Analysis of Nonlocal Games, Strategies, and Near-Optimal Bell Inequality Violations

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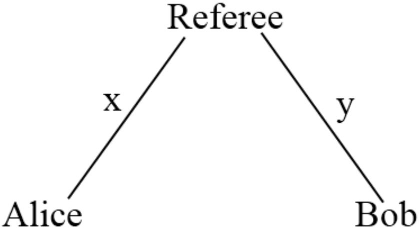
Introduction

- What is covered:
  - A summary of non-local games.
  - An analysis of the paper "Near-Optimal and Explicit Bell Inequality Violations" [1].
    - Hidden Matching Game [2]
    - Khot-Vishnoi Game [3]
- Goals and Motivations:
  - Present basic notion of nonlocal games (with examples).
  - Show classical and quantum protocols for Hidden Matching and Non-Local Hidden Matching games.
  - Present bounds for Hidden Matching and Khot-Vishnoi games.
  - Consider the violation achieved by the classical and quantum values for the Khot-Vishnoi game.

Outline

Nonlocal Games: Basic Premise

- Comprised of two or more players.
- Referee sends (x,y) from joint probability distribution π.
- Alice and Bob respond to these questions with some $a \in A$ and some $b \in B$.
- Some predicate V specified winning criteria.



Basic non-local game setup.

Nonlocal Games: Components

- Classical Strategy
  - Deterministic functions $A(x), B(y)$.
- Classical Value
  - $\omega(G)$: Maximum winning probability over all classical strategies.
- Quantum Strategy
  - $\{A_x^a\}\{B_y^b\}$: Perform measurement on their part of the system yielding outcome a or b.
- Quantum Value
  - $\omega^*(G)$: Supremum of expected winning probability taken over all quantum strategies.

Classical Versus Quantum Nonlocal Games

Classical Nonlocal Games

- $\omega(G)$
- A(x), B(y) (Deterministic)
- No entanglement

Quantum Nonlocal Games

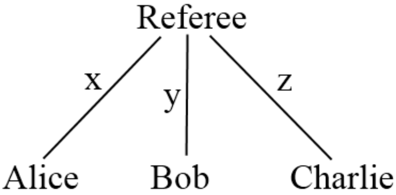
- $\omega^*(G)$
- $\{A_x^a\}, \{B_y^b\}$
- Entanglement

Bell Inequalities

- A Bell inequality is an upper bound on $\omega(G)$.
- A violation of a Bell inequality is whenever $\omega^*(G) > \omega(G)$.
- Quantified by the ratio $\frac{\omega^*(G)}{\omega(G)}$.
- CHSH game achieves a ratio of $\approx \frac{0.85}{0.75}$
  - Hence, the quantum strategy outperforms the classical one.

GHZ Game

- Related to the GHZ state (entangled state consisting of three sub-systems).
- Referee selects a 3-bit string $xyz \in \{000, 011, 101, 110\}$
- Predicate :
  - Win: $a \oplus b \oplus c = x \vee y \vee z$
  - Lose: otherwise



GHZ Game: Classical Strategy

- Maximum classical probability:
  - Consider a deterministic strategy for functions a_x, b_y and c_z.
  - Example: $a_0 = 1$ means Alice answers question 0 with 1.
- Winning Conditions:
  - $a_0 \oplus b_0 \oplus c_0 = 0$
  - $a_0 \oplus b_1 \oplus c_1 = 1$
  - $a_1 \oplus b_0 \oplus c_1 = 1$
  - $a_1 \oplus b_1 \oplus c_0 = 1$
- Classical value:
  - $\omega(G) = \frac{3}{4}$

GHZ Game: Probabilistic Strategy

- Can we do any better by adopting a probabilistic strategy?
  - No, because the probability that a probabilistic strategy wins is just an average of the probabilities that some collection of deterministic strategies win. (average of set of numbers $\ngtr$ initial set of numbers)
- Classically, we are stuck with $\omega(G) = \frac{3}{4}$.

