10. Sea $\mathcal{P}_n$ el conjunto de todos los polinomios de grado n, en x, con coeficientes reales:

$$|p_n\rangle \rightleftharpoons p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} = \sum_{i=0}^{n-1} a_i x^i .$$

(a). Demostrar que $\mathcal{P}_n$ es un espacio vectorial respecto a la suma de polinomios y a la multiplicación de polinomios por un número (número real).

(b). Si los coeficientes a_i son enteros ¿$\mathcal{P}_n$ será un espacio vectorial? ¿Por qué?

(c). ¿Cuál de los siguientes subconjuntos de $\mathcal{P}_n$ es un subespacio vectorial?

I. El polinomio cero y todos los polinomios de grado $n-1$.

II. El polinomio cero y todos los polinomios de grado par.

III. Todos los polinomios que tienen a x como un factor (grado $n > 1$).

IV. Todos los polinomios que tienen a $x-1$ como un factor.

Sea $|P_n\rangle = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$

$|Q_n\rangle = b_0 + b_1 x + \cdots + b_{n-1} x_{n-1} x^{n-1}$

entonces $|P_n\rangle + |Q_n\rangle = a_0 + a_1 x + \cdots + a_{n-1} x_{n-1} + b_0 + b_1 x + \cdots + b_{n-1} x_{n-1} x^{n-1}$

$= (a_0 + b_0) + (a_1 + b_1) x + \cdots + (a_{n-1} + b_{n-1}) x^{n-1}$

$\alpha |P_n\rangle = \alpha (a_0 + a_1 x + \cdots + a_{n-1} x_{n-1})$

$= \alpha a_0 + \alpha a_1 x + \cdots + \alpha a_{n-1} x_{n-1}$

Si $\beta \to \mathbb{Z}$ $\quad \beta = \frac{1}{c}$

$\beta |P_n\rangle = a_0 + a_1 x + \cdots a_{n-1} x^{n-1} \Rightarrow \frac{a_0}{c} + \frac{a_1}{c} x + \cdots + \frac{a_{n-1}}{c} x^{n-1}$

No Porque $\frac{a_i}{c}$ no seria cerrado en multiplicación

c]

I) No Contra ejemplo

$|P_n\rangle = x^{n-1} + 5 \qquad |Q_n\rangle = -x^{n-1} - 3 \to |P_n\rangle + |Q_n\rangle \to x^{n-1} + 5 - x^{n-1} - 3 = 2$

no se cuentra ni cero ni los de x^{n-1}

II $|P_n\rangle = a_0 + a_2 x^2 + a_4 x^4 + \cdots \qquad |Q_n\rangle = b_0 + b_2 x^2 + b_4 x^4 + \cdots$

$|P_n\rangle + |Q_n\rangle = a_0 + (a_2 + b_2) x^2 + (a_4 + b_4) x^4 + \cdots$

$\alpha |P_n\rangle = \alpha (a_0 + a_2 x^2 + a_4 x^4 + \cdots) \Rightarrow \alpha a_0 + \alpha a_2 x^2 + \alpha x^4 + \cdots$

III $|P_n\rangle = a_1 x + \cdots + a_{n-1} x^{n-1} \qquad |Q_n\rangle = b_1 x + \cdots + b_{n-1} x_{n-1} x^{n-1}$

$x(a_1 + a_2 x + \cdots + a_{n-1} x^{n-2}) \qquad = x(b_1 + b_2 x + \cdots + b_{n-1} x^{n-2})$

$|P_n\rangle + |Q_n\rangle \quad x(a_1 + a_2 x + \cdots + a_{n-1} x^{n-2}) + x(b_1 + b_2 x + \cdots + b_{n-1} x^{n-2})$

$x(a_1 + b_1 + (a_2 + b_2) x + \cdots + (a_{n-1} + b_{n-1}) x^{n-2})$

$$IV\rangle\ (x-1)(a_0+a_1x+\cdots+a_{n-1}x^{n-1}) \qquad (x-1)\,b_0+b_1x+\cdots+b_{n-1}x_{n-1}x^{n-1}$$

$$|P_n\rangle+|Q_n\rangle=(x-1)(a_0+a_1x+\cdots+a_{n-1}x^{n-1})+(x-1)(b_0+b_1x+\cdots+b_{n-1}x_{n-1}x^{n-1})$$

