y) = 0, z_y = -\sin y + \sin(x-y) = 0$
则得$\cos x = \sin y$，于是$y = \dfrac{\pi}{2} - x$，故$\cos x - \sin(x-y) = cos - \sin\left(2x - \dfrac{\pi}{2}\right) = 2\cos\dfrac{x}{2}\ \cos\dfrac{3}{2}x = 0$
因$0 \leqslant x \leqslant \dfrac{\pi}{2}$，则$\cos\dfrac{x}{2} \neq 0, \cos\dfrac{3}{2}x = 0$，于是$x = \dfrac{\pi}{3}, y = \dfrac{\pi}{6}$，即在点$\left(\dfrac{\pi}{3}, \dfrac{\pi}{6}\right)$处可能有极值
又$z_{x^2} = -\sin x - \cos(x-y), z_{xy} = \cos(x-y), z_{y^2} = -\cos y - \cos(x-y)$
则$A = -\sqrt{3} < 0, B = \dfrac{\sqrt{3}}{2}, C = -\sqrt{3}$，于是$H = \dfrac{9}{4} > 0$，即在点$\left(\dfrac{\pi}{3}, \dfrac{\pi}{6}\right)$处函数有极大值$z = \dfrac{3}{2}\sqrt{3}$.

(5) 令$z_x = y\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}} - \dfrac{x^2 y}{a^2\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}} = 0$, $z_y = x\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}} - \dfrac{xy^2}{b^2\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}} = 0$
则得$\begin{cases} x_1 = 0 \\ y_1 = 0 \end{cases} \begin{cases} x_2 = \dfrac{a}{\sqrt{3}} \\ y_2 = \dfrac{b}{\sqrt{3}} \end{cases} \begin{cases} x_3 = -\dfrac{a}{\sqrt{3}} \\ y_3 = -\dfrac{b}{\sqrt{3}} \end{cases} \begin{cases} x_4 = \dfrac{a}{\sqrt{3}} \\ y_4 = -\dfrac{b}{\sqrt{3}} \end{cases} \begin{cases} x_5 = -\dfrac{a}{\sqrt{3}} \\ y_5 = \dfrac{b}{\sqrt{3}} \end{cases}$
于是在点$P_1(0,0), P_2\left(\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}\right), P_3\left(-\dfrac{a}{\sqrt{3}}, -\dfrac{b}{\sqrt{3}}\right), P_4\left(\dfrac{a}{\sqrt{3}}, -\dfrac{b}{\sqrt{3}}\right), P_5\left(-\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}\right)$处可能取极值
又$z_{x^2} = \dfrac{-3a^2b^2xy + 2b^2x^3y + 3a^2xy^3}{a^4b^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{3}{2}}}$, $z_{xy} = \dfrac{a^4b^4 - 3a^2b^4x^2 + 2b^4x^4 - 3a^4b^2y^2 + 3a^2b^2x^2y^2 + 2a^4y^4}{a^4b^4\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{3}{2}}}$
$z_{y^2} = \dfrac{3b^2x^3y - 3a^2b^2xy + 2a^2xy^3}{a^2b^4\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{3}{2}}}$

在点$P_2\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}}\right),P_3\left(-\frac{a}{\sqrt{3}},-\frac{b}{\sqrt{3}}\right)$处，$A=-\frac{4\sqrt{3}\,b}{3a}<0,B=-\frac{2}{3}\sqrt{3},C=-\frac{4\sqrt{3}\,a}{3b}$

此时$H=4>0$，于是函数有极大值$z=\frac{\sqrt{3}}{9}ab$；

在点$P_4\left(\frac{a}{\sqrt{3}},-\frac{b}{\sqrt{3}}\right),P_5\left(-\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}}\right)$处，$A=\frac{4\sqrt{3}\,b}{3a}>0,B=-\frac{2}{3}\sqrt{3},C=\frac{4\sqrt{3}\,a}{3b}$

此时$H=4>0$，于是函数有极小值$z=-\frac{\sqrt{3}}{9}ab$；

在点$P_1(0,0)$处，$A=0,B=1,C=0$，此时$H=-1<0$，于是函数无极值.

2. 证明函数$z=(1+e^y)\cos x-ye^y$有无穷多个极大值，但无极小值.

证明：令$z_x=-(1+e^y)\sin x=0\quad z_y=e^y\cos x-e^y-ye^y=0$

因$1+e^y\neq 0$，则$\sin x=0$，于是$x=k\pi\ (k\in Z)$

又$e^y\neq 0$，则$\cos x-1-y=0$即有$y=\cos x-1$

当k为偶数时，$y=0$；当k为奇数时，$y=-2$，则可能的极值点为$\begin{cases}x_1=2k\pi\\y_1=0\end{cases}$ $\begin{cases}x_2=(2k+1)\pi\\y_2=-2\end{cases}$ $(k=0,\pm1,\pm2,\cdots)$

又$z_{x^2}=-(1+e^y)\cos x,z_{xy}=-e^y\sin x,z_{y^2}=e^y\cos x-2e^y-ye^y$

则在点$(2k\pi,0)$，$A=-2<0,B=0,C=-1$，此时$H=2>0$，则此时函数有极大值$z=2$

在点$((2k+1)\pi,-2)$，$A=1+\frac{1}{e^2},B=0,C=-\frac{1}{e^2}$，此时$H=-\frac{1}{e^2}\left(1+\frac{1}{e^2}\right)<0$，则此时函数无极值

综上可知，函数$z=(1+e^y)\cos x-ye^y$有无穷多个极大值，但无极小值.

3. 在已知周长为$2p$ 的一切三角形中求出面积最大的三角形.

解：设三角形的边长分别为x,y,z，则$C=x+y+z=2p$，$D=\frac{x+y+z}{2}=p$，于是$z=2p-x-y$

则$S=\sqrt{D(D-x)(D-y)(D-z)}=\sqrt{p(p-x)(p-y)(x+y-p)}$

考虑函数$u=S^2=p(p-x)(p-y)(x+y-p)$,$0<x,y<p$

S的极值均为u的极值且当u在点(x,y)取得的极值不为0时，S也在点(x,y)取得极值

令$u_x=p(p-y)(2p-2x-y)=0,u_y=p(p-x)(2p-x-2y)=0$

因$p\neq 0,0<x,y<p$，则解得$x=y=\frac{2}{3}p$，于是$z=\frac{2}{3}p$

则当$x=y=z=\frac{2}{3}p$ 时，u有极值即S有极值

从而当$x=y=z=\frac{2}{3}p$ 时，面积最大且值为$S=\frac{\sqrt{3}}{9}p^2$.

4. 曲面$z=\frac{1}{2}x^2-4xy+9y^2+3x-14y+\frac{1}{2}$在何处有最高点或最低点？

解：由$\begin{cases}z_x=x-4y+3=0\\z_y=-4x+18y-14=0\end{cases}$，解得$\begin{cases}x=1\\y=1\end{cases}$ 即在点$(1,1)$可能有极值

又$z_{x^2}=1,z_{xy}=-4,z_{y^2}=18$，则$A=1>0,B=-4,C=18$，于是$H=2>0$

则此时函数有极小值$z=-5$，从而曲面有最低点$(1,1,-5)$

又当$x^2+y^2\to+\infty$时，$z\to+\infty$，故曲面无最高点.

5. 已知$y=ax^2+bx+c$，现测得一组数据$(x_i,y_i),i=1,2,\cdots,n$，利用最小二乘法，求系数a,b,c所满足的三元一次方程组.

解：由已知，得$\varepsilon=\sum_{i=1}^{n}(y_i-ax_i^2-bx_i-c)^2$，为使总偏差$\varepsilon(a,b,c)$达到最小，由极值的必要条件，有

$$\frac{\partial\varepsilon}{\partial a}=-2\sum_{i=1}^{n}(y_i-ax_i^2-bx_i-c)x_i^2=0,\frac{\partial\varepsilon}{\partial b}=-2\sum_{i=1}^{n}(y_i-ax_i^2-bx_i-c)x_i=0,\frac{\partial\varepsilon}{\partial c}=-2\sum_{i=1}^{n}(y_i-ax_i^2-bx_i-c)=0$$

即a,b,c满足下列三元一次方程组：
$$\begin{cases}\left(\sum\limits_{i=1}^{n}x_i^4\right)a+\left(\sum\limits_{i=1}^{n}x_i^3\right)b+\left(\sum\limits_{i=1}^{n}x_i^2\right)c=\sum\limits_{i=1}^{n}x_i^2y_i\\\left(\sum\limits_{i=1}^{n}x_i^3\right)a+\left(\sum\limits_{i=1}^{n}x_i^2\right)b+\left(\sum\limits_{i=1}^{n}x_i\right)c=\sum\limits_{i=1}^{n}x_iy_i\\\left(\sum\limits_{i=1}^{n}x_i^2\right)a+\left(\sum\limits_{i=1}^{n}x_i\right)b+nc=\sum\limits_{i=1}^{n}y_i\end{cases}$$

6. 曲线$y=x^2$在$[0,1]$上要用什么样的直线$\eta=ax+b$来代替，才能使它的平方误差的积分

$J(a,b)=\int_0^1(y-\eta)^2\,\mathrm{d}x$为极小的意义下为最佳近似？

解: $J(a,b)=\int_0^1 (y-\eta)^2\,\mathrm{d}x=\int_0^1 (x^2-ax-b)^2\,\mathrm{d}x=\frac{1}{5}+\frac{a^2}{3}+b^2-\frac{a}{2}-\frac{2}{3}b+ab$
为了选择a,b使平方误差的积分$J(a,b)$达到极小，由极值的必要条件，有
令$\frac{\partial J}{\partial a}=-\frac{1}{2}+\frac{2}{3}a+b=0$，$\frac{\partial J}{\partial b}=-\frac{2}{3}+a+2b=0$
则$a=1,b=-\frac{1}{6}$
于是曲线$y=x^2$用直线$\eta=x-\frac{1}{6}$来代替，可达到最佳近似的要求.

§2. 条件极值

1. 求下列函数在所给条件下极值：

(1) $f = x + y$, 若$x^2 + y^2 = 1$；

(2) $f = x - 2y + 2z$, 若$x^2 + y^2 + z^2 = 1$；

(3) $f = xyz$, 若$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a}\ (x > 0, y > 0, z > 0, a > 0)$；

(4) $f = \frac{1}{x} + \frac{1}{y}$, 若$x + y = 2$；

(5) $f = xyz$, 若$x^2 + y^2 + z^2 = 1, x + y + z = 0$.

解：

(1) 作函数$L = x + y + \lambda(x^2 + y^2 - 1)$

解方程组$\begin{cases} L_x = 1 + 2\lambda x = 0 \\ L_y = 1 + 2\lambda y = 0 \\ L_\lambda = x^2 + y^2 - 1 = 0 \end{cases}$，得$\begin{cases} x_1 = \frac{\sqrt{2}}{2} \\ y_1 = \frac{\sqrt{2}}{2} \\ \lambda_1 = -\frac{\sqrt{2}}{2} \end{cases}$ $\begin{cases} x_2 = -\frac{\sqrt{2}}{2} \\ y_2 = -\frac{\sqrt{2}}{2} \\ \lambda_2 = \frac{\sqrt{2}}{2} \end{cases}$

又$L_{x^2} = 2\lambda, L_{xy} = 0, L_{y^2} = 2\lambda$

则$\mathrm{d}^2L\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\sqrt{2}(\mathrm{d}x^2 + \mathrm{d}y^2) < 0$，于是函数在$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$处取得极大值$\sqrt{2}$；

同理可得，函数在$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$处取得极小值$-\sqrt{2}$.

(2) 作函数$L = x - 2y + 2z + \lambda(x^2 + y^2 + z^2 - 1)$

解方程组$\begin{cases} L_x = 1 + 2\lambda x = 0 \\ L_y = -2 + 2\lambda y = 0 \\ L_z = 2 + 2\lambda z = 0 \\ L_\lambda = x^2 + y^2 + z^2 - 1 = 0 \end{cases}$，得$\begin{cases} x_1 = \frac{1}{3} \\ y_1 = -\frac{2}{3} \\ Z_1 = \frac{2}{3} \\ \lambda_1 = -\frac{3}{2} \end{cases}$ $\begin{cases} x_2 = -\frac{1}{3} \\ y_2 = \frac{2}{3} \\ z_2 = -\frac{2}{3} \\ \lambda_2 = \frac{3}{2} \end{cases}$

又$L_{x^2} = L_{y^2} = L_{z^2} = 2\lambda, L_{xy} == L_{xz} = L_{yz} = 0$

则$\mathrm{d}^2L(x_2, y_2, z_2) = 3(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) > 0$，于是函数在$\left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$处取得极小值$-3$；

同理可得，函数在$\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$处取得极大值3.

(3) 作函数$L = xyz + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{a}\right)$

解方程组$\begin{cases} L_x = yz - \frac{\lambda}{x^2} = 0 \\ L_y = xz - \frac{\lambda}{y^2} = 0 \\ L_z = xy - \frac{\lambda}{z^2} = 0 \\ L_\lambda = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{a} = 0 \end{cases}$，得$x = y = z = 3a, \lambda = 81a^4$

又$L_{x^2}(3a, 3a, 3a) = L_{y^2}(3a, 3a, 3a) = L_{z^2}(3a, 3a, 3a) = 6a$,
$L_{xy}(3a, 3a, 3a) = L_{xz}(3a, 3a, 3a) = L_{yz}(3a, 3a, 3a) = 3a$
则$\mathrm{d}^2L(3a, 3a, 3a) = 3a[(\mathrm{d}x + \mathrm{d}y + \mathrm{d}z)^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2] > 0$，于是函数在$(3a, 3a, 3a)$处取得极小值$27a^3$.

(4) 作函数$L = \frac{1}{x} + \frac{1}{y} + \lambda(x + y - 2)$

解方程组$\begin{cases} L_x = -\dfrac{1}{x^2} + \lambda = 0 \\ L_y = -\dfrac{1}{y^2} + \lambda = 0 \\ L_\lambda = x + y - 2 = 0 \end{cases}$，得$x = y = \lambda = 1$

又$L_{x^2}(1,1) = L_{y^2}(1,1) = 2, L_{xy}(1,1) = 0$

则$\mathrm{d}^2L(1,1) = 2(\mathrm{d}x^2 + \mathrm{d}y^2) > 0$，于是函数在$(1,1)$处取得极小值2.

(5) 作函数$L = xyz + u(x^2+y^2+z^2-1) + v(x+y+z)$

解方程组$\begin{cases} L_x = yz + 2ux + v = 0 \\ L_y = xz + 2uy + v = 0 \\ L_z = xy + 2uz + v = 0 \\ L_u = x^2 + y^2 + z^2 - 1 = 0 \\ L_v = x + y + z = 0 \end{cases}$，得

$$\begin{cases} x_1 = \dfrac{\sqrt{6}}{6} \\ y_1 = \dfrac{\sqrt{6}}{6} \\ z_1 = -\dfrac{\sqrt{6}}{3} \\ u_1 = \dfrac{\sqrt{6}}{12} \\ v_1 = \dfrac{1}{6} \end{cases} \begin{cases} x_2 = -\dfrac{\sqrt{6}}{6} \\ y_2 = -\dfrac{\sqrt{6}}{6} \\ z_2 = \dfrac{\sqrt{6}}{3} \\ u_2 = -\dfrac{\sqrt{6}}{12} \\ v_2 = \dfrac{1}{6} \end{cases} \begin{cases} x_3 = -\dfrac{\sqrt{6}}{3} \\ y_3 = \dfrac{\sqrt{6}}{6} \\ z_3 = \dfrac{\sqrt{6}}{6} \\ u_3 = \dfrac{\sqrt{6}}{12} \\ v_3 = \dfrac{1}{6} \end{cases} \begin{cases} x_4 = \dfrac{\sqrt{6}}{3} \\ y_4 = -\dfrac{\sqrt{6}}{6} \\ z_4 = -\dfrac{\sqrt{6}}{6} \\ u_4 = -\dfrac{\sqrt{6}}{12} \\ v_4 = \dfrac{1}{6} \end{cases} \begin{cases} x_5 = \dfrac{\sqrt{6}}{6} \\ y_5 = -\dfrac{\sqrt{6}}{3} \\ z_5 = \dfrac{\sqrt{6}}{6} \\ u_5 = \dfrac{\sqrt{6}}{12} \\ v_5 = \dfrac{1}{6} \end{cases} \begin{cases} x_6 = -\dfrac{\sqrt{6}}{6} \\ y_6 = \dfrac{\sqrt{6}}{3} \\ z_6 = -\dfrac{\sqrt{6}}{6} \\ u_6 = -\dfrac{\sqrt{6}}{12} \\ v_6 = \dfrac{1}{6} \end{cases}$$

又$\mathrm{d}^2L = 2u(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + 2(z\,\mathrm{d}x\,\mathrm{d}y + y\,\mathrm{d}x\,\mathrm{d}z + x\,\mathrm{d}y\,\mathrm{d}z)$

则在点(x_1,y_1,z_1)处，$\mathrm{d}^2L = \dfrac{\sqrt{6}}{6}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - 4\,\mathrm{d}x\,\mathrm{d}y + 2\,\mathrm{d}x\,\mathrm{d}z + 2\,\mathrm{d}y\,\mathrm{d}z)$

由$x^2+y^2+z^2=1$，得$2x\,\mathrm{d}x + 2y\,\mathrm{d}y + 2z\,\mathrm{d}z = 0$，则在点$(x_1,y_1,z_1)$处，有$\mathrm{d}x + \mathrm{d}y = 2\,\mathrm{d}z$

又由$x+y+z=0$，得$\mathrm{d}x + \mathrm{d}y + \mathrm{d}z = 0$，则$\mathrm{d}x = -\mathrm{d}y, \mathrm{d}z = 0$，于是$\mathrm{d}^2L(x_1,y_1,z_1) = \sqrt{6}\ \mathrm{d}x^2 > 0$，

则函数在$\left(\dfrac{\sqrt{6}}{6}, \dfrac{\sqrt{6}}{6}, -\dfrac{\sqrt{6}}{3}\right)$处取得极小值$-\dfrac{\sqrt{6}}{18}$

同理可得，函数在$(x_3,y_3,z_3),(x_5,y_5,z_5)$处取得极小值$-\dfrac{\sqrt{6}}{18}$

函数在$(x_2,y_2,z_2),(x_4,y_4,z_4),(x_6,y_6,z_6)$处取得极大值$\dfrac{\sqrt{6}}{18}$.

2. 求$f = x^m y^n z^p$在条件$x+y+z=a, a>0, m>0, n>0, p>0, x>0, y>0, z>0$之下的最大值.

解：因$x>0, y>0, z>0$，则$f = x^m y^n z^p$ 最大时，$\ln f = m\ln x + n\ln y + p\ln z$也最大，反之亦然，故只需求$\ln f$的极大点，它也是$f$的极大点

令$L = m\ln x + n\ln y + p\ln z + \lambda(x+y+z-a)$

则解方程$\begin{cases} L_x = \dfrac{m}{x} + \lambda = 0 \\ L_y = \dfrac{n}{y} + \lambda = 0 \\ L_z = \dfrac{p}{z} + \lambda = 0 \\ L_\lambda = x + y + z - a = 0 \end{cases}$，得$\begin{cases} x = \dfrac{ma}{m+n+p} \\ y = \dfrac{na}{m+n+p} \\ z = \dfrac{pa}{m+n+p} \\ \lambda = -\dfrac{m+n+p}{a} \end{cases}$

则$\left(\dfrac{ma}{m+n+p}, \dfrac{na}{m+n+p}, \dfrac{pa}{m+n+p}\right)$为可能极值点

又$L_{x^2} = -\dfrac{m}{x^2}, L_{xy} = L_{yz} = L_{xz} = 0, L_{y^2} = -\dfrac{n}{y^2}, L_{z^2} = -\dfrac{p}{z^2}, \mathrm{d}^2L = \left(-\dfrac{m}{x^2}\ \mathrm{d}x^2 - \dfrac{n}{y^2}\ \mathrm{d}y^2 - \dfrac{p}{z^2}\ \mathrm{d}z^2\right) < 0$

故在$\left(\dfrac{ma}{m+n+p}, \dfrac{na}{m+n+p}, \dfrac{pa}{m+n+p}\right)$处$\ln f$有极大值，即$f$有极大值$\dfrac{m^m n^n p^p}{(m+n+p)^{m+n+p}}\ a^{m+n+p}$

又$f = x^m n^n z^p$当(x,y,z)趋于边界$\begin{cases} x+y=a \\ z=0 \end{cases}$ $\begin{cases} x+z=a \\ y=0 \end{cases}$ $\begin{cases} y+z=a \\ x=0 \end{cases}$ 时，$f \to 0$，故f的唯一极大点也是它的最大点.

3. 求椭圆$x^2+3y^2=12$的内接等腰三角形，使其底边平行于椭圆的长轴，而面积最大.

解：由于题中三角形内接于椭圆$\dfrac{x^2}{(2\sqrt{3})^2} + \dfrac{y^2}{4} = 1$是等腰三角形，且底边平行于长轴

故其底边所对顶点必是短轴上椭圆的顶点$(0,\pm 2)$
设三角形的另一个顶点坐标为(x,y) $(x,y>0)$，则其内接等腰三角形底边长为$2x$，高为$y+2$
等腰三角形三顶点坐标为$A(0,-2),B(x,y),C(-x,y)$，由椭圆的对称性，得$A(0,2),B(x,-y),C(-x,-y)$也是其顶点
则$S=x(y+2)$，点(x,y)在椭圆$x^2+3y^2=12$上
又因此问题是求$S=x(y+2)$在限制条件$x^2+3y^2=12$上的最大值$(x,y>0)$
作函数$L=x(y+2)+\lambda(x^2+3y^2-12)$
则解方程$\begin{cases} L_x=y+2+2\lambda x=0 \\ L_y=x+6\lambda y=0 \\ L_\lambda=x^2+3y^2-12=0 \end{cases}$，得$\begin{cases} x=3 \\ y=1 \\ \lambda=-\dfrac{1}{2} \end{cases}$
于是其顶点坐标为$A(0,2),B(3,-1),C(-3,-1)$或$A(0,-2),B(3,1),C(-3,1)$
因此问题为实际问题，最大值必存在，则在$(0,2),(3,-1),(-3,-1)$或$(0,-2),(3,1),(-3,1)$处其面积最大，其值为9.

4. 试求抛物线$y^2=4x$上的点，使它与直线$x-y+4=0$相距最近.
解：设所求点坐标为(x,y)，则它到直线的距离为$d=\dfrac{1}{\sqrt{2}}\,|x-y+4|$，其中$y^2=4x$
直线$x-y+4=0$将平面分为左、右两部分，左面$x-y+4<0$，右面$x-y+4>0$
而抛物线$y^2=4x$在右面部分，因而点(x,y)到它的距离为$d=\dfrac{1}{\sqrt{2}}\,(x-y+4)$
令$L=\dfrac{1}{\sqrt{2}}\,(x-y+4)+\lambda(y^2-4x)$
则解方程组$\begin{cases} L_x=\dfrac{1}{\sqrt{2}}-4\lambda=0 \\ L_y=-\dfrac{1}{\sqrt{2}}+2\lambda y=0 \\ L_\lambda=y^2-4x=0 \end{cases}$，得$\begin{cases} x=1 \\ y=2 \\ \lambda=\dfrac{1}{4\sqrt{2}} \end{cases}$
又$L_{x^2}=0,L_{y^2}=\dfrac{1}{2\sqrt{2}},L_{xy}=0,\ \mathrm{d}^2L(1,2)=L_{x^2}\,\mathrm{d}x^2+2L_{xy}\,\mathrm{d}x\,\mathrm{d}y+L_{y^2}\,\mathrm{d}y^2=\dfrac{\mathrm{d}y^2}{2\sqrt{2}}>0$
故$(1,2)$为极小点，即点$(1,2)$到直线的距离最近.

5. 抛物面$z=x^2+y^2$被平面$x+y+z=1$截成一椭圆，求原点到这椭圆的最长与最短距离.
解：据题意，求距离$d=\sqrt{x^2+y^2+z^2}$在限制条件$z=x^2+y^2,x+y+z=1$的最值
即求$u=x^2+y^2+z^2$在限制条件下的最值
作$L=x^2+y^2+z^2+\lambda(z-x^2-y^2)+\gamma(x+y+z-1)$
则解方程组$\begin{cases} L_x=2x-2\lambda x+\gamma=0 \\ L_y=2y-2\lambda y+\gamma=0 \\ L_z=2z+\lambda+\gamma=0 \\ L_\lambda=z-x^2-y^2=0 \\ L_\gamma=x+y+z-1=0 \end{cases}$，得$\begin{cases} x_1=\dfrac{-1+\sqrt{3}}{2} \\ y_1=\dfrac{-1+\sqrt{3}}{2} \\ z_1=2-\sqrt{3} \\ \lambda_1=\dfrac{-5\sqrt{3}+9}{3} \\ \gamma_1=-7+\dfrac{11}{3}\sqrt{3} \end{cases}$，$\begin{cases} x_2=\dfrac{-1-\sqrt{3}}{2} \\ y_2=\dfrac{-1-\sqrt{3}}{2} \\ z_2=2+\sqrt{3} \\ \lambda_2=\dfrac{5\sqrt{3}+9}{3} \\ \gamma_2=-7-\dfrac{11}{3}\sqrt{3} \end{cases}$
于是$d(x_1,y_1,z_1)=\sqrt{9-5\sqrt{3}},d(x_2,y_2,z_2)=\sqrt{9+5\sqrt{3}}$
据问题的实际意义，最长、最短距离存在
则最长距离为原点到点$\left(-\dfrac{1+\sqrt{3}}{2},-\dfrac{1+\sqrt{3}}{2},2+\sqrt{3}\right)$的距离，为$\sqrt{9+5\sqrt{3}}$;
最短距离为原点到点$\left(\dfrac{-1+\sqrt{3}}{2},\dfrac{-1+\sqrt{3}}{2},2-\sqrt{3}\right)$的距离，为$\sqrt{9-5\sqrt{3}}$.

6. 求空间一点(a,b,c)到平面$Ax+By+Cz+D=0$的最短距离.
解：设(x,y,z)为平面$Ax+By+Cz+D=0$上任一点，则它与(a,b,c)点的距离为$d=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$，其中$(x,y,z)$满足$Ax+By+Cz+D=0$
因$d>0$，故d最大$\Longleftrightarrow d^2$最大
按题设，应求$d^2=(x-a)^2+(y-b)^2+(z-c)^2$在条件$Ax+By+Cz+D=0$下的极值
令$L=(x-a)^2+(y-b)^2+(z-c)^2+\lambda(Ax+By+Cz+D)$

则解方程组$\begin{cases} L_x = 2(x-a)+\lambda A = 0 \\ L_y = 2(y-b)+\lambda B = 0 \\ L_z = 2(z-c)+\lambda C = 0 \\ L_\lambda = Ax+By+Cz-D = 0 \end{cases}$，得$\begin{cases} x = a - \dfrac{1}{2}\lambda A \\ y = b - \dfrac{1}{2}\lambda B \\ z = c - \dfrac{1}{2}\lambda C \\ \lambda = \dfrac{2(Aa+Bb+Cc+D)}{A^2+B^2+C^2} \end{cases}$

于是$d = \dfrac{|Aa+Bb+Cc+D|}{\sqrt{A^2+B^2+C^2}}$

又当x,y,z中有任一趋于∞时，$d \to \infty$，因此在$\left\{(x,y,z)\middle|(x-a)^2+(y-b)^2+(z-c)^2 < d\right\}$内必取最小值

则点(a,b,c)到平面$Ax+By+Cz+D=0$的最短距离为$d = \dfrac{|Aa+Bb+Cc+D|}{\sqrt{A^2+B^2+C^2}}$.

第十六章　隐函数存在定理、函数相关

§1.　隐函数存在定理

1. 若在隐函数存在定理中条件改为：

   (1) 在区域$D:(x_0-a\leqslant x\leqslant x_0+1,y_0-b\leqslant y\leqslant y_0+b)$上连续；

   (2) $F(x_0,y_0)=0$；

   (3) 当x固定时，函数$F(x,y)$是y的单调函数；则可得到什么样的结论，试证明之.

   证明：结论及证明：

   (1) 在点(x_0,y_0)的某一邻域内，由方程$F(x,y)=0$能唯一确定$y=f(x)$是x的单调函数且$y_0=f(x_0)$.
   由条件(3)知，当x固定时，$F(x,y)$是y的严格单调函数.不妨设它是y的严格单增函数
   固定x_0，函数$F(x_0,y)$是y的严格增函数，且$F(x_0,y_0)=0$，因此有$F(x_0,y_0+b)>0,F(x_0,y_0-b)<0$
   由条件(1)知，$F(x,y)$在区域D上连续，因而存在$\eta>0$，使当$|x-x_0|<\eta$时，亦有
   $F(x,y_0+b)>0,F(x,y_0-b)<0$
   那末对$\forall\overline{x}\in O(x_0,\eta)$，由函数$F(\overline{x},y)$在$[y_0-b,y_0+b]$的连续性及$F(\overline{x},y_0+b)>0,F(\overline{x},y_0-b)<0$
   据零点存在定理，必存在$\overline{y}\in(y_0-b,y_0+b)$，使$F(\overline{x},\overline{y})=0$
   由于$F(\overline{x},\overline{y})$在$[\,y_0-b,y_0+b\,]$严格单调，从而当$y>\overline{y}$时，$F(\overline{x},y)>0$；当$y<\overline{y}$时，$F(\overline{x},y)<0$
   故上述$\overline{y}$是唯一的
   这表明对$\forall\overline{x}\in O(x_0,\eta)$，通过上述方法，有唯一的$\overline{y}$与$\overline{x}$对应，且满足$F(\overline{x},\overline{y})=0$，于是确定了定义在$O(x_0,\eta)$上的单值函数$y=f(x)$满足$F(x,f(x))=0$，特别有$F(x_0,y_0)=0$即$y_0=f(x_0)$.

   (2) $f(x)$是连续函数.
   $\forall x_1\in O(x_0,a)$，下证$y=f(x)$在x_1点连续.
   对$\forall\varepsilon>0(\varepsilon<b)$，设$y_1=f(x_1)$，于是$F(x_1,y_1)=0$
   又由条件(3)，$F(x,y)$是y的严格单增函数
   因此$F(x_1,y_1+\varepsilon)>0,F(x_1,y_1-\varepsilon)<0$
   则由F的连续性，知存在邻域$O(x_1,\delta)\subset O(x_0,a)$，使得当$x\in O(x_1,\delta)$时，恒有
   $F(x,y_1+\varepsilon)>0,F(x,y_1-\varepsilon)<0$
   于是据零点存在定理，得必有$y\in O(y_1,\varepsilon)$，使$F(x,y)=0$即$y=f(x)$
   即只要$|x-x_1|<\delta$，就有$|f(x)-f(x_1)|=|y-y_1|<\varepsilon$即$y=f(x)$在$x_1$点连续
   由$x_1\in O(x_0,a)$的任意性，得$f(x)$为连续函数.

2. 函数$F(x,y)\equiv y^2-x^2(1-x^2)=0$在哪些点近旁可唯一地决定单值连续，且有连续导数的函数$y=y(x)$.
   解：二元函数$F(x,y)=y^2-x^2(1-x^2)$在整个二维空间连续，它的偏导数$F_x=4x^3-2x,F_y=2y$也连续
   由$y^2-x^2(1-x^2)=0$，若$y=0$，则$x^2(1-x^2)=0$，解得$x=0,x=\pm1$
   又$y^2\geqslant0,x^2\geqslant0$，故$1-x^2\geqslant0$即$-1\leqslant x\leqslant1$
   当$y\neq0$时，$F_y\neq0$
   由隐函数存在定理1，在任何满足$\{(x,y)\big|\,|x|<1,x\neq0,y^2-x^2(1-x^2)=0\}$近旁可唯一地决定单值连续且有连续导数的函数$y=y(x)$.

3. 证明有唯一可导的函数$y=y(x)$满足方程 $\sin y+\sinh y=x$，并求出导数$y'(x)$.
   证明：二元函数$F(x,y)=\sin y+\sinh y-x$在整个二维空间连续，$F_x=-1,F_y=\cos y+\cosh y$也连续
   又$\cosh y=\dfrac{e^y+e^{-y}}{2}\geqslant1$ 且等号只在$y=0$时成立，而此时$\cos y=1$，在一般情况下$|\cos y|\leqslant1$
   则对一切点(x,y)，恒有$F_y=\cos y+\cosh y>0$，于是$F_y\neq0$
   由隐函数存在定理1，在任何满足上述方程的点(x,y)，有唯一可导的函数满足方程$\sin y+\sinh y=x$
   其导函数为$y'=-\dfrac{F_x}{F_y}=\dfrac{1}{\cos y+\cosh y}$.