GHZ Game: Quantum Strategy

- Players share some entangled state $|\psi\rangle$.
  - $|\psi\rangle = \frac{1}{2}|000\rangle - \frac{1}{2}|011\rangle - \frac{1}{2}|101\rangle - \frac{1}{2}|110\rangle$.
- Quantum Strategy:
  - If question to player $q = 0$.
    - Player performs Hadamard transform to their qubit.
  - If question to player $q = 1$.
    - Player performs Identity operator to their qubit.
  - Player measures qubit in standard basis and returns answer to referee.

GHZ Game: Quantum Strategy (2)

- Outcome cases for strategy:
  - Case 1 - $xyz = 000$:
    - Players measure qubits. Obviously results satisfy condition $a \oplus b \oplus c = 0$.
  - Case 2 - $xyz \in \{011, 101, 110\}$:
    - All possibilities will work the same via symmetry.
  - For case 2, consider scenario $xyz = 011$:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle\left(\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle\right)$$

$$(I \otimes H \otimes H)|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle|\psi^+\rangle - \frac{1}{\sqrt{2}}|1\rangle|\phi^-\rangle$$

$$\frac{1}{2}\left(|001\rangle + |010\rangle - |100\rangle + |111\rangle\right)$$

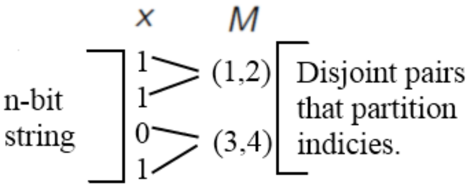
- Hence, when measurement occurs, results always satisfy $a \oplus b \oplus c = 1$. Thus the quantum strategy wins every time.

Hidden Matching Game

- Variant of Hidden Matching derived from communication complexity.
- Players have n outputs and entanglement dimension n.
- Violation of order $\frac{\sqrt{n}}{\log n}$ is achieved.
  - This violation is the same as previously achieved in [4], but in [1] is presented in explicit form.

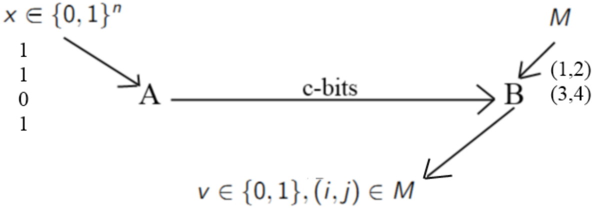
Hidden Matching Game: Inputs

- Consider the inputs for Alice and Bob:
  - Alice receives some n-bit binary string $x \in \{0,1\}^n$
  - Based on the input x, Bob receives a set of disjoint pairs, M, that partitions the set x.
  - Alice does not know M, and likewise Bob does not know x.



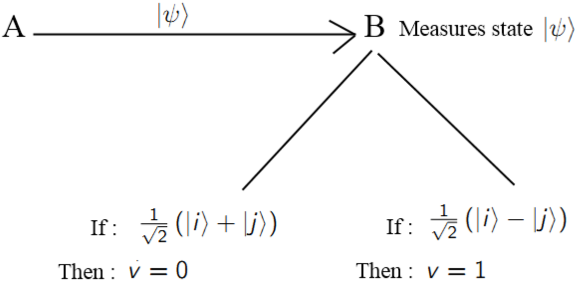
The Hidden Matching Game: Setup

- One-way communication between Alice and Bob, (Alice transmits c-bits to Bob).
- Bob outputs some bit $v \in \{0,1\}$, and some pair $(i,j) \in M$.
- Winning condition: $v = x_i \oplus x_j$.
- Example:
  - If Bob outputs (1,2), they win $\iff$ $v = 1$.
  - If Bob outputs (3,4), they win $\iff$ $v = 0$.
- Classical winning probability: $\frac{1}{2} + O(\frac{c}{\sqrt{n}})$



The Hidden Matching Game: Quantum Protocol

- $\exists$ protocol for HM with $\log n$ qubits of one-way communication that wins all the time.
- Protocol:
    - Alice and Bob share state $|\psi\rangle$.
    - Bob receives $|\psi\rangle$ and performs a measurement in n-element basis.