$$=(x-1)\left[(a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+\cdots(a_{n-1}+b_{n-1})x^{n-1}\right]$$

$$\alpha|P_n\rangle=\alpha\,(x-1)(a_0+a_1x+\cdots+a_{n-1}x^{n-1})$$

$$(x-1)\left[\alpha a_0+\alpha a_1x+\cdots\alpha a_{n-1}x^{n-1}\right]$$

$$|a\rangle = a^\alpha |q_\alpha\rangle = a^0 + a^i|q_i\rangle = a^0 + a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

$$|b\rangle^\dagger = b^0|q_0\rangle - b^j|q_j\rangle$$

Multiplicación entre cuaterniones

$\lvert q_i\rangle \odot \lvert q_j\rangle$	1	$\lvert q_1\rangle$	$\lvert q_2\rangle$	$\lvert q_3\rangle$
1	1	$\lvert q_1\rangle$	$\lvert q_2\rangle$	$\lvert q_3\rangle$
$\lvert q_1\rangle$	$\lvert q_1\rangle$	-1	$\lvert q_3\rangle$	$-\lvert q_2\rangle$
$\lvert q_2\rangle$	$\lvert q_2\rangle$	$-\lvert q_3\rangle$	-1	$\lvert q_1\rangle$
$\lvert q_3\rangle$	$\lvert q_3\rangle$	$\lvert q_2\rangle$	$-\lvert q_1\rangle$	-1

a) $|a\rangle = a^0 + a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$; $|b\rangle = b^0 + b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$$|a\rangle + |b\rangle = a^0 + a_x\hat{i} + a_y\hat{j} + a_z\hat{k} + b^0 + b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$$

$$= a^0 + b^0 + a_x\hat{i} + b_x\hat{i} + a_y\hat{j} + b_y\hat{j} + a_z\hat{k} + b_z\hat{k}$$

$$= a^0 + b^0 + (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j} + (a_z + b_z)\hat{k}$$

$$\alpha|a\rangle = \alpha(a^0 + a_x\hat{i} + a_y\hat{j} + a_z\hat{k})$$

$$= \alpha a^0 + \alpha a_x\hat{i} + \alpha a_y\hat{j} + \alpha a_z\hat{k}$$

b) $|d\rangle = |b\rangle \odot |r\rangle \rightarrow (d^0, d) = (b^0 r^0 - b\cdot r, r^0 b + b^0 r + b\times r)$

$$|b\rangle \odot |r\rangle = |b\rangle = (b^0 + b_x\hat{i} + b_y\hat{j} + b_z\hat{k} \odot r^0 + r_x\hat{i} + r_y\hat{j} + r_z\hat{k})$$

$$b^0 r^0 + b^0\underbrace{(r_x\hat{i} + r_y\hat{j} + r_z\hat{k})}_{r} + r^0\underbrace{(b_x\hat{i} + b_y\hat{j} + b_z\hat{k})}_{b} + b_x\hat{i}(r_x\hat{i} + r_y\hat{j} + r_z\hat{k}) + b_y\hat{j}(r_x\hat{i} + r_y\hat{j} + r_z\hat{k})$$

$$+ b_z\hat{k}(r_x\hat{i} + r_y\hat{j} + r_z\hat{k})$$

$$b^0 r^0 + b^0 r + r^0 b \underbrace{- b_x\hat{i} r_x\hat{i} - b_y\hat{j} r_y\hat{j} - b_z\hat{k} + r_z\hat{k}}_{-b\cdot r} + b_x\hat{i}(r_y\hat{j} + r_z\hat{k}) + b_y\hat{j}(r_x\hat{i} + r_z\hat{k})$$

$$+ b_z\hat{k}(r_x\hat{i} + r_y\hat{j})$$

$$b^0 r^0 + b^0 r + r^0 b - b\cdot r + b_x\hat{i}(r_y\hat{j} + r_z\hat{k}) + b_y\hat{j}(r_x\hat{i} + r_z\hat{k}) + b_z\hat{k}(r_x\hat{i} + r_y\hat{j})$$

$$\underbrace{+ b_x r_y\hat{k} - b_x r_z\hat{j} - b_y r_x\hat{k} + b_y r_z\hat{i} + b_z r_x\hat{j} - b_z r_y\hat{i}}_{b\times r}$$

