Homework 3 Due October 22, 2021

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Problem 3.1

Five people get on the elevator that stops at five floors. In how many ways they can get off? For example, one person gets off at the first floor, two will get off at the third, and the remaining two at the fifth floor. In how many ways they can get off at different floors? Now, consider that people in elevator have names, say *A, B, C, D,* and *E*, assuming that, for example, the case *A* on the first floor is different from the case *B* on thefirst floor. Answer the previous questions with this assumption.

$$People\ don't\ have\ names\ and\ get\ off$$

$$\binom{9}{4}$$

$$People\ don't\ have\ names\ and\ get\ off\ at\ different\ floors$$

$$Since\ there\ are\ 5\ people\ and\ 5\ floors\ there\ is\ just\ 1\ way\ for\ them\ to\ get\ off\ at\ different\ floors: 1\ per\ floor$$

$$People\ have\ names\ and\ get\ off$$

$$5^5 = 3125$$

$$People\ have\ names\ and\ get\ off\ at\ different\ floors$$

$$At\ each\ floor\ exactly\ one\ of\ A, \dots, E\ gets\ off. This\ can\ be\ done\ in\ 5\cdot 4\cdot 3\cdot 2 = 5! = 120\ ways$$

Problem 3.2

A lady wishes to color her fingernails on one hand using at most two of the colors: red, yellow, and blue. Inhow many ways she can do it?

$$The\ number\ of\ ways\ to\ color\ her\ fingernails\ using\ one\ color\ is$$

$$\binom{3}{1} = 3$$

$$The\ number\ of\ ways\ to\ color\ her\ fingernails\ using\ two\ colors\ is$$

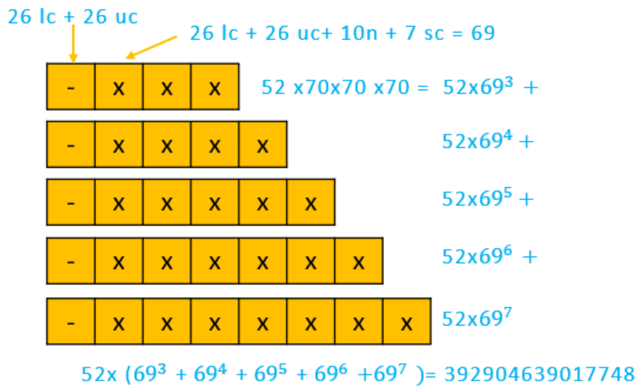
$$\binom{3}{2} \times (2^5 - 2) = 3(2^5 - 2) = 90$$

$$The\ total\ number\ of\ ways\ using\ at\ most\ two\ colors\ is$$

$$3 + 90 = 93$$

Problem 3.3

On a computer system the valid password should start with a letter (26 letters at all), it is case sensitive (upper case is different form lowercase), its length is from 4 up to 8, and symbols are {1,2,3,..,9,0,a,b,c,..z,A,B,C,..,Z,?,!,#,%,&,*,/}. How many such passwords can be created?



Problem 3.4

How many of the billion numbers in the range from 1 to 10^9 contain the digit 1?

$10^9 = 1000000000 = 1 + 999999999$

$\rightarrow 9^9$ *combinations not contain* 1, including 000000000, so $(10^9 - (9^9 - 1)) + 1 = 612579513$

Problem 3.5

Solve the following problems using the pigeonhole principle. For each problem, try to identify the pigeons, thepigeonholes, and a rule assigning each pigeon to a pigeonhole.

a) In every set of 100 integers, there exist two whose difference is a multiple of 37.

The pigeons are the 100 *integers. The pigeonholes are the numbers* 0 *to* 36. *Map integer* k *to* $rem(k, 37)$. *Since there are* 100 *pigeons and only* 37 *pigeonholes, two pigeons must go in the same pigeonhole.*

This means $rem(k1, 37) = rem(k2, 37,)$, *which implies that* $k1 - k2$ *is a multiple of* 37.

b) For any 5 points inside a unit square (not on the boundary), there are 2 points at distance less than $\frac{1}{2}$.

The pigeons are the points. The pigeonholes are the four subsquares of the unit square, each of side length 1/2. Pigeons are assigned to the subsquare that contains them, except that if the pigeon is on a boundary, it gets assigned to the leftmost and then lowest possible subsquare that includes it (so the point at (1/2, 1/2) is assigned to the lower left subsquare).

There are five pigeons and four pigeonholes, so more than one point must be in the same subsquare.

The diagonal of a subsquare is $1/\sqrt{2}$, *so two pigeons in the same hole are at most this distance.*

But pigeons must be inside the unit square, so two pigeons cannot be at the opposite ends of the same subsquare diagonal.

So at least one of them must be inside the subsquare, so their distance is less than the length of the diagonal.

c) Show that if $n + 1$ numbers are selected from $\{1, 2, 3, \ldots, 2n\}$, two must be consecutive, that is, equal to k and $k + 1$ for some k.

