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Monotonic Solutions of Cooperative Games[1])

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Abstract: The principle of monotonicity for cooperative games states that if a game changes so that some player's contribution to all coalitions increases or stays the same then the player's allocation should not decrease. There is a unique symmetric and efficient solution concept that is monotonic in this most general sense – the Shapley value. Monotonicity thus provides a simple characterization of the value without resorting to the usual "additivity" and "dummy" assumptions, and lends support to the use of the value in applications where the underlying "game" is changing, e.g. in cost allocation problems.

Monotonicity is a general principle of fair division which states that as the underlying data of a problem change, the solution should change in parallel fashion. It is particularly germane to applications in which allocations are not made once and for all, but are reassessed periodically as new information emerges. This is the case, for example, in dividing the joint benefits or costs of a cooperative enterprise fairly among the partners when the underlying structure of the enterprise is evolving over time. Such a situation can be modelled by a cooperative game.

We give several formulations of the monotonicity principle for cooperative games and characterize different solution concepts *via* this property. Some commonly advocated methods – including the nucleolus and the so-called separable costs remaining benefits method – fail even the weakest test of monotonicity. In a somewhat more comprehensive form, monotonicity is shown to be inconsistent with a solution staying in the core. Finally, in a still stronger form it is shown to be consistent with exactly one (symmetric and efficient) solution concept – the Shapley value.

A *cooperative game* with players $\{1, 2, \ldots, n\} = N$ is a real valued function $v(S)$ defined on all *coalitions* $S \subseteq N$ such that $v(\phi) = 0$. $v(S)$ is the *value* of S. Although we do not require it, v is often assumed to be *superadditive*

$$v(S \cup T) \geqslant v(S) + v(T) \qquad \text{for all disjoint } S,\ T \subseteq N. \tag{1}$$

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An *allocation procedure* is a function φ that to every cooperative game v on N associates an *allocation* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^N$ such that $\sum_N x_i = v(N)$. The latter requirement is sometimes called "efficiency."

An important application of cooperative games in which monotonicity plays a natural role is the cost allocation problem. Figure 1 illustrates a hypothetical example [see *Young/Okada/Hashimoto* for a detailed discussion of a real-world one]. Three nearby towns must invest in a municipal water supply system or other common service. By building a single distribution system as opposed to separate subsystems they may reap the benefits of increasing returns to scale. The problem is how to allocate the benefits of cooperation equitably among the towns. Let $c(S)$ represent the joint cost of supplying the members of S independently. Then c is subadditive and the benefits of cooperating as opposed to going alone are given by the characteristic function $v(S) = \sum_{i \in S} c(i) - c(S)$.

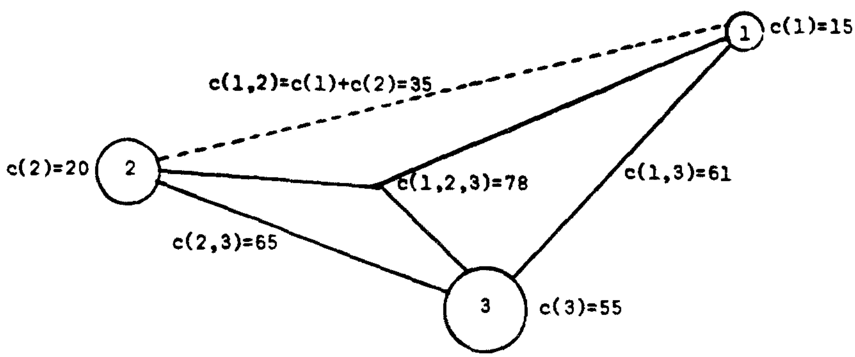

Fig. 1: A Cost Allocation Problem Among Three Towns.

The same type of problem arises in allocating the common overhead costs among different divisions in a firm. In this case $v(S)$ can be interpreted as the net *profit* (revenues minus costs) that each subset S of divisions could realize on a stand-alone basis [*Shubik*].