4. 设D是点$P_0:(x_0,y_0,z_0,u_0,v_0)$的邻域，若

   (1) $F(x_0,y_0,z_0,u_0,v_0)=0,G(x_0,y_0,z_0,u_0,v_0)=0$；

   (2) F,G关于一切变量的偏导数在D中连续；

   (3) $J=\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}$在$P_0$点不为零；

   则在(x_0,y_0,z_0)的邻域R内存在唯一的一对函数

$$u=f(x,y,z);v=g(x,y,z)$$

   满足：

(1) $u_0=f(x_0,y_0,z_0),v_0=g(x_0,y_0,z_0)$

(2) $F(x,y,z,f,g)\equiv 0,G(x,y,z,f,g)\equiv 0$

(3) $u=f(x,y,z),v=g(x,y,z)$在点(x_0,y_0,z_0)的邻域R内有对一切变量的偏导数，且
$\dfrac{\partial f}{\partial x}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(x,v)},\dfrac{\partial f}{\partial y}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(y,v)},\dfrac{\partial f}{\partial z}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(z,v)}$
$\dfrac{\partial g}{\partial x}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(u,x)},\dfrac{\partial g}{\partial y}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(u,y)},\dfrac{\partial g}{\partial z}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(u,z)}$

证明：由条件(3)知，F_u,F_v中至少有一个在P_0点不等于0
不妨设$F_v(P_0)\neq 0$，则由隐函数存在定理2，知在P_0点的某个邻域内可以把v从$F(x,y,z,u,v)=0$中解出来.
设$v=\varphi(x,y,z,u)$且$v_0=\varphi(x_0,y_0,z_0,u_0)$在$(x_0,y_0,z_0,u_0)$的某个邻域内是唯一的，具有关于$x,y,z,u$的连续偏导数
把$v=\varphi(x,y,z,u)$代入$G(x,y,z,u,v)$中得$G(x,y,z,u,\varphi(x,y,z,u))=\psi(x,y,z,u)$
故$\psi_u=G_u+G_v\cdot v_u=G_u+G_v\left(-\dfrac{F_u}{F_v}\right)=-\dfrac{J}{F_v}$
由假设$F_v(P_0)\neq 0$且在P_0点$J\neq 0$，故$\psi_u(x_0,y_0,z_0,u_0)\neq 0$
则由定理2，得在(x_0,y_0,z_0,u_0)的某邻域内可从方程$G=G(x,y,z,u,\varphi)\equiv\psi(x,y,z,u)=0$中解出$u$来.
设$u=f(x,y,z)$，它在(x_0,y_0,z_0)的某邻域内有连续偏导数，且$u_0=f(x_0,y_0,z_0)$
把$u=f(x,y,z)$代入$\varphi(x,y,z,u)$中得$v=\varphi(x,y,z,f(x,y,z))=g(x,y,z)$
则有$g(x_0,y_0,z_0)=\varphi(x_0,y_0,z_0,u_0)=v_0$
故$u=f(x,y,z),v=g(x,y,z)$即为所求
对方程组$\begin{cases}F(x,y,z,u,v)=0\\G(x,y,z,u,v)=0\end{cases}$两端关于$x$求导，得$\begin{cases}\dfrac{\partial F}{\partial x}+\dfrac{\partial F}{\partial u}\cdot\dfrac{\partial f}{\partial x}+\dfrac{\partial F}{\partial v}\cdot\dfrac{\partial g}{\partial x}=0\\\dfrac{\partial G}{\partial x}+\dfrac{\partial G}{\partial u}\cdot\dfrac{\partial f}{\partial x}+\dfrac{\partial G}{\partial v}\cdot\dfrac{\partial g}{\partial x}=0\end{cases}$
解之，得$\dfrac{\partial f}{\partial x}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(x,v)},\dfrac{\partial g}{\partial x}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(x,u)}$
同理可得$\dfrac{\partial f}{\partial y}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(y,v)},\dfrac{\partial f}{\partial z}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(z,v)},\dfrac{\partial g}{\partial y}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(u,y)},\dfrac{\partial g}{\partial z}=-\dfrac{1}{J}\ \dfrac{D(F,G)}{D(u,z)}$

5. 设$\varphi_i(x)(i=1,2,\cdots,n)$是$x$的连续可导函数，且
$G_i(x_1,\cdots,x_n)=F_i(\varphi_1(x_1),\cdots,\varphi_n(x_n))$
证明$\dfrac{\partial(G_1,G_2,\cdots,G_n)}{\partial(x_1,x_2,\cdots,x_n)}=\Delta(\varphi(x_1),\cdots,\varphi_n(x_n))\prod\limits_{i=1}^{n}\varphi_i{}'(x_i)$
其中$\Delta(x_1,x_2,\cdots,x_n)=\dfrac{D(F_1,F_2,\cdots,F_n)}{D(x_1,x_2,\cdots,x_n)}$
$\prod\limits_{i=1}^{n}\varphi_i{}'(x_i)=\varphi_1{}'(x_1)\varphi_2{}'(x_2)\cdots\varphi_n{}'(x_n)$.
证明：因$\varphi_i(x)(i=1,2,\cdots,n)$是$x$的连续可导函数，且$G_i(x_1,\cdots,x_n)=F_i(\varphi_1(x_1),\cdots,\varphi_n(x_n))$
则$\dfrac{\partial G_i}{\partial x_j}=\dfrac{\partial F_i}{\partial\varphi_j}\cdot\dfrac{\partial\varphi_j}{\partial x_j}=\dfrac{\partial F_i}{\partial\varphi_j}\ \varphi_i{}'(x_j)(i,j=1,2,\cdots,n)$

$$于是\frac{\partial(G_1,G_2,\cdots,G_n)}{\partial(x_1,x_2,\cdots,x_n)}=\begin{vmatrix}\frac{\partial G_1}{\partial x_1}&\frac{\partial G_1}{\partial x_2}&\cdots&\frac{\partial G_1}{\partial x_n}\\\frac{\partial G_2}{\partial x_1}&\frac{\partial G_2}{\partial x_2}&\cdots&\frac{\partial G_2}{\partial x_n}\\\cdots&\cdots&\cdots&\cdots\\\frac{\partial G_n}{\partial x_1}&\frac{\partial G_n}{\partial x_2}&\cdots&\frac{\partial G_n}{\partial x_n}\end{vmatrix}=\begin{vmatrix}\frac{\partial F_1}{\partial\varphi_1(x_1)}\varphi_1{}'(x_1)&\frac{\partial F_1}{\partial\varphi_2(x_2)}\varphi_2{}'(x_2)&\cdots&\frac{\partial F_1}{\partial\varphi_n(x_n)}\varphi_n{}'(x_n)\\\frac{\partial F_2}{\partial\varphi_1(x_1)}\varphi_1{}'(x_1)&\frac{\partial F_2}{\partial\varphi_2(x_2)}\varphi_2{}'(x_2)&\cdots&\frac{\partial F_2}{\partial\varphi_n(x_n)}\varphi_n{}'(x_n)\\\cdots&\cdots&\cdots&\cdots\\\frac{\partial F_n}{\partial\varphi_1(x_1)}\varphi_1{}'(x_1)&\frac{\partial F_n}{\partial\varphi_2(x_2)}\varphi_2{}'(x_2)&\cdots&\frac{\partial F_n}{\partial\varphi_n(x_n)}\varphi_n{}'(x_n)\end{vmatrix}=$$

$$\varphi_1{}'(x_1)\varphi_2{}'(x_2)\cdots\varphi_n{}'(x_n)\begin{vmatrix}\frac{\partial F_1}{\partial\varphi_1(x_1)}&\frac{\partial F_1}{\partial\varphi_2(x_2)}&\cdots&\frac{\partial F_1}{\partial\varphi_n(x_n)}\\\frac{\partial F_2}{\partial\varphi_1(x_1)}&\frac{\partial F_2}{\partial\varphi_2(x_2)}&\cdots&\frac{\partial F_2}{\partial\varphi_n(x_n)}\\\cdots&\cdots&\cdots&\cdots\\\frac{\partial F_n}{\partial\varphi_1(x_1)}&\frac{\partial F_n}{\partial\varphi_2(x_2)}&\cdots&\frac{\partial F_n}{\partial\varphi_n(x_n)}\end{vmatrix}=\prod_{i=1}^{n}\varphi_i{}'(x_i)\frac{D(F_1,F_2,\cdots,F_n)}{D(\varphi_1(x_1),\varphi_2(x_2),\cdots,\varphi_n(x_n))}=$$

$\Delta(\varphi(x_1),\cdots,\varphi_n(x_n))\prod\limits_{i=1}^{n}\varphi_i{}'(x_i)$.

6. 设$F(x,y,z)$有二阶连续偏导数，并由$F(x,y,z)=0$可确定$z=f(x,y)$.讨论$z=f(x,y)$的极值的必要和充分条

件.再求由

$$x^2+y^2+z^2-2x+2y-4z-10=0$$

所确定的$z=f(x,y)$的极值.

证明：因函数$z=f(x,y)$取得极值的必要条件为$\begin{cases} z_x=f_x(x,y)=0 \\ z_y=f_y(x,y)=0 \end{cases}$

又$z_x=-\dfrac{F_x}{F_z}, z_y=-\dfrac{F_y}{F_z}$，则$F(x,y,z)$取得极值的必要条件为$\begin{cases} F_x=0 \\ F_y=0 \end{cases}$

又隐函数取极值的的充分条件与显函数类同，只是求二阶偏导数时要用隐函数的高阶偏导数求法

令$\begin{cases} F_x=2x-2=0 \\ F_y=2y+2=0 \end{cases}$ 解得$\begin{cases} x=1 \\ y=-1 \end{cases}$，对应的$z$值为$z_1=-2, z_2=6$

又$\dfrac{\partial z}{\partial x}=\dfrac{x-1}{2-z}, \dfrac{\partial z}{\partial y}=\dfrac{1+y}{2-z}, \dfrac{\partial^2 z}{\partial x^2}=\dfrac{(x-1)^2+(2-z)^2}{(2-z)^3}, \dfrac{\partial^2 z}{\partial y^2}=\dfrac{(1+y)^2+(2-z)^2}{(2-z)^3}, \dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{(x-1)(1+y)}{(2-z)^3}$

于是在点$(1,-1,-2), \dfrac{\partial^2 z}{\partial x^2}=\dfrac{1}{4}, \dfrac{\partial^2 z}{\partial y^2}=\dfrac{1}{4}, \dfrac{\partial^2 z}{\partial x\partial y}=0$，由$\dfrac{1}{4}\cdot\dfrac{1}{4}-0=\dfrac{1}{16}>0$及$\dfrac{1}{4}>0$，则$z_1=-2$为极小值；

在点$(1,-1,6), \dfrac{\partial^2 z}{\partial x^2}=-\dfrac{1}{4}, \dfrac{\partial^2 z}{\partial y^2}=-\dfrac{1}{4}, \dfrac{\partial^2 z}{\partial x\partial y}=0$，由$\left(-\dfrac{1}{4}\right)\left(-\dfrac{1}{4}\right)-0=\dfrac{1}{16}>0$及$-\dfrac{1}{4}<0$，则$z_2=6$为极大值

§2. 函数行列式的性质、函数相关

1. 证明$\begin{cases} x = r\cos\theta\cos\varphi \\ y = r\cos\theta\sin\varphi \\ z = r\sin\theta \end{cases}$ 函数独立

证明: 因$\dfrac{D(x,y,z)}{D(r,\theta,\varphi)} = \begin{vmatrix} \cos\theta\cos\varphi & -r\sin\theta\cos\varphi & -r\cos\theta\sin\varphi \\ \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta & r\cos\theta & 0 \end{vmatrix} = -r^2\cos\theta$

则在$r \neq 0$且$\theta \neq k\pi + \dfrac{\pi}{2}$的区域$D$内$\dfrac{D(x,y,z)}{D(r,\theta,\varphi)} \neq 0$

于是据定理5，得原函数组在区域D内函数独立.

2. 证明$\begin{cases} y_1 = x_1 + x_2 + \cdots + x_n \\ y_2 = x_1^2 + x_2^2 + \cdots + x_n^2 \\ y_3 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \end{cases}$ 函数相关，并写出其函数关系式.

证明: 因存在函数$\varphi(t_1,t_2) = \dfrac{1}{2}(t_1^2 - t_2)$，使得$y_3 = \varphi(y_1,y_2) = \dfrac{1}{2}(y_1^2 - y_2)$在整个$n$维空间$(x_1,x_2,\cdots,x_n)$内成为恒等式

则函数组在整个n维空间中函数相关，其函数关系式为$y_3 = \dfrac{1}{2}(y_1^2 - y_2)$.

3. 下列函数是否相关?

(1) $\dfrac{x-y}{x-z}, \dfrac{y-z}{y-x}, \dfrac{z-x}{z-y}$

(2) $\dfrac{x}{1-x-y-z}, \dfrac{y}{1-x-y-z}, \dfrac{z}{1-x-y-z}$

解:

(1) 因$f_1 \cdot f_2 \cdot f_3 = -1$，则函数相关.

(2) 令$f_1(x,y,z) = \dfrac{x}{1-x-y-z}, f_2(x,y,z) = \dfrac{y}{1-x-y-z}, f_3(x,y,z) = \dfrac{z}{1-x-y-z}$

则Jocobi矩阵为$\begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} \dfrac{1-y-z}{(1-x-y-z)^2} & \dfrac{x}{(1-x-y-z)^2} & \dfrac{x}{(1-x-y-z)^2} \\ \dfrac{y}{(1-x-y-z)^2} & \dfrac{1-x-z}{(1-x-y-z)^2} & \dfrac{y}{(1-x-y-z)^2} \\ \dfrac{z}{(1-x-y-z)^2} & \dfrac{z}{(1-x-y-z)^2} & \dfrac{1-x-y}{(1-x-y-z)^2} \end{pmatrix}$

又此矩阵的秩为3，则据定理6，得函数组函数独立.

第三部分　含参变量的积分和广义积分

第十七章　含参变量的积分

1. 设$F(y)=\int_y^{y^2} e^{-x^2y}\,\mathrm{d}x$，计算$F'(y)$.

解：因定理4条件满足，应用定理4，有

$F'(y)=\int_y^{y^2}(-x^2)e^{-x^2y}\,\mathrm{d}x+2ye^{-y^5}-e^{-y^3}=\frac{5}{2}\,ye^{-y^5}-\frac{3}{2}e^{-y^3}-\frac{1}{2y}\,F(y).$

2. 设$F(y)=\int_0^y (x+y)f(x)\,\mathrm{d}x$，其中$f(x)$为可微函数，求$F''(y)$.

解：因$f(x)$为可微函数，则$f(x)$连续，于是$(x+y)f(x)$连续，则定理4条件满足

于是$F'(y)=\int_0^y f(x)\,\mathrm{d}x+2yf(y)$，$F''(y)=3f(y)+2yf'(y)$.

3. 若$F(y)=\int_0^1 \ln\sqrt{x^2+y^2}\,\mathrm{d}x$，直接计算积分，求出$F(y)$，再求出$F'(0)$，并检验应用定理4计算$F'(0)$的正确性.

解：当$y\neq 0$时，有$F(y)=\int_0^1 \ln\sqrt{x^2+y^2}\,\mathrm{d}x=x\ln\sqrt{x^2+y^2}\Big|_0^1-\int_0^1\frac{x^2}{x^2+y^2}\,\mathrm{d}x=\ln\sqrt{1+y^2}-1+$

$y\int_0^1\frac{\mathrm{d}\dfrac{x}{y}}{1+\left(\dfrac{x}{y}\right)^2}\,\mathrm{d}x=\ln\sqrt{1+y^2}-1+y\arctan\frac{1}{y}\,.$

因$F(0)=\int_0^1\ln x\,\mathrm{d}x=-1$

则$F_+{}'(0)=\lim\limits_{y\to+0}\frac{F(y)-F(0)}{y}=\frac{\pi}{2}\,,F_-{}'(0)=\lim\limits_{y\to-0}\frac{F(y)-F(0)}{y}=-\frac{\pi}{2}$

于是$F'(0)$不存在

另一方面，当$x>0$时，$\frac{\partial}{\partial y}\ln\sqrt{x^2+y^2}\Big|_{y=0}=\frac{y}{x^2+y^2}\Big|_{y=0}=0$，则$\int_0^1\left(\frac{\partial}{\partial y}\ln\sqrt{x^2+y^2}\right)\Big|_{y=0}\mathrm{d}x=0$

又$F_+{}'(0)=\frac{\pi}{2}\neq 0=\int_0^1\left(\frac{\partial}{\partial y}\ln\sqrt{x^2+y^2}\right)\Big|_{y=0}\mathrm{d}x, F_-{}'(0)=-\frac{\pi}{2}\neq 0=\int_0^1\left(\frac{\partial}{\partial y}\ln\sqrt{x^2+y^2}\right)\Big|_{y=0}\mathrm{d}x$

则当$y=0$时，不能在积分号下求导数，即使求左、右导数也不行.

4. 求函数

$E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\varphi}\,\mathrm{d}\varphi$和$F(k)=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}}\ (0<k<1)$

的导数且证明$E(k)$满足方程：

$$E''(k)+\frac{1}{k}\,E'(k)+\frac{E(k)}{1-k^2}=0$$

.

解：$E'(k)=-\int_0^{\frac{\pi}{2}}\frac{k\sin^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}\,\mathrm{d}\varphi=\frac{1}{k}\int_0^{\frac{\pi}{2}}\left[\sqrt{1-k^2\sin^2\varphi}-\frac{1}{\sqrt{1-k^2\sin^2\varphi}}\right]\mathrm{d}\varphi=\frac{1}{k}[E(k)-F(k)]$

$F'(k)=\int_0^{\frac{\pi}{2}}\frac{k\sin^2\varphi}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}}\,\mathrm{d}\varphi=-\frac{1}{k}\int_0^{\frac{\pi}{2}}\left[\frac{1}{\sqrt{1-k^2\sin^2\varphi}}-\frac{1}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}}\right]\mathrm{d}\varphi=-\frac{1}{k}\,F(k)+\frac{1}{k}\int_0^{\frac{\pi}{2}}\frac{1}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}}\,\mathrm{d}\varphi$

因$\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{k^2\sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right)=\frac{k^2(\cos^2\varphi-\sin^2\varphi)(1-k^2\sin^2\varphi)+k^4\sin^2\varphi\cos^2\varphi}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}}=\frac{k^2-1+(1-k^2\sin^2\varphi)^2}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}}=$

$\frac{k^2-1}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}}+\sqrt{1-k^2\sin^2\varphi}$

则$\frac{1}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}}=\frac{1}{1-k^2}\,\sqrt{1-k^2\sin^2\varphi}-\frac{1}{1-k^2}\,\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{k^2\sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right)$

于是$\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}}=\frac{1}{1-k^2}\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\varphi}\,\mathrm{d}\varphi=\frac{1}{1-k^2}\,E(k)$

则$F'(k)=\dfrac{1}{k(1-k^2)}\,E(k)-\dfrac{1}{k}\,F(k)$

于是$E''(k)=\dfrac{(E'(k)-F'(k))k-(E(k)-F(k))}{k^2}=-\dfrac{E(k)}{k^2(1-k^2)}-\dfrac{F(k)}{k^2}$

从而$E''(k)+\dfrac{1}{k}\,E'(k)+\dfrac{E(k)}{1-k^2}=-\dfrac{E(k)}{k^2(1-k^2)}+\dfrac{F(k)}{k^2}+\dfrac{E(k)-F(k)}{k^2}+\dfrac{E(k)}{1-k^2}=0.$

5. 研究函数

$$F(y)=\int_0^1\frac{yf(x)}{x^2+y^2}\,\mathrm{d}x,(y\geqslant 0)$$

的连续性，其中$f(x)$是$[0,1]$上连续且为正的函数.

解：设$d>c>0$，取$y\in[c,d]$，则被积函数$\dfrac{yf(x)}{x^2+y^2}$在$[0,1]\times[c,d]$上连续

由定理1，得$F(y)=\displaystyle\int_0^1\frac{yf(x)}{x^2+y^2}\,\mathrm{d}x$在$[c,d]$上连续，由$c,d$的任意性，得$F(y)$在$y>0$连续

又$f(x)$是$[0,1]$上连续且为正的函数，则$f(x)$在$[0,1]$上必有最小值$m>0$

由于$F(y)\geqslant m\displaystyle\int_0^1\frac{y}{x^2+y^2}\,\mathrm{d}x=m\arctan\frac{1}{y}$及$\displaystyle\lim_{y\to+0}\arctan\frac{1}{y}=\frac{\pi}{2}$，则$\displaystyle\lim_{y\to+0}F(y)\geqslant\frac{m\pi}{2}>0$

又$F(0)=0$，则$F(y)$当$y=0$时不连续.

6. 应用对参数求导法计算积分：

(1) $\displaystyle\int_0^{\frac{\pi}{2}}\ln(a^2-\sin^2x)\,\mathrm{d}x\ (a>1)$(不必定常数，若计算时出现无界情况，取极限计算)；

(2) $\displaystyle\int_0^{\pi}\ln(1-2a\cos x+a^2)\,\mathrm{d}x\ (|a|<1)$

解：

(1) 设$I(a)=\displaystyle\int_0^{\frac{\pi}{2}}\ln(a^2-\sin^2x)\,\mathrm{d}x$

因被积函数$\ln(a^2-\sin^2x)$在$\left[0,\dfrac{\pi}{2}\right]\times[1,+\infty]$上不连续

则不能用定理2，为了能用定理，缩小范围$\left[0,\dfrac{\pi}{2}\right]\times[b,c](b>1,c\to+\infty)$

这时$f(x,a)=\ln(a^2-\sin^2x)$及$f_a=\dfrac{2a}{a^2-\sin^2x}$都在闭矩形$\left[0,\dfrac{\pi}{2}\right]\times[b,c]$上连续

由定理2，有$I'(a)=\displaystyle\int_0^{\frac{\pi}{2}}\frac{2a}{a^2-\sin^2x}\,\mathrm{d}x=\frac{2}{\sqrt{a^2-1}}\left[\arctan\frac{a+1}{\sqrt{a^2-1}}+\arctan\frac{a-1}{\sqrt{a^2-1}}\right]=\frac{\pi}{\sqrt{a^2-1}}$

对a积分，得$I(a)=\pi\ln|a+\sqrt{a^2-1}|+C$

因$a\in[b,c]$，由b,c的任意性，则$I(a)=\pi\ln|a+\sqrt{a^2-1}|+C$

(2) 设$I(a)=\displaystyle\int_0^{\pi}\ln(1-2a\cos x+a^2)\,\mathrm{d}x$

当$|a|<1$时，由于$1-2a\cos x+a^2\geqslant 1-2|a|+a^2=(1-|a|)^2>0$

则$f(x,a)=\ln(1-2a\cos x+a^2)$及$f_a=\dfrac{-2\cos x+2a}{1-2a\cos x+a^2}$都在闭矩形$[0,\pi]\times[-b,b]$上连续$(|a|\leqslant b<1)$

由定理2，有$I'(a)=\displaystyle\int_0^{\pi}\frac{-2\cos x+2a}{1-2a\cos x+a^2}\,\mathrm{d}x=\frac{\pi}{a}-\frac{1-a^2}{a}\int_0^{\pi}\frac{\mathrm{d}x}{(1+a^2)-2a\cos x}=\frac{\pi}{a}-\frac{1-a^2}{a(1+a^2)}\int_0^{\pi}\frac{\mathrm{d}x}{1+\left(-\frac{2a}{a^2+1}\right)\cos x}$

作代换$t=\tan\dfrac{x}{2}$

则$\displaystyle\int_0^{\pi}\frac{\mathrm{d}x}{1+\left(-\frac{2a}{a^2+1}\right)\cos x}=2\int_0^{+\infty}\frac{1+a^2}{(1-a)^2+(1+a)^2t^2}\,\mathrm{d}t=\frac{1+a^2}{1-a^2}\,\pi$

于是$I'(a)=0$，从而$I(a)=C$

又$I(0)=0$，则$C=0$，于是$I(a)=0$

因$a\in[-b\,b]$，由b的任意性，得当$|a|<1$时，$I(a)=0.$

7. 应用积分号下求积分方法计算积分：

$$\int_0^1\sin\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}\,\mathrm{d}x\ (a>0,b>0)$$

(若出现无界情况与前面同样处理).

解: 不妨设$a<b$

因$\displaystyle\int_0^1 \sin\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}\,\mathrm{d}x=\int_0^1 \sin\left(\ln\frac{1}{x}\right)\mathrm{d}x\int_a^b x^y\,\mathrm{d}y=\int_a^b \mathrm{d}y\int_0^1 \sin\left(\ln\frac{1}{x}\right)x^y\,\mathrm{d}x.$

这里，当$x=0$时，$\sin\left(\ln\dfrac{1}{x}\right)x^y$理解为0，从而$\sin\left(\ln\dfrac{1}{x}\right)x^y$在$0\leqslant x\leqslant 1,a\leqslant y\leqslant b$上连续

则可应用积分号下的积分法交换积分次序

作代换$x=e^{-t}$，可得$\displaystyle\int_0^1 \sin\left(\ln\frac{1}{x}\right)x^y\,\mathrm{d}x=\int_0^{+\infty} e^{-(y+1)t}\sin t\,\mathrm{d}t=\frac{1}{1+(1+y)^2}$

于是$\displaystyle\int_0^1 \sin\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}\,\mathrm{d}x=\int_a^b \frac{\mathrm{d}y}{1+(1+y)^2}=\arctan(1+b)-\arctan(1+a)=\arctan\frac{b-a}{1+(1+b)(1+a)}$

8. 证明$\displaystyle\int_0^1 \mathrm{d}x\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}y\neq\int_0^1 \mathrm{d}y\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}x.$

证明: 因$\displaystyle\int_0^1 \mathrm{d}x\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}y=\int_0^1 \frac{\mathrm{d}x}{1+x^2}=\frac{\pi}{4}$，$\displaystyle\int_0^1 \mathrm{d}y\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}x=-\int_0^1 \frac{\mathrm{d}y}{1+y^2}=-\frac{\pi}{4}$

则$\displaystyle\int_0^1 \mathrm{d}x\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}y\neq\int_0^1 \mathrm{d}y\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}x.$

9. 设函数$f(x,y)$在$D=[a,A;b,B]$有界，除去D内有限条连续曲线$y=\varphi_i(x)$，f在D连续，证明：

$$F(x)=\int_b^B f(x,y)\,\mathrm{d}y$$

在$[a,A]$连续.

证明: 不妨设只有一条连续曲线$y=\varphi_1(x)$，$f(x,y)$在这条曲线上间断

因$f(x,y)$有界，记$M=\sup\limits_{[a,A;b,B]}|f(x,y)|$

任取$x_0\in[\alpha,\beta]\subset[a,A],y_0=\varphi_1(x_0)\in[b,B]$

下证$F(x)=\displaystyle\int_b^B f(x,y)\,\mathrm{d}y$在$x_0$点连续，即证$\forall\varepsilon>0,\exists\delta>0$，当$|x-x_0|<\delta$时，有

$|F(x)-F(x_0)|=\left|\displaystyle\int_b^B f(x,y)\,\mathrm{d}y-\int_b^B f(x_0,y)\,\mathrm{d}y\right|<\varepsilon$

由于$y=\varphi_1(x)$在x_0点连续，则对$\forall\varepsilon_1>0,\exists\delta_1>0$，当$|x-x_0|<2\delta_1$时，有$|y-y_0|=|\varphi_1(x)-\varphi_1(x_0)|<\varepsilon_1$

于是在$x_0-\delta_1\leqslant x\leqslant x_0+\delta_1,b\leqslant y\leqslant B$的带域内使$f(x,y)$间断的点只含于以$(x_0,y_0)$为中心的矩形域$x_0-\delta_1\leqslant x\leqslant x_0+\delta_1,y_0-\varepsilon_1<y<y_0+\varepsilon_1$在这带域的上、下两侧(若$y_0-\varepsilon_1$恰好等于$b$或$y_0+\varepsilon_1$恰好等于$B$，则只有上侧或下侧)，闭域中$f(x,y)$为连续

因而在这两个(或一个)闭域中$f(x,y)$为一致连续，特别对$\forall\varepsilon_2>0,\exists\delta_2>0$，当$|x-x_0|<\delta_2$时，有

$|f(x,y)-f(x_0,y)|<\varepsilon_2$

现取$\delta=\min(\delta_1,\delta_2)$，当$|x'-x_0|<\delta$时，有

$|F(x')-F(x)|=\left|\displaystyle\int_b^B f(x',y)\,\mathrm{d}y-\int_b^B f(x_0,y)\,\mathrm{d}y\right|\leqslant\int_b^{y_0-\varepsilon_1}|f(x',y)-f(x_0,y)|\,\mathrm{d}y+\int_{y_0-\varepsilon_1}^{y_0+\varepsilon_1}|f(x',y)|\,\mathrm{d}y+$

$\displaystyle\int_{y_0-\varepsilon_1}^{y_0+\varepsilon_1}|f(x_0,y)|\,\mathrm{d}y+\int_{y_0+\varepsilon_1}^{B}|f(x',y)-f(x_0,y)|\,\mathrm{d}y\leqslant\varepsilon_2(B-b)+4\varepsilon_1 M$

若取$\varepsilon_1=\dfrac{\varepsilon}{8M}$ $,\varepsilon_1=\dfrac{\varepsilon}{2(B-b)}$

则当$|x'-x_0|<\delta$时，有$|F(x')-F(x_0)|<\varepsilon$

当$x_0\in[a,\alpha]$或$x_0\in[\beta,A]$时，$F(x)$在x_0连续，故$F(x)$在$[a,A]$连续

若$f(x,y)$有间断的连续曲线有几条，则只需把使$f(x,y)$可能成为间断的点用至多几个小矩形隔开就行了

其余论证相同

由于$f(x,y)$有界且至多有几条间断线，则$F(x)=\displaystyle\int_b^B f(x,y)\,\mathrm{d}y$存在且在$[a,A]$连续.

第十八章　含参变量的广义积分

1. 证明：若在$[a,\infty;c,d]$内成立$|f(x,y)|\leqslant F(x,y)$，并且关于$y\in[c,d]$积分$\int_a^{+\infty}F(x,y)\,\mathrm{d}x$一致收敛，则$\int_a^{+\infty}f(x,y)\,\mathrm{d}x$关于$y\in[c,d]$亦一致收敛，且绝对收敛.

证明：因积分$\int_a^{+\infty}F(x,y)\,\mathrm{d}x$关于$y\in[c,d]$一致收敛，则由含参变量的广义积分的柯西一致收敛原理，得

对$\forall\varepsilon>0$，存在与y无关的$A_0(\varepsilon)>a$，当$A,A'\geqslant A_0$时，对一切$y\in[c,d]$，有$\left|\int_A^{A'}F(x,y)\,\mathrm{d}x\right|<\varepsilon$

而$\left|\int_A^{A'}f(x,y)\,\mathrm{d}x\right|\leqslant\left|\int_A^{A'}|f(x,y)|\,\mathrm{d}x\right|\leqslant\left|\int_A^{A'}F(x,y)\,\mathrm{d}x\right|<\varepsilon$对一切$y\in[c,d]$都成立

由无穷限含参变量广义积分的柯西一致收敛原理，$\int_a^{+\infty}f(x,y)\,\mathrm{d}x$关于$y\in[c,d]$一致收敛，$\int_a^{+\infty}|f(x,y)|\,\mathrm{d}x$关于$y\in[c,d]$一致收敛

则$\int_a^{+\infty}f(x,y)\,\mathrm{d}x$关于$y\in[c,d]$一致收敛且绝对收敛.

2. 证明下列积分在所给定的区间内一致收敛：

(1) $\int_0^{+\infty}\dfrac{\cos xy}{x^2+y^2}\ \mathrm{d}x\ (y\geqslant a>0)$

(2) $\int_0^{+\infty}\dfrac{\cos xy}{x^2+1}\ \mathrm{d}x\ (-\infty<y<+\infty)$

(3) $\int_0^1\ln xy\,\mathrm{d}x\left(\dfrac{1}{b}\leqslant y\leqslant b,b>1\right)$

证明：

(1) 因$y\geqslant a>0$，则$\left|\dfrac{\cos xy}{x^2+y^2}\right|\leqslant\dfrac{1}{x^2+a^2}$ 而$\int_0^{+\infty}\dfrac{\mathrm{d}x}{x^2+a^2}=\dfrac{\pi}{2a}$ 收敛

于是由魏氏判别法，得$\int_0^{+\infty}\dfrac{\cos xy}{x^2+y^2}\ \mathrm{d}x$ 关于y在$[a,+\infty)(a>0)$内一致收敛.

(2) 因$y\in(-\infty,+\infty)$, $\left|\dfrac{\cos xy}{x^2+1}\right|\leqslant\dfrac{1}{x^2+1}$ 而$\int_0^{+\infty}\dfrac{\mathrm{d}x}{x^2+1}=\dfrac{\pi}{2}$ 收敛

于是由魏氏判别法，得$\int_0^{+\infty}\dfrac{\cos xy}{x^2+1}\ \mathrm{d}x$ 关于y在$(-\infty,+\infty)$内一致收敛.

(3) $x=0$是奇点，当$\dfrac{1}{b}\leqslant y\leqslant b,b>1,0<x\leqslant1$时，$|\ln xy|\leqslant|\ln x|+|\ln y|\leqslant-\ln x+\ln b=\ln\dfrac{b}{x}$

因$\lim\limits_{x\to+0}x^{\frac14}\ln\dfrac{b}{x}=\lim\limits_{x\to+0}\dfrac{\ln\dfrac{b}{x}}{x^{-\frac14}}=0$，则由无界函数广义积分判别法的极限形式，得$\int_0^1\ln\dfrac{b}{x}\ \mathrm{d}x$收敛

从而由魏氏判别法，得$\int_0^1\ln xy\,\mathrm{d}x$关于$y$在$[\dfrac{1}{b},b](b>1)$上一致收敛.

3. 设$f(x,y)$在$[a,+\infty;c,d]$连续，对$[c,d)$上每一个y，$\int_a^{+\infty}f(x,y)\,\mathrm{d}x$收敛，但积分在$y=d$发散.

证明这积分在$[c,d]$非一致收敛.

证明：由$\int_a^{+\infty}f(x,d)\,\mathrm{d}x$发散，得$\exists\varepsilon_0>0,\forall A_0>a,\exists A',A''\geqslant A_0$，使$\left|\int_{A'}^{A''}f(x,d)\,\mathrm{d}x\right|\geqslant\varepsilon_0$

这表明对$y=d\in[c,d]$有$\left|\int_{A'}^{A''}f(x,y)\,\mathrm{d}x\right|\geqslant\varepsilon_0$，说明$\int_a^{+\infty}f(x,y)\,\mathrm{d}x$在$[c,d]$非一致收敛.