$$b^0 r^0 + b^0 r + r^0 b - b\cdot r + b\times r$$

$$(S^{\alpha j}\delta^0_\alpha + S^{j\alpha}\delta^0_\alpha)|q_j\rangle$$

$$(A^{jki} b_j r_k - A^{kji} b_j r_k)|q_i\rangle$$

c) $|b\rangle = b^\alpha|q_\alpha\rangle$ y $r^\alpha|q_\alpha\rangle$ si $|d\rangle = |b\rangle \odot |r\rangle$

$$|d\rangle = |b\rangle \odot |r\rangle = a|q_0\rangle + S^{(\alpha j)}\delta^0_\alpha|q_j\rangle + A^{[jk]i} b_j r_k |q_i\rangle$$

$$(b^0 + b^i|q_i\rangle) \odot (r^0 + r^j|q_j\rangle)$$

$$b^0 r^0 + b^0 r^j|q_j\rangle + r^0|b^i|q_i\rangle + b^i|q_i\rangle \odot r^j|q_j\rangle$$

$b^0 r^0 + b^0 r^j |q_j\rangle + r^0 |b^i |q_i\rangle + b^i r^j \left[|q_i\rangle |q_j\rangle \right]$ $\Big\{$ genera $-\delta^i_j$ Por que $i=j \to -1$; ε^{kij} un vector cíclico $+$, Anti. $-$

$b^0 r^0 + b^0 r^j |q_j\rangle + r^0\, b^i |q_i\rangle + b^i r^j \left(-\delta^i_j + \varepsilon^{kij} |q_k\rangle \right)$

$b^0 r^0 + b^0 r^j |q_j\rangle + r^0 b^i |q_i\rangle - b^i r^j \delta^i_j + \varepsilon^{kij} b^i r^j |q_k\rangle$

$\underbrace{b^0 r^0 - b^i r^i}_{\alpha} + b^0 r^i |q_i\rangle + r^0 b^i |q_i\rangle + \varepsilon^{kij} b^i r^j |q_k\rangle$

$\alpha |q_0\rangle + \underbrace{b^\alpha r^i \delta^0_\alpha + r^\alpha b^i \delta^0_\alpha}_{S^{\alpha j}\delta^0_\alpha + S^{j\alpha}\delta^0_\alpha} |q_i\rangle + \underbrace{\varepsilon^{kij} b^i r^j |q_k\rangle}$

$\varepsilon^{kij} = -\varepsilon^{kji} \to \left(A^{jki} b_j r_k - A^{kji} b_j r_k \right)$

$\alpha |q_0\rangle + S^{\alpha j}\delta^0_\alpha + S^{j\alpha}\delta^0_\alpha + \left(A^{jki} b_j r_k - A^{kji} b_j r_k \right) |q_k\rangle$

$\alpha |q_0\rangle + S^{(\alpha j)} \delta^0_\alpha |q_j\rangle + A^{[jk]i} b_j r_k |q_k\rangle$

d) $\alpha \longrightarrow$ Parte escalar $\longrightarrow b^0 r^0 - b^i r^i$
Parte simetrica $S^{(ij)} \longrightarrow b^0 r^i + b^i r^0 |q\rangle \longrightarrow$ parte vectorial
Parte Antisimetrica $A^{[jki]} \longrightarrow \varepsilon^{kij} b^i r^j |q_k\rangle \longrightarrow$ parte pseudovectorial

No es ninguno de los dos ni vector ni pseudovector sino una mezcla de todos los anteriores

e) $\left\{ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ base para Cuaterniones $|b\rangle = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}$

$z = x + yi \quad w = a + bi$

$$\alpha\, \sigma_0 + \beta\, \sigma_1 + \gamma\, \sigma_2 + \mu\, \sigma_3 = 0$$

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left.\begin{aligned} \alpha + \mu &= 0 \\ \beta - i\gamma &= 0 \\ \beta + \gamma &= 0 \\ \alpha - \mu &= 0 \end{aligned}\right\} \quad \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -i & 0 & 0 \\ 0 & 1 & i & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \xrightarrow[-F_2+F_3]{} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \to \alpha = \beta = \gamma = \mu = 0$$