A frequently encountered form of monotonicity is *aggregate montonicity*. This principle states that if the value of the coalition of the whole increases, while the value of all other coalitions remains fixed, then no player should get less than before:

$$v(N) \geqslant w(N) \text{ and } v(S) = w(S) \quad \text{for all } S \subset N,$$
$$\text{implies } \varphi_i(v) \geqslant \varphi_i(w) \quad \text{for all } i. \tag{2}$$

This concept was first defined for cooperative games by [*Megiddo*], and is well-known in other problems of fair division, e.g. "monotonicity" in the bargaining problem *Kalai/Smorodinsky*, and "house monotonicity" in apportionment *Balinski/Young* [1982]. It has the following natural interpretation in the cost allocation context. Suppose that all players agree to cooperate and undertake an investment project most efficient for the whole group with a specified allocation of estimated costs. If a cost overrun occurs, how should it be allocated? Since the alternative projects were not tried, the available data are the cost of the project actually undertaken and the previously estimated costs of those not undertaken. In other words, only the datum $v(N)$ has changed. *Aggregate monotonicity* says that, however the overrun may be allocated, no one should *benefit* by having his assessment *reduced*. In the context of cost allocation in a firm, it says that an increase in aggregate overhead costs should not benefit any division.

Several well-known allocation procedures obey this principle; others do not.

The *egalitarian rule* divides the savings from the grand coalition equally among the players irrespective of the value of other coalitions:

$$\varphi_i(v) = v(N) / |N|.$$

This method is obviously monotonic in the aggregate – indeed is monotonic in a more comprehensive sense, as will presently be shown.

Another method that is monotonic in the aggregate is the *Shapley value*:

$$\varphi_i(v) = \sum_{S:\, i \in S} \frac{(|S|-1)!\,|N-S|!}{|N|!} v^i(S).$$

Here $v^i(S)$ represents the *marginal contribution of i to S*, and v^i is called the *derivative* of v with respect to i:

$$v^i(S) = \begin{cases} v(S) - v(S-i) & \text{if } i \in S \\ v(S+i) - v(S) & \text{if } i \notin S. \end{cases} \tag{3}$$

A well-known method in the cost benefit literature for allocating the costs of multiple-purpose reservoir projects [*James/Lee*] is the *separable costs remaining benefits* (SCRB) method. It can be given the following simple game theoretic formulation:

$$\varphi_i(v) = \frac{v^i(N)}{\sum_{j \in N} v^j(N)} v(N).$$

This method is *not* monotonic in the aggregate: In the example of Figure 1, if there is a cost overrun of 1 unit, total benefits decrease from 12 to 11. The marginal values $(v^1(N), v^2(N), v^3(N))$ decrease from (2, 3, 12) to (1, 2, 11), and the allocation changes from (1 7/17, 2 2/17, 8 8/17) to (11/14, 1 8/14, 8 9/14). Thus player 3 benefits by a cost overrun. This casts doubt on the reasonableness of the SCRB method for allocating costs in public projects, as is now commonly done.

Another well-known method that suffers from the same defect is the nucleolus. The *nuclelous* of *Schmeidler* [1969] is the allocation x that lexicographically minimizes the vector of "excesses" $e(S, \mathbf{x}) = v(S) - \sum_S x_i$, $\phi \underset{\neq}{\subset} S \underset{\neq}{\subset} N$, when these are arranged in order of descending magnitude. The nucleolus is a *core* solution concept *Gillies* [1959] in the sense that $\varphi(v) \in \{x: \sum_S x_i \geqslant v(S)$ for all S, $\Sigma x_i = v(N)\}$ whenever the latter is nonempty. As *Megiddo* [1974] showed, the nucleolus is not monotonic in the aggregate. It is natural to ask whether there are any *core* solution concepts that are. The answer is affirmative. One example is the *per capita* (or "normalized") *nucleolus* of *Grotte* [1970], which is the analog of the nucleolus with excesses defined on a per capita basis: $e(S, x) = (v(S) - \sum_{i \in S} x_i) / |S|$ for $\phi \underset{\neq}{\subset} S \underset{\neq}{\subset} N$.