4. 讨论下列积分在指定区间的一致收敛性：

(1) $\int_1^{+\infty}x^\alpha e^{-x}\,\mathrm{d}x\ (a\leqslant\alpha\leqslant b;a,b$为任意实数$)$

(2) $\int_0^{+\infty}\sqrt{\alpha}e^{-\alpha x^2}\,\mathrm{d}x\ (0<\alpha<+\infty)$

(3) $\int_{-\infty}^{+\infty} e^{-(x-\alpha)^2}\,\mathrm{d}x$

(i) $a<\alpha<b$

(ii) $-\infty<\alpha<+\infty$

(4) $\int_0^1 x^{p-1}\ln^2 x\,\mathrm{d}x$

(i) $p\geqslant p_0>0$

(ii) $p>0$

(5) $\int_0^{+\infty} e^{-\alpha x}\sin x\,\mathrm{d}x\ (\alpha>0)$

解:

(1) 因$\alpha\in[a,b], x\in(1,+\infty)$，则$0<|x^\alpha e^{-x}|\leqslant x^b e^{-x}$

又$\lim\limits_{x\to+\infty} x^2\cdot x^b e^{-x}=0$，则据无穷限广义积分的柯西判别法的极限形式，得$\int_1^{+\infty} x^b e^{-x}\,\mathrm{d}x$一致收敛

于是由魏氏判别法，得$\int_1^{+\infty} x^\alpha e^{-x}\,\mathrm{d}x$关于$\alpha\in[a,b]$($a,b$为任意实数)一致收敛.

(2) $\int_0^{+\infty}\sqrt{\alpha}e^{-\alpha x^2}\,\mathrm{d}x=\dfrac{\sqrt{\pi}}{2}$收敛，但它在$(0,+\infty)$关于$\alpha$非一致收敛

对$\forall A>0$，因$\lim\limits_{\alpha\to+0}\int_A^{+\infty}\sqrt{\alpha}e^{-\alpha x^2}\,\mathrm{d}x=\lim\limits_{\alpha\to+0}\int_{\sqrt{\alpha}A}^{+\infty} e^{-t^2}\,\mathrm{d}t=\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t=\dfrac{\sqrt{\pi}}{2}$

故对于$0<\varepsilon_0<\dfrac{\sqrt{\pi}}{2}$，必存在$\alpha_0>0$，使得$\left|\int_A^{+\infty}\sqrt{\alpha_0}e^{-\alpha_0 x^2}\,\mathrm{d}x\right|=\int_A^{+\infty}\sqrt{\alpha_0}e^{-\alpha_0 x^2}\,\mathrm{d}x>\varepsilon_0$，

即$\int_0^{+\infty}\sqrt{\alpha}e^{-\alpha x^2}\,\mathrm{d}x$关于$\alpha$在$(0,+\infty)$上不一致收敛.

(3) 对任意固定的$\alpha\in(-\infty,+\infty)$，积分$\int_{-\infty}^{+\infty} e^{-(x-\alpha)^2}\,\mathrm{d}x$都收敛，且$\int_{-\infty}^{+\infty} e^{-(x-\alpha)^2}\,\mathrm{d}x=\sqrt{\pi}$

(i) $|x|$充分大时，对一切$a<\alpha<b$，有$0<e^{-(x-\alpha)^2}<2e^{-\frac{x^2}{4}}$

因$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{4}}\,\mathrm{d}x=2\int_0^{+\infty} e^{-\frac{x^2}{4}}\,\mathrm{d}x$收敛

则由魏氏判别法，得$\int_{-\infty}^{+\infty} e^{-(x-\alpha)^2}\,\mathrm{d}x$对$a<\alpha<b$一致收敛.

(ii) 对$\forall A>0$，有$\lim\limits_{\alpha\to+\infty}\int_A^{+\infty} e^{-(x-\alpha)^2}\,\mathrm{d}x=\lim\limits_{\alpha\to+\infty}\int_{A-\alpha}^{+\infty} e^{-t^2}\,\mathrm{d}t=\sqrt{\pi}$

则当α充分大时，$\int_A^{+\infty} e^{-(x-\alpha)^2}\,\mathrm{d}x>\dfrac{\sqrt{\pi}}{2}$

由此，得$\int_0^{+\infty} e^{-(x-\alpha)^2}\,\mathrm{d}x$在$-\infty<\alpha<+\infty$上非一致收敛

从而$\int_{-\infty}^{+\infty} e^{-(x-\alpha)^2}\,\mathrm{d}x$在$-\infty<\alpha<+\infty$上非一致收敛.

(4) (i) $|x^{p-1}\ln^2 x|=x^{p-1}\ln^2 x\leqslant x^{p_0-1}\ln^2 x\ (p\geqslant p_0>0, 0\leqslant x\leqslant 1)$

积分$\int_0^1 x^{p-1}\ln^2 x\,\mathrm{d}x=\int_0^{+\infty} e^{-p_0 z}z^2\,\mathrm{d}z$

$\lim\limits_{z\to+\infty} z^2\cdot e^{-p_0 z}z^2=\lim\limits_{z\to+\infty}\dfrac{z^4}{e^{p_0 z}}=0\ (p_0>0)$

则由柯西判别法的极限形式$\int_0^{+\infty} e^{-p_0 z}z^2\,\mathrm{d}z$收敛，于是$\int_0^1 x^{p_0-1}\ln^2 x\,\mathrm{d}x$收敛

从而由魏氏判别法，得$\int_0^1 x^{p-1}\ln^2 x\,\mathrm{d}x$关于$p$在$p\geqslant p_0>0$上一致收敛.

(ii) 因当$x\in\left(0,\dfrac{1}{e}\right)$, $\ln^2 x\geqslant 1$

故有$\int_0^1 x^{p-1}\ln^2 x\,\mathrm{d}x>\int_0^{\frac{1}{e}} x^{p-1}\ln^2 x\,\mathrm{d}x>\int_0^{\frac{1}{e}} x^{p-1}\,\mathrm{d}x=\dfrac{1}{p}\left(\dfrac{1}{e}\right)^p\to+\infty\ (p\to+0)$

于是$\int_0^1 x^{p-1}\ln^2 x\,\mathrm{d}x$在$p>0$时不一致收敛.

(5) 用反证法.

假设$\int_0^{+\infty} e^{-\alpha x}\sin x\,\mathrm{d}x$关于$\alpha > 0$一致收敛，则对$\forall\varepsilon > 0, \exists M > 0$，当$A'' > A' > M$时，对一切$\alpha > 0$成立$\left|\int_{A'}^{A''} e^{-\alpha x}\sin x\,\mathrm{d}x\right| < \varepsilon$

从而对于$\alpha\in(0,1)$亦成立$\left|\int_{A'}^{A''} e^{-\alpha x}\sin x\,\mathrm{d}x\right| < \varepsilon$

在不等式两边令$\alpha\to 0$，则有$\left|\int_{A'}^{A''}\sin x\,\mathrm{d}x\right| \leqslant \varepsilon$，从而$\int_0^{+\infty}\sin x\,\mathrm{d}x$收敛

而$\int_0^{A}\sin x\,\mathrm{d}x = 1-\cos A$，当$A\to+\infty$时，$\cos A$的极限不存在，于是$\int_0^{+\infty}\sin x\,\mathrm{d}x$发散，则矛盾，故假设不成立

从而$\int_0^{+\infty} e^{-\alpha x}\sin x\,\mathrm{d}x$关于$\alpha > 0$不一致收敛.

5. 证明：

(1) $\int_0^{+\infty}\dfrac{\alpha}{x^2+\alpha^2}\,\mathrm{d}x$在不含$\alpha = 0$的任何区间上是连续函数；

(2) $F(p) = \int_0^{\pi}\dfrac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x$在$(0,2)$内连续.

证明：

(1) 设$F(\alpha) = \int_0^{+\infty}\dfrac{\alpha}{x^2+\alpha^2}\,\mathrm{d}x$.

对任何$\alpha_0\neq 0$，不妨设$\alpha_0 > 0$，今取$\delta > 0$，使得$\alpha_0-\delta > 0$，下证$F(\alpha)$在$[\alpha_0-\delta,\alpha_0+\delta]$内一致收敛

事实上，当$\alpha\in[\alpha_0-\delta,\alpha_0+\delta]$时，$\dfrac{\alpha}{x^2+\alpha^2}\leqslant\dfrac{\alpha_0+\delta}{x^2+(\alpha_0-\delta)^2}$

因积分$\int_0^{+\infty}\dfrac{\alpha_0+\delta}{(\alpha_0-\delta)^2+x^2}\,\mathrm{d}x$收敛，则由魏氏判别法，得$F(\alpha)$在$[\alpha_0-\delta,\alpha_0+\delta]$上关于$\alpha$一致收敛

于是由连续定理，得$F(\alpha)$在该区间上是α的连续函数，特别在α_0点连续

由于$\alpha_0\neq 0$的任意性，得$\int_0^{+\infty}\dfrac{\alpha}{x^2+\alpha^2}\,\mathrm{d}x$对任何$\alpha\neq 0$连续，由此可知$F(\alpha)$在任何不含$\alpha = 0$的区间上都连续

但由$\lim\limits_{\alpha\to+0}\int_0^{+\infty}\dfrac{\alpha}{\alpha^2+x^2}\,\mathrm{d}x = \dfrac{\pi}{2}$, $\lim\limits_{\alpha\to-0}\int_0^{+\infty}\dfrac{\alpha}{\alpha^2+x^2}\,\mathrm{d}x = -\dfrac{\pi}{2}$

得$F(\alpha)$在$\alpha = 0$处不连续，则$\int_0^{+\infty}\dfrac{\alpha}{x^2+\alpha^2}\,\mathrm{d}x$在不含$\alpha = 0$的任何区间上是连续函数.

(2) 任取$p\in(0,2)$，则存在$0 < p_1, p_2 < 2$，使$0 < p_1\leqslant p\leqslant p_2 < 2$

因0和π均可能是奇点，将积分分为三段

$$\int_0^{\pi}\frac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x = \int_0^{1}\frac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x + \int_1^{\pi-1}\frac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x + \int_{\pi-1}^{\pi}\frac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x$$

对于$\int_0^{1}\dfrac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x$

因$\dfrac{\sin x}{x^p(\pi-x)^{2-p}}\leqslant\dfrac{\sin x}{x^{p_2}(\pi-x)^{2-p_2}}\ (0\leqslant x\leqslant 1, 0 < p_1\leqslant p\leqslant p_2 < 2)$

且$\lim\limits_{x\to+0} x^{p_2-1}\dfrac{\sin x}{x^{p_2}(\pi-x)^{2-p_2}} = \dfrac{1}{\pi^{2-p_2}}$

因$p_2 < 2$，则$p_2-1 < 1$，于是由柯西判别法的极限形式，得$\int_0^{1}\dfrac{\sin x}{x^{p_2}(\pi-x)^{2-p_2}}\,\mathrm{d}x$收敛

从而由魏氏判别法，得$\int_0^{1}\dfrac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x$关于$p\in[p_1,p_2]$一致收敛

又被积函数$\dfrac{\sin x}{x^p(\pi-x)^{2-p}}$在$(0,1]\times[p_1,p_2]$上连续，则由连续性定理，得$\int_0^{1}\dfrac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x$在$[p_1,p_2]$连续

$\int_1^{\pi-1}\dfrac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x$是含参变量的常义积分

因被积函数$\dfrac{\sin x}{x^p(\pi-x)^{2-p}}$在$[1,\pi-1]\times[p_1,p_2]$连续，则由连续性定理，得$\displaystyle\int_1^{\pi-1}\frac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x$在$[p_1,p_2]$连续

对于$\displaystyle\int_{\pi-1}^{\pi}\frac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x$

因$\dfrac{\sin x}{x^p(\pi-x)^{2-p}}\leqslant\dfrac{\sin(\pi-x)}{x^{p_1}(\pi-x)^{2-p_1}}\ (\pi-1\leqslant x\leqslant\pi,0<p_1\leqslant p\leqslant p_2<2)$

且$\displaystyle\lim_{x\to\pi-0}(\pi-x)^{1-p_1}\frac{\sin(\pi-x)}{x^{p_1}(\pi-x)^{2-p_1}}=\frac{1}{\pi^{p_1}}$

因$p_1>0$，则$1-p_1<1$，于是由柯西判别法的极限形式，得$\displaystyle\int_{\pi-1}^{\pi}\frac{\sin(\pi-x)}{x^{p-1}(\pi-x)^{2-p_1}}\,\mathrm{d}x$收敛

从而由魏氏判别法，得$\displaystyle\int_{\pi-1}^{\pi}\frac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x$关于$p\in[p_1,p_2]$一致收敛

又被积函数$\dfrac{\sin x}{x^p(\pi-x)^{2-p}}$在$[\pi-1,\pi)\times[p_1,p_2]$上连续，则由连续性定理，得$\displaystyle\int_{\pi-1}^{\pi}\frac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x$在$[p_1,p_2]$连续

综合以上，得$F(p)$在$[p_1,p_2]$连续，从而在其上任一点p连续

又由$p\in(0,2)$的任意性，得$\displaystyle F(p)=\int_0^{\pi}\frac{\sin x}{x^p(\pi-x)^{2-p}}\,\mathrm{d}x$在$(0,2)$内连续..

6. 设$f(t)$当$t>0$时连续.如果$\displaystyle\int_0^{+\infty}t^{\lambda}f(t)\,\mathrm{d}t$当$\lambda=a,\lambda=b$时都收敛，那末$\displaystyle\int_0^{+\infty}t^{\lambda}f(t)\,\mathrm{d}t$关于$\lambda$在$[a,b]$上一致收敛.

证明: 因$f(t)$当$t>0$时连续，则被积函数$t^{\lambda}f(t)$的奇点只可能是0

于是$\displaystyle\int_0^{+\infty}t^{\lambda}f(t)\,\mathrm{d}t=\int_0^1 t^{\lambda}f(t)\,\mathrm{d}t+\int_1^{+\infty}t^{\lambda}f(t)\,\mathrm{d}t$

对于$\displaystyle\int_0^1 t^{\lambda}f(t)\,\mathrm{d}t=\int_0^1 t^{\lambda-a}t^af(t)\,\mathrm{d}t$

因$\displaystyle\int_0^{+\infty}t^af(t)\,\mathrm{d}t$收敛，则$\displaystyle\int_0^1 t^af(t)\,\mathrm{d}t$收敛，从而关于$\lambda\in(-\infty,+\infty)$一致收敛，而$t^{\lambda-a}$对于$\lambda\geqslant a$单调减且$|t^{\lambda-a}|\leqslant 1\ (0\leqslant t\leqslant 1,\lambda\geqslant a)$

则由Abel判别法，得$\displaystyle\int_0^1 t^{\lambda}f(t)\,\mathrm{d}t$关于$\lambda\geqslant a$一致收敛

对于$\displaystyle\int_1^{+\infty}t^{\lambda}f(t)\,\mathrm{d}t=\int_1^{+\infty}t^{\lambda-b}t^bf(t)\,\mathrm{d}t$

因$\displaystyle\int_0^{+\infty}t^bf(t)\,\mathrm{d}t$收敛，则$\displaystyle\int_1^{+\infty}t^bf(t)\,\mathrm{d}t$收敛，从而关于$\lambda\in(-\infty,+\infty)$一致收敛，而$t^{\lambda-b}$对于$\lambda\leqslant b$单调减且$|t^{\lambda-b}|\leqslant 1\ (1\leqslant t<+\infty,\lambda\leqslant b)$

则由Abel判别法，得$\displaystyle\int_1^{+\infty}t^{\lambda}f(t)\,\mathrm{d}t$关于$\lambda\leqslant b$一致收敛

于是$\displaystyle\int_0^{+\infty}t^{\lambda}f(t)\,\mathrm{d}t$关于$\lambda$在$[a,b]$上一致收敛.

7. 从等式$\displaystyle\frac{e^{-ax}-e^{-bx}}{x}=\int_a^b e^{-xy}\,\mathrm{d}y$出发，计算积分

$$\int_0^{+\infty}\frac{e^{-ax}-e^{-bx}}{x}\,\mathrm{d}x(b>a>0)$$

解: 因$\displaystyle\frac{e^{-ax}-e^{-bx}}{x}=\int_a^b e^{-xy}\,\mathrm{d}y$，则$\displaystyle\int_0^{+\infty}\frac{e^{-ax}-e^{-bx}}{x}\,\mathrm{d}x=\int_0^{+\infty}\mathrm{d}x\int_a^b e^{-xy}\,\mathrm{d}y$

函数e^{-xy}在$[0,+\infty)\times[a,b]$上连续

又对$y\in[a,b](a>0)$，则$|e^{-xy}|\leqslant e^{-ax}$且$\displaystyle\int_0^{+\infty}e^{-ax}\,\mathrm{d}x=\frac{1}{a}$收敛

于是由魏氏判别法，得$\displaystyle\int_0^{+\infty}e^{-xy}\,\mathrm{d}x$关于$y$在$[a,b]$一致收敛

由积分交换顺序定理，得$\displaystyle\int_0^{+\infty}\frac{e^{-ax}-e^{-bx}}{x}\,\mathrm{d}x=\int_0^{+\infty}\mathrm{d}x\int_a^b e^{-xy}\,\mathrm{d}y=\int_a^b\mathrm{d}y\int_0^{+\infty}e^{-xy}\,\mathrm{d}x=\int_a^b\frac{\mathrm{d}y}{y}=\ln\frac{b}{a}\ (b>a>0)$

8. 试证明$\Gamma(s)$的导数存在，求出$\Gamma'(s)$的积分表达式，说明推导过程是合理的.

证明: $\Gamma(s)=\int_0^{+\infty} x^{s-1}e^{-x}\,\mathrm{d}x$

因$x^{s-1}e^{-x}$及$\dfrac{\partial}{\partial s}\left(x^{s-1}e^{-x}\right)=x^{s-1}e^{-x}\ln x$在$0<x<+\infty, s>0$上连续

$\int_0^{+\infty} x^{s-1}e^{-x}\,\mathrm{d}x=\int_0^1 x^{s-1}e^{-x}\,\mathrm{d}x+\int_1^{+\infty} x^{s-1}e^{-x}\,\mathrm{d}x$

对于任意的$s>0$，总可取$0<s_0\leqslant s\leqslant S_0$

$x^{s-1}e^{-x}\leqslant x^{s_0-1}e^{-x}(0\leqslant x\leqslant 1)$

因若$s_0<1$，0为奇点，由$\lim\limits_{x\to+0} x^{1-s_0}x^{s_0-1}e^{-x}=1$及柯西判别法的极限形式，得$\int_0^1 x^{s_0-1}e^{-x}x\,\mathrm{d}x$收敛；

若$s_0\geqslant 1$，则$\int_0^1 x^{s_0-1}e^{-x}\,\mathrm{d}x$为常义积分，故收敛

总之$\int_0^1 x^{s_0-1}e^{-x}\,\mathrm{d}x$收敛，从而由魏氏判别法，得$\int_0^1 x^{s-1}e^{-x}\,\mathrm{d}x$关于$s$在$s\geqslant s_0$上一致收敛

又$x^{s-1}e^{-x}\leqslant x^{S_0-1}e^{-x}(1\leqslant x<+\infty)$

因$\lim\limits_{x\to+\infty} x^2x^{S_0-1}e^{-x}=0$，则由柯西判别法的极限形式，得$\int_1^{+\infty} x^{S_0-1}e^{-x}\,\mathrm{d}x$收敛

于是由魏氏判别法，得$\int_1^{+\infty} x^{s-1}e^{-x}\,\mathrm{d}x$关于$s$在$s\leqslant S_0$上一致收敛

从而$\int_0^{+\infty} x^{s-1}e^{-x}\,\mathrm{d}x$在$[\,s_0,S_0\,]$上一致收敛，故收敛.

$\int_0^{+\infty} x^{s-1}e^{-x}\ln x\,\mathrm{d}x=\int_0^1 x^{s-1}e^{-x}\ln x\,\mathrm{d}x+\int_1^{+\infty} x^{s-1}e^{-x}\ln x\,\mathrm{d}x$

对上面的$0<s_0\leqslant s\leqslant S_0$，$|x^{s-1}e^{-x}\ln x|\leqslant x^{s_0-1}|\ln x|\ (0<x\leqslant 1)$

因$\lim\limits_{x\to+0} x^{1-\frac{s_0}{2}}x^{s_0-1}\ln x=\lim\limits_{x\to+0}\dfrac{\ln x}{x^{-\frac{s_0}{2}}}=0$，则由柯西判别法的极限形式，得

$\int_0^1 x^{s_0-1}e^{-x}|\ln x|\,\mathrm{d}x=-\int_0^1 x^{s_0-1}e^{-x}\ln x\,\mathrm{d}x$收敛

于是由魏氏判别法，得$\int_0^1 x^{s-1}e^{-x}\ln x\,\mathrm{d}x$在$s\geqslant s_0$上一致收敛

又$x^{s-1}e^{-x}\ln x=x^se^{-x}\dfrac{\ln x}{x}<x^{S_0}e^{-x}\ (1\leqslant x<+\infty)$

因$\lim\limits_{x\to+\infty} x^2x^{S_0}e^{-x}=\lim\limits_{x\to+\infty}\dfrac{x^{S_0+2}}{e^x}=0$，则由柯西判别法的极限形式，得$\int_1^{+\infty} x^{S_0}e^{-x}\,\mathrm{d}x$收敛，于是由魏氏判别法，得$\int_1^{+\infty} x^{s-1}e^{-x}\ln x\,\mathrm{d}x$在$s\leqslant S_0$上一致收敛

从而$\int_0^{+\infty} x^{s-1}e^{-x}\ln x\,\mathrm{d}x$在$[\,s_0,S_0\,]$上一致收敛

则由积分号下求导定理，得$\Gamma(s)$在$[\,s_0,S_0\,]$上可导，当然在s可导，且$\Gamma'(s)=\int_0^{+\infty} x^{s-1}e^{-x}\ln x\,\mathrm{d}x$

再由$s>0$的任意性，得$\Gamma(s)$在$s>0$可导且$\Gamma'(s)=\int_0^{+\infty} x^{s-1}e^{-x}\ln x\,\mathrm{d}x$.

9. (1) 从$\int_0^{+\infty} e^{-y^2}\,\mathrm{d}y=\dfrac{\sqrt{\pi}}{2}$推出$L(c)=\int_0^{+\infty} e^{-y^2-\frac{c^2}{y^2}}\,\mathrm{d}y=\dfrac{\sqrt{\pi}}{2}e^{-2c}$；

(2) 利用积分号下求导的法则引出$\dfrac{\mathrm{d}L}{\mathrm{d}c}=-2L$来求得同一结果，并推出$\int_0^{+\infty} e^{-ay^2-\frac{b}{y^2}}\,\mathrm{d}y\ (a>0,b>0)$之值.

证明:

(1) $L(c)=\int_0^{+\infty} e^{-y^2-\frac{c^2}{y^2}}\,\mathrm{d}y=\int_0^{+\infty} e^{-\left(y-\frac{c}{y}\right)^2-2c}\,\mathrm{d}y=e^{-2c}\int_0^{+\infty} e^{-\left(y-\frac{c}{y}\right)^2}\,\mathrm{d}y=$
$e^{-2c}\int_0^{+\infty} e^{-\left(y-\frac{c}{y}\right)^2}\,\mathrm{d}\left(y-\dfrac{c}{y}\right)+e^{-2c}\int_0^{+\infty} e^{-\left(y-\frac{c}{y}\right)^2}\,\mathrm{d}\dfrac{c}{y}$

在前一积分中令$u=y-\dfrac{c}{y}$，在后一积分中令$v=\dfrac{c}{y}$

则$L(c)=e^{-2c}\int_{-\infty}^{+\infty} e^{-u^2}\,\mathrm{d}u-e^{-2c}\int_0^{+\infty} e^{-(v-\frac{c}{v})^2}\,\mathrm{d}v=\sqrt{\pi}e^{-2c}-L(c)$

于是$L(c)=\int_0^{+\infty} e^{-y^2-\frac{c^2}{y^2}}\,\mathrm{d}y=\dfrac{\sqrt{\pi}}{2}\,e^{-2c}$.

(2) $L(c)=\int_0^{+\infty} e^{-y^2-\frac{c^2}{y^2}}\,\mathrm{d}y,\ \dfrac{\mathrm{d}L}{\mathrm{d}c}=2\int_0^{+\infty} e^{-y^2-\frac{c^2}{y^2}}\left(-\dfrac{c}{y^2}\right)\mathrm{d}y$

令$v=\dfrac{c}{y}$，则$\dfrac{\mathrm{d}L}{\mathrm{d}c}=-2\int_0^{+\infty} e^{-v^2-\frac{c^2}{v^2}}\,\mathrm{d}v=-2L(c)$

于是$\ln L=-2c+\ln c_0$即$\ln\dfrac{L}{c_0}=-2c$亦即$L=c_0e^{-2c}$

又$L(0)=\int_0^{+\infty} e^{-y^2}\,\mathrm{d}y=\dfrac{\sqrt{\pi}}{2}$，则$c_0=\dfrac{\sqrt{\pi}}{2}$，于是$L(c)=\dfrac{\sqrt{\pi}}{2}\,e^{-2c}$

则令$u=\sqrt{a}y$，有

$$\int_0^{+\infty} e^{-ay^2-\frac{b}{y^2}}\,\mathrm{d}y=\frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-u^2-\frac{(\sqrt{ab})^2}{u^2}}\,\mathrm{d}u=\frac{1}{\sqrt{a}}\cdot\frac{\sqrt{\pi}}{2}e^{-2\sqrt{ab}}=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}\ (a>0,b>0).$$

第四部分　多变量积分学

第十九章　积分(二重、三重积分，第一类曲线、曲面积分)的定义和性质

§2.　积分的性质

1. 证明中值定理：若$f(M),g(M)$在Ω上连续，$g(M)$在Ω不变号，则

$$\int_\Omega f(M)g(M)\,\mathrm{d}\Omega=f(P)\int_\Omega g(M)\,\mathrm{d}\Omega$$

其中$P\in\Omega$.

证明：设Ω是有界闭区域且有度量
因$f(M),g(M)$在Ω上连续，$g(M)$在Ω不变号
则$f(M),g(M)$在Ω上可积，且可设$g(M)\geqslant 0$，$M=\max\limits_{M\in\Omega}\{f(M)\},m=\min\limits_{M\in\Omega}\{f(M)\}$
由性质4，得$m\int_\Omega g(M)\,\mathrm{d}\Omega\leqslant\int_\Omega f(M)g(M)\,\mathrm{d}\Omega\leqslant M\int_\Omega g(M)\,\mathrm{d}\Omega$
若$\int_\Omega g(M)\,\mathrm{d}\Omega=0$，由于$g(M)\geqslant 0$且连续，则必有$g(M)\equiv 0,M\in\Omega$，从而$\int_\Omega f(M)g(M)\,\mathrm{d}\Omega=0$即要证不等式成立；
若$\int_\Omega g(M)\,\mathrm{d}\Omega>0$，则$m\leqslant\dfrac{\int_\Omega f(M)g(M)\,\mathrm{d}\Omega}{\int_\Omega g(M)\,\mathrm{d}\Omega}\leqslant M$
由连续函数的介值定理，得必存在$P\in\Omega$，使$\dfrac{\int_\Omega f(M)g(M)\,\mathrm{d}\Omega}{\int_\Omega g(M)\,\mathrm{d}\Omega}=f(P)$
即$\int_\Omega f(M)g(M)\,\mathrm{d}\Omega=f(P)\int_\Omega g(M)\,\mathrm{d}\Omega$
同理，当$g(M)\leqslant 0$时，亦有$\int_\Omega f(M)g(M)\,\mathrm{d}\Omega=f(P)\int_\Omega g(M)\,\mathrm{d}\Omega$.

2. 证明：若$f(M)$在Ω上连续，$f(M)\geqslant 0$，但$f(M)\not\equiv 0$，则

$$\int_\Omega f(M)\,\mathrm{d}\Omega>0$$

.
证明：因$f(M)\geqslant 0$，$f(M)\not\equiv 0$，则至少存在一点$M_0\in\Omega$，使得$f(M_0)>0$
又$f(M)$在Ω上连续，当然在M_0连续，则必存在$\delta>0$，当$M\in O(M_0,\delta)$时，有$f(M)>0$
于是$\int_\Omega f(M)\,\mathrm{d}\Omega=\int_{\Omega\setminus O(M_0,\delta)}f(M)\,\mathrm{d}\Omega+\int_{O(M_0,\delta)}f(M)\,\mathrm{d}\Omega\geqslant\int_{O(M_0,\delta)}f(M)\,\mathrm{d}\Omega>0$

3. 证明：若$f(M)$在Ω上连续，在Ω的任何部分区域$\Omega'\subseteq\Omega$上

$$\int_{\Omega'}f(M)\,\mathrm{d}\Omega=0$$

则$f(M)\equiv 0$
由此证明：若$f(M),g(M)$在Ω上连续，在Ω的任何部分区域$\Omega'\subseteq\Omega$上成立：

$$\int_{\Omega'}f(M)\,\mathrm{d}\Omega=\int_{\Omega'}g(M)\,\mathrm{d}\Omega$$

则在Ω上成立：$f(M)\equiv g(M)$.
证明：用反证法.若存在点$M'\in\Omega$，使$f(M')\neq 0$，不妨设$f(M')>0$
由于$f(M)$在Ω上连续，则存在M'的邻域$\Omega'=O(M',\delta)\subset\Omega(\delta>0)$，使得$f(M)>\dfrac{f(M')}{2}>0,\forall M\in\Omega'$
于是有$\int_{\Omega'}f(M)\,\mathrm{d}\Omega\geqslant\dfrac{f(M')}{2}\,||\Omega'||>0$与题设$\int_{\Omega'}f(M)\,\mathrm{d}\Omega=0$矛盾

则假设不成立，即有$f(M)\equiv 0$

令$F(M)=f(M)-g(M)$，则在Ω的任何部分区域$\Omega'\subseteq\Omega$上$\int_{\Omega'}F(M)\,\mathrm{d}\Omega=\int_{\Omega'}f(M)\,\mathrm{d}\Omega-\int_{\Omega'}g(M)\,\mathrm{d}\Omega=0$

从而由上面所证结论，有$F(M)\equiv 0$，即$f(M)-g(M)\equiv 0$亦即$f(M)\equiv g(M)$.

4. 若$|f(M)|$在Ω上可积，那末$f(M)$在Ω上是否可积？考察函数$f(x,y)=-1$，当x和y中至少有一个是无理数时；$f(x,y)=1$，当x和y都是有理数时，在$[0,1;0,1]$上的积分.

解：未必.

事实上，$f(x,y)$在$[0,1;0,1]$上的上和、下和分别为$S'=\sum_{i_k}M_{i_k}\Delta\Omega_{i_k}=1, S=\sum_{i_k}m_{i_k}\Delta\Omega_{i_k}=-1$

其中$M_{i_k}=\max\limits_{[0,1;0,1]}f(x,y)=1, m_{i_k}=\min\limits_{[0,1;0,1]}f(x,y)=-1$

从而$f(x,y)$在$[0,1;0,1]$上不可积

然而$|f(x,y)|\equiv 1$在$[0,1;0,1]$上可积.