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} x + yi & a + ib \\ -a + bi & x - iy \end{pmatrix}$$

$$\left.\begin{aligned} \alpha + \mu &= x + iy \\ \alpha - \mu &= x - iy \end{aligned}\right\} \qquad \begin{aligned} \alpha + \mu &= x + iy \\ \alpha - \mu &= x - iy \\ \hline 2\alpha &= 2x \\ \alpha = x &\longrightarrow \mu = iy \end{aligned}$$

$$\begin{aligned} \beta - i\gamma &= a + ib \\ \beta + i\gamma &= -a + ib \end{aligned} \qquad \begin{aligned} \beta - i\gamma &= a + ib \\ \beta + i\gamma &= -a + ib \\ \hline 2\beta \quad 0 &= 2ib \\ \beta &= ib \end{aligned} \qquad \begin{aligned} i\gamma &= -a - ib + ib \\ \gamma &= \frac{-a}{i} \end{aligned}$$

Por lo tanto las matrices de Pauli si generan un Cuaternion.

f) $I\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$; $|q_1\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ $|q_2\rangle = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$; $|q_3\rangle = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

Comprobacion por regla de multiplicación

$$|q_1\rangle \cdot |q_1\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -I$$

$$|q_2\rangle \cdot |q_2\rangle = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -I$$

$$|q_3\rangle \cdot |q_3\rangle = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -I$$

$$|q_1\rangle \cdot |q_2\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = |q_3\rangle$$

$$|q_1\rangle \cdot q_3\rangle \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = -|q_2\rangle$$

$$|q_2\rangle \cdot |q_3\rangle = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = |q_1\rangle$$

Al ser matrices $n \times n$ $|q_2\rangle \cdot |q_1\rangle = -|q_3\rangle$; $|q_3\rangle \cdot |q_1\rangle = |q_2\rangle$; $|q_3\rangle \cdot |q_2\rangle = -|q_1\rangle$

G) $\langle a \tilde{|} b\rangle = |a\rangle^* \odot |b\rangle$

Comprobar dos propiedades $\langle v|v\rangle \overset{?}{=} ||v|| \geq 0 \;\wedge\; \langle v|w\rangle \overset{?}{=} \langle w|v\rangle$ y linealidad

$\langle au+v|w\rangle \overset{?}{=} a\langle u|w\rangle + \langle v|w\rangle$

$a u^0 + a u^j|q_j\rangle + v^0 + v^j|q_j\rangle \odot w^0 + w^k|q_k\rangle \rightarrow a u^0 + v^0 - (a u^i + v^i)|q_i\rangle \cdot w^0 + w^k q_k\rangle$

$(a u^0 + v^0) w^0)|q_0\rangle + (a u^0 + v^0) w^k|q_k\rangle - w^0(a u^i + v^i)|q_j\rangle + [(-a u^i + v^i) w^k\; |q_i\rangle|q_k\rangle]$

$(a u^0 + v^0) w^0\; |q_0\rangle + (a u^0 + v^0) w^k|q_k\rangle - w^0(a u^i + v^i)|q_i\rangle + [(-a u^i + v^i) w^k\; (-\delta^i_k + \varepsilon^{jik}|q_j\rangle)$

$[a(u^0 w^0) + v^0 w^0] q_0\rangle + a(u^0 w^k) + v^0 w^k|q_k\rangle(-w^0 a u^i - w^0 v^i)|q_i\rangle + [-a u^i w^k(-\delta^i_k + \varepsilon^{ijk}|q_j\rangle) + v^i w^k(-\delta^i_k + \varepsilon^{ijk}|q_j\rangle)$

$$\underbrace{a(u^0 w^0) + a(u^0 w^k)(-w^0 a u^i)|q_i\rangle + [-a u^i w^k(-\delta^i_k + \varepsilon^{ijk}|q_j\rangle)]}_{a\langle u|w\rangle}$$

$$\underbrace{(u^0 w^0) + (v^0 w^k)(-w^0 v^i)|q_i\rangle + [-v^i w^k(-\delta^i_k + \varepsilon^{ijk}|q_j\rangle)]}_{\langle v|w\rangle}$$

$\Big\}\; \langle au+v|w\rangle = a\langle u|w\rangle + \langle v|w\rangle$

$\langle v|v\rangle \overset{?}{=} ||v||$

$v^0 - v^i|q_i\rangle \odot v^0 + v^i|q_i\rangle = v^0 v^0 - v^i|q_i\rangle v^i|q_i\rangle = (v^0)^2 - (v^i)^2 \overset{-1}{\cancel{|q_i\rangle|q_i\rangle}}$

$(v^0)^2 + (v^i)^2 = \alpha$ — siempre positivo por suma de cuadrados.

