Homework 2

Due March 25, 19:00 Cristian Brinza FAF-212

Problem 2.1

The **error function** (also called Gauss error function) is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Use Taylor series to approximate erf(x) with a polynomial $T_n(x)$. What is n, if the desired accuracy is 10^{-5}?Using this approximation, plot the graph of erf(x) on $[-3, 3]$.

Maclaurin series: $\sum_{n=0}^{\infty} \frac{f(n)(a)}{n!}(x-a)^n \rightarrow : e^{-t} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = 1 + \left(\frac{-t^2}{1}\right) + \left(\frac{-t^4}{2}\right) + \left(\frac{-t^6}{6}\right)$

Integration (term by term) : $\int_0^x e\, dt = x; \int_0^x -t^2\, dt = -\frac{x^3}{3}; \frac{1}{2}\int_0^x -t^4\, dt = \frac{x^5}{10}; \frac{1}{6}\int_0^x -t^6\, dt = \frac{-x^7}{42}$

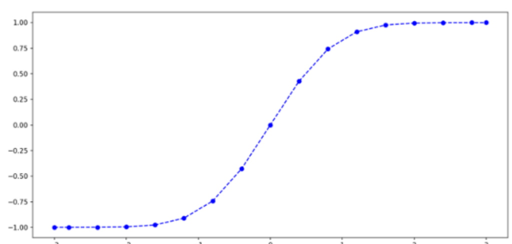
As we obtain : $\int_0^x e^{-t^2}\, dt = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$ **which in series form look like :** $\operatorname{erf}(x) = T_n(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{n!(2n-1)}$

Calculating (finding a number $\underline{n}$, such that the approximation accuracy will be 10^{-5} (finding first term))

Looking at the interval [-3;3], we can choose the biggest value -> x=3, as for results,
n = 28 - 0.0001…; n = 29 - -3.05 10^{-5} ; n = 30 - 8.86 10^{-6} -> n= 29 corresponds to our needs, - so our n will become 29-1=28 (the next will become reminder)

Taylor Polynomial: $T_n = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{n!(2n-1)}$ **– so the polynomial is**

$$\boldsymbol{T_{28} = l^x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots - \frac{x^{55}}{27! \cdot 55} + \frac{x^{57}}{28! \cdot 57}}$$



Problem 2.2

Consider the sequence of Fibonacci numbers:

$$F_0 = 1, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n = 2, 3, \ldots$$

Let $R_n = \frac{F_{n+1}}{F_n}$. It can be shown that

$$\lim_{n\to\infty} R_n = \frac{1+\sqrt{5}}{2} \equiv \phi,$$

which is known as **golden ratio**. Write a code that will compute numerically the first 40 terms of the sequence R_n together with errors $\varphi\ R_n$. In computations make sure that you are using IEEE double precision. Comment your results. What can be said on the order of convergence?

R1: 2.0	Error: 0.3819660112501051
R2: 1.5	Error: 0.1180339887498949
…	
R38: 1.6180339887498947	Error: 2.220446049250313e-16
R39: 1.618033988749895	Error: 0.0
R40: 1.618033988749895	Error: 0.0

(observe that this approximation converges φ so, by using formula we can deal with *Linear Order of Convergence*)

Problem 2.3

Thermistors are temperature-measuring devices based on the principle that the thermistor material exhibits a change in electrical resistance with a change in temperature. By measuring the resistance of the thermistor material, one can then determine the temperature. For a 10K3A Betatherm thermistor, the relationship between

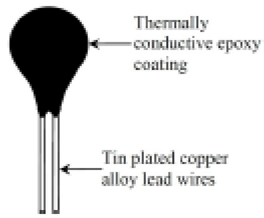


Figure 1: A typical thermistor

the resistance R of the thermistor and the temperature T is given by

$$\frac{1}{T} = 1.129241 \times 10^{-3} + 2.341077 \times 10^{-4} \log R + 8.775468 \times *10^{-8} \left(\log R \right)^3$$

where T is in Kelvin and R is in Ohms, and log denotes the natural logarithm. A thermistor error of no more than $\pm 0.01^o C$ is acceptable. To find the range of the resistance that is within this acceptable limit at $19^o C$, we need to solve

$$\frac{1}{19.01 + 273.15} = 1.129241 \times 10^{-3} + 2.341077 \times 10^{-4} \log R + 8.775468 \times *10^{-8} \left(\log R \right)^3 \qquad (1)$$

$$\frac{1}{18.99 + 273.15} = 1.129241 \times 10^{-3} + 2.341077 \times 10^{-4} \log R + 8.775468 \times *10^{-8} \left(\log R \right)^3 \qquad (2)$$