第二十章　重积分的计算及应用

§1.　二重积分的计算

1. 化二重积分

$$I=\iint\limits_D f(x,y)\,\mathrm{d}\sigma$$

为二次积分(分别列出对两个变量先后次序不同的二次积分)，其中积分域D分别为：

(1) D是由x轴与$x^2+y^2=r^2(y>0)$所围成的区域；

(2) D是由$y=0,y=x^2(x>0)$及$x+y=2$所围成的区域；

(3) D是由$y=x,x=2$及$y=\dfrac{1}{x}\,(x>0)$所围成的区域；

(4) D是圆环$1\leqslant x^2+y^2\leqslant 4$

解：

(1) $I=\displaystyle\int_{-r}^{r}\mathrm{d}x\int_0^{\sqrt{r^2-x^2}}f(x,y)\,\mathrm{d}y=\int_0^r\mathrm{d}y\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}f(x,y)\,\mathrm{d}x$

(2) $I=\displaystyle\int_0^1\mathrm{d}y\int_{\sqrt[3]{y}}^{2-y}f(x,y)\,\mathrm{d}x=\int_0^1\mathrm{d}x\int_0^{x^3}f(x,y)\,\mathrm{d}y+\int_1^2\mathrm{d}x\int_0^{2-x}f(x,y)\,\mathrm{d}y$

(3) $I=\displaystyle\int_{\frac{1}{2}}^{1}\mathrm{d}y\int_{\frac{1}{y}}^{2}f(x,y)\,\mathrm{d}x+\int_1^2\mathrm{d}y\int_y^2 f(x,y)\,\mathrm{d}x=\int_1^2\mathrm{d}x\int_{\frac{1}{x}}^{x}f(x,y)\,\mathrm{d}y$

(4) $I=\displaystyle\int_{-2}^{-1}\mathrm{d}x\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}f(x,y)\,\mathrm{d}y+\int_{-1}^{1}\mathrm{d}x\left[\int_{-\sqrt{4-x^2}}^{-\sqrt{1-x^2}}f(x,y)\,\mathrm{d}y+\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}}f(x,y)\,\mathrm{d}y\right]+\int_1^2\mathrm{d}x\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}f(x,y)\,\mathrm{d}y=$
$\displaystyle\int_{-2}^{-1}\mathrm{d}y\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}f(x,y)\,\mathrm{d}x+\int_{-1}^{1}\mathrm{d}y\left[\int_{-\sqrt{4-y^2}}^{-\sqrt{1-y^2}}f(x,y)\,\mathrm{d}x+\int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}}f(x,y)\,\mathrm{d}x\right]+\int_1^2\mathrm{d}y\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}f(x,y)\,\mathrm{d}x$

2. 设$f(x,y)$在区域D上连续，其中D是由$y=x,y=a$及$x=b(b>a)$所围成的，证明

$$\int_a^b\mathrm{d}x\int_a^x f(x,y)\,\mathrm{d}y=\int_a^b\mathrm{d}y\int_y^b f(x,y)\,\mathrm{d}x$$

证明：由$f(x,y)$在D上连续，故可积

令$\overline{f}(x,y)=\begin{cases} f(x,y), & (x,y)\in D\\ 0, & (x,y)\in[a,b;a,b]\backslash D\end{cases}$ 除$y=x$外连续，故必可积

则$\displaystyle\iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int\limits_{[a,b;\,a,b]}\overline{f}(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_a^b\mathrm{d}x\int_a^b\overline{f}(x,y)\,\mathrm{d}y=\int_a^b\mathrm{d}x\int_a^x f(x,y)\,\mathrm{d}y$

$\displaystyle\iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int\limits_{[a,b;\,a,b]}\overline{f}(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_a^b\mathrm{d}y\int_a^b\overline{f}(x,y)\,\mathrm{d}x=\int_a^b\mathrm{d}y\int_y^b f(x,y)\,\mathrm{d}x.$

3. 在下列积分中改变逐次积分的次序：

(1) $\displaystyle\int_0^{2a}\mathrm{d}x\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}}f(x,y)\,\mathrm{d}y$；

(2) $\displaystyle\int_0^{2\pi}\mathrm{d}x\int_0^{\sin x}f(x,y)\,\mathrm{d}y$；

(3) $\displaystyle\int_0^1\mathrm{d}y\int_0^{2y}f(x,y)\,\mathrm{d}x+\int_1^3\mathrm{d}y\int_0^{3-y}f(x,y)\,\mathrm{d}x$；

(4) $\displaystyle\int_0^1\mathrm{d}x\int_0^{x^2}f(x,y)\,\mathrm{d}y+\int_1^2\mathrm{d}x\int_0^{\sqrt{1-(x-1)^2}}f(x,y)\,\mathrm{d}y.$

解：

(1) $\displaystyle\int_0^{2a}\mathrm{d}x\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}}f(x,y)\,\mathrm{d}y=\int_0^a\mathrm{d}y\left[\int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}}f(x,y)\,\mathrm{d}x+\int_{a+\sqrt{a^2-y^2}}^{2a}f(x,y)\,\mathrm{d}x\right]+\int_a^{2a}\mathrm{d}y\int_{\frac{y^2}{2a}}^{2a}f(x,y)\,\mathrm{d}x.$

(2) $\int_0^{2\pi} \mathrm{d}x \int_0^{\sin x} f(x,y)\,\mathrm{d}y = \int_0^{\pi} \mathrm{d}x \int_0^{\sin x} f(x,y)\,\mathrm{d}y + \int_{\pi}^{2\pi} \mathrm{d}x \int_0^{\sin x} f(x,y)\,\mathrm{d}y =$
$\int_0^{\pi} \mathrm{d}x \int_0^{\sin x} f(x,y)\,\mathrm{d}y - \int_{\pi}^{2\pi} \int_{\sin x}^{0} f(x,y)\,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_{\arcsin y}^{\pi-\arcsin y} f(x,y)\,\mathrm{d}x - \int_{-1}^{0} \mathrm{d}y \int_{\pi-\arcsin y}^{2\pi+\arcsin y} f(x,y)\,\mathrm{d}x.$

(3) $\int_0^1 \mathrm{d}y \int_0^{2y} f(x,y)\,\mathrm{d}x + \int_1^3 \mathrm{d}y \int_0^{3-y} f(x,y)\,\mathrm{d}x = \int_0^2 \mathrm{d}x \int_{\frac{x}{2}}^{3-x} f(x,y)\,\mathrm{d}y.$

(4) $\int_0^1 \mathrm{d}x \int_0^{x^2} f(x,y)\,\mathrm{d}y + \int_1^2 \mathrm{d}x \int_0^{\sqrt{1-(x-1)^2}} f(x,y)\,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_{\sqrt{y}}^{1+\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x.$

4. 计算下列二重积分：

(1) $\iint\limits_{[a,b;\,c,d]} xye^{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$；

(2) $\iint\limits_{\Omega} xy^2\,\mathrm{d}x\,\mathrm{d}y, \Omega$是由抛物线$y^2=2px$和直线$x=\dfrac{\rho}{2}\ (\rho>0)$所界的区域；

(3) $\iint\limits_{\Omega} \dfrac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{2a-x}}\ (a>0), \Omega$是由圆心在点$(a,a)$半径为$a$且与坐标轴相切的圆周的较短一段弧和坐标轴所围成的区域；

(4) $\iint\limits_{\Omega} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y, \Omega$是以$y=x, y=x+a, y=a$和$y=3a\ (a>0)$为边的区域.

解：

(1) $\iint\limits_{[a,b;\,c,d]} xye^{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_a^b xe^{x^2}\,\mathrm{d}x \int_c^d ye^{y^2}\,\mathrm{d}y = \dfrac{1}{4}\,(e^{b^2}-e^{a^2})(e^{d^2}-e^{c^2}).$

(2) $\iint\limits_{\Omega} xy^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\rho}{2}} x\,\mathrm{d}x \int_{-\sqrt{2p\,x}}^{\sqrt{2p\,x}} y^2\,\mathrm{d}y = \dfrac{p\,\rho^3}{21}\,\sqrt{p\,\rho}.$

(3) $\iint\limits_{\Omega} \dfrac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{2a-x}} = \int_0^a \dfrac{\mathrm{d}x}{\sqrt{2a-x}} \int_0^{a-\sqrt{2ax-x^2}} \mathrm{d}y = \left(2\sqrt{2}-\dfrac{8}{3}\right)\ a\sqrt{a}.$

(4) $\iint\limits_{\Omega} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_a^{3a} \mathrm{d}y \int_{y-a}^{y} (x^2+y^2)\,\mathrm{d}x = 14a^4.$

5. 证明

$$J=\int_a^b \mathrm{d}x \int_a^x f(y)\,\mathrm{d}y = \int_a^b f(y)(b-y)\,\mathrm{d}y = \int_a^b f(x)(b-x)\,\mathrm{d}x$$

证明：将$\int_a^b \mathrm{d}x \int_a^x f(y)\,\mathrm{d}y$逐项积分，得$\iint\limits_{\Omega} f(y)\,\mathrm{d}x\,\mathrm{d}y$，其中$\Omega$是$x=b, x=y, y=a$所围成的区域对此积分可化为先对$x$后对$y$的积分，则得
$\int_a^b \mathrm{d}x \int_a^x f(y)\,\mathrm{d}y = \int_a^b \mathrm{d}y \int_y^b f(y)\,\mathrm{d}x = \int_a^b f(y)(b-y)\,\mathrm{d}y = \int_a^b f(x)(b-x)\,\mathrm{d}x.$

6. 设平面上区域D在x轴和y轴上的投影长度为l_x, l_y, D的面积为$|D|$，(α,β)为D内任一点，证明：

(1) $\left|\iint\limits_{D} (x-\alpha)(y-\beta)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant l_x l_y |D|$；

(2) $\left|\iint\limits_{D} (x-\alpha)(y-\beta)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant \dfrac{l_x^2 l_y^2}{4}$.

证明：

(1) 由于$(x-\alpha)(y-\beta)$在D上连续，故由积分中值定理，存在$(\xi,\eta)\in D$，使得
$$\left|\iint\limits_{D} (x-\alpha)(y-\beta)\,\mathrm{d}x\,\mathrm{d}y\right| = \left|(\xi-\alpha)(\eta-\beta)\iint\limits_{D}\ \mathrm{d}x\,\mathrm{d}y\right| \leqslant l_x l_y |D|$$

(2) 设$l_x=b-a, l_y=d-c$，则

$$\left|\iint\limits_D (x-\alpha)(y-\beta)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant \iint\limits_D |x-\alpha||y-\beta|\,\mathrm{d}x\,\mathrm{d}y \leqslant \iint\limits_{[a,b;\,c,d]} |x-\alpha||y-\beta|\,\mathrm{d}x\,\mathrm{d}y =$$

$$\int_a^b |x-\alpha|\,\mathrm{d}x \int_c^d |y-\beta|\,\mathrm{d}y = \left(\int_a^\alpha (\alpha-x)\,\mathrm{d}x + \int_\alpha^b (x-\alpha)\,\mathrm{d}x\right)\left(\int_c^\beta (\beta-y)\,\mathrm{d}y + \int_\beta^d (y-\beta)\,\mathrm{d}y\right) =$$

$$\frac{(b-\alpha)^2+(\alpha-a)^2}{2}\cdot\frac{(d-\beta)^2+(\beta-c)^2}{2} \leqslant \frac{(b-a)^2}{2}\cdot\frac{(d-c)^2}{2} = \frac{l_x^2 l_y^2}{4}$$

7. 用极坐标计算$\iint\limits_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$时，积分限如何配置(写出下列区域上的两种逐次积分)？

(1) Ω :半圆$x^2+y^2\leqslant a^2, y\geqslant 0$;

(2) Ω :半环$a^2\leqslant x^2+y^2\leqslant b^2, x\geqslant 0$;

(3) Ω :圆$x^2+y^2\leqslant ay\ (a>0)$;

(4) Ω :正方形：$0\leqslant x\leqslant a, 0\leqslant y\leqslant a$.

解:

(1) $$\iint\limits_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\pi \mathrm{d}\theta \int_0^{|a|} f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r = \int_0^{|a|} r\,\mathrm{d}r \int_0^\pi f(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta.$$

(2) $$\iint\limits_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{|a|}^{|b|} f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r = \int_{|a|}^{|b|} r\,\mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta.$$

(3) $$\iint\limits_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\pi \mathrm{d}\theta \int_0^{a\sin\theta} f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r = \int_0^a r\,\mathrm{d}r \int_{\arcsin\frac{r}{a}}^{\pi-\arcsin\frac{r}{a}} f(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta.$$

(4) $$\iint\limits_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_0^{\frac{a}{\cos\theta}} f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\frac{a}{\sin\theta}} f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r$$

$$= \int_0^a r\,\mathrm{d}r \int_0^{\frac{\pi}{2}} f(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta + \int_a^{\sqrt{2}\,a} r\,\mathrm{d}r \int_{\arccos\frac{a}{r}}^{\arcsin\frac{a}{r}} f(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta.$$

8. 在下列积分中引进新变量u, v，变换下列积分.

(1) $\displaystyle\int_a^b \mathrm{d}x \int_{\alpha x}^{\beta x} f(x,y)\,\mathrm{d}y\ (0<a<b, 0<\alpha<\beta)$，若$\begin{cases} u=x, \\ v=\dfrac{y}{x}\end{cases}$;

(2) $\displaystyle\int_0^2 \mathrm{d}x \int_{1-x}^{2-x} f(x,y)\,\mathrm{d}y$，若$u=x+y, v=x-y$;

(3) $\displaystyle\iint\limits_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$，其中$\Omega$是由曲线$\sqrt{x}+\sqrt{y}=\sqrt{a}$与坐标轴所界的区域.若$\begin{cases} x=u\cos^4 v \\ y=u\sin^4 v\end{cases}$

解:

(1) 因$\begin{cases} x=u \\ y=uv\end{cases}$，则$|J|=\left|\dfrac{D(x,y)}{D(u,v)}\right|=u>0$

于是$$\int_a^b \mathrm{d}x \int_{\alpha x}^{\beta x} f(x,y)\,\mathrm{d}y = \int_a^b u\,\mathrm{d}u \int_\alpha^\beta f(u,uv)\,\mathrm{d}v$$

(2) 因$\begin{cases} x=\dfrac{u+v}{2} \\ y=\dfrac{u-v}{2}\end{cases}$，则$|J|=\left|\dfrac{D(x,y)}{D(u,v)}\right|=\dfrac{1}{2}$

于是$$\int_0^2 \mathrm{d}x \int_{1-x}^{2-x} f(x,y)\,\mathrm{d}y = \frac{1}{2}\int_1^2 \mathrm{d}u \int_{-u}^{4-u} f\left(\frac{u+v}{2}, \frac{u-v}{2}\right)\mathrm{d}v$$

(3) 因$\begin{cases} x=u\cos^4 v \\ y=u\sin^4 v\end{cases}$，则$|J|=\left|\dfrac{D(x,y)}{D(u,v)}\right|=\dfrac{u\sin^3 2v}{2}$

于是$$\iint\limits_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int_0^a u\,\mathrm{d}u \int_0^{\frac{\pi}{2}} \sin^3 2v f(u\cos^4 v, u\sin^4 v)\,\mathrm{d}v = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^3 2v\,\mathrm{d}v \int_0^a u f(u\cos^4 v, u\sin^4 v)\,\mathrm{d}u.$$

9. 应用极坐标计算下列二重积分：

(1) $\displaystyle\iint\limits_{x^2+y^2\leqslant R^2} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y$；

(2) $\displaystyle\iint\limits_{\pi^2\leqslant x^2+y^2\leqslant 4\pi^2} \sin\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$；

(3) $\displaystyle\iint\limits_{\Omega}(x+y)\,\mathrm{d}x\,\mathrm{d}y$, ($\Omega$是圆$x^2+y^2\leqslant x+y$的内部).

解：

(1) $\displaystyle\iint\limits_{x^2+y^2\leqslant R^2} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y=\int_0^{2\pi}\mathrm{d}\theta\int_0^R re^{-r^2}\,\mathrm{d}r=\pi(1-e^{-R^2})$.

(2) $\displaystyle\iint\limits_{\pi^2\leqslant x^2+y^2\leqslant 4\pi^2} \sin\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y=\int_0^{2\pi}\mathrm{d}\theta\int_{\pi}^{2\pi} r\sin r\,\mathrm{d}r=-6\pi^2$

(3) 作变换$x=\dfrac{1}{2}+r\cos\theta, y=\dfrac{1}{2}+r\sin\theta$，则$|J|=r$

于是$\displaystyle\iint\limits_{\Omega}(x+y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{1}{\sqrt{2}}}[r+r^2(\sin\theta+\cos\theta)]\,\mathrm{d}r=\frac{\pi}{2}$.

10. 求由锥面$z=\dfrac{h}{R}\sqrt{x^2+y^2}$、平面$z=0$及圆柱面$x^2+y^2=R^2$所围的立体体积.

解：锥面$z=\dfrac{h}{R}\sqrt{x^2+y^2}$、平面$z=0$及圆柱面$x^2+y^2=R^2$所围的立体在$XOY$平面上的射影域是圆域$\Omega=\{(x,y)\big|\,x^2+y^2\leqslant R^2\}$，在第一象限部分记为$\Omega_1$

则利用对称性，得所求立体体积为

$\displaystyle V=\iint\limits_{\Omega}z\,\mathrm{d}x\,\mathrm{d}y=4\iint\limits_{\Omega_1}z\,\mathrm{d}x\,\mathrm{d}y=\frac{4h}{R}\iint\limits_{\Omega_1}\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y=\frac{4h}{R}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R r^2\,\mathrm{d}r=\frac{2}{3}\pi R^2h.$

11. 求球面$x^2+y^2+z^2=a^2$与圆柱面$x^2+y^2=ax\ (a>0)$公共部分的体积.

解：由对称性，得$\displaystyle V=2\iint\limits_{\Omega}\sqrt{a^2-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{a\cos\theta}r\sqrt{a^2-r^2}\,\mathrm{d}r=\frac{2}{3}a^3\left(\pi-\frac{4}{3}\right)$.

12. 求由抛物线$y^2=mx, y^2=nx\ (0<m<n)$和直线$y=\alpha x, y=\beta x\ (0<\alpha<\beta)$所围成区域的面积.

解：作变换：$u=\dfrac{y^2}{x}, v=\dfrac{y}{x}$，则$|J|=\left|\dfrac{D(x,y)}{D(u,v)}\right|=\dfrac{1}{\left|\dfrac{D(u,v)}{D(x,y)}\right|}=\dfrac{1}{\frac{y^2}{x^3}}=\dfrac{u}{v^4}$

于是所求面积为$\displaystyle D=\iint\limits_{\Omega}\mathrm{d}x\,\mathrm{d}y=\int_{\alpha}^{\beta}\frac{\mathrm{d}v}{v^4}\int_m^n u\,\mathrm{d}u=\frac{1}{6}(n^2-m^2)\left(\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)$.

13. 求曲线$\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)^2=\dfrac{xy}{c^2}$所围的面积.

解：此曲线只在1、3象限且关于原点对称，故只需计算图形在第一象限中的面积，再2倍即可

令$x=ar\cos\theta, y=br\sin\theta$，则$|J|=|ab|r, r=\dfrac{\sqrt{|ab|}}{|c|}\sqrt{\sin\theta\cos\theta}$

于是$\displaystyle D=\iint\limits_{D}\mathrm{d}x\,\mathrm{d}y=2\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{\frac{\sqrt{|ab|}}{|c|}\sqrt{\sin\theta\cos\theta}}|ab|r\,\mathrm{d}r=\frac{a^2b^2}{2c^2}$.

14. 求一物体的体积，此物体的界面为：平面$z=0$，抛物面$2z=\dfrac{x^2}{a}+\dfrac{y^2}{b}$，以及以球面$x^2+y^2+(z-c)^2=c^2$与这个抛物面的交线为准线的正柱面($a,b,c>0$).

解：将$z=\dfrac{x^2}{2a}+\dfrac{y}{2b^2}$代入球方程，得$x^2+y^2+\left(\dfrac{x^2}{2a}+\dfrac{y^2}{2b}-c\right)^2=c^2$

令$x=\sqrt{a}\ r\cos\theta, y=\sqrt{b}\ r\sin\theta$，则$r=2\sqrt{c-(a\cos^2\theta+b\sin^2\theta)}, |J|=\sqrt{ab}\ r$

于是$V=\iint\limits_D \frac{1}{2}\left(\frac{x^2}{a}+\frac{y^2}{b}\right)\mathrm{d}x\,\mathrm{d}y=4\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\sqrt{c-(a\cos^2\theta+b\sin^2\theta)}}\frac{\sqrt{ab}}{2}r^3\,\mathrm{d}r=4\sqrt{ab}\pi\left(\frac{3}{8}a^2+\frac{3}{8}b^2+\frac{1}{4}ab-ac-bc+c^2\right).$

15. 求边长为a的正方形薄板的质量，设薄板上每一点的密度与该点距正方形某一顶点的距离成正比，且在正方形的中点处密度为ρ_0.

解：设某一顶点为原点$(0,0)$，则$\rho=k\sqrt{x^2+y^2}$且当$x=y=\frac{a}{2}$时，$\rho=\rho_0$，于是$k=\frac{\sqrt{2}\ \rho_0}{a}$

则密度函数为$\rho(x,y)=\frac{\sqrt{2}}{a}\rho_0\sqrt{x^2+y^2}$

于是利用第7题(4)，得

$$m=\iint\limits_{[0\ ,a;\ 0,a]}\frac{\sqrt{2}\ \rho_0}{a}\sqrt{x^2+y^2}\,\mathrm{d}\,\mathrm{d}y=\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{\frac{a}{\cos\theta}}\frac{\sqrt{2}\ \rho_0}{a}r^2\,\mathrm{d}r+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{\frac{a}{\sin\theta}}\frac{\sqrt{2}\ \rho_0}{a}r^2\,\mathrm{d}r$$

$$=\frac{\rho_0a^2}{3}[2+\sqrt{2}\ \ln(1+\sqrt{2})].$$

§2. 三重积分的计算

1. 计算下列三重积分：

(1) $\iiint\limits_V xy^2z^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, V$：由曲面$z=xy, y=x, z=0, x=1$所围成；

(2) $\iiint\limits_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, V$：由曲面$x^2+y^2+z^2=1, x\geqslant 0, y\geqslant 0, z\geqslant 0$围成.

解；

(1) $\iiint\limits_V xy^2z^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_0^1 x\,\mathrm{d}x\int_0^x y^2\,\mathrm{d}y\int_0^{xy} z^3\,\mathrm{d}z=\dfrac{1}{364}$.

(2) $\iiint\limits_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_0^1 x\,\mathrm{d}x\int_0^{\sqrt{1-x^2}} y\,\mathrm{d}y\int_0^{\sqrt{1-x^2-y^2}} z\,\mathrm{d}z=\dfrac{1}{48}$.

2. 指示下列三重积分的区域V的形状并改变积分次序：

(1) $\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int_0^{x+y}f(x,y,z)\,\mathrm{d}z$；

(2) $\int_0^1\mathrm{d}x\int_0^{x}\mathrm{d}y\int_0^{xy}f(x,y,z)\,\mathrm{d}z$；

(3) $\int_1^2\mathrm{d}x\int_0^{1}\mathrm{d}y\int_{1-x-y}^{0}f(x,y,z)\,\mathrm{d}z$；

(4) $\int_{-1}^1\mathrm{d}x\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\mathrm{d}y\int_{\sqrt{x^2+y^2}}^{1}f(x,y,z)\,\mathrm{d}z$；

(5) $\int_0^1\mathrm{d}x\int_0^{1}\mathrm{d}y\int_0^{x^2+y^2}f(x,y,z)\,\mathrm{d}z$.

解；

(1)
$$\begin{aligned}
&\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int_0^{x+y}f(x,y,z)\,\mathrm{d}z=\int_0^1\mathrm{d}y\int_0^{1-y}\mathrm{d}x\int_0^{x+y}f(x,y,z)\,\mathrm{d}z\\
&=\int_0^1\mathrm{d}x\int_0^{x}\mathrm{d}z\int_0^{1-x}f(x,y,z)\,\mathrm{d}y+\int_0^1\mathrm{d}x\int_x^{1}\mathrm{d}z\int_{z-x}^{1-x}f(x,y,z)\,\mathrm{d}y\\
&=\int_0^1\mathrm{d}z\int_0^{z}\mathrm{d}x\int_{z-x}^{1-x}f(x,y,z)\,\mathrm{d}y+\int_0^1\mathrm{d}z\int_z^{1}\mathrm{d}x\int_0^{1-x}f(x,y,z)\,\mathrm{d}y\\
&=\int_0^1\mathrm{d}z\int_0^{z}\mathrm{d}y\int_{z-y}^{1-y}f(x,y,z)\,\mathrm{d}x+\int_0^1\mathrm{d}z\int_z^{1}\mathrm{d}y\int_0^{1-y}f(x,y,z)\,\mathrm{d}x\\
&=\int_0^1\mathrm{d}y\int_0^{y}\mathrm{d}z\int_0^{1-y}f(x,y,z)\,\mathrm{d}x+\int_0^1\mathrm{d}y\int_y^{1}\mathrm{d}z\int_{z-y}^{1-y}f(x,y,z)\,\mathrm{d}x
\end{aligned}$$

(2)
$$\begin{aligned}
&\int_0^1\mathrm{d}x\int_0^{x}\mathrm{d}y\int_0^{xy}f(x,y,z)\,\mathrm{d}z=\int_0^1\mathrm{d}y\int_y^{1}\mathrm{d}x\int_0^{xy}f(x,y,z)\,\mathrm{d}z\\
&=\int_0^1\mathrm{d}x\int_0^{x^2}\mathrm{d}z\int_{\frac{z}{x}}^{x}f(x,y,z)\,\mathrm{d}y=\int_0^1\mathrm{d}z\int_{\sqrt{z}}^{1}\mathrm{d}x\int_{\frac{z}{x}}^{x}f(x,y,z)\,\mathrm{d}y\\
&=\int_0^1\mathrm{d}z\int_{\sqrt{z}}^{1}\mathrm{d}y\int_y^{1}f(x,y,z)\,\mathrm{d}x+\int_0^1\mathrm{d}z\int_z^{\sqrt{z}}\mathrm{d}y\int_{\frac{z}{y}}^{1}f(x,y,z)\,\mathrm{d}x\\
&=\int_0^1\mathrm{d}y\int_0^{y^2}\mathrm{d}z\int_y^{1}f(x,y,z)\,\mathrm{d}x+\int_0^1\mathrm{d}y\int_{y^2}^{y}\mathrm{d}z\int_{\frac{z}{y}}^{1}f(x,y,z)\,\mathrm{d}x
\end{aligned}$$

(3)
$$\begin{aligned}
&\int_1^2\mathrm{d}x\int_0^{1}\mathrm{d}y\int_{1-x-y}^{0}f(x,y,z)\,\mathrm{d}z=\int_0^1\mathrm{d}y\int_1^{2}\mathrm{d}x\int_{1-x-y}^{0}f(x,y,z)\,\mathrm{d}z\\
&=\int_0^1\mathrm{d}y\int_{-y}^{0}\mathrm{d}z\int_1^{2}f(x,y,z)\,\mathrm{d}x+\int_0^1\mathrm{d}y\int_{-1-y}^{-y}\mathrm{d}z\int_{1-y-z}^{2}f(x,y,z)\,\mathrm{d}x\\
&=\int_{-2}^{-1}\mathrm{d}z\int_{-1-z}^{1}\mathrm{d}y\int_{1-y-z}^{2}f(x,y,z)\,\mathrm{d}x+\int_{-1}^{0}\mathrm{d}z\int_0^{-z}\mathrm{d}y\int_{1-y-z}^{2}f(x,y,z)\,\mathrm{d}x+\int_{-1}^{0}\mathrm{d}z\int_{-z}^{1}\mathrm{d}y\int_1^{2}f(x,y,z)\,\mathrm{d}x
\end{aligned}$$

$$=\int_{-2}^{-1}\mathrm{d}z\int_{-z}^{2}\mathrm{d}x\int_{1-x-z}^{1}f(x,y,z)\,\mathrm{d}y+\int_{-1}^{0}\mathrm{d}z\int_{1}^{1-z}\mathrm{d}x\int_{1-x-z}^{1}f(x,y,z)\,\mathrm{d}y+\int_{-1}^{0}\mathrm{d}z\int_{1-z}^{2}\mathrm{d}x\int_{0}^{1}f(x,y,z)\,\mathrm{d}y$$

$$=\int_{1}^{2}\mathrm{d}x\int_{1-x}^{2}\mathrm{d}z\int_{0}^{1}f(x,y,z)\,\mathrm{d}y+\int_{1}^{2}\mathrm{d}x\int_{-x}^{1-x}\mathrm{d}z\int_{1-x-z}^{1}f(x,y,z)\,\mathrm{d}y$$

(4) $$\int_{-1}^{1}\mathrm{d}x\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\mathrm{d}y\int_{\sqrt{x^2+y^2}}^{1}f(x,y,z)\,\mathrm{d}z=\int_{-1}^{1}\mathrm{d}y\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\mathrm{d}x\int_{\sqrt{x^2+y^2}}^{1}f(x,y,z)\,\mathrm{d}z$$

$$=\int_{-1}^{1}\mathrm{d}x\int_{|x|}^{1}\mathrm{d}z\int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}}f(x,y,z)\,\mathrm{d}y=\int_{0}^{1}\mathrm{d}z\int_{-z}^{z}\mathrm{d}x\int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}}f(x,y,z)\,\mathrm{d}y$$

$$=\int_{0}^{1}\mathrm{d}z\int_{-z}^{z}\mathrm{d}y\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}}f(x,y,z)\,\mathrm{d}x=\int_{-1}^{1}\mathrm{d}y\int_{|y|}^{1}\mathrm{d}z\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}}f(x,y,z)\,\mathrm{d}x$$

(5) $$\int_{0}^{1}\mathrm{d}x\int_{0}^{1}\mathrm{d}y\int_{0}^{x^2+y^2}f(x,y,z)\,\mathrm{d}z=\int_{0}^{1}\mathrm{d}y\int_{0}^{1}\mathrm{d}x\int_{0}^{x^2+y^2}f(x,y,z)\,\mathrm{d}z$$

$$=\int_{0}^{1}\mathrm{d}x\int_{0}^{x^2}\mathrm{d}z\int_{0}^{1}f(x,y,z)\,\mathrm{d}y+\int_{0}^{1}\mathrm{d}x\int_{x^2}^{x^2+1}\mathrm{d}z\int_{\sqrt{z-x^2}}^{1}f(x,y,z)\,\mathrm{d}y$$

$$=\int_{0}^{1}\mathrm{d}z\int_{0}^{\sqrt{z}}\mathrm{d}x\int_{\sqrt{z-x^2}}^{1}f(x,y,z)\,\mathrm{d}y+\int_{0}^{1}\mathrm{d}z\int_{\sqrt{z}}^{1}\mathrm{d}x\int_{0}^{1}f(x,y,z)\,\mathrm{d}y+\int_{1}^{2}\mathrm{d}z\int_{\sqrt{z-1}}^{1}\mathrm{d}x\int_{\sqrt{z-x^2}}^{1}f(x,y,z)\,\mathrm{d}y$$

$$=\int_{0}^{1}\mathrm{d}z\int_{0}^{\sqrt{z}}\mathrm{d}y\int_{\sqrt{z-y^2}}^{1}f(x,y,z)\,\mathrm{d}x+\int_{0}^{1}\mathrm{d}z\int_{\sqrt{z}}^{1}\mathrm{d}y\int_{0}^{1}f(x,y,z)\,\mathrm{d}x+\int_{1}^{2}\mathrm{d}z\int_{\sqrt{z-1}}^{1}\mathrm{d}y\int_{\sqrt{z-y^2}}^{1}f(x,y,z)\,\mathrm{d}x$$

$$=\int_{0}^{1}\mathrm{d}y\int_{0}^{y^2}\mathrm{d}z\int_{0}^{1}f(x,y,z)\,\mathrm{d}x+\int_{0}^{1}\mathrm{d}y\int_{y^2}^{y^2+1}\mathrm{d}z\int_{\sqrt{z-y^2}}^{1}f(x,y,z)\,\mathrm{d}x$$

3. 计算下列三重积分：

(1) $\iiint\limits_V z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$，其中积分区域$V$是由球面$x^2+y^2+z^2=4$与抛物面$z=\dfrac{1}{3}(x^2+y^2)$所围成的立体；

(2) $\iiint\limits_V (x^2+y^2+z^2)\,\mathrm{d}V$，其中$V$是$x^2+y^2+z^2\leqslant 1$；

(3) $\iiint\limits_V z^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, V$由两个球$x^2+y^2+z^2\leqslant R^2, x^2+y^2+z^2\leqslant 2Rz$的公共部分所组成；

(4) $\iiint\limits_V \sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, V$为椭球$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\leqslant 1.$

解；

(1) 利用柱面坐标，得$\iiint\limits_V z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{3}}\mathrm{d}r\int_{\frac{r^2}{3}}^{\sqrt{4-r^2}}rz\,\mathrm{d}z=\dfrac{13}{4}\pi$

(2) 利用球面坐标，得$\iiint\limits_V (x^2+y^2+z^2)\,\mathrm{d}V=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\int_0^1\rho^4\,\mathrm{d}\rho=\dfrac{4}{5}\pi$

(3) 利用球面坐标，得

$$\iiint\limits_V z^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{3}}\cos^2\varphi\sin\varphi\,\mathrm{d}\varphi\int_0^R\rho^4\,\mathrm{d}\rho+\int_0^{2\pi}\mathrm{d}\theta\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^2\varphi\sin\varphi\,\mathrm{d}\varphi\int_0^{2R\cos\varphi}\rho^4\,\mathrm{d}\rho$$

$$=\frac{59}{480}\pi R^5$$

(4) 由广义球面坐标，得

$$\iiint\limits_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=abc\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\int_0^1\rho^2\sqrt{1-\rho^2}\,\mathrm{d}\rho=\frac{\pi^2}{4}abc.$$

4. 利用球面坐标或柱面坐标计算下列曲面所界体积：

(1) $x^2+y^2+z^2=4R^2$的内部被$x^2+y^2=2Rx$所划出的部分；

(2) $(x^2+y^2+z^2)^3=3xyz.$

解;

(1) 利用柱面坐标$x=r\cos\theta, y=r\sin\theta, z=z$且$|J|=r$，在此变换下，曲面方程变为：
$r^2+z^2=4R^2, r=2r\cos\theta$
则$V=\iiint\limits_V \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\,\mathrm{d}\theta\int_0^{2R\cos\theta} r\,\mathrm{d}r\int_0^{\sqrt{4R^2-r^2}}\,\mathrm{d}z=\dfrac{16}{3}\,R^3\left(\pi-\dfrac{4}{3}\right)$

(2) 由题知立体在第一、第三、第六及第八卦限内，对于这些卦限分别有
$x,y,z\geqslant 0; x,y\leqslant 0, z\geqslant 0; x,z\leqslant 0, y\geqslant 0; x\geqslant 0, y,z\leqslant 0$ 因原式左端及右端当x,y,z中任两个同时变号时等式仍成立，故立体在这四个卦限内的各部分，一对一对地对称于坐标轴之一.
由球面坐标$x=\rho\sin\varphi\cos\theta, y=\rho\sin\varphi\sin\theta, z=\rho\cos\varphi, |J|=\rho^2\sin\varphi$
曲面方程变为：$\rho^6=3\rho^3\sin^2\varphi\cos\varphi\sin\theta\cos\theta$即$\rho^3=3\sin^2\varphi\cos\varphi\sin\theta\cos\theta$，
且在第一卦限内，$\rho\geqslant 0, 0\leqslant\theta\leqslant\dfrac{\pi}{2}, 0\leqslant\varphi\leqslant\dfrac{\pi}{2}$
于是$V=4\int_0^{\frac{\pi}{2}}\,\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\sin\varphi\,\mathrm{d}\varphi\int_0^{\sqrt[3]{3\sin^2\varphi\cos\varphi\sin\theta\cos\theta}}\rho^2\,\mathrm{d}\rho=\dfrac{1}{2}$.