$\langle v|w\rangle \overset{?}{=} \langle w|v\rangle^*$

$\langle v|w\rangle = (v_0 - v^i|q_i\rangle) \odot (|w_0 + w_j\; |q_j\rangle) = v^0 w^0 + v^0 w^j|q_j\rangle - w^0 v^i|q_i\rangle + v^i w^i$

$\langle w|v\rangle^* = (w^0 - w^i|q_i\rangle) \odot (|v_0 + v^j\; |q_j\rangle) = v^0 w^0 + w^0 v^j|q_j\rangle - v^0 w^j|q_j\rangle + v^i w^i$

$v^0 w^0 - w^0 v^j|q_j + v^0 w^j|q_j\rangle + v^i w^i$

H) $\langle a|b\rangle = \frac{1}{2}[\langle a\tilde{|}b\rangle - |q_1\rangle \odot \langle a\tilde{|}b\rangle \odot |q_1\rangle]$

$\sqrt{\langle v|v\rangle} = \frac{1}{2}[\alpha - |q_1\rangle \odot \alpha\;\; |q_1\rangle] = \frac{1}{2}[\alpha - \alpha|q_1\rangle|q_1\rangle] = \frac{1}{2}[\alpha + \alpha] = \alpha$

$\langle au+v|w\rangle \overset{?}{=} a\langle u|w\rangle + \langle v|w\rangle$

$\frac{1}{2}[\widetilde{\langle au+v|w\rangle} - |q_1\rangle \odot \langle au+v|w\rangle \odot |q_1\rangle] = \frac{1}{2}[\alpha\langle u\tilde{|}w\rangle + \langle v\tilde{|}w\rangle - |q_1\rangle \odot (\alpha\langle u\tilde{|}w\rangle + \langle v\tilde{|}w\rangle)|q_1\rangle

$$\frac{1}{2}\left[\bar{a}\langle\widetilde{w|w}\rangle + \langle\widetilde{v|w}\rangle - a|q_1\rangle \odot \langle\widetilde{w|w}\rangle \odot |q_1\rangle - |q_1\rangle \odot \langle\widetilde{b|c}\rangle \odot q_1\rangle\right.$$

$$\frac{1}{2} a\left[\langle\widetilde{w|v}\rangle - |q_1\rangle \odot \langle\widetilde{w|w}\rangle \odot |q_1\rangle\right] + \frac{1}{2}\left[\langle\widetilde{v|w}\rangle - |q_1\rangle \odot \langle\widetilde{v|w}\rangle \odot |q_1\rangle\right.$$

$$= a\langle w|w\rangle + \langle v|w\rangle$$

$$\langle v|w\rangle \overset{?}{=} \langle w|v\rangle^*$$

$$\langle v|w\rangle = (a^0 b^0 + a^i b^i) + (a_0 b_1 - b_0 a_1 |q_1\rangle$$

$$\langle w|v\rangle = (b^0 a^0 + b^i a^i) + (b^0 a_1 - a_0 b_1 |q_1\rangle)$$

$$\langle\widetilde{w|v}\rangle = \langle a^0 b^0 + a^i b^i) + (a^0 b^1 - b^0 a^1 |q_1\rangle)$$

I) $n(|b\rangle) = \||a\rangle\| = \sqrt{\langle a|a\rangle} = \sqrt{|a\rangle^* \odot |a\rangle}$

$$\langle a|a\rangle = a_0 - a^i|q_i\rangle \odot |a_0 + a_i|q_i\rangle \rightarrow (a^0)^2 - a^i|q_i\rangle(a^0) + a^0 a^i|q_i\rangle - a_i|q_i\rangle a^i|q_i\rangle$$

$$= (a^0)^2 - a^0 a^i|q_i\rangle + a_0 a^i|q_i\rangle - (a^i)^2$$

$$= (a^0)^2 - a^{i^2}(-1) = (a^0)^2 + (a^i)^2$$

$$\||a|v^i\rangle\| = |a|\,\||v_i\rangle\|$$

$$= \||a v^i\rangle\|^2 = \|a(b^0 + b^i|q_i\rangle = (a b^0 + b^i|q_i\rangle)^* \odot (a b^0 + a b^i|q_i\rangle$$

$$= \alpha^2(b^0)^2 + \alpha^2 b_0 b^i|q_i\rangle - a^2 b^i|q_i\rangle \odot b^i|q_i\rangle$$