	Egalitarian	SCRB	Shapley	Nuc.	PC Nuc.
Player					
1	4	1 7/17	2 1/6	1	2/3
2	4	2 2/17	2 4/6	1 1/2	1 1/6
3	4	8 8/17	7 1/6	9 1/2	10 1/6

Tab. 1: Allocation by Five Procedures

It is straightforward to prove that the per capita nucleolus distributes *changes* in the value $v(N)$ equally among the players [*Young/Okada/Hashimoto*]. The same is evidently true for the Shapley value and egalitarian rule. The allocations given by the five methods for the example are compared in Table 1.

A more comprehensive view of monotonicity, however, shows that no doctoring of the nucleolus or the core can be entirely satisfactory. A method satisfies *coalitional monotonicity* if an increase in the value of a particular coalition implies, *ceteris paribus*, no descrease in the allocation to any member of that coalition:

$$v(T) \geqslant w(T) \text{ for some } T \text{ and } v(S) = w(S) \quad \text{for all } S \neq T$$

$$\text{implies } \varphi_i(v) \geqslant \varphi_i(w) \quad \text{for all } i \in T. \tag{4}$$

Successive application of (4) shows that it is equivalent to: for all i, v, w,

$$\text{if } v(S) \geqslant w(S) \text{ for all } S \text{ containing } i \text{ and}$$

$$v(S) = w(S) \quad \text{for all } S \text{ not containing } i \text{ then } \varphi_i(v) \geqslant \varphi_i(w). \tag{5}$$

Thus coalitional monotonicity can also be understood as saying that if a particular player gains in the strength of his claim because the alternatives available to him increase in value whereas the value of the alternatives not involving him stay fixed, then he does not lose.

Monotonicity in the sense of (5) is particularly relevant to cost allocation in the firm, and was first proposed in that context by *Shubik* [1962]. Since a division manager's control is essentially over the costs of his production process, increasing the efficiency of his operations (by lowering the common costs of all coalitions containing him) should never *damage* his individual profit statement. Otherwise "individual rational actions based on the cost assignment may add up to corporate idiocy" [*Shubik*]. The Shapley value satisfies (5) (as Shubik noted) and so does the egalitarian rule.

Theorem 1: For $|N| \geqslant 5$ no core allocation rule is coalitionally monotonic.

Proof: Consider the 5-player game defined on $N = \{1, 2, 3, 4, 5\}$ as follows:

$$w(S_1) = w(3,5) = 3, \qquad w(S_2) = w(1,2,3) = 3,$$

$$w(S_3) = w(1,3,4) = 9, \qquad w(S_4) = w(2,4,5) = 9,$$

$$w(S_5) = w(1,2,4,5) = 9, \qquad w(N) = w(1,2,3,4,5) = 11,$$

and for $S \neq S_1, \ldots, S_5$, N, $w(S) = \max\limits_{S_k \subseteq S} w(S_k)$ or $w(S) = 0$ if S contains no S_k.

If $\mathbf{x}$ is in the core of w, then $\sum\limits_{S_k} x_i \geqslant w(S_k)$ for $1 \leqslant k \leqslant 5$; adding these relations we find that $\sum\limits_k \sum\limits_{S_k} x_i = 3 \sum\limits_N x_i \geqslant 33$ whence $\sum\limits_N x_i \geqslant 11$. But $\sum\limits_N x_i = 11$ by definition, so all inequalities $\sum\limits_{S_k} x_i \geqslant w(S_k)$ must be equalities. These have a unique solution, $\bar{\mathbf{x}} = (0, 1, 2, 7, 1)$, which constitutes the core of w.