The Hidden Matching Game: Quantum Protocol (2)

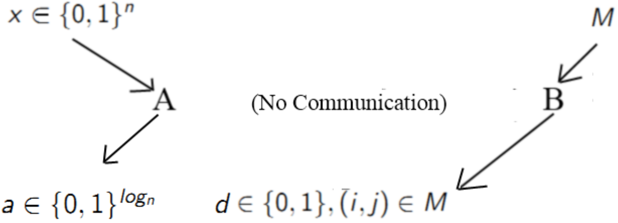
- $\forall (i,j) \in M$, probability of $\frac{1}{\sqrt{2}}(|i\rangle + |j\rangle) = \frac{2}{n}$ if $x_i \oplus x_j = 0$.
- $\forall (i,j) \in M$, probability of $\frac{1}{\sqrt{2}}(|i\rangle - |j\rangle) = \frac{2}{n}$ if $x_i \oplus x_j = 1$.
- Otherwise 0.
- Hence, Bob's output is always correct.

The Hidden Matching Non-Local Game

- Similar to communication complexity version except:
  - Players are space-like separated; no communication between parties takes place.
  - Alice now outputs a $\log n$ bit string $a \in \{0,1\}^{\log n}$.
- Winning condition: $(a \cdot (i \oplus j)) \oplus d = x_i \oplus x_j$.
  - Inner product between two $\log n$-bit strings $\oplus$ bit d yields a bit.

The Hidden Matching Non-Local Game (2)

- Winning probability 1 with n-dimensional entanglement.
- Classical bound $\frac{1}{2} + O\left(\frac{\log(n)}{\sqrt{n}}\right)$

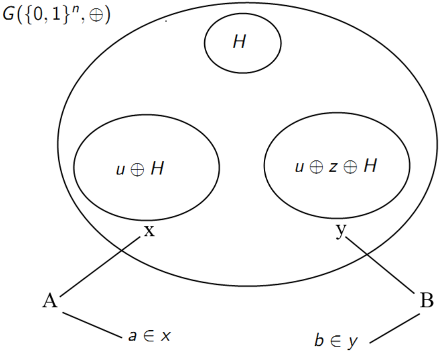


The Hidden Matching Non-Local Game: Classical Bounds

- Classical bound obtained from the Non-Local game follows from classical bound obtained from original communication complexity HM game.
- Why?
  - Non-local strategy can be reduced to a communication strategy to win the communication game.
  - If players can win the non-local game with some probability p, Alice can send the $\log n$-bit string to Bob.
  - Bob now possesses all the necessary information to compute v.
  - Therefore Bob can win.

Khot-Vishnoi Game: Setup

- Consider some group G consisting of all n-bit strings and bit-wise addition modulo-two.
- Also consider some subgroup of G, H, consisting of all n Hadamard codewords.
  - This can be thought of as the rows of an $n \times n$ Hadamard matrix translated from +/-1 basis to the 0,1 basis.



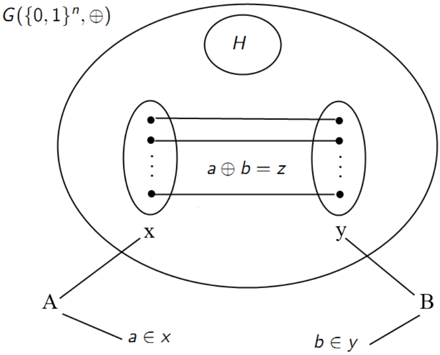
Khot-Vishnoi Game: Setup (2)

- Players are provided strings unknown to them:
  - $u \in \{0,1\}^n$ (Uniformly random string.)
  - $z \in_\eta \{0,1\}^n$ (η-random string; $\eta \in [0, \frac{1}{2}]$)

  We also have two cosets of H; x and y. These labels of x and y are then given to the players, with u and z remaining unknown to them.
  - Coset x: $u \oplus H$
  - Coset y: $u \oplus z \oplus H$
- Output: $A \rightarrow a \in x$ and $B \rightarrow b \in y$
- Winning condition: $a \oplus b = z$

Khot-Vishnoi Game: Bijection

- Condition $a \oplus b = z$ presents some bijection between cosets x and y.



Khot-Vishnoi Game: Bijection (2)

- If we consider a quantum strategy:
  - Entanglement can be exploited to take advantage of bijection.
- If we consider a classical strategy:
  - Finding corresponding element in coset B from A will require us to, at best, output one element at random.
  - This provides probability of $\frac{1}{n}$ for classical players.