5. 利用适当的坐标变换计算下列曲面所围体积：

(1) $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)^2=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$

(2) $\left(\dfrac{x}{a}+\dfrac{y}{b}\right)^2+\left(\dfrac{z}{c}\right)^2=1$, $(x>0, y>0, z>0, a,b,c>0)$

(3) $z=x^2+y^2, z=2(x^2+y^2), xy=a^2, xy=2a^2, x=2y, 2x=y$，(其中$x,y>0$)

解;

(1) 由广义球面坐标:$x=a\rho\sin\varphi\cos\theta, y=b\rho\sin\varphi\sin\theta, z=c\rho\cos\varphi$，其中$\rho\geqslant 0, 0\leqslant\theta\leqslant 2\pi, 0\leqslant\varphi\leqslant\pi$，这时$|J|=abc\rho^2\sin\varphi$
曲面方程变为：$\rho=\sin\varphi$
则$V=abc\int_0^{2\pi}\,\mathrm{d}\theta\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\int_0^{\sin\varphi}\rho^2\,\mathrm{d}\rho=\dfrac{\pi^2}{4}\,abc$

(2) 作变换：$x=ar\cos^2\theta\cos\varphi, y=br\sin^2\theta\cos\varphi, z=cr\sin\varphi$，其中$0\leqslant r\leqslant 1, 0\leqslant\theta\leqslant\dfrac{\pi}{2}, 0\leqslant\varphi\leqslant\dfrac{\pi}{2}$，
这时$|J|=2abcr^2\cos\theta\sin\theta\cos\varphi$
则$V=\int_0^{\frac{\pi}{2}}\,\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\,\mathrm{d}\varphi\int_0^1(2abcr^2\cos\theta\sin\theta\cos\varphi)\,\mathrm{d}r=\dfrac{abc}{3}$

(3) 令$z=u(x^2+y^2), xy=v, x=wy$，则$x=\sqrt{wv}, y=\sqrt{\dfrac{v}{w}}, z=u\left(wv+\dfrac{v}{w}\right)$
此时$|J|=\dfrac{v}{2}+\dfrac{v}{2w^2}$，且$1\leqslant u\leqslant 2, a^2\leqslant v\leqslant 2a^2, \dfrac{1}{2}\leqslant w\leqslant 2$
于是$V=\int_1^2\,\mathrm{d}u\int_{a^2}^{2a^2}v\,\mathrm{d}v\int_{\frac{1}{2}}^2\left(\dfrac{1}{2}+\dfrac{1}{2w^2}\right)\,\mathrm{d}w=\dfrac{9}{4}\,a^4$

6. 求具有单位体积$0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1, 0\leqslant z\leqslant 1$的物体的质量，若物体在点$M(x,y,z)$的密度为$\mu=x+y+z$.

解; $m=\iiint\limits_V(x+y+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_0^1\,\mathrm{d}x\int_0^1\,\mathrm{d}y\int_0^1(x+y+z)\,\mathrm{d}z=\dfrac{3}{2}$.

§3. 积分在物理上的应用

1. 求下列曲线所界薄板的质心坐标：

(1) $ay = x^2, x + y = 2a\ (a > 0)$

(2) $r = a(1 + \cos\varphi)\ (0 \leqslant \varphi \leqslant \pi)$

解：

(1) 密度ρ为常数，则$x_G = \dfrac{\iint\limits_{\Omega} x\,\mathrm{d}\Omega}{\iint\limits_{\Omega}\ \mathrm{d}\Omega}, y_G = \dfrac{\iint\limits_{\Omega} y\,\mathrm{d}\Omega}{\iint\limits_{\Omega}\ \mathrm{d}\Omega}$

由$\displaystyle\iint\limits_{\Omega}\ \mathrm{d}\Omega = \int_{-2a}^{a}\ \mathrm{d}x \int_{\frac{x^2}{a}}^{2a-x}\ \mathrm{d}y = \frac{9}{2}\,a^2$

$$\iint\limits_{\Omega} x\,\mathrm{d}\Omega = \int_{-2a}^{a} x\,\mathrm{d}x \int_{\frac{x^2}{a}}^{2a-x}\ \mathrm{d}y = -\frac{9}{4}\,a^3$$

$$\iint\limits_{\Omega} y\,\mathrm{d}\Omega = \int_{-2a}^{a}\ \mathrm{d}x \int_{\frac{x^2}{a}}^{2a-x} y\,\mathrm{d}y = \frac{36}{5}\,a^3$$

则$x_G = -\dfrac{a}{2}, y_G = \dfrac{8}{5}\,a$

(2) 密度ρ为常数，则$x_G = \dfrac{\iint\limits_{\Omega} r\cos\varphi\,\mathrm{d}\Omega}{\iint\limits_{\Omega}\ \mathrm{d}\Omega}, y_G = \dfrac{\iint\limits_{\Omega} r\sin\varphi\,\mathrm{d}\Omega}{\iint\limits_{\Omega}\ \mathrm{d}\Omega}$

由$\displaystyle\iint\limits_{\Omega}\ \mathrm{d}\Omega = \int_{0}^{\pi}\ \mathrm{d}\varphi \int_{0}^{a(1+\cos\varphi)} r\,\mathrm{d}r = \frac{3}{4}\,a^2\pi$

$$\iint\limits_{\Omega} r\cos\varphi\,\mathrm{d}\Omega = \int_{0}^{\pi}\ \mathrm{d}\varphi \int_{0}^{a(1+\cos\varphi)} r^2\cos\varphi\,\mathrm{d}r = \frac{5}{8}\,a^3\pi$$

$$\iint\limits_{\Omega} r\sin\varphi\,\mathrm{d}\Omega = \int_{0}^{\pi}\ \mathrm{d}\varphi \int_{0}^{a(1+\cos\varphi)} r^2\sin\varphi\,\mathrm{d}r = \frac{4}{3}\,a^3$$

则$x_G = \dfrac{5}{6}\,a, y_G = \dfrac{16a}{9\pi}$.

2. 求由下列曲面所界的物体的质心：

(1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1, x \geqslant 0, y \geqslant 0, z \geqslant 0$

(2) $z = x^2 + y^2, x + y = a, x = 0, y = 0, z = 0$

解：

(1) 密度ρ为常数，则$x_G = \dfrac{\iiint\limits_{V} x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\iiint\limits_{V}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}, y_G = \dfrac{\iiint\limits_{V} y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\iiint\limits_{V}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}, z_G = \dfrac{D\iiint\limits_{V} z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\iiint\limits_{V}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}$

令$x = a\rho\sin\varphi\cos\theta, y = b\rho\sin\varphi\sin\theta, z = c\rho\cos\varphi$，其中$0 \leqslant \rho \leqslant 1, 0 \leqslant \theta \leqslant \dfrac{\pi}{2}, 0 \leqslant \varphi \leqslant \dfrac{\pi}{2}$，此时$|J| = abc\rho^2\sin\varphi$

则$\displaystyle\iiint\limits_{V}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = abc\int_{0}^{\frac{\pi}{2}}\ \mathrm{d}\theta \int_{0}^{\frac{\pi}{2}} \sin\varphi\,\mathrm{d}\varphi \int_{0}^{1} \rho^2\,\mathrm{d}\rho = \frac{\pi}{6}\,abc$

$$\iiint\limits_{V} x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = a^2bc\int_{0}^{\frac{\pi}{2}} \cos\theta\,\mathrm{d}\theta \int_{0}^{\frac{\pi}{2}} \sin^2\varphi\,\mathrm{d}\varphi \int_{0}^{1} \rho^3\,\mathrm{d}\rho = \frac{\pi}{16}\,a^2bc$$

于是$x_G = \dfrac{3}{8}\,a$，由对称性，得$y_G = \dfrac{3}{8}\,b, z_G = \dfrac{3}{8}\,c$

(2) 密度ρ为常数，则$x_G = \dfrac{\iiint\limits_{V} x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\iiint\limits_{V}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}, y_G = \dfrac{\iiint\limits_{V} y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\iiint\limits_{V}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}, z_G = \dfrac{\iiint\limits_{V} z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\iiint\limits_{V}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}$

由$\displaystyle\iiint\limits_{V}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{0}^{a}\ \mathrm{d}x \int_{0}^{a-x}\ \mathrm{d}y \int_{0}^{x^2+y^2}\ \mathrm{d}z = \frac{a^4}{6}$

$$\iiint\limits_V x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_0^a x\,\mathrm{d}x\int_0^{a-x}\,\mathrm{d}y\int_0^{x^2+y^2}\,\mathrm{d}z=\frac{a^5}{15}$$

$$\iiint\limits_V y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\frac{a^5}{15}\,,\iiint\limits_V z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\frac{7}{180}\,a^6$$

则$x_G=\dfrac{2}{5}\,a,y_G=\dfrac{2}{5}\,a,z_G=\dfrac{7}{30}\,a^2.$

3. 求均匀分布于两个圆$r=2\sin\theta$及$r=4\sin\theta$之间的区域上的质量的质心.

解：由对称性，得$\overline{x}=0$

又$\overline{y}=\dfrac{1}{3\pi}\displaystyle\int_0^\pi\,\mathrm{d}\theta\int_{2\sin\theta}^{4\sin\theta}r^2\sin\theta\,\mathrm{d}r=\frac{7}{3}$，则所求形心为$\left(0,\dfrac{7}{3}\right)$.

4. 在某一生产过程中，要在半圆形的直边上添上一个边与直径等长的矩形，使整个平面图形的质心落在圆心上，试求矩形的另一边长.

解：设密度ρ为常数，矩形的另一边长为l，圆心在坐标原点$(0,0)$，取圆位于x轴上方，取矩形位于x轴下方

于是$\overline{x}=\dfrac{\rho\displaystyle\int_{-R}^{R}x\,\mathrm{d}x\int_{-l}^{\sqrt{R^2-x^2}}\,\mathrm{d}y}{\rho\left(\frac{1}{2}\,\pi R^2+2Rl\right)}=0$

$$\overline{y}=\frac{\rho\displaystyle\int_{-R}^{R}\,\mathrm{d}x\int_{-l}^{\sqrt{R^2-x^2}}y\,\mathrm{d}y}{\rho\left(\frac{1}{2}\,\pi R^2+2Rl\right)}==\frac{2}{\pi R+4l}\,\left(\frac{2}{3}\,R^2-l^2\right)$$

令$\overline{y}=0$，则得$l=\dfrac{\sqrt{6}}{3}\,R.$

5. 求均匀分布在由$y=x^2$与$y=1$所围成的平面图形上的质量关于直线$y=-1$的转动惯量.

解：$I_{y=1}=\displaystyle\iint\limits_\Omega(y+1)^2\,\mathrm{d}\Omega=\int_{-1}^{1}\,\mathrm{d}x\int_{x^2}^{1}(y+1)^2\,\mathrm{d}y=\frac{368}{105}\,.$

6. 求由下列曲面所界均匀体对于所示轴的转动惯量：

(1) $z=x^2+y^2,x+y=\pm1,x-y=\pm1,z=0$关于$z$轴；

(2) 长方体关于它的一棱.

解：

(1) 曲面所界均匀物体对于OZ轴的转动惯量记为I_{OZ}

则$I_{OZ}=\displaystyle\iiint\limits_V(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

$$=\int_0^1\,\mathrm{d}x\int_{x-1}^{1-x}\,\mathrm{d}y\int_0^{x^2+y^2}(x^2+y^2)\,\mathrm{d}z+\int_{-1}^{0}\,\mathrm{d}x\int_{-(1+x)}^{x+1}\,\mathrm{d}y+\int_0^{x^2+y^2}(x^2+y^2)\,\mathrm{d}z=\frac{14}{45}$$

(2) 设长方体$0\leqslant z\leqslant c,0\leqslant y\leqslant b,0\leqslant x\leqslant a$

关于z轴的转动惯量为$I_{OZ}=\displaystyle\int_0^a\,\mathrm{d}x\int_0^b\,\mathrm{d}y\int_0^c(x^2+y^2)\,\mathrm{d}z=\frac{abc}{3}\,(a^2+b^2).$

7. 求均匀薄片$x^2+y^2\leqslant R^2,z=0$对于z轴上一点$(0,0,c)$ $(c>0)$处单位质量的引力.

解：引力在OX,OY轴上的射影为0，即$F_x=F_y=0$，设$\rho=\rho_0$

则$F_z=k\displaystyle\iint\limits_\Omega\rho_0\frac{c}{d^3}\,\mathrm{d}\Omega=k\rho_0\int_0^{2\pi}\,\mathrm{d}\theta\int_0^R\frac{cr}{(r^2+c^2)^{\frac{3}{2}}}\,\mathrm{d}r=2k\rho_0\pi c\left[\frac{1}{c}\,-\,\frac{1}{\sqrt{R^2+c^2}}\right].$

8. 求均匀柱体$x^2+y^2\leqslant a^2,0\leqslant z\leqslant h$对于$p\,(0,0,c)$ $(c>h)$点处的单位质量的引力.

解：设$\rho=\rho_0$，由对称性，得引力在OX,OY轴上的射影为0，即$F_x=F_y=0$

利用柱面坐标，得引力在OZ轴上的射影为：

$$F_z=k\rho_0\iint\limits_\Omega\,\mathrm{d}x\,\mathrm{d}y\int_0^h\frac{z-c}{(x^2+y^2+(z-c)^2)^{\frac{3}{2}}}\,\mathrm{d}z=k\rho_0\int_0^{2\pi}\,\mathrm{d}\theta\int_0^a r\,\mathrm{d}r\int_0^h\frac{z-c}{[r^2+(z-c)^2]^{\frac{3}{2}}}\,\mathrm{d}z$$

$$=2\pi k\rho_0(\sqrt{a^2+c^2}-\sqrt{a^2+(c-h)^2}-h).$$

§4. 广义重积分

1. 计算下列广义重积分之值：

(1) $\displaystyle\iint\limits_{\substack{xy\geqslant 1\\ x\geqslant 1}} \frac{\mathrm{d}x\,\mathrm{d}y}{x^p y^q}$

(2) $\displaystyle\iint\limits_{x^2+y^2\leqslant 1} \frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{1-x^2-y^2}}$

(3) $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y$.并由此证明概率积分

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x = 1$$

解：

(1) 由于被积函数非负，故$I=\displaystyle\iint\limits_{\substack{xy\geqslant 1\\ x\geqslant 1}} \frac{\mathrm{d}x\,\mathrm{d}y}{x^p y^q}=\int_1^{+\infty}\frac{\mathrm{d}x}{x^p}\int_{\frac{1}{x}}^{+\infty}\frac{\mathrm{d}y}{y^q}$

当$q\leqslant 1$时，由$x\geqslant 1$，知$0<\dfrac{1}{x}\leqslant 1$，则得积分$\displaystyle\int_{\frac{1}{x}}^{+\infty}\frac{\mathrm{d}y}{y^q}$发散且有$\displaystyle\int_{\frac{1}{x}}^{+\infty}\frac{\mathrm{d}y}{y^q}=+\infty$，

于是$I=\displaystyle\iint\limits_{\substack{xy\geqslant 1\\ x\geqslant 1}} \frac{\mathrm{d}x\,\mathrm{d}y}{x^p y^q}=+\infty$

当$q>1$时，$\displaystyle\int_{\frac{1}{x}}^{+\infty}\frac{\mathrm{d}y}{y^q}=\frac{x^{q-1}}{q-1}$

此时，当$p>q$时，$I=\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{x^p}\int_{\frac{1}{x}}^{+\infty}\frac{\mathrm{d}y}{y^q}=\frac{1}{q-1}\int_1^{+\infty}x^{q-p-1}\,\mathrm{d}x=\frac{1}{(q-1)(p-q)}$

当$p\leqslant q$时，$(p+1)-q\leqslant 1$，则积分$\displaystyle\frac{1}{q-1}\int_1^{+\infty}x^{q-p-1}\,\mathrm{d}x=+\infty$，从而得$I=\displaystyle\iint\limits_{\substack{xy\geqslant 1\\ x\geqslant 1}} \frac{\mathrm{d}x\,\mathrm{d}y}{x^p y^q}=+\infty$

综上可知，当$p>q>1$时，$I=\displaystyle\iint\limits_{\substack{xy\geqslant 1\\ x\geqslant 1}} \frac{\mathrm{d}x\,\mathrm{d}y}{x^p y^q}=\frac{1}{(q-1)(p-q)}$，其余情况$I=+\infty$.

(2) $\displaystyle\iint\limits_{x^2+y^2\leqslant 1} \frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{1-x^2-y^2}}=\lim_{\varepsilon\to 1}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\varepsilon}\frac{r}{\sqrt{1-r^2}}\,\mathrm{d}r=2\pi.$

(3) 作变换$x=r\cos\theta, y=r\sin\theta\ (0\leqslant\theta\leqslant 2\pi, r>0), |J|=r$

则$\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y=\int_0^{2\pi}\mathrm{d}\theta\int_0^{+\infty}re^{-r^2}\,\mathrm{d}r=\pi.$

由$\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y=\int_{-\infty}^{+\infty}e^{-x^2}\,\mathrm{d}x\int_{-\infty}^{+\infty}e^{-y^2}\,\mathrm{d}y$且$\displaystyle\int_{-\infty}^{+\infty}e^{-x^2}\,\mathrm{d}x=\int_{-\infty}^{+\infty}e^{-y^2}\,\mathrm{d}y$为某一值

则$\displaystyle\int_{-\infty}^{+\infty}e^{-x^2}\,\mathrm{d}x=\sqrt{\pi}$即$\dfrac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-x^2}\,\mathrm{d}x=1.$

2. 讨论下面广义重积分的收敛性：

(1) $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+|x|^p)(1+|y|^q)}$

(2) $\displaystyle\iint\limits_{0\leqslant y\leqslant 1}\frac{\varphi(x,y)}{(1+x^2+y^2)^p}\,\mathrm{d}x\,\mathrm{d}y,\quad 0<m\leqslant|\varphi(x,y)|\leqslant M$

(3) $\displaystyle\int_0^a\int_0^a\frac{\varphi(x,y)}{|x-y|^p}\,\mathrm{d}x\,\mathrm{d}y,\quad 0<m\leqslant|\varphi(x,y)|\leqslant M$

(4) $\displaystyle\iint\limits_{x^2+y^2\leqslant 1} \frac{\varphi(x,y)}{(x^2+xy+y^2)^p}\,\mathrm{d}x\,\mathrm{d}y,\quad 0<m\leqslant|\varphi(x,y)|\leqslant M$

解:

(1) 因被积函数为正且关于OX,OY轴对称，则$\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+|x|^p)(1+|y|^q)}=4\int_0^{+\infty}\int_0^{+\infty}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^p)(1+y^q)}$

又$\displaystyle\lim_{x\to+\infty}x^p\frac{1}{1+x^p}=1$，则由无穷限广义积分柯西判别法的极限形式，得

当$p>1$时，积分$\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{1+x^p}$收敛；当$p\leqslant 1$时，积分$\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{1+x^p}$发散

同理可得，当$q>1$时，积分$\displaystyle\int_0^{+\infty}\frac{\mathrm{d}y}{1+y^q}$收敛；当$q\leqslant 1$时，积分$\displaystyle\int_0^{+\infty}\frac{\mathrm{d}y}{1+y^q}$发散

综上可知，当$p>1$且$q>1$时，积分$\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+|x|^p)(1+|y|^q)}$收敛，其余情况均发散.

(2) 因$\displaystyle\frac{m}{(1+x^2+y^2)^p}\leqslant\frac{\varphi(x,y)}{(1+x^2+y^2)^p}\leqslant\frac{M}{(1+x^2+y^2)^p}$

则由广义重积分的比较判别法及广义重积分性质，得$\displaystyle\iint\limits_{0\leqslant y\leqslant 1}\frac{\varphi(x,y)}{(1+x^2+y^2)^p}\,\mathrm{d}x\,\mathrm{d}y$与$\displaystyle\iint\limits_{0\leqslant y\leqslant 1}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2+y^2)^p}$

同敛散

由被积函数的对称性及非负性，得$\displaystyle\iint\limits_{0\leqslant y\leqslant 1}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2+y^2)^p}=2\int_0^1\mathrm{d}y\int_0^{+\infty}\frac{\mathrm{d}x}{(1+x^2+y^2)^p}$

由于$0\leqslant y\leqslant 1$，则

若$p\geqslant 0$，则$\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{(2+x^2)^p}\leqslant\int_0^{+\infty}\frac{\mathrm{d}x}{(1+x^2+y^2)^p}\leqslant\int_0^{+\infty}\frac{\mathrm{d}x}{(1+x^2)^p}$

若$p<0$，则$\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{(2+x^2)^p}\geqslant\int_0^{+\infty}\frac{\mathrm{d}x}{(1+x^2+y^2)^p}\geqslant\int_0^{+\infty}\frac{\mathrm{d}x}{(1+x^2)^P}$

对于$\alpha>0$，由于$\displaystyle\lim_{x\to+\infty}x^{2p}\frac{1}{(\alpha^2+x^2)^p}=1$，则积分$\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{(\alpha^2+x^2)^p}$当$p>\dfrac{1}{2}$时收敛；当$p\leqslant\dfrac{1}{2}$时发散

于是$\displaystyle\iint\limits_{0\leqslant y\leqslant 1}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2+y^2)^p}$当$p>\dfrac{1}{2}$时收敛；当$p\leqslant\dfrac{1}{2}$时发散

从而$\displaystyle\iint\limits_{0\leqslant y\leqslant 1}\frac{\varphi(x,y)}{(1+x^2+y^2)^p}\,\mathrm{d}x\,\mathrm{d}y$当$p>\dfrac{1}{2}$时收敛；当$p\leqslant\dfrac{1}{2}$时发散.

(3) 因$\displaystyle 0<\frac{m}{|x-y|^p}\leqslant\frac{\varphi(x,y)}{|x-y|^p}\leqslant\frac{M}{|x-y|^p}$

则由广义重积分的比较判别法及广义重积分性质，得$\displaystyle\int_0^a\int_0^a\frac{\varphi(x,y)}{|x-y|^p}\,\mathrm{d}x\,\mathrm{d}y$与$\displaystyle\int_0^a\int_0^a\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^p}$同敛散

由被积函数的对称性及非负性，得$\displaystyle\int_0^a\int_0^a\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^p}\,\mathrm{d}x\,\mathrm{d}y=2\int_0^a\mathrm{d}x\int_0^x\frac{\mathrm{d}y}{(x-y)^p}$

当$p<1$时，$\displaystyle\int_0^a\mathrm{d}x\int_0^x\frac{\mathrm{d}y}{(x-y)^p}=\frac{a^{2-p}}{(2-p)(1-p)}$

则$\displaystyle\int_0^a\int_0^a\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^p}=2\int_0^a\mathrm{d}x\int_0^x\frac{\mathrm{d}y}{(x-y)^p}=\frac{2a^{2-p}}{(2-p)(1-p)}$

当$p\geqslant 1$时，$\displaystyle\int_0^a\mathrm{d}x\int_0^x\frac{\mathrm{d}y}{(x-y)^p}=\lim_{\varepsilon\to+0}\int_\varepsilon^a\mathrm{d}x\int_0^{x-\varepsilon}\frac{\mathrm{d}y}{(x-y)^p}$

当$p=1$时，$\displaystyle\int_0^a\mathrm{d}x\int_0^x\frac{\mathrm{d}y}{(x-y)^p}=\lim_{\varepsilon\to+0}\int_\varepsilon^a\mathrm{d}x\int_0^{x-\varepsilon}\frac{\mathrm{d}y}{x-y}=\lim_{\varepsilon\to+0}(a\ln a-a+\varepsilon-a\ln\varepsilon)=+\infty$

则$\displaystyle\int_0^a\int_0^a\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^p}$当$p=1$时发散；

当$p=2$时，$\displaystyle\int_0^a\mathrm{d}x\int_0^x\frac{\mathrm{d}y}{(x-y)^p}=\lim_{\varepsilon\to+0}\int_\varepsilon^a\mathrm{d}x\int_0^{x-\varepsilon}\frac{\mathrm{d}y}{(x-y)^2}=\lim_{\varepsilon\to+0}\left(\frac{a+\varepsilon\ln\varepsilon}{\varepsilon}-1-\ln a\right)=-\infty$

则$\displaystyle\int_0^a\int_0^a\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^p}$当$p=2$时发散；

当$p>1$且$p\neq 2$时，$\displaystyle\int_0^a \mathrm{d}x\int_0^x \frac{\mathrm{d}y}{(x-y)^p}=\lim_{\varepsilon\to+0}\int_\varepsilon^a \mathrm{d}x\int_0^{x-\varepsilon}\frac{\mathrm{d}y}{(x-y)^p}$

$$=\lim_{\varepsilon\to+0}\left[\frac{1}{(p-1)\varepsilon^{p-1}}\left(a-\frac{p-1}{p-2}\,\varepsilon\right)+\frac{1}{(p-1)(p-2)a^{p-2}}\right]=+\infty$$

则$\displaystyle\int_0^a\int_0^a \frac{\mathrm{d}x\,\mathrm{d}y)}{|x-y|^p}$ 当$p>1$且$p\neq 2$时发散

综上可知$\displaystyle\int_0^a\int_0^a \frac{\varphi(x,y)}{|x-y|^p}\,\mathrm{d}x\,\mathrm{d}y$当$p<1$ 时收敛；当$p\geqslant 1$时发散.

(4) $(0,0)$是奇点，由于$x^2++xy+y^2>0$(当$(x,y)\neq(0,0)$)，则

$$\frac{m}{(x^2+xy+y^2)^p}\leqslant\frac{\varphi(x,y)}{(x^2+xy+y^2)^p}\leqslant\frac{M}{(x^2+xy+y^2)^p}$$

由广义重积分的比较判别法及广义重积分性质，得$\displaystyle\int_0^a\int_0^a\frac{\varphi(x,y)}{(x^2+xy+y^2)^p}\,\mathrm{d}x\,\mathrm{d}y$与$\displaystyle\int_0^a\int_0^a\frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+xy+y^2)^p}$同敛散

$$\iint\limits_{x^2+y^2\leqslant 1}\frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+xy+y^2)^p}=\lim_{\varepsilon\to+0}\int_\varepsilon^1\frac{\mathrm{d}r}{r^{2p-1}}\int_0^{2\pi}\frac{\mathrm{d}\theta}{(1+\sin\theta\cos\theta)^p}=N\lim_{\varepsilon\to+0}\int_\varepsilon^1\frac{\mathrm{d}r}{r^{2p-1}}$$

$$=\begin{cases}N\displaystyle\lim_{\varepsilon\to+0}(-\ln\varepsilon)=+\infty, & p=1\\ N\displaystyle\lim_{\varepsilon\to+0}\frac{1-\varepsilon^{2-2p}}{2-2p}=\begin{cases}\dfrac{N}{2-2p}, & p<1\\ \infty, & p>1\end{cases}\end{cases}\quad\left(\text{其中}N=\int_0^{2\pi}\frac{\mathrm{d}\theta}{(1+\sin\theta\cos\theta)^p}\text{为常义积分，为常量}\right)$$

总之，$\displaystyle\iint\limits_{x^2+y^2\leqslant 1}\frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+xy+y^2)^p}$ 当$p<1$时收敛；当$p\geqslant 1$时发散

从而$\displaystyle\iint\limits_{x^2+y^2\leqslant 1}\frac{\varphi(x,y)}{(x^2+xy+y^2)^p}\,\mathrm{d}x\,\mathrm{d}y$当$p<1$ 时收敛；当$p\geqslant 1$时发散.

3. 证明 设$\mathcal{D}$是由在第一象限的抛物线$y=x^2$，圆周$x^2+y^2=1$及x轴所围成的区域，则$\displaystyle\iint\limits_{\mathcal{D}}\frac{\mathrm{d}x\,\mathrm{d}y}{x^2+y^2}$ 存在.

证明：$(0,0)$是奇点

$$\iint\limits_{\mathcal{D}}\frac{\mathrm{d}x\,\mathrm{d}y}{x^2+y^2}=\int_0^{\theta_0}\mathrm{d}\theta\int_{\frac{\sin\theta}{\cos^2\theta}}^1\frac{\mathrm{d}r}{r}=\int_0^{\theta_0}\ln\frac{\cos^2\theta}{\sin\theta}\,\mathrm{d}\theta,0\text{是奇点}$$

$$\left(\text{其中}\theta_0\text{满足}\frac{\sin\theta_0}{\cos^2\theta_0}=1\text{即}\sin\theta_0=\frac{\sqrt{5}-1}{2}\right)$$

因$\displaystyle\lim_{\theta\to+0}\theta^{\frac{1}{2}}\ln\frac{\cos^2\theta}{\sin\theta}=0$，则由柯西判别法的极限形式，得$\displaystyle\int_0^{\theta_0}\ln\frac{\cos^2\theta}{\sin\theta}\,\mathrm{d}\theta$ 收敛

从而原积分$\displaystyle\iint\limits_{\mathcal{D}}\frac{\mathrm{d}x\,\mathrm{d}y}{x^2+y^2}$ 存在.

4. 求均匀正圆锥体关于在它的顶点处的质量为m的质点的引力.

解：引力在OX,OY轴上的射影为0，即$F_x=F_y=0$，

$$F_z=\iiint\limits_V\frac{mz}{gr^3}\,\mathrm{d}V=\frac{m}{g}\int_0^{2\pi}\mathrm{d}\theta\int_0^R\mathrm{d}\rho\int_0^{\frac{h}{R}\rho}\frac{\rho z}{(\rho^2+z^2)^{\frac{3}{2}}}\,\mathrm{d}z=\frac{2mR\pi}{gl}(l-g).$$

第二十一章　曲线积分和曲面积分的计算

§1.　第一类曲线积分的计算

1. 计算$\int_l (x+y)\,\mathrm{d}s$，$l$是以$O(0,0),A(1,0),B(0,1)$为顶点的三角形.

解：$I=\int_l (x+y)\,\mathrm{d}s=\left\{\int_{\overline{OA}}+\int_{\overline{AB}}+\int_{\overline{BO}}\right\}(x+y)\,\mathrm{d}s$

在直线段$\overline{OA}$上，$y=0,\mathrm{d}s=\mathrm{d}x$，则$\int_{\overline{OA}}(x+y)\,\mathrm{d}s=\int_0^1 x\,\mathrm{d}x=\frac{1}{2}$；

在直线段$\overline{AB}$上，$y=1-x,\mathrm{d}s=\sqrt{2}\ \mathrm{d}x$，则$\int_{\overline{AB}}(x+y)\,\mathrm{d}s=\int_0^1\sqrt{2}\,\mathrm{d}x=\sqrt{2}$；

在直线段$\overline{BO}$上，$x=0,\mathrm{d}s=\mathrm{d}y$，则$\int_{\overline{BO}}(x+y)\,\mathrm{d}s=\int_0^1 y\,\mathrm{d}y=\frac{1}{2}$

于是$I=1+\sqrt{2}$.

2. 计算$\int_l (x^2+y^2)\,\mathrm{d}s$，$l$是以原点为中心，半径为$R$的左半圆周.

解：因$l:x=R\cos\theta,y=R\sin\theta,\frac{\pi}{2}\leqslant\theta\leqslant\frac{3}{2}\pi$，则$\mathrm{d}s=\sqrt{x_\theta^2+y_\theta^2}\ \mathrm{d}\theta=R\,\mathrm{d}\theta$

于是$\int_l (x^2+y^2)\,\mathrm{d}s=\pi R^3$.

3. 计算$\int_l (x^2+y^2+z^2)\,\mathrm{d}s$，$l$是圆螺旋线：$x=a\cos t,y=a\sin t,z=bt\ (0\leqslant t\leqslant 2\pi)$.

解：因$\mathrm{d}s=\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\ \mathrm{d}t=\sqrt{a^2+b^2}\ \mathrm{d}t$

则$I=\int_l (x^2+y^2+z^2)\,\mathrm{d}s=\frac{2}{3}\pi(3a^2+4\pi^2b^2)\sqrt{a^2+b^2}$.

4. 计算$\int_l x^2\,\mathrm{d}s$，l是球面$x^2+y^2+z^2=a^2$与平面$x+y+z=0$相交的圆周.

解：由对称性，得$\int_l x^2\,\mathrm{d}s=\int_l y^2\,\mathrm{d}s=\int_l x^2\,\mathrm{d}s$，则$\int_l x^2\,\mathrm{d}s=\frac{1}{3}\int_l (x^2+y^2+z^2)\,\mathrm{d}s=\frac{a^2}{3}\int_l \mathrm{d}s=\frac{2}{3}\pi a^3$.

5. 计算$\int_l \frac{z^2}{x^2+y^2}\ \mathrm{d}s$，$l$是螺线：$x=a\cos t,y=a\sin t,z=at,(0\leqslant t\leqslant 2\pi)$.

解：因$\mathrm{d}s=\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\ \mathrm{d}t=\sqrt{2}\ \mathrm{d}t$，则$I=\int_l \frac{z^2}{x^2+y^2}\ \mathrm{d}s=\frac{8\sqrt{2}}{3}\pi^3 a$.

6. 设一金属丝l的方程为：

$$x=e^t\cos t,y=e^t\sin t,z=e^t,(0\leqslant t\leqslant t_0)$$

它在每一点的密度与该点的矢径平方成反比，且在点$(1,0,1)$处为1，求它的质量.

解：因$\rho=\frac{k}{x^2+y^2+z^2}$且在点$(1,0,1)$处$\rho=1$，则$k=2$，于是$\rho=\frac{2}{x^2+y^2+z^2}=e^{-2t}$

又$\mathrm{d}s=\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\ \mathrm{d}t=\sqrt{3}e^t\,\mathrm{d}t$，则$m=\int_l \rho\,\mathrm{d}s=\sqrt{3}(1-e^{-t_0})$.