The pigeonholes will be the n *sets* $\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2n - 1, 2n\}$.

The pigeons will be the $n + 1$ *selected numbers.*

A pigeon is assigned to the unique pigeon hole of which it is a member. By the Pigeonhole Principle, two pigeons must assigned to some hole, and these are the two consecutive numbers required. Notice that we've actually shown a bit more: there will be two consecutive numbers with the smaller being odd.

Problem 3.6

Here are the solutions to the next 8 questions, in no particular order.

$$n^m, \quad m^n, \quad \frac{n!}{(n-m)!}, \quad \binom{n+m}{m}, \quad \binom{n-1+m}{m}, \quad \binom{n-1+m}{n}, \quad 2^{mn}$$

1. 2. 3. 4. 5. 6. 7.

Answer all the questions. Justify your answers.

(a) How many solutions over the natural numbers are there to the inequality

$$x_1 + x_2 + \ldots + x_n \leq m \ ?$$

$$\binom{n+m}{m}$$

This is the same as the number of solutions to the equation the equality $x1 + x2 + \cdots + xn + y = m$,

and which has a bijection to sequences with m stars and n bars.

(b) How many length m words can be formed from an n letter alphabet, if no letter is used more thanonce?

$$\frac{n!}{(n-m)!}$$

There are n choices for the first letter, $n - 1$ *choices for the second letter,* $\ldots n - m + 1$ *choices for*
the mth letter, so by the Generalized Product rule,
the number of words is $n \cdot (n-1) \cdots (n-m+1)$.

(c) How many length m words can be formed from an n letter alphabet, if letters can be reused?

n^m *by the Product Rule.*

(d) In how many ways you can connect elements from set A to elements from set B when $|A| = m$ and
$|B| = n$? One element from A can be connected to as many elements from B as possible.

2^{mn}

The graph of a binary relations from A to B is a subset of $A \times B$. *There are on 2mn such subsets because* $|A \times B| = mn$. .

(e) How many injections are there from set A to set B, where $|A| = m$ and $|B| = n$, and $n > m$?

$$\frac{n!}{(n-m)!}$$

There is a bijection between the injections and the length m sequences of distinct elements of B.
By the Generalized Product rule, the number of such sequences is
$n \cdot (n-1) \cdots (n-m+1)$.

(f) How many ways are there to place a total of m **distinguishable** balls into n **distinguishable** urns,with some urns possibly empty or with several balls?

n^m

There is a bijection between a placement of the balls and length m sequence whose ith element is the urn where the ith ball is placed. So the number of placements is the same as the number of length m sequences of elements from a size $- n$ *set.*

(g) How many ways are there to place a total of m **indistinguishable** balls into n **distinguishable** urns,with some urns possibly empty or with several balls?

$$\binom{n-1+m}{m}$$

$$\textit{This is the same as the number of selections of m donuts with n possible flavors, which is the number of sequences with m stars and } n-1 \textit{ bars.}$$

(h) How many ways are there to put a total of m distinguishable balls into n distinguishable urns with at most one ball in each urn?

$$\frac{n!}{(n-m)!}$$

There is a bijection between a placement of balls and a length m sequence whose ith element is the urn containing the ith ball. So the number of ball placements is the same as number of length m sequences of distinct elements from a set of n elements..

Problem 3.7

Bonus Problem (for extra credit)

How many times will the code *B* in the following program be executed?

```
For k := 1 to n
     For j := 1 to k
         For i := 1 to j
               B(i,j,k)
         end for
     end for
end for
```

As the program will combine 3 numbers from an logic array (1,2,3,4, … , n), the program will do an algebric sequence time of executions:

$$a_n = \frac{1}{6}n(n+1)(n+2) - (closed\ from)$$

$$If\ n = 0 \to 0\ iterations$$

$$If\ n = 1 \to 1\ itteration\ and\ B\ will\ recive\ values$$

$$k = 1, j = 1, i = 1$$

$$If\ n = 2 \to 4\ itterations\ and\ B\ will\ recive\ values$$

$$k = 1, j = 1, i = 1$$
$$k = 1, j = 1, i = 2$$
$$k = 1, j = 2, i = 2$$
$$k = 2, j = 2, i = 2$$

$$If\ n = 3 \to 10\ itterations\ and\ B\ will\ recive\ values$$

$$k = 1, j = 1, i = 1$$
$$k = 1, j = 1, i = 2$$
$$k = 1, j = 2, i = 2$$
$$k = 2, j = 2, i = 2$$
$$k = 1, j = 1, i = 3$$
$$k = 1, j = 2, i = 3$$
$$k = 2, j = 2, i = 3$$
$$k = 1, j = 3\ i = 3$$
$$k = 2, j = 3\ i = 3$$
$$k = 3, j = 3\ i = 3$$

$$If\ n = 4 \to 20\ itterations\ ,\ If\ n = 5 \to 35\ itterations,\ \ If\ n = 6 \to 56\ itterations$$

0,1,4,10,20,35,56,84,120,265,220,286,364,455,560,680,826, ….