Compare the game v which is identical to w except that $v(S_5) = v(N) = 12$. A similar argument shows that the unique core element is now $\bar{\bar{\mathbf{x}}} = (3, 0, 0, 6, 3)$. Thus the allocation to both 2 and 4 *decreased,* even though the value of the sets containing them monotonically *increased*. This shows that no core allocation procedure is monotonic for $|N| = 5$ and by obvious extension for $|N| \geqslant 5$. □

Coalitional monotonicity refers to monotonic changes in the *absolute* value of the coalitions containing a given player. More likely to occur in practice are situations where the value of the coalitions containing a given player i increase *relative* to the value of the coalitions not containing i. In these cases i's allocation should certainly not decrease either. For example, if several division managers simultaneously take steps to increase efficiency by decreasing joint costs, but one division makes a greater relative improvement in the sense that its marginal contribution to all possible coalitions increases, then that division should not be penalized. A procedure φ satisfies *strong monotonicity* if

$$v^i(S) \geqslant w^i(S) \text{ for all } S \text{ implies } \varphi_i(v) \geqslant \varphi_i(w). \tag{6}$$

The procedure φ is *symmetric* if for all permutations π of N, $\varphi_{\pi i}(\pi v) = \varphi_i(v)$, where $\pi v(S) = v(\pi S)$ for all S.

Theorem 2: The Shapley value is the unique symmetric allocation procedure that is strongly monotonic.

Proof: It is clear that the Shapley value is strongly monotonic. To prove the converse, note that strong monotonicity means that for any two games, v, w on N,

$$v^i(S) = w^i(S) \text{ for all } S \text{ implies } \varphi_i(v) = \varphi_i(w). \tag{7}$$

Next consider the symmetric game w on N which is identically zero on all coalitions, so that $w^i(S) = 0$ for all i, S. By symmetry $\varphi_i(w) = \varphi_j(w)$ for all $i \neq j$, and by definition of an allocation procedure, $\sum_N \varphi_i(w) = 0$, hence $\varphi_i(w) = 0$ for all i. By (7) it follows that for any game v on N and any $i \in N$,

$$v^i(S) = 0 \quad \text{for all } S \text{ implies } \varphi_i(v) = 0. \tag{8}$$

That is, dummy players get nothing.

We now exploit the fact noted by Shapley that every game v can be expressed as a sum of *primitive games*

$$v = \sum_{\phi \neq R \subseteq N} c_R v_R, \tag{9}$$

where

$$c_R v_R(S) = \begin{cases} c_R & \text{if } R \subseteq S \\ 0 & \text{if } R \nsubseteq S. \end{cases}$$

The Shapley value can be expressed as $\varphi_i(v) = \sum_{\phi \neq R \subseteq N} \varphi_i(c_R v_R) = \sum_{R: i \in R} c_R / |R|$.

Define the *index* I of v to be the minimum number of non-zero terms in some expression for v of form (9). The theorem is proved by induction on I.

If $I = 0$, every i is a dummy so by (8) $\varphi_i(v) = 0$. If $I = 1$, $v = c_R v_R$ for some $R \subseteq N$. For $i \notin R$, $v^i(S) = 0$ for all S, so by (8) $\varphi_i(v) = 0$. For all $i, j \in R$, symmetry implies that $\varphi_i(v) = \varphi_j(v)$; combined with the requirement that $\sum_N \varphi_k(v) = v(N)$ we conclude that $\varphi_i(v) = c_R / |R|$ for all $i \in R$. Therefore $\varphi(v)$ is the Shapley value whenever the index of v is 0 or 1.