$$= a^2(b^0)^2 - a^2(b^i)^2 = \alpha^2\left[(b^0)^2 - (b^i)^2\right)$$

$$\||v_i\rangle + |v_j\rangle\| \leq \||v_j\rangle\| + \||v_j\rangle|$$

$$\||v_i\rangle + |v_j\rangle\|^2 = \|(a^0 + b^0) + (a^i + b^i|q_i\rangle)\| = (a^0 + b^0) - (a^i + b^i|q_i\rangle) \odot (a^0 + b^0) + (a^i + b^i)|q_i\rangle$$

$$\Rightarrow (a^0 + b^0)^2 + \cancel{(a^0 + b^0)(a^i + b^i)|q_i\rangle} - \cancel{(a^0 + b^0)(a^i + b^i)|q_i\rangle} - (a^i + b^i)|q_i\rangle \odot (a^i + b^i)|q_i\rangle$$

$$= (a^0 + b^0)^2 + (a^i + b^i) \Rightarrow (a^0)^2 + (b^0)^2 + 2a^0 b^0 + (a^i)^2 + (b^i)^2 + 2b^i a^i = \||v_i\rangle\|^2 + \|v_j\|^2 + 2\langle v_i|v_j\rangle$$

J) $<v|v> \overset{?}{=} ||v||$

$$v^0 - v^i|q_i\rangle \odot v^0 + v^i|q_i\rangle = v^0 v^0 - v^i|q_i\rangle v^i|q_i\rangle = (v^0)^2 - (v^i)^2 \cancel{|q_i\rangle|q_i\rangle}^{-1}$$

$$(v^0)^2 + (v^i)^2 = V^2$$

$|\bar{a}\rangle \odot |a\rangle = |a\rangle \odot |\bar{a}\rangle = (\Delta)$ elemento neutro

$$|\bar{a}\rangle = \frac{|a\rangle^*}{|||a\rangle||} = \left(\frac{a^0 - a^i|q_i\rangle}{(a)^2}\right)$$

$$|\bar{a}\rangle \odot |a\rangle = \left(\frac{a_0 - a_i|q_i\rangle}{(a)^2}\right) \odot (a^0 + a^i|q_i\rangle = \frac{a^0}{(a)^2} - \frac{a^i|q_i\rangle}{(a)^2} \odot a_0 + a^i|q_i\rangle$$

$$= \frac{(a_0)^2}{(a)^2} + \cancel{\frac{a_0}{(a)^2} a_i|q_i\rangle} - \cancel{\frac{a_i|q_i\rangle}{(a)^2}(a^0)} - \frac{a^i|q_i\rangle}{(a)^2}(q^i|q_i\rangle) = \frac{(a^0)^2}{(a)^2} - \frac{(a^i)^2}{(a)^2}(-1) = \frac{(a^0)^2}{(a)^2} + \frac{(a^i)^2}{(a)^2} = 1$$

Si es el inverso

K) $(a^0 + a^i|b_i\rangle) \odot (b^0 + b^i|q^i\rangle = a^0 b^0 + a^0 b^i|q_i\rangle + b^0 a^i|q_i\rangle + a^i b^j|q_j\rangle + a_j|q_i\rangle b^j|q_j\rangle$

$$= (a^0 b^0 - \vec{a}\cdot\vec{b},\ b_0\vec{a} + a^0\vec{b} + \vec{a}\times\vec{b}$$

$$(|a\rangle \odot |b\rangle) \odot |c\rangle = (a^0 b^0 - \vec{a}\cdot\vec{b},\ b^0\vec{a} + a^0\vec{b} + \vec{a}\times\vec{b}) \odot (c^0 + \vec{c})$$

$$(a^0 b^0 c^0 - c^0(\vec{a}\cdot\vec{b}) - b^0(\vec{a}\cdot\vec{c}) - a^0(\vec{b}\cdot\vec{c}) - (\vec{a}\times\vec{b})\cdot\vec{c},\ a^0 b^0\cdot\vec{c} - (\vec{a}\cdot\vec{b}) + c^0 b^0\vec{a} + c^0 a^0\vec{b} + c^0(\vec{a}\times\vec{b}) +$$

$$+ b^0(\vec{a}\times\vec{c}) + a^0(\vec{b}\times\vec{c}) + (\vec{a}\times\vec{b})\times\vec{c}$$