7. 求椭圆$x=a\cos t,y=b\sin t$周界的质量$(0\leqslant t\leqslant 2\pi)$，若曲线在点$M(x,y)$的线性密度为$\rho=|y|$.

解：$M=\int_l |y|\,\mathrm{d}s$，其中$l$为椭圆$x=a\cos t,y=b\sin t(0\leqslant t\leqslant 2\pi)$

(1) 设$a>b$，则$\mathrm{d}s=\sqrt{x'^2(t)+y'^2(t)}\ \mathrm{d}t=a\sqrt{1-\varepsilon_1^2\cos^2 t}\ \mathrm{d}t$，其中$\varepsilon_1=\frac{\sqrt{a^2-b^2}}{a}$

于是$M=\int_l |y|\,\mathrm{d}s=\int_0^\pi ab\sin t\sqrt{1-\varepsilon_1\cos^2 t}\ \mathrm{d}t+\int_\pi^{2\pi} a(-b\sin t)\sqrt{1-\varepsilon_1^2\cos^2 t}\ \mathrm{d}t=$

$2ab\sqrt{1-\varepsilon_1^2}+\frac{2ab}{\varepsilon_1}\ \arcsin\varepsilon_1=2b^2+\frac{2ab}{\varepsilon_1}\ \arcsin\varepsilon_1$

(2) 设$a<b$，则$\mathrm{d}s=\sqrt{x'^2(t)+y'^2(t)}\ \mathrm{d}t=a\sqrt{1+\varepsilon_2^2\cos^2 t}\ \mathrm{d}t$，其中$\varepsilon_2=\frac{\sqrt{b^2-a^2}}{a}$

于是$M=\int_l |y|\,\mathrm{d}s=\int_0^\pi ab\sin t\sqrt{1+\varepsilon_2^2\cos^2 t}\ \mathrm{d}t+\int_\pi^{2\pi} a(-b\sin t)\sqrt{1+\varepsilon_2^2\cos^2 t}\ \mathrm{d}t=$

$2ab\sqrt{1+\varepsilon_2^2}+\frac{2ab}{\varepsilon_2}\ \ln(\varepsilon_2+\sqrt{1+\varepsilon_2^2})=2b^2+\frac{2ab}{\varepsilon_2}\ \ln(\varepsilon_2+\sqrt{1+\varepsilon_2^2})$

(3) 若$a=b$，则$\mathrm{d}s=\sqrt{x'^2(t)+y'^2(t)}\ \mathrm{d}t=a\,\mathrm{d}t$，于是$M=\int_l |y|\,\mathrm{d}s=\int_0^\pi a^2\sin t\,\mathrm{d}t+\int_\pi^{2\pi} a(-a\sin t)\,\mathrm{d}t=4a^2$

从而$M=\begin{cases} 2b^2+\dfrac{2ab}{\varepsilon_1}\ \arcsin\varepsilon_1\ , & a>b \\ 4a^2, & a=b \\ 2b^2+\dfrac{2ab}{\varepsilon_2}\ \ln(\varepsilon_2+\sqrt{1+\varepsilon_2^2})\ , & a<b \end{cases}$

§2. 第一类曲面积分的计算

1. 计算下列曲面面积：

(1) $z = axy$包含在圆柱$x^2 + y^2 = a^2$内的部分；

(2) 锥面$x^2 + y^2 = \frac{1}{3}z^2$与平面$x + y + z = 2a(a > 0)$所界部分的表面；

(3) 柱面$x^2 + y^2 = a^2$被二平面$x + z = 0, x - z = 0(x > 0, y > 0)$所截部分.

解：

(1) 由$z_x = ay, z_y = ax$，得$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + a^2x^2 + a^2y^2}$
由对称性，并利用柱面坐标，得
$S = 4\iint\limits_{\sigma_{xy}} \sqrt{1 + a^2x^2 + a^2y^2}\,\mathrm{d}x\,\mathrm{d}y = 4\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^a \sqrt{1 + a^2r^2}\,r\,\mathrm{d}r = \frac{2}{3a^2}\pi\left[(1 + a^4)^{\frac{3}{2}} - 1\right].$

(2) 曲面的交线在xoy平面上的射影为$3x^2 + 3y^2 = (2a - x - y)^2$即$x^2 + y^2 - xy + 2a(x + y) = 2a^2$
令$x = \frac{1}{\sqrt{2}}(x' - y'), y = \frac{1}{\sqrt{2}}(x' + y')$，则方程变为$\frac{(x' + 2\sqrt{2}\,a)^2}{(2\sqrt{3}\,a)^2} + \frac{y'^2}{(2a)^2} = 1$
由此可见，曲面所界的物体在xoy平面上的射影域为以$2a$为短半轴，$2\sqrt{3}\,a$为长半轴的椭圆
物体的表面积由截面和截出的锥面两部分组成
对于$z = 2a - x - y, z = \sqrt{3x^2 + 3y^2}$分别有$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{3}, \sqrt{1 + z_x^2 + z_y^2} = 2$
于是物体的表面积为$S = \iint\limits_{\Omega}\sqrt{3}\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{\Omega} 2\,\mathrm{d}x\,\mathrm{d}y = (\sqrt{3} + 2)\pi\cdot 2a\cdot 2\sqrt{3}\,\pi = 4\pi(3 + 2\sqrt{3})a^2.$

(3) 由$y_x = -\frac{x}{y}, y_z = 0$，得$\sqrt{1 + y_x^2 + y_z^2} = \sqrt{1 + \left(\frac{x}{y}\right)^2} = \frac{|a|}{\sqrt{a^2 - x^2}}$
则$S = \iint\limits_{\sigma_{xz}}\frac{|a|}{\sqrt{a^2 - x^2}}\,\mathrm{d}x\,\mathrm{d}z = \int_0^{|a|}\mathrm{d}x\int_{-x}^{x}\frac{|a|}{\sqrt{a^2 - x^2}}\,\mathrm{d}z = 2a^2.$

2. 计算第一类曲面积分：

(1) $\iint\limits_S (x + y + z)\,\mathrm{d}S, S$:左半球面$x^2 + y^2 + z^2 = a^2, y \leqslant 0$；

(2) $\iint\limits_S x\,\mathrm{d}S, S$:螺旋面$x = u\cos v, y = u\sin v, z = cv$上的一部分$0 \leqslant u \leqslant a, 0 \leqslant v \leqslant 2\pi$；

(3) $\iint\limits_S \mathrm{d}S, S$:球面$x^2 + y^2 + z^2 = 2cz(c > 0)$夹在锥面$x^2 + y^2 = z^2$内的部分；

(4) $\iint\limits_S (x^2 + y^2)\,\mathrm{d}S, S$:体积$\sqrt{x^2 + y^2} \leqslant z \leqslant 1$的边界；

(5) $\iint\limits_S \frac{\mathrm{d}S}{r^2}, S$为圆柱面$x^2 + y^2 = R^2$介于$z = 0$和$z = H$之间的部分，其中$r$为曲面上的点到原点的距离.

解：

(1) 将$x^2 + y^2 + z^2 = a^2$投影到xoz平面，此时有$y = -\sqrt{a^2 - x^2 - z^2}$
则$y_x = \frac{x}{\sqrt{a^2 - x^2 - z^2}}, y_z = \frac{z}{\sqrt{a^2 - x^2 - z^2}}$，于是$\sqrt{1 + y_x^2 + y_z^2} = \frac{a}{\sqrt{a^2 - x^2 - z^2}}$
于是$\iint\limits_S (x + y + z)\,\mathrm{d}S = \int_{-a}^{a}\mathrm{d}x\int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}}\frac{a}{\sqrt{a^2 - x^2 - z^2}}(x - \sqrt{a^2 - x^2 - z^2} + z)\,\mathrm{d}z = -\pi a^3$

(2) $E = x_u^2 + y_u^2 + z_u^2 = 1, F = x_ux_v + y_uy_v + z_uz_v = 0, G = x_v^2 + y_v^2 + z_v^2 = u^2 + c^2$
则$\iint\limits_S x\,\mathrm{d}S = \iint\limits_{\Sigma} u\cos v\sqrt{u^2 + c^2}\,\mathrm{d}u\,\mathrm{d}v = \int_0^a u\sqrt{u^2 + c^2}\,\mathrm{d}u\int_0^{2\pi}\cos v\,\mathrm{d}v = 0$

(3) 因$x^2+y^2+z^2=2cz$，则$x^2+y^2+(z-c)^2=c^2, z=c+\sqrt{c^2-x^2-y^2}$

于是$z_x=-\dfrac{x}{\sqrt{c^2-x^2-y^2}}, z_y=-\dfrac{y}{\sqrt{c^2-x^2-y^2}}$，则$\sqrt{1+z_x^2+z_y^2}=\dfrac{c}{\sqrt{c^2-x^2-y^2}}$

于是$\displaystyle\iint\limits_S \mathrm{d}S=\iint\limits_\sigma \frac{c}{\sqrt{c^2-x^2-y^2}}\,\mathrm{d}\sigma=\int_0^{2\pi}\mathrm{d}\theta\int_0^c\frac{cr}{\sqrt{c^2-r^2}}\,\mathrm{d}r=2\pi c^2.$

(4) 分为两部分：

第一部分：$z=1, \sqrt{1+z_x^2+z_y^2}=1$；第二部分：$z=\sqrt{x^2+y^2}, \sqrt{1+z_x^2+z_y^2}=\sqrt{2}$

则$\displaystyle\iint\limits_S(x^2+y^2)\,\mathrm{d}S=\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^3\,\mathrm{d}r+\int_0^{2\pi}\mathrm{d}\theta\int_0^1\sqrt{2}\,r^3\,\mathrm{d}r=\frac{\pi}{2}(1+\sqrt{2}).$

(5) $x=R\cos\theta, y=R\sin\theta, z=z\ (0\leqslant\theta 2\pi, 0\leqslant z\leqslant H)$

则$E=x_\theta^2+y_\theta^2+z_\theta^2=R^2, F=x_\theta x_z+y_\theta y_z+z_\theta z_z=0, G=x_z^2+y_z^2+z_z^2=1$

于是$\sqrt{EG-F^2}=R$，从而$\displaystyle\iint\limits_S\frac{\mathrm{d}S}{r^2}=\int_0^{2\pi}\mathrm{d}\theta\int_0^H\frac{R}{R^2+z^2}\,\mathrm{d}z=2\pi\arctan\frac{H}{R}.$

3. 求抛物面壳$z=\dfrac{1}{2}(x^2+y^2), 0\leqslant z\leqslant 1$的质量.此壳的密度为$\rho=z$.

解：因$z=\dfrac{1}{2}(x^2+y^2)$，则$z_x=x, z_y=y$，于是$\sqrt{1+z_x^2+z_y^2}=\sqrt{1+x^2+y^2}$

则质量$\displaystyle M=\iint\limits_S\rho\,\mathrm{d}S=\frac{1}{2}\iint\limits_{x^2+y^2\leqslant 2}(x^2+y^2)\sqrt{1+x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y=\frac{1}{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{2}}r^3\sqrt{1+r^2}\,\mathrm{d}r=\frac{2(1+6\sqrt{3})}{15}\pi.$

§3. 第二类曲线积分

1. 计算下列第二类曲线积分：

(1) $\int_l (x^2-2xy)\,\mathrm{d}x+(y^2-2xy)\,\mathrm{d}y, l$为$y=x^2$从$(1,1)$到$(-1,1)$；

(2) $\oint_l (x^2+y^2)\,\mathrm{d}x+(x^2-y^2)\,\mathrm{d}y, l$为以$A(1,0),B(2,0)C(2,1),D(1,1)$为顶点的正方形，正向；

(3) $\int_l (2a-y)\,\mathrm{d}x+\,\mathrm{d}y, l$为旋轮线$x=a(t-\sin t), y=a(1-\cos t)$，从$(0,0)$到$(2\pi,0)$；

(4) $\int_l y\,\mathrm{d}x-x\,\mathrm{d}y+(x^2+y^2)\,\mathrm{d}z, l$为曲线$x=e^t, y=e^{-t}, z=at$从$(1,1,0)$到$(e,e^{-1},a)$

解：

(1) $\int_l (x^2-2xy)\,\mathrm{d}x+(y^2-2xy)\,\mathrm{d}y=\int_1^{-1}[x^2-2x^3+2x(x^4-2x^3)]\,\mathrm{d}x=\frac{14}{15}$.

(2) $I=\oint_l (x^2+y^2)\,\mathrm{d}x+(x^2-y^2)\,\mathrm{d}y=\left(\int_{\overline{AB}}+\int_{\overline{BC}}+\int_{\overline{CD}}+\int_{\overline{DA}}\right)(x^2+y^2)\,\mathrm{d}x+(x^2-y^2)\,\mathrm{d}y$

沿$\overline{AB}$，$y=0$，故$\int_{\overline{AB}}(x^2-y^2)\,\mathrm{d}y=0$

同样，有$\int_{\overline{BC}}(x^2+y^2)\,\mathrm{d}x=\int_{\overline{CD}}(x^2-y^2)\,\mathrm{d}y=\int_{\overline{DA}}(x^2+y^2)\,\mathrm{d}x=0$

则$I=\int_1^2 x^2\,\mathrm{d}x+\int_0^1(4-y^2)\,\mathrm{d}y+\int_2^1(x^2+1)\,\mathrm{d}x=\int_1^0(1-y^2)\,\mathrm{d}y=2.$

(3) $\int_l (2a-y)\,\mathrm{d}x+\,\mathrm{d}y=\int_0^{2\pi}[(a+a\cos t)\cdot a(1-\cos t)+a\sin t]\,\mathrm{d}t=a^2\pi.$

(4) $\int_l y\,\mathrm{d}x-x\,\mathrm{d}y+(x^2+y^2)\,\mathrm{d}z=\int_0^1[e^{-t}\cdot e^t-e^t(-e^{-t})+(e^{2t}+e^{-2t})a]\,\mathrm{d}t=2+\frac{a}{2}(e^2-e^{-2}).$

2. 求积分

$$J=\int_{(0,0,0)}^{(1,1,1)}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\x&y&z\end{vmatrix}\cdot\mathrm{d}\mathbf{r}$$

其中$\mathrm{d}\mathbf{r}$为矢径方向，积分路径分别为：

(1) 沿直线；

(2) 沿曲线$\mathbf{r}=\mathbf{i}\sin\varphi+\mathbf{j}(1-\cos\varphi)+\mathbf{k}\,\frac{2\varphi}{\pi}, \left(0\leqslant\varphi\leqslant\frac{\pi}{2}\right)$.

解：

(1) 直线方程为：$x=y=z$

则$J=\int_{(0,0,0)}^{(1,1,1)}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\x&y&z\end{vmatrix}\cdot\mathrm{d}\mathbf{r}=\int_{(0,0,0)}^{(1,1,1)}(z-y)\,\mathrm{d}x+(x-z)\,\mathrm{d}y+(y-x)\,\mathrm{d}z=$

$\int_0^1(x-x)\,\mathrm{d}x+\int_0^1(y-y)\,\mathrm{d}y+\int_0^1(z-z)\,\mathrm{d}z=0.$

(2) $J=\int_{(0,0,0)}^{(1,1,1)}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\x&y&z\end{vmatrix}\cdot\mathrm{d}\mathbf{r}=\int_0^{\frac{\pi}{2}}\left\{\left[\frac{2\varphi}{\pi}-(1-\cos\varphi)\right]\cos\varphi+\left(\sin\varphi-\frac{2\varphi}{\pi}\right)\sin\varphi+[(1-\cos\varphi)-\sin\varphi]\cdot\frac{2}{\pi}\right\}\mathrm{d}\varphi=$

$1-\frac{\pi}{2}-\frac{8}{\pi}$.

3. 设光滑闭曲线L在光滑曲面S上，S的方程为$z=f(x,y)$，曲线L在XY面上的投影曲线为l，函数$P(x,y,z)$在L上连续，证明

$$\oint_L P(x,y,z)\,\mathrm{d}x=\oint_l P[x,y,f(x,y)]\,\mathrm{d}x$$

证明：不妨设S为曲面的上侧，$z=f(x,y),(x,y)\in D$

则曲面的边界L在XY平面上的投影应是逆时针方向的曲线$l: x=\varphi(t), y=\psi(t), a<b, a\leqslant t\leqslant b$

空间曲线L的方程随之可表为$L: x=\varphi(t), y=\psi(t), z=\omega(t)=f[\varphi(t),\psi(t)], a\leqslant t\leqslant b$

于是$\oint_L P(x,y,z)\,\mathrm{d}x=\int_a^b P\left(\varphi(t),\psi(t),f[\varphi(t),\psi(t)]\right)\varphi'(t)\,\mathrm{d}t=\oint_l P[x,y,f(x,y)]\,\mathrm{d}x.$

4. 证明：对于曲线积分的估计式为

$$\left|\int_l P\,\mathrm{d}x+Q\,\mathrm{d}y\right|\leqslant LM\ ,\ (\text{式中}L\text{为积分曲线段长度})$$

$$M=\max_{(x,y)\in l}\sqrt{P^2+Q^2}$$

利用这个不等式估计：

$$I_R=\oint_{x^2+y^2=R^2}\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{(x^2+xy+y^2)^2}$$

并证明$\lim\limits_{R\to\infty}I_R=0$.

证明:

(1) $\left|\int_l P\,\mathrm{d}x+Q\,\mathrm{d}y\right|=\left|\int_l [P\cos\alpha+Q\sin\alpha]\,\mathrm{d}S\right|\leqslant\int_l|(P,Q)\cdot(\cos\alpha,\sin\alpha)|\ \mathrm{d}S\leqslant\int_l|(P,Q)||(\cos\alpha,\sin\alpha)|\,\mathrm{d}S=\int_l\sqrt{P^2+Q^2}\,\mathrm{d}S=\sqrt{P^2(\xi,\eta)+Q^2(\xi,\eta)}\int_l\mathrm{d}S\leqslant ML.$

(2) 因$P=\dfrac{y}{(x^2+xy+y^2)^2}$,$Q=\dfrac{-x}{(x^2+xy+y^2)^2}$，则$\sqrt{P^2+Q^2}=\dfrac{\sqrt{x^2+y^2}}{(x^2+xy+y^2)^2}=\dfrac{R}{(R^2+xy)^2}$

于是$M=\max\limits_{(x,y)\in l}\sqrt{P^2+Q^2}=\dfrac{R}{(R^2+xy)^2}\Bigg|_{\substack{x=\pm\frac{\sqrt{2}}{2}R\\ y=\mp\frac{\sqrt{2}}{2}}}=\dfrac{4}{R^3}$

则$0\leqslant|I_R|=\left|\oint_{x^2+y^2=R^2}\dfrac{y\,\mathrm{d}x-x\,\mathrm{d}y}{(x^2+xy+y^2)^2}\right|\leqslant LM=\dfrac{8\pi}{R^2}$

又$\lim\limits_{R\to\infty}\dfrac{8\pi}{R^2}=0$，则$\lim\limits_{R\to\infty}I_R=0$.

5. 设平面区域D由一条连续闭曲线L所围成，区域D的面积设为S，推导用曲线积分计算面积S的公式：

$$S=\frac{1}{2}\oint_L x\,\mathrm{d}y-y\,\mathrm{d}x$$

证明:

(1) 首先考虑图形$D=PQRS$，其中$QR,PS\parallel Y$轴，$PQ:y=y_0(x);SR:y=y_1(x)$且在$[a,b]$上连续
这时$L:PSRQP$
将D的面积看作两曲边梯形$abPS$和$abQP$的面积之差(其中a,b分为SP,RQ与X轴的交点)
于是有$S=\int_a^b[\,y_1(x)-y_0(x)\,]\,\mathrm{d}x$
另一方面，据II型曲线计算公式，有$\int_{\widehat{PQ}}y\,\mathrm{d}x=\int_a^b y_0(x)\,\mathrm{d}x,\int_{\widehat{SR}}y\,\mathrm{d}x=\int_a^b y_1(x)\,\mathrm{d}x$
并注意到$\int_{\overline{PS}}y\,\mathrm{d}x=\int_{\overline{PQ}}y\,\mathrm{d}x=0$
则$-\int_L y\,\mathrm{d}x=\int_{PSRQP}y\,\mathrm{d}x=\left(\int_{\overline{PS}}+\int_{\widehat{SR}}+\int_{\overline{RQ}}+\int_{\widehat{QR}}\right)y\,\mathrm{d}x=\int_a^b y_1(x)\,\mathrm{d}x-\int_a^b y_0(x)\,\mathrm{d}x=S$
即$S=-\int_L y\,\mathrm{d}x.$

(2) 对于区域$D=PQRS$，其中$PQ,RS\parallel X$轴，同理，有$\int_L x\,\mathrm{d}y=S.$

(3) 对于更复杂的区域情形可化为上两种情形，同样计算诸小块面积，然后相加，注意重复路线相互抵消，同样可得上两种结果.

综上所述，有$S=\dfrac{1}{2}\oint_L x\,\mathrm{d}y-y\,\mathrm{d}x.$

6. 计算下列曲线所围区域的面积：

(1) 椭圆：$x=a\cos t,y=b\sin t,(0\leqslant t\leqslant 2\pi)$；

(2) 星形线：$x=a\cos^3 t,y=a\sin^3 t,(0\leqslant t\leqslant 2\pi)$.

解：

(1) $S = \dfrac{1}{2}\displaystyle\oint_L x\,\mathrm{d}y - y\,\mathrm{d}x = \dfrac{1}{2}\int_0^{2\pi} ab\,\mathrm{d}t = \pi ab.$

(2) $S = \dfrac{1}{2}\displaystyle\oint_L x\,\mathrm{d}y - y\,\mathrm{d}x = \dfrac{3}{2}\,a^2\int_0^{2\pi} \sin^2 t\cos^2 t\,\mathrm{d}t = \dfrac{3}{8}\,\pi a^2.$

§4. 第二类曲面积分

1. 计算$\iint\limits_S (x+y)\,\mathrm{d}y\,\mathrm{d}z+(y+z)\,\mathrm{d}z\,\mathrm{d}x+(z+x)\,\mathrm{d}x\,\mathrm{d}y$

S是以原点为中心的正方体(每边长度为2)的边界，指向外侧.

解：$I=\iint\limits_S (x+y)\,\mathrm{d}y\,\mathrm{d}z+(y+z)\,\mathrm{d}z\,\mathrm{d}x+(z+x)\,\mathrm{d}x\,\mathrm{d}y=\iint\limits_S (x+y)\,\mathrm{d}y\,\mathrm{d}z+\iint\limits_S (y+z)\,\mathrm{d}z\,\mathrm{d}x+\iint\limits_S (z+x)\,\mathrm{d}x\,\mathrm{d}y$

计算$I=\iint\limits_S (x+y)\,\mathrm{d}y\,\mathrm{d}z$

因正方体六个面中有四个面垂直于YOZ平面，则此四个面的面积为0

于是$\iint\limits_S (x+y)\,\mathrm{d}y\,\mathrm{d}z=\iint\limits_{\substack{-1\leqslant y\leqslant 1\\-1\leqslant z\leqslant 1}}(1+y)\,\mathrm{d}y\,\mathrm{d}z-\iint\limits_{\substack{-1\leqslant y\leqslant 1\\-1\leqslant z\leqslant 1}}(-1+y)\,\mathrm{d}y\,\mathrm{d}z=2\iint\limits_{\substack{-1\leqslant y\leqslant 1\\-1\leqslant z\leqslant 1}}\mathrm{d}y\,\mathrm{d}z=8$

同理可得$\iint\limits_S (y+z)\,\mathrm{d}z\,\mathrm{d}x=8,\iint\limits_S (z+x)\,\mathrm{d}x\,\mathrm{d}y=8$

于是$I=\iint\limits_S (x+y)\,\mathrm{d}y\,\mathrm{d}z+(y+z)\,\mathrm{d}z\,\mathrm{d}x+(z+x)\,\mathrm{d}x\,\mathrm{d}y=24.$

2. 计算$\iint\limits_S f(x)\,\mathrm{d}y\,\mathrm{d}z+g(y)\,\mathrm{d}x\,\mathrm{d}z+h(z)\,\mathrm{d}x\,\mathrm{d}y$

式中f,g,h为连续函数，S为平行六面体$(0\leqslant x\leqslant a,0\leqslant y\leqslant b,0\leqslant z\leqslant c)$的边界，指向外侧.

解：设$S_1:x=a;S_2:x=0;S_3:y=b;S_4:y=0;S_5:z=c;S_6:z=0$

则$I=\iint\limits_S f(x)\,\mathrm{d}y\,\mathrm{d}z+g(y)\,\mathrm{d}x\,\mathrm{d}z+h(z)\,\mathrm{d}x\,\mathrm{d}y=$

$$\left(\iint\limits_{S_1}+\iint\limits_{S_2}+\iint\limits_{S_3}+\iint\limits_{S_4}+\iint\limits_{S_5}+\iint\limits_{S_6}\right)f(x)\,\mathrm{d}y\,\mathrm{d}z+g(y)\,\mathrm{d}x\,\mathrm{d}z+h(z)\,\mathrm{d}x\,\mathrm{d}y$$

因$\iint\limits_{S_1} f(x)\,\mathrm{d}y\,\mathrm{d}z+g(y)\,\mathrm{d}x\,\mathrm{d}z+h(z)\,\mathrm{d}x\,\mathrm{d}y=\iint\limits_{S_1} f(x)\,\mathrm{d}y\,\mathrm{d}z=f(a)bc$

$\iint\limits_{S_2} f(x)\,\mathrm{d}y\,\mathrm{d}z+g(y)\,\mathrm{d}x\,\mathrm{d}z+h(z)\,\mathrm{d}x\,\mathrm{d}y=\iint\limits_{S_2} f(x)\,\mathrm{d}y\,\mathrm{d}z=-f(0)bc$

$\iint\limits_{S_3} f(x)\,\mathrm{d}y\,\mathrm{d}z+g(y)\,\mathrm{d}x\,\mathrm{d}z+h(z)\,\mathrm{d}x\,\mathrm{d}y=\iint\limits_{S_3} g(y))\,\mathrm{d}z\,\mathrm{d}x=g(b)ac$

$\iint\limits_{S_4} f(x)\,\mathrm{d}y\,\mathrm{d}z+g(y)\,\mathrm{d}x\,\mathrm{d}z+h(z)\,\mathrm{d}x\,\mathrm{d}y=\iint\limits_{S_4} g(y)\,\mathrm{d}z\,\mathrm{d}x=-g(0)ac$

$\iint\limits_{S_5} f(x)\,\mathrm{d}y\,\mathrm{d}z+g(y)\,\mathrm{d}x\,\mathrm{d}z+h(z)\,\mathrm{d}x\,\mathrm{d}y=\iint\limits_{S_5} h(z)\,\mathrm{d}x\,\mathrm{d}y=h(c)ab$

$\iint\limits_{S_6} f(x)\,\mathrm{d}y\,\mathrm{d}z+g(y)\,\mathrm{d}x\,\mathrm{d}z+h(z)\,\mathrm{d}x\,\mathrm{d}y=\iint\limits_{S_6} h(z)\,\mathrm{d}x\,\mathrm{d}y=-h(0)ab$

则$I=\iint\limits_S f(x)\,\mathrm{d}y\,\mathrm{d}z+g(y)\,\mathrm{d}x\,\mathrm{d}z+h(z)\,\mathrm{d}x\,\mathrm{d}y=abc\left[\dfrac{f(a)-f(0)}{a}+\dfrac{g(b)-g(0)}{b}+\dfrac{h(c)-h(0)}{c}\right].$

3. 计算$\iint\limits_S yz\,\mathrm{d}z\,\mathrm{d}x$

$S:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$的上半表面的上侧.

解：将椭球面表为参数(φ,θ)形式：$x=a\sin\varphi\cos\theta,y=b\sin\varphi\cos\theta,z=c\cos\varphi\left(0\leqslant\varphi\leqslant\dfrac{\pi}{2},0\leqslant\theta\leqslant 2\pi\right)$

$I=\iint\limits_S yz\,\mathrm{d}z\,\mathrm{d}x=\pm\iint\limits_\Omega bc\sin\varphi\cos\varphi\sin\theta\cdot B\,\mathrm{d}\varphi\,\mathrm{d}\theta$，其中$\Omega$为$\varphi\theta$平面上的区域$0\leqslant\varphi\leqslant\dfrac{\pi}{2},0\leqslant\theta\leqslant 2\pi$

且$B=z_\varphi x_\theta-x_\varphi z_\theta=ac\sin^2\varphi\sin\theta$

因积分沿上侧，应取正号，即得$I=abc^2\displaystyle\int_0^{\frac{\pi}{2}}\sin^3\varphi\cos\varphi\,\mathrm{d}\varphi\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta=\dfrac{\pi}{4}abc^2$

4. 计算$\iint\limits_S z\,\mathrm{d}x\,\mathrm{d}y + x\,\mathrm{d}y\,\mathrm{d}z + y\,\mathrm{d}x\,\mathrm{d}z$

S为柱面$x^2+y^2=1$被平面$z=0$及$z=3$所截部分的外侧.

解：由于柱面$x^2+y^2=1$在XOY平面上的投影为一圆周，故其面积为0，从而$\iint\limits_S z\,\mathrm{d}x\,\mathrm{d}y = 0$

又$\iint\limits_S x\,\mathrm{d}y\,\mathrm{d}z = \left(\iint\limits_{S_{前}} + \iint\limits_{S_{后}}\right) x\,\mathrm{d}y\,\mathrm{d}z = \iint\limits_{S_{yz}} \sqrt{1-y^2}\,\mathrm{d}y\,\mathrm{d}z - \iint\limits_{S_{yz}} (-\sqrt{1-y^2})\,\mathrm{d}y\,\mathrm{d}z =$

$2\int_0^2 \mathrm{d}z \int_{-1}^1 \sqrt{1-y^2}\,\mathrm{d}y = 3\pi$

$\iint\limits_S y\,\mathrm{d}x\,\mathrm{d}z = \left(\iint\limits_{S_{右}} + \iint\limits_{S_{左}}\right) y\,\mathrm{d}x\,\mathrm{d}z = 2\int_0^3 \mathrm{d}z \int_{-1}^1 \sqrt{1-x^2}\,\mathrm{d}x = 3\pi$

则$\iint\limits_S z\,\mathrm{d}x\,\mathrm{d}y + x\,\mathrm{d}y\,\mathrm{d}z + y\,\mathrm{d}x\,\mathrm{d}z = 6\pi$.

5. 计算$\iint\limits_S x^3\,\mathrm{d}y\,\mathrm{d}z + y^3\,\mathrm{d}z\,\mathrm{d}x + z^3\,\mathrm{d}x\,\mathrm{d}y$

S为球面$x^2+y^2+z^2=a^2$的外侧.

解：据轮换对称，只需计算$\iint\limits_S x^3\,\mathrm{d}y\,\mathrm{d}z$，且$\iint\limits_S x^3\,\mathrm{d}y\,\mathrm{d}z = \iint\limits_{S_1} x^3\,\mathrm{d}y\,\mathrm{d}z + \iint\limits_{S_2} x^3\,\mathrm{d}y\,\mathrm{d}z$

其中S_1及S_2分别表示下半球面及上半球面，即$S_2: x=\sqrt{a^2-y^2-z^2}$应取上侧；$S_1: x=-\sqrt{a^2-y^2-z^2}$应取下侧

则$\iint\limits_S x^3\,\mathrm{d}y\,\mathrm{d}z = \iint\limits_{S_2} x^3\,\mathrm{d}y\,\mathrm{d}z + \iint\limits_{S_1} x^3\,\mathrm{d}y\,\mathrm{d}z = 2\iint\limits_{y^2+z^2\leqslant a^2} (a^2-y^2-z^2)^{\frac{3}{2}}\,\mathrm{d}y\,\mathrm{d}z =$

$2\int_0^{2\pi} \mathrm{d}\theta \int_0^a r(a^2-r^2)^{\frac{3}{2}}\,\mathrm{d}r = \frac{4}{5}\pi a^5$

于是$\iint\limits_S x^3\,\mathrm{d}y\,\mathrm{d}z + y^3\,\mathrm{d}z\,\mathrm{d}x + z^3\,\mathrm{d}x\,\mathrm{d}y = \frac{12}{5}\pi a^5$.

第二十二章　各种积分间的联系和场论初步

§1.　各种积分间的联系

1. 利用格林公式计算曲线积分：

(1) $\oint_l xy^2\,\mathrm{d}x - x^2y\,\mathrm{d}y, l$：圆周$x^2+y^2=a^2$；

(2) $\oint_l (x+y)\,\mathrm{d}x-(x-y)\,\mathrm{d}y, l$：椭圆周$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$；

(3) $\oint_l (x+y)^2\,\mathrm{d}x-(x^2+y^2)\,\mathrm{d}y, l$：顶点为$A(1,1),B(3,2),C(2,5)$的三角形的边界；

(4) $\int_{A\overset{\frown}{M}O}(e^x\sin y-my)\,\mathrm{d}x+(e^x\cos y-m)\,\mathrm{d}y$，
其中$A\overset{\frown}{M}O$为由点$A(a,0)$至点$O(0,0)$经过上半圆周$x^2+y^2=ax$的道路；

(5) $\oint_l e^x[(1-\cos y)\,\mathrm{d}x-(y-\sin y)\,\mathrm{d}y], l$：区域$0<x<\pi, 0<y<\sin x$的边界.