Write a computer routine implementing Newton's method and solve equations (1) and (2) using Newton's method with initial guess R_0 = 15000 and error tolerance of 10^{-5}.
What is the obtained range for resistance values?

Newton's method

$$(1)\ 1.129241 \times 10^{-3} + 2.341077 \times\ 10^{-4} \log \mathrm{R} + 8.775468 \times 10^{-8} \times (\log \mathrm{R})^3 - \frac{1}{19.01\ +\ 273.15} = 0$$

Output

R_1: 12932.85373451167

R_2: 13065.867732409068

R_3: 13066.540718572338

R_4: 13066.540735568875

R_5: 13066.540735568875

$$(2)\ 1.129241 \times 10^{-3} + 2.341077 \times\ 10^{-4} \log \mathrm{R} + 8.775468 \times 10^{-8} \times (\log \mathrm{R})^3 - \frac{1}{18.99\ +\ 273.15} = 0$$

Output

R_1: 12946.457909523315

R_2: 13077.773356259226

R_3: 13078.428544712671

R_4: 13078.42856080743

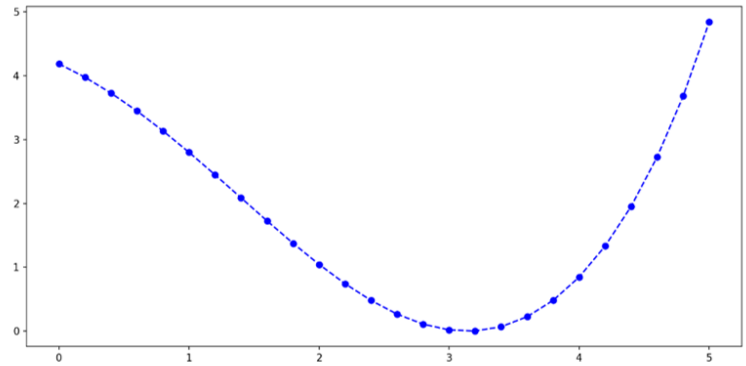
R_5: 13078.428560807408

(So the obtained range for resistance values is *13066.540735568875 – 13078.428560807408)*

Problem 2.4

Consider the function $f(x) = e^{x-\pi} + \cos x - x + \pi$.

a) Plot its graph on interval $[0, 5]$.



b) Apply Newton's method routine you developed in **Problem 2.3** to solve equation $f(x) = 0$ on interval $[0, 5]$. What can be said about its order of convergence? Argue why this is happening? How its convergence order can be improved?

R1 :2.6570843953178036 Error: 0.4845082582719895

R2 :2.9108040796341474 Error: 0.23078857395564567

R3 :3.028571074564518 Error: 0.1130215790252751

…

R19 :3.1415909617802273 Error: 1.6918095657736387e-06

R20 :3.1415918076662614 Error: 8.459235316671254e-07 **-20 iterations**

As in the *Problem 2.4 a)* graph, the root is located In the interval [2;4], as initial value (R0) was chosen min variable of the interval -> R_0=2

c) Write a modified routine that will ensure **quadratic** convergence and apply it.+

d) Instead of solving $f(x) = 0$, try to apply the fixed point iterations $x_{n+1} = e^{x_n-\pi} + \cos x_n + \pi$. Comment on your results.

R1 :3.0447558833494806 Error: 0.09683677024031256

R2 :3.0539818291901977 Error: 0.08761082439959544

R3 :3.061545365543905 Error: 0.08004728804588801

…

R39 :3.121458781819971 Error: 0.020133871769822065

R40 :3.1218627942977903 Error: 0.01972985929200277 **-40 iterations**

R_0=2, the results were obtained using *fixed point iterations* and *Newton's method*

Problem 2.5

a) Compute the fixed point $x_{n+1} = \cos x_n - 1 + x_n$ with initial guess $x_0 = 0.1$.