$$|a\rangle \odot (|b\rangle \cdot |c\rangle) = |a\rangle \odot (b^0 c^0 - \vec{c},\ b^0\vec{c} + c^0\vec{b} + \vec{b}\times\vec{c})$$

$$= (a^0 b^0 c^0 - a^0(\vec{b}\cdot\vec{c}) - b^0(\vec{a}\cdot\vec{b}) - c^0(\vec{a}\cdot\vec{b}) - \vec{a}\cdot(\vec{b}\cdot\vec{c}),\ a^0 b^0\vec{c} + a^0 c^0\vec{b} + a_0(b\times c) + b^0 c^0\vec{a} - (\vec{b}\cdot\vec{c})\vec{a}$$
$$+ b^0(\vec{a}\times\vec{c}) + c^0(\vec{a}\times\vec{b}) + \vec{a}\times(\vec{b}\times\vec{c})$$

Si es asociativa

$$1 \odot |a\rangle = a_0(1), (1)(\vec{a}) = (a^0, \vec{a})$$
$$|a\rangle \odot 1 = (a_0(1), (1)\vec{a}) = (a^0, \vec{a})$$

No hay grupo porque no hay elemento inverso que haga zero entre cuaterniones

TABLA

$\|v\rangle \odot \|w\rangle$	1	-1	i	$-i$	$\hat{j}$	$-\hat{j}$	$\hat{k}$	$-\hat{k}$
1	1	-1	$\hat{i}$	$-\hat{i}$	$\hat{j}$	$-\hat{j}$	k	$-\hat{k}$
-1	-1	1	$-\hat{i}$	$\hat{i}$	$-\hat{j}$	$\hat{j}$	$-k$	$\hat{k}$
i	$\hat{i}$	$-\hat{i}$	-1	1	$\hat{k}$	$-\hat{k}$	$-\hat{j}$	$\hat{j}$
$-i$	$-\hat{i}$	$\hat{i}$	1	-1	$-\hat{k}$	k	$\hat{j}$	$-\hat{j}$
j	$\hat{j}$	$-\hat{j}$	$-\hat{k}$	k	-1	1	$\hat{i}$	$\hat{i}$
$-j$	$-\hat{j}$	$\hat{j}$	$\hat{k}$	$-k$	1	-1	$-\hat{j}$	$\hat{j}$
k	$\hat{k}$	$-\hat{k}$	$\hat{j}$	$-\hat{j}$	$-\hat{i}$	$\hat{i}$	-1	1
$-k$	$-k$	k	$-\hat{j}$	$\hat{j}$	i	$-i$	1	-1

4) $v^0 = 0 \rightarrow |v\rangle = v^j |q_j\rangle$

$|v'\rangle = \widetilde{|a\rangle} \odot |v\rangle \odot |a\rangle$

$\||v'\rangle\|^2 = (v^{1'})^2 + (v^{2'})^2 + (v^{3'})^3 = (v^1)^2 + (v^2)^2 + (V^3)^3$

$\| |v\rangle \odot |q\rangle \|^2 = \| v\rangle \odot |q\rangle)^* \odot (|v\rangle \odot |q\rangle)$

$= (|q\rangle^* \odot |v\rangle \odot (|v\rangle \odot |q\rangle$

$= |q\rangle^* \odot \||v\rangle\|^2 \odot |q\rangle$

$= \|v\|^2 \odot \||q\rangle\|^2$

$|v\rangle \odot |q\rangle$

$(v^0 q^0 - \vec{p} \cdot \vec{v},\ p^0 \vec{v} + {}_0\vec{p} + \vec{v} \times \vec{q})$

$| v^0 q^0 - \vec{v} \cdot \vec{q},\ - v^0 \vec{q},\ q^0 (-\vec{v}) + p^0 (\vec{q})$

$+ (-\vec{q}) \times (\vec{v})$

$= v^0 q^0 - \vec{q} \cdot \vec{v},\ q^0(\vec{V}) - p^0 \vec{q}$

$\| |v'\rangle \| = \left\| \dfrac{|a\rangle^*}{\|a\|^2} \odot |v\rangle \odot |a\rangle \right\|$

$\left\| \dfrac{|a\rangle^*}{\|a\|^2} \right\| \|v\| \, \| |a\rangle \| = \|v\| \dfrac{\|a\|}{\|a\|^2} = \|v\rangle\|$

$\|v'\| = \||v\rangle\|$