解:

(1) 由格林公式，此时$P=xy^2, Q=-x^2y$
则$\oint_l xy^2\,\mathrm{d}x-x^2y\,\mathrm{d}y=\iint\limits_D(-2xy-2xy)\,\mathrm{d}x\,\mathrm{d}y=-4\int_{-a}^a x\,\mathrm{d}x\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}y\,\mathrm{d}y=0$

(2) $P=x+y, Q=-(x-y)$，则$\oint_l(x+y)\,\mathrm{d}x-(x-y)\,\mathrm{d}y=-2\iint\limits_D\,\mathrm{d}x\,\mathrm{d}y=-2\pi ab$

(3) AB,BC,CA的方程分别为：$AB: x-2y+1=0; BC: 3x+y-11=0; CA: 4x-y-3=0$
$P=(x+y)^2, Q=-(x^2+y^2)$
则$I=\oint_l(x+y)^2\,\mathrm{d}x-(x^2+y^2)\,\mathrm{d}y=-2\iint\limits_D(2x+y)\,\mathrm{d}x\,\mathrm{d}y$
$=-2\left[\int_1^2\,\mathrm{d}x\int_{\frac{x+1}{2}}^{4x-3}(2x+y)\,\mathrm{d}y+\int_2^3\,\mathrm{d}x\int_{\frac{x+1}{2}}^{11-3x}(2x+y)\,\mathrm{d}y\right]=-46\dfrac{3}{2}$

(4) 在Ox轴上连接点$O(0,0)$与$A(a,0)$，这样便构成封闭的半圆形$AMOA$，且在线段OA上
$\int_{OA}(e^x\sin y-my)\,\mathrm{d}x+(e^x\cos y-m)\,\mathrm{d}y=0$
则$\int_{A\overset{\frown}{M}OA}(e^x\sin y-my)\,\mathrm{d}x+(e^x\cos y-m)\,\mathrm{d}y=\int_{A\overset{\frown}{M}O}(e^x\sin y-my)\,\mathrm{d}x+(e^x\cos y-m)\,\mathrm{d}y$
利用格林公式，得$\int_{A\overset{\frown}{M}OA}(e^x\sin y-my)\,\mathrm{d}x+(e^x\cos y-m)\,\mathrm{d}y=m\iint\limits_D\,\mathrm{d}x\,\mathrm{d}y=\dfrac{\pi m}{8}\,a^2$
于是$\int_{A\overset{\frown}{M}O}(e^x\sin y-my)\,\mathrm{d}x+(e^x\cos y-m)\,\mathrm{d}y=\dfrac{\pi m^2}{8}\,a^2$

(5) $P=e^x(1-\cos y), Q=(y-\sin y)(-e^x)$
则$\oint_l e^x[(1-\cos y)\,\mathrm{d}x-(y-\sin y)\,\mathrm{d}y]=-\iint\limits_D ye^x\,\mathrm{d}x\,\mathrm{d}y=-\int_0^\pi e^x\,\mathrm{d}x\int_0^{\sin x}y\,\mathrm{d}y=-\dfrac{1}{5}\,(e^\pi-1).$

2. 利用格林公式计算下列曲线所围面积：

(1) 星形线：$x=a\cos^3 t, y=b\sin^3 t$；

(2) 抛物线：$(x+y)^2=ax(a>0)$和x轴

解:

(1) 由格林公式，面积D为$D=\iint\limits_D\,\mathrm{d}x\,\mathrm{d}y=\dfrac{1}{2}\oint_l x\,\mathrm{d}y-y\,\mathrm{d}x$

又$x=a\cos^3 t, y=b\sin^3 t\ (0\leqslant t\leqslant 2\pi)$，则$D=\dfrac{1}{2}\oint_l x\,\mathrm{d}y-y\,\mathrm{d}x=\dfrac{3}{8}\,ab\int_0^{2\pi}\sin^2 2t\,\mathrm{d}t=\dfrac{3}{8}\,\pi ab$

(2) 作代换$y=tx$，则原方程化为$x^2(1+t)^2=ax\ (a>0,x>0)$

于是得曲线参数方程$x=\dfrac{a}{(1+t)^2}, y=\dfrac{at}{(1+t)^2}\ (0\leqslant t<+\infty)$

它与Ox轴的交点为$(a,0)$与$(0,0)$

在Ox轴上从$(0,0)$点到$(a,0)$点的一段上有$x\,\mathrm{d}y-y\,\mathrm{d}x=0$；在抛物线上有$x\,\mathrm{d}y-y\,\mathrm{d}x=\dfrac{a^2}{(1+t)^4}\ \mathrm{d}t$

于是面积$D=\dfrac{1}{2}\oint_l x\,\mathrm{d}y-y\,\mathrm{d}x=\dfrac{a^2}{2}\int_0^{+\infty}\dfrac{\mathrm{d}t}{(1+t)^4}=\dfrac{a^2}{6}.$

3. 证明若C为平面上封闭曲线，$\mathbf{l}$为任意方向则

$$\oint_c \cos(\mathbf{l},\mathbf{n})\,\mathrm{d}s=0$$

式中$\mathbf{n}$为C的外法线方向.

证明：不妨设C的方向为逆时针方向

因$(\mathbf{l},\mathbf{n})=(\mathbf{l},x)-(\mathbf{n},x)$，则$\cos(\mathbf{l},\mathbf{n})=\cos(\mathbf{l},x)\cos(\mathbf{n},x)+\sin(\mathbf{l},x)\sin(\mathbf{n},x)$

又$\sin(\mathbf{n},x)=-\cos(\mathbf{t},x),\cos(\mathbf{n},x)=\sin(\mathbf{t},x)$且$\cos(\mathbf{t},x)=\dfrac{\mathrm{d}x}{\mathrm{d}s},\sin(\mathbf{t},x)=\dfrac{\mathrm{d}y}{\mathrm{d}s}$

则$\cos(\mathbf{l},\mathbf{n})\,\mathrm{d}s=\cos(\mathbf{l},x)\,\mathrm{d}y-\sin(\mathbf{l},x)\,\mathrm{d}x$

于是$\displaystyle\oint_C\cos(\mathbf{l},\mathbf{n})\,\mathrm{d}s=\oint_C[-\sin(\mathbf{l},x)\,\mathrm{d}x+\cos(\mathbf{l},x)\,\mathrm{d}y]$

由$P=-\sin(\mathbf{l},x),Q=\cos(\mathbf{l},x)$，得$\dfrac{\partial P}{\partial y}=0=\dfrac{\partial Q}{\partial x}$

于是$\displaystyle\oint_C\cos(\mathbf{l},\mathbf{n})\,\mathrm{d}s=\iint\limits_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y=0$

4. 设$u(x,y),v(x,y)$是具有二阶连续偏导数的函数，并设

$$\Delta u\equiv\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}$$

证明：

(1) $\displaystyle\iint\limits_\sigma\Delta u\,\mathrm{d}x\,\mathrm{d}y=\int_l\frac{\partial u}{\partial n}\ \mathrm{d}s$

(2) $\displaystyle\iint\limits_\sigma v\Delta u\,\mathrm{d}x\,\mathrm{d}y=-\iint\limits_\sigma\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y+\oint_l v\frac{\partial u}{\partial n}\ \mathrm{d}s$

(3) $\displaystyle\iint\limits_\sigma(u\Delta v-v\Delta u)\,\mathrm{d}x\,\mathrm{d}y=-\int_l\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)\mathrm{d}s$

其中σ为闭曲线l所围的平面区域，$\dfrac{\partial u}{\partial n},\dfrac{\partial v}{\partial n}$为沿$l$外法线方向导数.

证明：

(1) $\displaystyle\int_l\frac{\partial u}{\partial n}\ \mathrm{d}s=\int_l\left(\frac{\partial u}{\partial x}\ \cos(\mathbf{n},x)+\frac{\partial u}{\partial y}\ \sin(\mathbf{n},x)\right)\mathrm{d}s=\int_l\frac{\partial u}{\partial x}\ \sin(\mathbf{t},x)\,\mathrm{d}s-\int_l\frac{\partial u}{\partial y}\ \cos(\mathbf{t},x)\,\mathrm{d}s$

$\displaystyle=\int_l\frac{\partial u}{\partial x}\ \mathrm{d}y-\int_l\frac{\partial u}{\partial y}\ \mathrm{d}x=\iint\limits_\sigma\left[\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right]\mathrm{d}x\,\mathrm{d}y=\iint\limits_\sigma\Delta u\,\mathrm{d}x\,\mathrm{d}y$

(2) 因$\displaystyle\oint_l v\frac{\partial u}{\partial n}\ \mathrm{d}s=\oint_l v\left(\frac{\partial u}{\partial x}\ \cos(\mathbf{n},x)+\frac{\partial u}{\partial y}\ \sin(\mathbf{n},x)\right)\mathrm{d}s=\oint_l\left[v\frac{\partial u}{\partial x}\ \sin(\mathbf{t},x)-v\frac{\partial u}{\partial y}\ \cos(\mathbf{t},x)\right]\mathrm{d}s$

$\displaystyle=\oint_l v\frac{\partial u}{\partial x}\ \mathrm{d}y-v\frac{\partial u}{\partial y}\ \mathrm{d}x=\iint\limits_\sigma\left[\frac{\partial}{\partial x}\ \left(v\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\ \left(-v\frac{\partial u}{\partial y}\right)\right]\mathrm{d}x\,\mathrm{d}y$

$\displaystyle=\iint\limits_\sigma\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y+\iint\limits_\sigma v\Delta u\,\mathrm{d}x\,\mathrm{d}y$

则$\displaystyle\iint\limits_\sigma v\Delta u\,\mathrm{d}x\,\mathrm{d}y=-\iint\limits_\sigma\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y+\oint_l v\frac{\partial u}{\partial n}\ \mathrm{d}s$

(3) 由(2)，得$\iint\limits_{\sigma} u\Delta v\,\mathrm{d}x\,\mathrm{d}y = -\iint\limits_{\sigma}\left(\dfrac{\partial v}{\partial x}\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\dfrac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y+\oint_l u\dfrac{\partial v}{\partial n}\,\mathrm{d}s$

则$\iint\limits_{\sigma}(u\Delta v - v\Delta u)\,\mathrm{d}x\,\mathrm{d}y = -\int_l\left(v\dfrac{\partial u}{\partial n} - u\dfrac{\partial v}{\partial n}\right)\mathrm{d}s$

5. 求以下积分之值

$$I=\oint_l[x\cos(\mathbf{n},x)+y\cos(\mathbf{n},y)]\,\mathrm{d}s$$

l：包围有界区域的简单封闭曲线，$\mathbf{n}$为它的外法线方向.

解：$I=\oint_l[x\cos(\mathbf{n},x)+y\cos(\mathbf{n},y)]\,\mathrm{d}s=\oint_l[x\sin(\mathbf{t},x)-y\cos(\mathbf{t},x)]\,\mathrm{d}s$

$=\oint_l x\,\mathrm{d}y-y\,\mathrm{d}x=\iint\limits_{\sigma}\left(\dfrac{\mathrm{d}x}{\mathrm{d}x}+\dfrac{\mathrm{d}y}{\mathrm{d}y}\right)\mathrm{d}x\,\mathrm{d}y=2\iint\limits_{\sigma}\mathrm{d}x\,\mathrm{d}y=2S$

6. 证明：

$$\oint_l\frac{\cos(r,\mathbf{n})}{r}\,\mathrm{d}s=0$$

其中l是一单连通区域σ的边界而r是l上的一点到σ外某一定点的距离.若r表示l上一点到σ内某一定点的距离，那末这积分之值等于2π.

证明：设$\mathbf{r}$为点$A(x,y)$到l上的点$M(\xi,\eta)$的向量，$\mathbf{n},\mathbf{r}$与Ox轴的夹角分别为α,β

则$(\mathbf{r},\mathbf{n})=\alpha-\beta$，于是$\cos(\mathbf{r},\mathbf{n})=\cos\alpha\cos\beta+\sin\alpha\sin\beta=\dfrac{\xi-x}{r}\cos\alpha+\dfrac{\eta-y}{r}\sin\alpha$

则$\oint_l\dfrac{\cos(\mathbf{r},\mathbf{n})}{r}\,\mathrm{d}s=\oint_l\left(\dfrac{\eta-y}{r^2}\sin\alpha+\dfrac{\xi-x}{r^2}\cos\alpha\right)\mathrm{d}s=\oint_l\dfrac{\xi-x}{r^2}\,\mathrm{d}\eta-\dfrac{\eta-y}{r^2}\,\mathrm{d}\xi$

因$P=-\dfrac{\eta-y}{r^2}$,$Q=\dfrac{\xi-x}{r^2}$，则$\dfrac{\partial P}{\partial\eta}=\dfrac{-(\xi-x)^2+(\eta-y)^2}{r^4}=\dfrac{\partial Q}{\partial\xi}$

因而P,Q的偏导数除去点A(此处$r=0$)外，在全平面上是连续的，且$\dfrac{\partial Q}{\partial\xi}=\dfrac{\partial P}{\partial\eta}$

于是利用格林公式，知当点A在曲线 l 之外时，$\oint_l\dfrac{\cos(r,\mathbf{n})}{r}\,\mathrm{d}s=0$

当点在曲线 l 之内时，$P,Q,\dfrac{\partial P}{\partial\eta},\dfrac{\partial Q}{\partial\xi}$ 均在(x,y)不连续，则不能直接使用格林公式，为此在 l 所包围的区域σ内，以A为圆心，R为半径作一圆，以其圆周作为曲线 l'，并使其包围的区域$\sigma'\subset\sigma$，再将σ扩大为σ''，使$\sigma\subset\sigma''$

因$P,Q,\dfrac{\partial P}{\partial\eta},\dfrac{\partial Q}{\partial\xi}$ 均在除(x,y)外的整个平面上连续且$\dfrac{\partial P}{\partial\eta}=\dfrac{\partial Q}{\partial\xi}$

则在复连通区域$\sigma''\backslash(x,y)$中连续且$\dfrac{\partial P}{\partial\eta}=\dfrac{\partial Q}{\partial\xi}$

这时$\oint_l\dfrac{\xi-x}{r^2}\,\mathrm{d}\eta-\dfrac{\eta-y}{r^2}\mathrm{d}\xi=\oint_{l'}\dfrac{\xi-x}{r^2}\,\mathrm{d}\eta-\dfrac{\eta-y}{r^2}\mathrm{d}\xi$

而$\oint_{l'}\dfrac{\xi-x}{r^2}\,\mathrm{d}\eta-\dfrac{\eta-y}{r^2}\mathrm{d}\xi=\int_0^{2\pi}\mathrm{d}\theta=2\pi$

则$\oint_l\dfrac{\cos(r,\mathbf{n})}{r}\,\mathrm{d}s=2\pi$

(当点A在 l 上时，$\oint_l\dfrac{\cos(r,\mathbf{n})}{r}\,\mathrm{d}s=\pi$)

7. 利用高斯公式变换以下积分：

(1) $\iint\limits_S xy\,\mathrm{d}x\,\mathrm{d}y+xz\,\mathrm{d}x\,\mathrm{d}z+yz\,\mathrm{d}y\,\mathrm{d}z$

(2) $\iint\limits_S\left(\dfrac{\partial u}{\partial x}\cos\alpha+\dfrac{\partial u}{\partial y}\cos\beta+\dfrac{\partial u}{\partial z}\cos\gamma\right)\mathrm{d}S$

其中$\cos\alpha,\cos\beta,\cos\gamma$是曲面的外法线方向余弦.

解：

(1) 因$P=yz, Q=xz, R=xy$，则$P_x=Q_y=R_z=0$

于是由高斯公式，得$\iint\limits_S xy\,\mathrm{d}x\,\mathrm{d}y+xz\,\mathrm{d}x\,\mathrm{d}z+yz\,\mathrm{d}y\,\mathrm{d}z=0$

(2) 因$P=u_x, Q=u_y, R=u_z$，则由高斯公式，得

$$\iint\limits_S\left(\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma\right)\mathrm{d}S=\iiint\limits_V\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

8. 利用高斯公式计算曲面积分：

(1) $\iint\limits_S x^2\,\mathrm{d}y\,\mathrm{d}z+y^2\,\mathrm{d}x\,\mathrm{d}z+z^2\,\mathrm{d}x\,\mathrm{d}y, S$：立方体$0\leqslant x,y,z\leqslant a$的外表面；

(2) $\iint\limits_S x^3\,\mathrm{d}y\,\mathrm{d}z+y^3\,\mathrm{d}x\,\mathrm{d}z+z^3\,\mathrm{d}x\,\mathrm{d}y, S$：单位球外表面；

(3) $\iint\limits_S(x^2\cos\alpha+y^2\cos\beta+z^2\cos\gamma)\,\mathrm{d}S, S$：$x^2+y^2=z^2, 0\leqslant z\leqslant h; \cos\alpha,\cos\beta,\cos\gamma$为此曲面外法线方向余弦.

解:

(1) 因$P=x^2, Q=y^2, R=z^2$，

则由高斯公式，得$\iint\limits_S x^2\,\mathrm{d}y\,\mathrm{d}z+y^2\,\mathrm{d}x\,\mathrm{d}z+z^2\,\mathrm{d}x\,\mathrm{d}y=2\int_0^a\mathrm{d}x\int_0^a\mathrm{d}y\int_0^a(x+y+z)\,\mathrm{d}z=3a^4$

(2) 因$P=x^3, Q=y^3, R=z^3$

则由高斯公式，得$\iint\limits_S x^3\,\mathrm{d}y\,\mathrm{d}z+y^3\,\mathrm{d}x\,\mathrm{d}z+z^3\,\mathrm{d}x\,\mathrm{d}y=3\iiint\limits_V(x^2+y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

$$=3\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\int_0^1 r^4\,\mathrm{d}r=\frac{12}{5}\pi$$

(3) 由高斯公式，得

$$\iint\limits_S(x^2\cos\alpha+y^2\cos\beta+z^2\cos\gamma)\,\mathrm{d}S=2\iiint\limits_V(x+y+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$=2\int_0^{2\pi}\mathrm{d}\varphi\int_0^h r\,\mathrm{d}r\int_r^h[r(\cos\varphi+\sin\varphi)+z]\,\mathrm{d}z=\frac{\pi h^4}{2}.$$

9. 证明：若

$$\Delta u\equiv\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

S是V的边界曲面，则成立下面公式：

(1) $\iiint\limits_V\Delta u\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\iint\limits_S\frac{\partial u}{\partial n}\,\mathrm{d}S$

(2) $\iint\limits_S u\frac{\partial u}{\partial n}\,\mathrm{d}S=\iiint\limits_V\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z+\iiint\limits_V u\Delta u\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

式中u在$V+S$上有连续二阶导数，$\dfrac{\partial u}{\partial n}$为沿曲面$S$外法线方向的导数.

证明:

(1) $\iint\limits_S\frac{\partial u}{\partial n}\,\mathrm{d}S=\iint\limits_S\left[\frac{\partial u}{\partial x}\cos(\mathbf{n},x)+\frac{\partial u}{\partial y}\cos(\mathbf{n},y)+\frac{\partial u}{\partial z}\cos(\mathbf{n},z)\right]\mathrm{d}S=\iiint\limits_V\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=$
$\iiint\limits_V\Delta u\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

(2) $\iint\limits_S u\frac{\partial u}{\partial n}\,\mathrm{d}S=\iint\limits_S\left[u\frac{\partial u}{\partial x}\cos(\mathbf{n},x)+u\frac{\partial u}{\partial y}\cos(\mathbf{n},y)+u\frac{\partial u}{\partial z}\cos(\mathbf{n},z)\right]\mathrm{d}S$

$$=\iiint\limits_V\left[\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(u\frac{\partial u}{\partial z}\right)\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \iiint_V \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iiint_V u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \iiint_V \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iiint_V u\Delta u\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

10. 证明由曲面S所包围的体积等于

$$V = \frac{1}{3}\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,\mathrm{d}S$$

式中$\cos\alpha, \cos\beta, \cos\gamma$为曲面$S$的外法线的方向余弦.

证明: 由高斯公式，得

$$V = \iiint_V \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{3}\iint_S (x\,\mathrm{d}y\,\mathrm{d}z + y\,\mathrm{d}x\,\mathrm{d}z + z\,\mathrm{d}x\,\mathrm{d}y) = \frac{1}{3}\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,\mathrm{d}S.$$

11. 利用斯托克司公式计算曲线积分：

(1) $\oint_l y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z, l$：圆周$\begin{cases} x^2+y^2+z^2=a^2 \\ x+y+z=0 \end{cases}$ 从x轴正向看去圆周是逆时针方向的；

(2) $\oint_l (z-y)\,\mathrm{d}x + (x-z)\,\mathrm{d}y + (y-x)\,\mathrm{d}z, l$是从$(a,0,0)$经$(0,a,0)$和$(0,0,a)$回到$(a,0,0)$的三角形.

解:

(1) 把平面$x+y+z=0$上l所包围的区域记为σ，则σ的法线方向为$(1,1,1)$

则其方向余弦为$\cos\alpha = \cos\beta = \cos\gamma = \dfrac{1}{\sqrt{3}}$

于是$\oint_l y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\iint_S (\sqrt{3}\,\mathrm{d}S = -\sqrt{3}\,\pi a^2$

(2) 把l所包围的区域记为σ，则σ的法线方向为$(1,1,1)$，则其方向余弦为$\cos\alpha = \cos\beta = \cos\gamma = \dfrac{1}{\sqrt{3}}$

又$P = z-y, Q = x-z, R = y-x$，则$\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z} = 2, \dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x} = 2, \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 2$

于是$\oint_l (z-y)\,\mathrm{d}x + (x-z)\,\mathrm{d}y + (y-x)\,\mathrm{d}z = 2\sqrt{3}\iint_S \mathrm{d}S = 3a^2$

§2. 曲线积分和路径的无关性

1. 设在某闭矩形区域D内$\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$，试证

$$U(x,y)=\int_{x_0}^{x}P(x,y)\,\mathrm{d}x+\int_{y_0}^{y}Q(x_0,y)\,\mathrm{d}y+C$$

为$P\,\mathrm{d}x+Q\,\mathrm{d}y$的原函数，其中$C=U(x_0,y_0)$.

证明：因$U(x,y)=\displaystyle\int_{x_0}^{x}P(x,y)\,\mathrm{d}x+\int_{y_0}^{y}Q(x_0,y)\,\mathrm{d}y+C$

则$\dfrac{\partial U}{\partial x}=P(x,y),\dfrac{\partial U}{\partial y}=\displaystyle\int_{x_0}^{x}P_y(x,y)\,\mathrm{d}x+Q(x_0,y)=\int_{x_0}^{x}Q_x(x,y)\,\mathrm{d}x+Q(x_0,y)=Q(x,y)$

于是$\mathrm{d}U=P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y$

又$U(x_0,y_0)=C$，则$U(x,y)=\displaystyle\int_{x_0}^{x}P(x,y)\,\mathrm{d}x+\int_{y_0}^{y}Q(x_0,y)\,\mathrm{d}y+C$为$P\,\mathrm{d}x+Q\,\mathrm{d}y$的原函数，其中$C=U(x_0,y_0)$

2. 计算下列全微分式的线积分：

(1) $\displaystyle\int_{(0,0)}^{(1,1)}(x-y)(\,\mathrm{d}x-\,\mathrm{d}y)$

(2) $\displaystyle\int_{(0,0)}^{(a,b)}f(x+y)(\,\mathrm{d}x+\,\mathrm{d}y)$，式中$f(u)$是连续函数；

(3) $\displaystyle\int_{(2,1)}^{(1,2)}\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{x^2}$，沿不和$Oy$轴相交的途径；

(4) $\displaystyle\int_{(1,2,3)}^{(0,1,1)}yz\,\mathrm{d}x+xz\,\mathrm{d}y+xy\,\mathrm{d}z$

(5) $\displaystyle\int_{(1,0)}^{(6,8)}\frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{\sqrt{x^2+y^2}}$，沿不通过原点的途径；

(6) $\displaystyle\int_{(2,1)}^{(1,2)}\varphi(x)\,\mathrm{d}x+\psi(y)\,\mathrm{d}y$，其中$\varphi,\psi$为连续函数.

解：

(1) 因$(x-y)(\,\mathrm{d}x-\,\mathrm{d}y)=\mathrm{d}\dfrac{(x-y)^2}{2}$，则$\displaystyle\int_{(0,0)}^{(1,1)}(x-y)(\,\mathrm{d}x-\,\mathrm{d}y)=\left.\frac{(x-y)^2}{2}\right|_{(0,0)}^{(1,1)}=0$

(2) 因$P+Q=f(x+y)$，则$\dfrac{\partial P}{\partial y}=f'(x+y)=\dfrac{\partial Q}{\partial x}$，于是从$(0,0)$到$(a,b)$积分与路径无关

取$(0,0)\to(a,0)\to(a,b)$，

则$\displaystyle\int_{(0,0)}^{(a,b)}f(x+y)(\,\mathrm{d}x+\,\mathrm{d}y)=\int_0^a f(x+0)\,\mathrm{d}x+\int_0^b f(a+y)\,\mathrm{d}y=\int_0^{a+b}f(u)\,\mathrm{d}u$

(3) 当$x\neq 0$时，$\dfrac{y\,\mathrm{d}x-x\,\mathrm{d}y}{x^2}=\mathrm{d}\left(-\dfrac{y}{x}\right)$，则$\displaystyle\int_{(2,1)}^{(1,2)}\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{x^2}=\left.-\frac{y}{x}\right|_{(2,1)}^{(1,2)}=-\frac{3}{2}$

(4) 因$yz\,\mathrm{d}x+xz\,\mathrm{d}y+xy\,\mathrm{d}z=\mathrm{d}xyz$，则$\displaystyle\int_{(1,2,3)}^{(0,1,1)}yz\,\mathrm{d}x+xz\,\mathrm{d}y+xy\,\mathrm{d}z=0$

(5) 当$(x,y)\neq(0,0)$时，$\dfrac{x\,\mathrm{d}x+y\,\mathrm{d}y}{\sqrt{x^2+y^2}}=\mathrm{d}\sqrt{x^2+y^2}$，则$\displaystyle\int_{(1,0)}^{(6,8)}\frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{\sqrt{x^2+y^2}}=9$

(6) 由于φ,ψ是连续函数，故有$\varphi(x)\,\mathrm{d}x+\psi(y)\,\mathrm{d}y=\mathrm{d}(F(x)+G(x))$

其中$F(x)=\displaystyle\int_2^x\varphi(u)\,\mathrm{d}u,G(y)=\int_1^y\psi(v)\,\mathrm{d}v$

于是有$\displaystyle\int_{(2,1)}^{(1,2)}\varphi(x)\,\mathrm{d}x+\psi(y)\,\mathrm{d}y=(F(x)+G(y))\Big|_{(2,1)}^{(1,2)}=\int_2^1\varphi(u)\,\mathrm{d}u+\int_1^2\psi(v)\,\mathrm{d}v=\int_1^2[\psi(x)-\varphi(x)]\,\mathrm{d}x$

3. 求原函数u：

(1) $(x^2+2xy-y^2)\,\mathrm{d}x+(x^2-2xy-y^2)\,\mathrm{d}y$
(2) $(2x\cos y-y^2\sin x)\,\mathrm{d}x+(2y\cos x-x^2\sin y)\,\mathrm{d}y$
(3) $\dfrac{a}{z}\ \mathrm{d}x+\dfrac{b}{z}\ \mathrm{d}y+\dfrac{-by-ax}{z^2}\ \mathrm{d}z$
(4) $(x^2-2yz)\,\mathrm{d}x+(y^2-2xz)\,\mathrm{d}y+(z^2-2xy)\,\mathrm{d}z$
(5) $e^x[e^y(x-y+2)+y]\,\mathrm{d}x+e^x[e^y(x-y)+1]\,\mathrm{d}y$

解:

(1) 因$\dfrac{\partial P}{\partial y}=2x-2y,\dfrac{\partial Q}{\partial x}=2x-2y$，则$\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$
于是$u=\displaystyle\int_0^x(x^2+2xy-y^2)\,\mathrm{d}x+\int_0^y(-y^2)\,\mathrm{d}y+C=\frac{x^3}{3}+x^2y-xy^2-\frac{y^3}{3}+C$

(2) 因$(2x\cos y-y^2\sin x)\,\mathrm{d}x+(2y\cos x-x^2\sin y)\,\mathrm{d}y=\cos y\,\mathrm{d}x^2+x^2\,\mathrm{d}\cos y+y^2\,\mathrm{d}\cos x+\cos x\,\mathrm{d}y^2=\mathrm{d}(x^2\cos y+y^2\cos x)$
则$u=x^2\cos y+y^2\cos x+C$

(3) 因$\dfrac{a}{z}\ \mathrm{d}x+\dfrac{b}{z}\ \mathrm{d}y+\dfrac{-by-ax}{z^2}\ \mathrm{d}z=a\dfrac{z\,\mathrm{d}x-x\,\mathrm{d}z}{z^2}+b\dfrac{z\,\mathrm{d}y-y\,\mathrm{d}z}{z^2}=\mathrm{d}\dfrac{ax+by}{z}$
则$u=\dfrac{1}{z}\,(ax+by)+C$

(4) 因$(x^2-2yz)\,\mathrm{d}x+(y^2-2xz)\,\mathrm{d}y+(z^2-2xy)\,\mathrm{d}z=\dfrac{1}{3}\,(\mathrm{d}x^3+\mathrm{d}y^3+\mathrm{d}z^3)-2(yz\,\mathrm{d}x+xz\,\mathrm{d}y+xy\,\mathrm{d}z)=\mathrm{d}\left(\dfrac{x^3+y^3+z^3}{3}-2xyz\right)$
则$u=\dfrac{1}{3}\,(x^3+y^3+z^3)-2xyz+C$

(5) 因$e^x[e^y(x-y+2)+y]\,\mathrm{d}x+e^x[e^y(x-y)+1]\,\mathrm{d}y=(x-y)e^{x+y}(\mathrm{d}x+\mathrm{d}y)+2e^{x+y}\,\mathrm{d}x+\mathrm{d}(ye^x)=\mathrm{d}((x-y)e^{x+y})+e^{x+y}\,\mathrm{d}(x+y)+\mathrm{d}(ye^x)=\mathrm{d}((x-y+1)e^{x+y})+\mathrm{d}(ye^x)$
则$u=(x-y+1)e^{x+y}+ye^x+C$

4. 验证：

$$P\,\mathrm{d}x+Q\,\mathrm{d}y=\frac{1}{2}\,\frac{x\,\mathrm{d}y-y\,\mathrm{d}x}{Ax^2+2Bxy+Cy^2}$$

$(A,B,C$为常数，且$AC-B^2>0)$适合条件：

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

求：关于奇点$(0,0)$的循环常数.

证明: 因$P=-\dfrac{y}{2(Ax^2+2Bxy+Cy^2)},Q=\dfrac{x}{2(Ax^2+2Bxy+Cy^2)}$
则$\dfrac{\partial P}{\partial y}=\dfrac{Cy^2-Ax^2}{2(Ax^2+2Bxy+Cy^2)^2}=\dfrac{\partial Q}{\partial x}$
于是$\omega=\displaystyle\oint_{x^2+y^2=1}P\,\mathrm{d}x+Q\,\mathrm{d}y=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\mathrm{d}t}{2(A\cos^2t+2B\sin t\cos t+C\sin^2t)}=\frac{1}{\sqrt{AC-B^2}}\ \arctan\frac{C\left(\tan t+\frac{B}{C}\right)}{\sqrt{AC-B^2}}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$
$=\begin{cases}\dfrac{\pi}{\sqrt{AC-B^2}}, & C>0\\ -\dfrac{\pi}{\sqrt{AC-B^2}}, & C<0\end{cases}$

5. 证明：

$$\int\frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{x^2+y^2}$$

关于奇点$(0,0)$的循环常数为0，从而$\dfrac{x\,\mathrm{d}x+y\,\mathrm{d}y}{x^2+y^2}$的积分与路径无关.

证明: 因$P=\dfrac{x}{x^2+y^2},Q=\dfrac{y}{x^2+y^2}$，则$\dfrac{\partial P}{\partial y}=-\dfrac{2xy}{(x^2+y^2)^2}=\dfrac{\partial Q}{\partial x}$
于是$\omega=\displaystyle\oint_{x^2+y^2=1}P\,\mathrm{d}x+Q\,\mathrm{d}y=0$

(1) 若闭路l不包围$(0,0)$点，可将奇点$(0,0)$与区域D的边界用一条曲线C连接起来，于是复连通区域变成了单连通区域

又$\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$，则由等价条件，得$\displaystyle\oint_l \frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{x^2+y^2}=0$

(2) 若闭路l包围奇点$(0,0)$，因沿环绕奇点的任一闭路的积分等于循环常数，则$\displaystyle\oint_l \frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{x^2+y^2}=0$

总之$\dfrac{x\,\mathrm{d}x+y\,\mathrm{d}y}{x^2+y^2}$的积分与路径无关.