Assume now that $\varphi(v)$ is the Shapley value whenever the index of v is at most I, and let v have index $I + 1$ with expression

$$v = \sum_{k=1}^{I+1} c_{R_k} v_{R_k}, \quad \text{all } c_{R_k} \neq 0.$$

Let $R = \bigcap_{k=1}^{I+1} R_k$ and suppose that $i \notin R$. Define the game

$$w = \sum_{k: i \in R_k} c_{R_k} v_{R_k}.$$

The index of w is at most I and $w^i(S) = v^i(S)$ for all S, so by induction and strong monotonicity it follows that

$$\varphi_i(v) = \varphi_i(w) = \sum_{k=i\in R_k} c_{R_k} / |R_k| \tag{10}$$

which is the Shapley value of i.

It remains to show that $\varphi_i(v)$ is the Shapley value when $i \in R = \bigcap_{k=1}^{I+1} R_k$.

By symmetry, $\varphi_i(v)$ is a constant c for all members of R; likewise the Shapley value is some constant c' for all members of R. Since both allocations sum to $v(N)$ and are equal for all i not in R, it follows that $c = c'$. □

If it is desired to stay entirely within the class of superadditive games, the above proof may be modified as follows. Given an expression $v = \sum_{\phi\neq R\subseteq N} c_R v_R$, for each r $(1 \leqslant r \leqslant n)$ define $c_r = \max_{R:|R|=r} c_R$ and $\tilde{c}_R = c_{|R|} - c_R \geqslant 0$. Let $u = \sum_{\phi\neq R\subseteq N} c_{|R|} v_R$ and write v in the form $v = u - \sum_{\phi\neq R\subseteq N} \tilde{c}_R v_R$.

Consider all such expressions for v in which u is superadditive and *symmetric* (i.e. $v(S)$ depends only on S) and $\tilde{c}_R > 0$ for all R. Let the index of v be the minimum number of terms in such an expression. Proceed by induction on this index, noting that the deletion of any term $\tilde{c}_R v_R$ leaves a superadditive game.

The above proof only requires the assumption that a player's value *depends* only on the vector of his marginal contributions (7), which is weaker than strong monotonicity.

Strictly speaking, (7) is a type of *independence* condition rather than a monotonicity condition. It highlights the general point that monotonicity and independence are closely related since a statement about the direction in which the solution should change as the underlying data change (montonicity) may well imply a statement about circumstances under which the solution does not change even though the underlying data do change (independence). The reason for emphasizing the somewhat stronger notion of monotonicity is that it seems to be the most relevant to actual applications.

It is interesting to note that Shapley's original axiom scheme employs an independence condition which is just a weaker variant of (7), namely the *dummy axiom*:

$$v^i(S) = 0, \text{ for all } S \subseteq N \text{ implies } \varphi_i(v) = 0. \tag{11}$$

Combined with Shapley's additivity axiom, (11) implies (7), because $v^i(S) = w^i(S)$ is equivalent to saying that i is a dummy in the game $v - w$, hence $\varphi_i(v - w) = 0$, so by additivity $\varphi_i(v) = \varphi_i(w)$.

However, independence is by no means equivalent to additivity: the function $\varphi_i(v) = [v^i(N)]^2$ satisfies independence (and strong monotonicity) but is not additive. Thus "efficiency" (and symmetry) play a crucial role.

An analogous result to Theorem 2 [*Young*, 1985] applies in the case of cost functions $f(x_1, \ldots, x_n)$ which are defined for variable levels of output $\mathbf{x} = (x_1, \ldots, x_n) \geqslant 0$. Here $f(\mathbf{x})$ is the joint cost of production and the object is to find unit prices $(p_1, \ldots, p_n)$ as a function of output $\bar{x}_1, \ldots, \bar{x}_n$ and f such that $\sum_{i=1}^{n} p_i \bar{x}_i = f(\bar{\mathbf{x}})$. In this case strong monotonicity is phrased in terms of the partial derivatives of f, and uniquely characterizes the Aumann-Shapley prices defined as

$$p_i = \int_0^1 \frac{\partial f}{\partial x_i}(t\bar{\mathbf{x}})\, dt.$$

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