R1: 0.09500416527802583
R2: 0.09049466291815525
R3: 0.08640281449600365
…
R903: 0.002157858950687902
R904: 0.0021555307739658247

b) What can be said about the speed of convergence? Compare it with bisection method.

It is extremely slowly; the results obtained using fixed point iterations are converging 0, the real solution is R = 0. Fixed point iteration is not always faster than bisection. But in this specific example/ problem it is (a lot) as in bisection method, the error is divided by 2 with each iteration -> converge much faster

c) Write a modified computer routine that will speed up the convergence.

R2: -1.0 <- **initial guess is equal to -1, as numbers approach 0**
R3: 0.11208576142904647
R4: 0.12747620839147253
…
R21: 2.8923297643062033e-05 **to speed up the convergence Secant method was used**
R22: 1.787558245118042e-05 **(it converges much faster, even it is not a quadratic convergence)**

Problem 2.6

Newton's method is used to find the root α of $f(x) = 0$. The first 10 iterates are shown in the table below.

(1) What can be said about the order of convergence? Is it slower or faster than bisection method?
(2) What can be said about the root α to explain this convergence?
(3) Knowing function $f(x)$, how would you speed up the convergence?

n	x_n	$x_n - x_{n-1}$
0	2.0	
1	2.1248	0.124834
2	2.2148	0.089944
3	2.2805	0.065698
4	2.3289	0.048386
5	2.3647	0.035827
6	2.3913	0.026624
7	2.4111	0.019835
8	2.4260	0.014803
9	2.4370	0.011062
10	2.4453	0.0082745

1) $Let\ \varepsilon_n = \alpha - x_n$

$$\frac{\varepsilon_n}{\varepsilon_{n-1}} = \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}} - ratio\ convergence - for\ n = 2 \rightarrow \frac{0.089944}{0.124834} \approx 0;72 > 0.5$$

2)the root multiplicity is higher than 1-> $\frac{\lambda-1}{\lambda} \approx 0 - 72 \rightarrow \lambda \approx 4$

3) In order to speed – we take the third derivate of Newton Method

Problem 2.7

For solving the equation $x + \ln x = 0$, there were proposed three methods:

$$\begin{aligned}(a)\quad x &= -\ln x\\ (b)\quad x &= e^{-x}\\ (c)\quad x &= \frac{x+e^{-x}}{2}\end{aligned}$$

Formulas were tested with fixed point iteration alghoritm:

(1) Which of the formulas can be used?

(A) and (C) as (B) is not usable

(2) Which of the formulas should be used?

(C) as it is the fastest and the most efficient

(3) Give an even better formula!

Problem 2.8

Consider the following table of iterates from an iteration method which is convergent to a fixed point α of the function $g(x)$:

n	x_n	$x_n - x_{n-1}$
0	1.00	
1	0.36788	$-6.3212E-01$
2	0.69220	$3.2432E-01$
3	0.50047	$-1.9173E-01$
4	0.60624	$1.0577E-01$
5	0.54540	$-6.0848E-02$
6	0.57961	$3.4217E-02$

$$p = \frac{\left(\frac{0.0656698}{0.089944}\right)}{\left(\frac{0.089944}{0.124834}\right)} \approx 0.958$$

(1) Show that this is a linearly convergent iteration method.

$$M_2 = \frac{x_2 - x_1}{x_1 - x_0} \approx 0{,}72 \ \rightarrow M_{10} = \frac{x_{10} - x_9}{x_9 - x_8} \approx 0{,}748 \rightarrow M = \frac{\frac{M_2 + M_{10}}{2}}{x_1 - x_0} \approx 0{,}734 \rightarrow$$

(2) Find its rate of linear convergence. Is this method faster or slower than bisection method?

Therefore the rate of linear convergence is approximately 0.734

(3) Propose a way to accelerate the convergence of this method?

One way to accelerate the convergence of this method is to use Aitken extrapolation