§3.　场论初步

1. 设$\mathbf{H}(t)=e^t\mathbf{a}+e^{-t}\mathbf{b}$，其中$\mathbf{a},\mathbf{b}$为常向量，$t$为参数，

(1) 求$\dfrac{\mathrm{d}\mathbf{H}}{\mathrm{d}t}$

(2) 证明$\dfrac{\mathrm{d}^2\mathbf{H}}{\mathrm{d}t^2}=\mathbf{H}$

解：因$\mathbf{H}(t)=e^t\mathbf{a}+e^{-t}\mathbf{b}$，$\mathbf{a},\mathbf{b}$为常向量，则

(1) $\dfrac{\mathrm{d}\mathbf{H}}{\mathrm{d}t}=e^t\mathbf{a}-e^{-t}\mathbf{b}$

(2) $\dfrac{\mathrm{d}^2\mathbf{H}}{\mathrm{d}t^2}=\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}\mathbf{H}}{\mathrm{d}t}\right)=e^t\mathbf{a}+e^{-t}\mathbf{b}=\mathbf{H}$

2. 证明：$\dfrac{\mathrm{d}}{\mathrm{d}t}[\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})]=\dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\cdot(\mathbf{B}\times\mathbf{C})+\mathbf{A}\cdot\left(\dfrac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\times\mathbf{C}\right)+\mathbf{A}\cdot\left(\mathbf{B}\times\dfrac{\mathrm{d}\mathbf{C}}{\mathrm{d}t}\right)$

证明：设$\mathbf{A}=A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k},\mathbf{B}=B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k},\mathbf{C}=C_x\mathbf{i}+C_y\mathbf{j}+C_z\mathbf{k}$

则$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$，对等式两端求导，右端用对行列式求导法则，得

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})]=\begin{vmatrix}A_{x_t} & A_{y_t} & A_{z_t}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}+\begin{vmatrix}A_x & A_y & A_z\\ B_{x_t} & B_{y_t} & B_{z_t}\\ C_x & C_y & C_z\end{vmatrix}+\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_{x_t} & C_{y_t} & C_{z_t}\end{vmatrix}$$

$$=\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\cdot(\mathbf{B}\times\mathbf{C})+\mathbf{A}\cdot\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\times\mathbf{C}\right)+\mathbf{A}\cdot\left(\mathbf{B}\times\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t}\right)$$

3. 设$\mathbf{a}=3\mathbf{i}+20\mathbf{j}-15\mathbf{k}$，对下列数量场$\phi$分别求出$\mathrm{grad}\phi$及$\mathrm{div}(\phi\mathbf{a})$.

(1) $\phi=(x^2+y^2+z^2)^{-\frac{1}{2}}$

(2) $\phi=x^2+y^2+z^2$

(3) $\phi=\ln(x^2+y^2+z^2)$

解：

(1) $\mathrm{grad}\phi=\phi_x\mathbf{i}+\phi_y\mathbf{j}+\phi_z\mathbf{k}=\dfrac{-x}{(x^2+xy+y^2)^{\frac{3}{2}}}\mathbf{i}+\dfrac{-y}{(x^2+xy+y^2)^{\frac{3}{2}}}\mathbf{j}+\dfrac{-z}{(x^2+xy+y^2)^{\frac{3}{2}}}\mathbf{k}$

$\mathrm{div}(\phi\mathbf{a})=\phi\mathrm{div}\mathbf{a}+\mathrm{grad}\phi\cdot\mathbf{a}=\mathrm{grad}\phi\cdot\mathbf{a}=\dfrac{-3x-20y+15z}{(x^2+xy+y^2)^{\frac{3}{2}}}$

(2) $\mathrm{grad}\phi=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k},\mathrm{div}(\phi\mathbf{a})=6x+40y-30z$

(3) $\mathrm{grad}\phi=\dfrac{2x}{x^2+y^2+z^2}\mathbf{i}+\dfrac{2y}{x^2+y^2+z^2}\mathbf{j}+\dfrac{2z}{x^2+y^2+z^2}\mathbf{k},\mathrm{div}(\phi\mathbf{a})=\dfrac{6x+40y-30z}{x^2+y^2+z^2}$

4. 设$U(x,y,z)=xyz$

(1) 求$U(x,y,z)$在点$P_1(0,0,0),P_2(1,1,1)$及$P_3(2,1,1)$处沿$\mathbf{b}=2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$的方向导数；

(2) 在上述三点处，$U(M)$的最大方向导数为何值？

(3) 在上述三点处，求$\mathrm{divgrad}U(M)$及$\mathrm{rotgrad}U(M)$.

解：

(1) 因$\mathbf{b}$的方向余弦为$\cos\alpha=\dfrac{2}{\sqrt{29}},\cos\beta=\dfrac{3}{\sqrt{29}},\cos\gamma=-\dfrac{4}{\sqrt{29}}$

则$\dfrac{\partial U}{\partial b}=yz\cos\alpha+xz\cos\beta+xy\cos\gamma=\dfrac{1}{\sqrt{29}}(2yz+3xz-4xy)$

于是在$P_1(0,0,0)$点$\dfrac{\partial U}{\partial b}=0$；在$P_2(1,1,1)$点$\dfrac{\partial U}{\partial b}=\dfrac{\sqrt{29}}{29}$；在$P_3(2,1,1)$点$\dfrac{\partial U}{\partial b}=0$

(2) 因$\dfrac{\partial U}{\partial b}=\mathrm{grad}U\cdot\mathbf{b_0}=|\mathrm{grad}U|\cos(\mathrm{grad}U,\mathbf{b_0})$，其中$\mathbf{b_0}$是$\mathbf{b}$方向的单位向量

则$U(M)$的最大方向导数为$|\mathrm{grad}U|=\sqrt{y^2z^2+x^2z^2+x^2y^2}$

于是在$P_1(0,0,0)$点$|\mathrm{grad}U|=0$；在$P_2(1,1,1)$点$|\mathrm{grad}U|=\sqrt{3}$；在$P_3(2,1,1)$点$|\mathrm{grad}U|=3$

(3) 因$\mathrm{grad}U = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

则$\mathrm{divgrad}U = \dfrac{\partial(yz)}{\partial x} + \dfrac{\partial(xz)}{\partial y} + \dfrac{\partial(xy)}{\partial z} = 0$

$\mathrm{rotgrad}U = \left(\dfrac{\partial(xy)}{\partial y} - \dfrac{\partial(xz)}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial(yz)}{\partial z} - \dfrac{\partial(xy)}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial(xz)}{\partial x} - \dfrac{\partial(yz)}{\partial y}\right)\mathbf{k} = \mathbf{0}$

于是在上述三点处，$\mathrm{divgrad}U(M) = 0, \mathrm{rotgrad}U(M) = \mathbf{0}$.

5. 求向量$\mathbf{a} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$穿过球$x^2 + y^2 + z^2 = 1, x > 0, y > 0, z > 0$的流量.

解：$\Phi = \iint\limits_S x^2\,\mathrm{d}y\,\mathrm{d}z + y^2\,\mathrm{d}x\,\mathrm{d}z + z^2\,\mathrm{d}x\,\mathrm{d}y$，$\iint\limits_S z^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1(1-r^2)r\,\mathrm{d}r = \dfrac{\pi}{8}$

类似地，分别向XOZ, YOZ平面投影，可得$\iint\limits_S y^2\,\mathrm{d}x\,\mathrm{d}z = \iint\limits_S x^2\,\mathrm{d}y\,\mathrm{d}z = \dfrac{\pi}{8}$，于是$\Phi = \dfrac{3}{8}\pi$.

6. 求$\mathbf{a} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$通过$S$的流量，设

(1) S为圆柱体$x^2 + y^2 \leqslant a^2, 0 \leqslant z \leqslant h$的侧面；

(2) S为(1)中圆柱体的上底面；

(3) S为(1)中圆柱体的表面.

解：

(3)$\iint\limits_S a_n\,\mathrm{d}S = \iiint\limits_V \mathrm{div}\mathbf{a}\,\mathrm{d}V = \iiint\limits_V \left[\dfrac{\partial(yz)}{\partial x} + \dfrac{\partial(xz)}{\partial y} + \dfrac{\partial(xy)}{\partial z}\right]\mathrm{d}V = 0$

于是向量$\mathbf{a}$穿过圆柱体表面的流量为0

(2)因在圆柱体的上、下底面$a_n = xy$，则$\iint\limits_{S_{上}} a_n\,\mathrm{d}S = \iint\limits_{x^2+y^2\leqslant a^2} xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\mathrm{d}\theta\int_0^a r^3\sin\theta\cos\theta\,\mathrm{d}r = 0$

同理$\iint\limits_{S_{下}} a_n\,\mathrm{d}S = 0$，于是向量$\mathbf{a}$穿过圆柱体上底面的流量为0

(1)因$\iint\limits_S a_n\,\mathrm{d}S = \iint\limits_{S_{侧}} a_n\,\mathrm{d}S + \iint\limits_{S_{上}} a_n\,\mathrm{d}S + \iint\limits_{S_{下}} a_n\,\mathrm{d}S$，则$\iint\limits_{S_{侧}} a_n\,\mathrm{d}S = 0$.

7. 求$\mathbf{a} = \mathrm{grad}\left(\arctan\dfrac{y}{x}\right)$沿曲线$l$的环流量：

(1) l为$(x-2)^2 + (y-2)^2 = 1, z = 0$；

(2) l为$x^2 + y^2 = 4, z = 1$.

解：

(1) 由已知，有$\mathbf{a} = \mathrm{grad}\left(\arctan\dfrac{y}{x}\right) = -\dfrac{y}{x^2+y^2}\,\mathbf{i} + \dfrac{x}{x^2+y^2}\,\mathbf{j}$

则$\mathrm{rot}\mathbf{a} = \left[\dfrac{\partial\left(\frac{x}{x^2+y^2}\right)}{\partial x} - \dfrac{\partial\left(-\frac{y}{x^2+y^2}\right)}{\partial y}\right]\mathbf{k} = 0$(除$x = y = 0$即$Oz$轴上的点)

因$l : (x-2)^2 + (y-2)^2 = 1, z = 0$是不围绕$z$轴的曲线，故可于$l$上张一曲面$S$，使$S$与$Oz$轴不相交

则据斯托克司公式，有环流量$\oint_l \mathbf{a}\,\mathrm{d}\mathbf{l} = \iint\limits_S \mathbf{n}\cdot\mathrm{rot}\mathbf{a}\,\mathrm{d}S = 0$

(2) 因$l : x^2 + y^2 = 4, z = 1$，此时l正好围绕Oz轴旋转一周，取常数$c > 0$充分小($c < 2$)，使l位于平面$z = c$的上方，在平面$z = c$上围绕Oz轴取一圆周$l_r : x^2 + y^2 = r^2, z = c$，$r$充分小，使$r$小于2，以$l$与$l_r$为边界张上一曲面$S$，使$S$与$Oz$轴不相交

由斯托克司公式，得$\oint_l \mathbf{a}\cdot\,\mathrm{d}\mathbf{r} + \oint_{-l_r}\mathbf{a}\cdot\,\mathrm{d}\mathbf{r} = \iint\limits_S \mathbf{n}\cdot\mathrm{rot}\mathbf{a}\,\mathrm{d}S = 0$，其中$-l_r$表示沿顺时针方向

于是环流量$\oint_l \mathbf{a}\cdot\,\mathrm{d}\mathbf{r} = \oint_{l_r}\mathbf{a}\cdot\,\mathrm{d}\mathbf{r}$

又取l_r的参数方程$x = r\cos\theta, y = r\sin\theta, z = c$，得$\oint_{l_r}\mathbf{a}\cdot\,\mathrm{d}\mathbf{r} = \int_0^{2\pi}\mathrm{d}\theta = 2\pi$

从而$\oint_l \mathbf{a}\cdot\,\mathrm{d}\mathbf{r} = 2\pi$.

8. 求向量$\mathbf{a}=-y\mathbf{i}+x\mathbf{j}+c\mathbf{k}$($c$为常数)的环流量:

   (1) 沿圆周$x^2+y^2=1, z=0$;

   (2) 沿圆周$(x-2)^2+y^2=1, z=0$.

   解:

   (1) 因$l: x^2+y^2=1, z=0$,则$\mathbf{l}=\cos t\mathbf{i}+\sin t\mathbf{j}+0\mathbf{k}(0\leqslant t\leqslant 2\pi)$
   于是$\mathbf{a}\cdot \mathrm{d}\mathbf{l}=\mathrm{d}t$,从而所求环流量为$\oint_l \mathbf{a}\cdot \mathrm{d}\mathbf{l}=\int_0^{2\pi}\mathrm{d}t=2\pi$

   (2) 因$l: (x-2)^2+y^2=1, z=0$,则$\mathbf{l}=(2+\cos t)\mathbf{i}+\sin t\mathbf{j}+0\mathbf{k}(0\leqslant t\leqslant 2\pi)$
   于是$\mathbf{a}\cdot \mathrm{d}\mathbf{l}=(2\cos t+1)\,\mathrm{d}t$,从而所求环流量为$\oint_l \mathbf{a}\cdot \mathrm{d}\mathbf{l}=\int_0^{2\pi}(2\cos t+1)\,\mathrm{d}t=2\pi$

9. 证明:

   (1) $\mathrm{rot}(u\mathbf{A})=u\mathrm{rot}\mathbf{A}+\mathrm{grad}u\times\mathbf{A}$;

   (2) $\mathrm{div}(\phi\mathbf{a})=\phi\mathrm{div}\mathbf{a}+\mathrm{grad}\phi\cdot\mathbf{a}$

   (3) $\mathrm{graddiv}\mathbf{a}-\mathrm{rotrot}\mathbf{a}=\Delta\mathbf{a}$

   证明:

   (1) 因$\mathrm{rot}_x(u\mathbf{A})=u\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\left(A_z\frac{\partial u}{\partial y}-A_y\frac{\partial u}{\partial z}\right)=u\mathrm{rot}_x\mathbf{A}+(\mathrm{grad}u\times\mathbf{A})_x$
   同法可得,$\mathrm{rot}_y(u\mathbf{A})=u\mathrm{rot}_y\mathbf{A}+(\mathrm{grad}u\times\mathbf{A})_y, \mathrm{rot}_z(u\mathbf{A})=u\mathrm{rot}_z\mathbf{A}+(\mathrm{grad}u\times\mathbf{A})_z$
   于是$\mathrm{rot}(u\mathbf{A})=u\mathrm{rot}\mathbf{A}+\mathrm{grad}u\times\mathbf{A}$

   (2) 因$\frac{\partial(\phi a_x)}{\partial x}=\phi\frac{\partial a_x}{\partial x}+a_x\frac{\partial\phi}{\partial x}, \frac{\partial(\phi a_y)}{\partial y}=\phi\frac{\partial a_y}{\partial y}+a_y\frac{\partial\phi}{\partial y}, \frac{\partial(\phi a_z)}{\partial z}=\phi\frac{\partial a_z}{\partial z}+a_z\frac{\partial\phi}{\partial z}$
   则$\mathrm{div}(\phi\mathbf{a})=\phi\mathrm{div}\mathbf{a}+\mathrm{grad}\phi\cdot\mathbf{a}$

   (3) 因$\mathrm{graddiv}\mathbf{a}=\mathrm{grad}\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right)$
   $=\left(\frac{\partial^2 a_x}{\partial x^2}+\frac{\partial^2 a_y}{\partial x\partial y}+\frac{\partial^2 a_z}{\partial x\partial z}\right)\mathbf{i}+\left(\frac{\partial^2 a_x}{\partial x\partial y}+\frac{\partial^2 a_y}{\partial y^2}+\frac{\partial^2 a_z}{\partial y\partial z}\right)\mathbf{j}+\left(\frac{\partial^2 a_x}{\partial x\partial z}+\frac{\partial^2 a_y}{\partial y\partial z}+\frac{\partial^2 a_z}{\partial z^2}\right)\mathbf{k}$
   $\mathrm{rotrot}\mathbf{a}=\mathrm{rot}\left[\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)\mathbf{k}\right]$
   $=\left(\frac{\partial^2 a_y}{\partial x\partial y}+\frac{\partial^2 a_z}{\partial x\partial z}-\frac{\partial^2 a_x}{\partial y^2}-\frac{\partial^2 a_x}{\partial z^2}\right)\mathbf{i}+\left(\frac{\partial^2 a_x}{\partial x\partial y}+\frac{\partial^2 a_z}{\partial y\partial z}-\frac{\partial^2 a_y}{\partial x^2}-\frac{\partial^2 a_y}{\partial z^2}\right)\mathbf{j}+\left(\frac{\partial^2 a_x}{\partial x\partial z}+\frac{\partial^2 a_y}{\partial y\partial z}-\frac{\partial^2 a_z}{\partial x^2}-\frac{\partial^2 a_z}{\partial y^2}\right)\mathbf{k}$
   则$\mathrm{graddiv}\mathbf{a}-\mathrm{rotrot}\mathbf{a}=\Delta a_x\mathbf{i}+\Delta a_y\mathbf{j}+\Delta a_z\mathbf{k}=\Delta\mathbf{a}$

10. 求$\mathrm{rot}(\mathbf{w}\times\mathbf{r})$,$\mathbf{w}$为常矢量,$\mathbf{r}$为矢径向量.
解: 设$\mathbf{w}=(w_1,w_2,w_3), \mathbf{r}=(x,y,z)$
于是$\mathbf{w}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ w_1&w_2&w_3\\ x&y&z\end{vmatrix}=(w_2z-w_3y)\mathbf{i}+(w_3x-w_1z)\mathbf{j}+(w_1y-w_2x)\mathbf{k}$
$\mathrm{rot}(\mathbf{w}\times\mathbf{r})=2w_1\mathbf{i}+2w_2\mathbf{j}+2w_2\mathbf{k}=2\mathbf{w}$

11. 证明$\mathbf{a}=yz(2x+y+z)\mathbf{i}+xz(x+2y+z)\mathbf{j}+xy(x+y+2z)\mathbf{k}$为保守场,并求其势函数.
证明: 对空间任一点(x,y,z),有
$\mathrm{rot}\mathbf{a}=\left\{\frac{\partial}{\partial y}[xy(x+y+2z)]-\frac{\partial}{\partial z}[xz(x+2y+z)]\right\}\mathbf{i}$
$+\left\{\frac{\partial}{\partial z}[yz(2x+y+z)]-\frac{\partial}{\partial x}[xy(x+y+2z)]\right\}\mathbf{j}$
$+\left\{\frac{\partial}{\partial x}[xz(x+2y+z)]-\frac{\partial}{\partial y}[yz(2x+y+z)]\right\}\mathbf{k}$
$=0$
则$\mathbf{a}$为保守场
由于势ϕ满足$\mathrm{d}\phi=\mathbf{a}\cdot\mathrm{d}\mathbf{l}=a_x\,\mathrm{d}x+a_y\,\mathrm{d}y+a_z\,\mathrm{d}z=\mathrm{d}[(xyz(x+y+z)]$
则其势函数为$u(x,y,z)=xyz(x+y+z)+C$,其中C为任意常数.

12. 求向量$\mathbf{a}=\mathbf{r}$沿螺线$\mathbf{r}=a\cos t\mathbf{i}+a\sin t\mathbf{j}+bt\mathbf{k}(0\leqslant t\leqslant 2\pi)$的一段所作的功.

解：因$\mathbf{a}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}, l: x=a\cos t, y=a\sin t, z=bt(0\leqslant 2\pi)$

则所求的功为$W=\int_l x\,\mathrm{d}x+y\,\mathrm{d}y+z\,\mathrm{d}z=2b^2\pi^2$

13. 设ϕ为可微函数，计算：$\mathrm{grad}\phi(r), \mathrm{div}(\phi(r)\mathbf{r})$及$\mathrm{rot}(\phi(r)\mathbf{r})$.

解：$\mathrm{grad}\phi(r)=\phi'(r)\mathrm{grad}r=\phi'(r)\cdot\dfrac{\mathbf{r}}{r}$

$\mathrm{div}(\phi(r)\mathbf{r})=\phi(r)\mathrm{div}\mathbf{r}+\mathbf{r}\cdot\mathrm{grad}\phi(r)=3\phi(r)+r\phi'(r)$

$\mathrm{rot}(\phi(r)\mathbf{r})=\phi(r)\mathrm{rot}\mathbf{r}+\mathrm{grad}\phi(r)\times\mathbf{r}=\mathbf{0}$

14. 求满足条件$\mathrm{div}(\phi(r)\mathbf{r})=0$的函数$\phi(r)$.

解：由上题，得$\mathrm{div}(\phi(r)\mathbf{r})=3\phi(r)+r\phi'(r)$

要使$\mathrm{div}(\phi(r)\mathbf{r})=0$，只要$3\phi(r)+r\phi'(r)=0$即要$\dfrac{\phi'(r)}{\phi(r)}=-\dfrac{3}{r}$

则得$\phi(r)=\dfrac{c}{r^3}$ (c为常数)

15. 求以下各向量的散度及旋度($\mathbf{a},\mathbf{b}$为常向量)：

(1) $(\mathbf{a}\cdot\mathbf{r})\mathbf{b}$

(2) $\mathbf{a}\times\mathbf{r}$

(3) $\phi(r)(\mathbf{a}\times\mathbf{r})$

(4) $\mathbf{r}\times(\mathbf{a}\times\mathbf{r})$

解：

(1) $\mathrm{div}[(\mathbf{a}\cdot\mathbf{r})\mathbf{b}]=(\mathbf{a}\cdot\mathbf{r})\mathrm{div}\mathbf{b}+\mathrm{grad}(\mathbf{a}\cdot\mathbf{r})\cdot\mathbf{b}=\mathrm{grad}(\mathbf{a}\cdot\mathbf{r})\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{b}$

$\mathrm{rot}[(\mathbf{a}\cdot\mathbf{r})\mathbf{b}]=(\mathbf{a}\cdot\mathbf{r})\mathrm{rot}\mathbf{b}+\mathrm{grad}(\mathbf{a}\cdot\mathbf{r})\times\mathbf{b}=\mathrm{grad}(\mathbf{a}\cdot\mathbf{r})\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$

(2) 因$(\mathbf{a}\times\mathbf{r})_x=a_yz-a_zy, (\mathbf{a}\times\mathbf{r})_y=a_zx-a_xz, (\mathbf{a}\times\mathbf{r})_z=a_xy-a_yx$

则$\mathrm{div}(\mathbf{a}\times\mathbf{r})=\dfrac{\partial}{\partial x}(a_yz-a_zy)+\dfrac{\partial}{\partial y}(a_zx-a_xz)+\dfrac{\partial}{\partial z}(a_xy-a_yx)=0$

$$\mathrm{rot}(\mathbf{a}\times\mathbf{r})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ a_yz-a_zy & a_zx-a_xz & a_xy-a_yx\end{vmatrix}=2\mathbf{a}$$

(3) $\mathrm{div}[\phi(r)(\mathbf{a}\times\mathbf{r})]=\phi(r)\mathrm{div}(\mathbf{a}\times\mathbf{r})+\mathrm{grad}(\phi(r))(\mathbf{a}\times\mathbf{r})=\phi(r)(\mathbf{r}\cdot\mathrm{rot}\mathbf{a}-\mathbf{a}\cdot\mathrm{rot}\mathbf{r})+(\mathbf{r}\phi'(r)\cdot(\mathbf{a}\times\mathbf{r})=\phi'(r)\dfrac{\mathbf{r}}{r}\cdot(\mathbf{a}\times\mathbf{r})=\dfrac{\phi'(r)}{r}[\mathbf{r}\cdot(\mathbf{a}\times\mathbf{r})]=0$

$\mathrm{rot}[\phi(r)(\mathbf{a}\times\mathbf{r})]=\phi(r)\mathrm{rot}(\mathbf{a}\times\mathbf{r})+\mathrm{grad}(\phi(r))\times(\mathbf{a}\times\mathbf{r})=$

$\phi(r)[-(a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k})+3\mathbf{a}]+\left(\dfrac{\mathbf{r}}{r}\phi'(r)\right)\times(\mathbf{a}\times\mathbf{r})=2\phi(r)\mathbf{a}+\dfrac{\phi'(r)}{r}[\mathbf{r}^2\mathbf{a}-(\mathbf{a}\cdot\mathbf{r})\mathbf{r}]$

(4) 因$\mathbf{r}\times(\mathbf{a}\times\mathbf{r})=|\mathbf{r}|^2\mathbf{a}-(\mathbf{a}\cdot\mathbf{r})\mathbf{r}$

则$\mathrm{div}[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})]=|\mathbf{r}|^2\mathrm{div}\mathbf{a}+\mathrm{grad}|\mathbf{r}|^2\cdot\mathbf{a}-[(\mathbf{r}\cdot\mathbf{a})\mathrm{div}\mathbf{r}+\mathbf{r}\cdot\mathrm{grad}(\mathbf{a}\cdot\mathbf{r})]=(a_x+a_y+a_z)-4(xa_x+ya_y+za_z)$

$\mathrm{rot}[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})]=|\mathbf{r}|^2\mathrm{rot}\mathbf{r}+\mathrm{grad}|\mathbf{r}|^2\times\mathbf{a}-(\mathbf{r}\cdot\mathbf{a})\mathrm{rot}\mathbf{r}-\mathrm{grad}(\mathbf{r}\cdot\mathbf{a})\times\mathbf{r}=\dfrac{1}{r}(\mathbf{r}\times\mathbf{a})-\mathbf{a}\times\mathbf{r}.$

16. 证明以下等式：

(1) $\mathrm{grad}(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\times\mathrm{rot}\mathbf{b}+\mathbf{b}\times\mathrm{rot}\mathbf{a}+(\mathbf{b}\cdot\nabla)\mathbf{a}+(\mathbf{a}\cdot\nabla)\mathbf{b}$

(2) $\mathrm{rot}(\mathbf{a}\times\mathbf{b})=(\mathbf{b}\cdot\nabla)\mathbf{a}-(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathrm{div}\mathbf{b})\mathbf{a}-(\mathrm{div}\mathbf{a})\mathbf{b}$

(3) $\mathbf{c}\cdot\mathrm{grad}(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot(\mathbf{c}\cdot\nabla)+\mathbf{b}\cdot(\mathbf{c}\cdot\nabla)\mathbf{a}$

(4) $(\mathbf{c}\cdot\nabla)(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times(\mathbf{c}\cdot\nabla)\mathbf{b}-\mathbf{b}\times(\mathbf{c}\cdot\nabla)\mathbf{a}$

证明：

(1) $\mathrm{grad}(\mathbf{a}\cdot\mathbf{b})=\mathrm{grad}(a_xb_x+a_yb_y+a_zb_z)=$

$\left(a_x\dfrac{\partial b_x}{\partial x}+b_x\dfrac{\partial a_x}{\partial x}+a_y\dfrac{\partial b_y}{\partial x}+b_y\dfrac{\partial a_y}{\partial x}+a_z\dfrac{\partial b_z}{\partial x}+b_z\dfrac{\partial a_z}{\partial x}\right)\mathbf{i}+$

$\left(a_x\dfrac{\partial b_x}{\partial y}+b_x\dfrac{\partial a_x}{\partial y}+a_y\dfrac{\partial b_y}{\partial y}+b_y\dfrac{\partial a_y}{\partial y}+a_z\dfrac{\partial b_z}{\partial y}+b_z\dfrac{\partial a_z}{\partial y}\right)\mathbf{j}+$

$$\left(a_x\frac{\partial b_x}{\partial z}+b_x\frac{\partial a_x}{\partial z}+a_y\frac{\partial b_y}{\partial z}+b_y\frac{\partial a_y}{\partial z}+a_z\frac{\partial b_z}{\partial z}+b_z\frac{\partial a_z}{\partial z}\right)\mathbf{k}$$
$$=\mathbf{a}\times\mathrm{rot}\mathbf{b}+\mathbf{b}\times\mathrm{rot}\mathbf{a}+(\mathbf{b}\cdot\nabla)\mathbf{a}+(\mathbf{a}\cdot\nabla)\mathbf{b}$$

(2) $\mathrm{rot}(\mathbf{a}\times\mathbf{b})=\left(b_x\frac{\partial}{\partial x}+b_y\frac{\partial}{\partial y}+b_z\frac{\partial}{\partial z}\right)\mathbf{a}-\left(a_x\frac{\partial}{\partial x}+a_y\frac{\partial}{\partial y}+a_z\frac{\partial}{\partial z}\right)\mathbf{b}+(\mathrm{div}\mathbf{b})\mathbf{a}-(\mathrm{div}\mathbf{a})\mathbf{b}$
$=(\mathbf{b}\cdot\nabla)\mathbf{a}-(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathrm{div}\mathbf{b})\mathbf{a}-(\mathrm{div}\mathbf{a})\mathbf{b}$

(3) $\mathbf{c}\cdot\mathrm{grad}(\mathbf{a}\cdot\mathbf{b})=$

$$c_x\left(a_x\frac{\partial b_x}{\partial x}+b_x\frac{\partial a_x}{\partial x}+a_y\frac{\partial b_y}{\partial x}+b_y\frac{\partial a_y}{\partial x}+a_z\frac{\partial b_z}{\partial x}+b_z\frac{\partial a_z}{\partial x}\right)+$$
$$c_y\left(a_x\frac{\partial b_x}{\partial y}+b_x\frac{\partial a_x}{\partial y}+a_y\frac{\partial b_y}{\partial y}+b_y\frac{\partial a_y}{\partial y}+a_z\frac{\partial b_z}{\partial y}+b_z\frac{\partial a_z}{\partial y}\right)+$$
$$c_z\left(a_x\frac{\partial b_x}{\partial z}+b_x\frac{\partial a_x}{\partial z}+a_y\frac{\partial b_y}{\partial z}+b_y\frac{\partial a_y}{\partial z}+a_z\frac{\partial b_z}{\partial z}+b_z\frac{\partial a_z}{\partial z}\right)$$
$$=\mathbf{a}\cdot(\mathbf{c}\cdot\nabla)+\mathbf{b}\cdot(\mathbf{c}\cdot\nabla)\mathbf{a}$$

(4) $(\mathbf{c}\cdot\nabla)(\mathbf{a}\times\mathbf{b})=\left(c_x\frac{\partial}{\partial x}+c_y\frac{\partial}{\partial y}+c_z\frac{\partial}{\partial z}\right)[(a_yb_z-a_zb_y)\mathbf{i}+(a_zb_x-a_xb_z)\mathbf{j}+(a_xb_y-a_yb_x)\mathbf{k}]$

$$=\left(b_zc_x\frac{\partial a_y}{\partial x}+a_yc_x\frac{\partial b_z}{\partial x}-b_yc_x\frac{\partial a_z}{\partial x}-a_zc_x\frac{\partial b_y}{\partial x}+b_zc_y\frac{\partial a_y}{\partial y}+a_yc_y\frac{\partial b_z}{\partial y}-b_yc_y\frac{\partial a_z}{\partial y}-a_zc_y\frac{\partial b_y}{\partial y}+\right.$$
$$\left.b_zc_z\frac{\partial a_y}{\partial z}+a_yc_z\frac{\partial b_z}{\partial z}-b_yc_z\frac{\partial a_z}{\partial z}-a_zc_z\frac{\partial b_y}{\partial z}\right)\mathbf{i}+$$
$$\left(b_xc_x\frac{\partial a_z}{\partial x}+a_zc_x\frac{\partial b_x}{\partial x}-b_zc_x\frac{\partial a_x}{\partial x}-a_xc_x\frac{\partial b_z}{\partial x}+b_xc_y\frac{\partial a_z}{\partial y}+a_zc_y\frac{\partial b_x}{\partial y}-b_zc_y\frac{\partial a_x}{\partial y}-a_xc_y\frac{\partial b_z}{\partial y}+\right.$$
$$\left.b_xc_z\frac{\partial a_z}{\partial z}+a_zc_z\frac{\partial b_x}{\partial z}-b_zc_z\frac{\partial a_x}{\partial z}-a_xc_z\frac{\partial b_z}{\partial z}\right)\mathbf{j}+$$
$$\left(b_yc_x\frac{\partial a_x}{\partial x}+a_xc_x\frac{\partial b_y}{\partial x}-b_xc_x\frac{\partial a_y}{\partial x}-a_yc_x\frac{\partial b_x}{\partial x}+b_yc_y\frac{\partial a_x}{\partial y}+a_xc_y\frac{\partial b_y}{\partial y}-b_xc_y\frac{\partial a_y}{\partial y}-a_yc_y\frac{\partial b_x}{\partial y}+\right.$$
$$\left.b_yc_z\frac{\partial a_x}{\partial z}+a_xc_z\frac{\partial b_y}{\partial z}-b_xc_z\frac{\partial a_y}{\partial z}-a_yc_z\frac{\partial b_x}{\partial z}\right)\mathbf{k}$$
$$=\mathbf{a}\times(\mathbf{c}\cdot\nabla)\mathbf{b}-\mathbf{b}\times(\mathbf{c}\cdot\nabla)\mathbf{a}$$

17. 试证$\mathrm{divgrad}\sin^2 r$可表示成$F(r)$的形式，并写出$F(x)$.

证明: $\mathrm{divgrad}\sin^2 r=\Delta\sin^2 r=\frac{\partial^2\sin^2 r}{\partial x^2}+\frac{\partial^2\sin^2 r}{\partial y^2}+\frac{\partial^2\sin^2 r}{\partial z^2}=$

$$\frac{\partial}{\partial x}\left(\sin 2r\cdot\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\sin 2r\cdot\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\sin 2r\cdot\frac{z}{r}\right)=2\cos 2r+2\sin 2r\cdot\frac{1}{r}=\frac{2}{r}\left(\sin 2r+r\cos 2r\right)$$

即$F(r)=\frac{2}{r}\left(\sin 2r+r\cos 2r\right)$

则$F(x)=\mathrm{divgrad}\sin^2 xr=\frac{\mathrm{d}^2\sin^2 x}{\mathrm{d}x^2}=2\cos 2x$

18. 证明：当$|\mathbf{a}|^2=$常数时，有$(\mathbf{a}\cdot\nabla)\mathbf{a}=-\mathbf{a}\times\mathrm{rot}\mathbf{a}$.

证明: 因$|\mathbf{a}|^2=$常数，则$\mathrm{grad}|\mathbf{a}|^2=0$

由16题(1)，得$\mathrm{grad}(\mathbf{a}\cdot\mathbf{a})=2[\mathbf{a}\times\mathrm{rot}\mathbf{a}+(\mathbf{a}\cdot\nabla)\mathbf{a}]=0$

则$(\mathbf{a}\cdot\nabla)\mathbf{a}=-\mathbf{a}\times\mathrm{rot}\mathbf{a}$.