Khot-Vishnoi Game: Classical Value

Theorem: Every classical strategy $\leq 1/n^{\eta/(1-\eta)}$

- Consider deterministic functions $A, B : \{0,1\}^n \to \{0,1\}$
  - $A(u) = 1 \iff$ Alice's output on coset $u \oplus H = u$
  - $B(u) = 1 \iff$ Bob's output on coset $u \oplus z \oplus H = u$
- $\mathbb{E}_u[A(u)] = \frac{1}{n}$ (Alice chooses one element per coset).

Khot-Vishnoi Game: Winning Conditions

- Players win $\iff \forall u, z$:
  - $\gamma = \sum_{h \in H} A(u \oplus h) B(u \oplus z \oplus h)$
- $\gamma = 1 \iff$ players win on input pair $(u \oplus H, u \oplus, z \oplus H)$
- $\gamma = 0$ otherwise.

Khot-Vishnoi Game: Winning Probability

- Winning probability is:
  - $\mathbb{E}_{u,z}\left[\sum_{h\in H} A(u\oplus h)B(u\oplus z\oplus h)\right]$
- Using linearity of expectation, we can move the sum outside.
  - $\sum_{h\in H} \mathbb{E}_{u,z}\left[A(u\oplus h)B(u\oplus z\oplus h)\right]$
- Since we shift u and $u\oplus z$ by the same h:
  - $n\ \mathbb{E}_{u,z}\left[A(u)B(u\oplus z)\right]$

Khot-Vishnoi Game: Achieving our Bound

- We can bound the expectation by (propsed in paper):
  - $\mathbb{E}_{u,z}\left[A(u) \oplus B(u \oplus z)\right] \leq 1/n^{1/(1-\eta)}$
- The above inequality is proved via hypercontractivity.
- All we need to satisfy our theorem is to notice that:
  - $n \cdot 1/n^{1/(1-\eta)} = 1/n^{\eta/(1-\eta)}$
- Thus we achieve our bound and find the classical value:

$$\omega(KV) = 1/n^{\eta/(1-\eta)} \approx \frac{1}{n}$$

Khot-Vishnoi Game: Quantum Value

We now consider the probability for a quantum strategy:

- *Assume* $\exists$ a quantum strategy that wins with probability at least $(1-2\eta)^2$ $\forall$ n and $\eta \in \left[0, \frac{1}{2}\right]$.

Khot-Vishnoi Game: Defining Vectors

- In order to find this corresponding strategy, we need to define some vectors; $|v^a\rangle$ and $|v^b\rangle$ with $a, b \in \{0,1\}^n$:

$$|v^a\rangle = \left(\frac{(-1)^{a_i}}{\sqrt{n}}\right), \; |v^b\rangle = \left(\frac{(-1)^{b_i}}{\sqrt{n}}\right)$$

Khot-Vishnoi Game: Defining Vectors

- These vectors possess specific properties that we shall use in the analysis of our bound:
  - Inner product of v^a and v^b is equal to one minus twice the Hamming distance between a and b over n.

$$\forall a, b \left\langle v^a, v^b \right\rangle = 1 - 2d(a, b)/n$$

  - Following vectors form an orthonormal basis of $\mathbb{R}^n$

$$\{v^a | a \in x\}, \ \{v^b | b \in y\}$$

Khot-Vishnoi Game: Quantum Strategy

- Our quantum strategy is as follows:
  1. Players start with some maximally entangled state $|\psi\rangle$
  2. Players perform projective measurements:
     - Alice: On input x, performs a projective measurement $\{v^a | a \in x\}$
     - Bob: On input y, performs a projective measurement $\{v^b | b \in y\}$
  3. Players output result from measurement; a and b.

Khot-Vishnoi Game: Obtaining Quantum Value

- The probability to obtain a, b is $\frac{\langle v^a, v^b \rangle}{n}$. This is because we are using the maximally entangled state $|\psi\rangle$ in our quantum strategy.
- For inputs x, y the winning probability is:

$$\frac{1}{n} \sum_{a \in x} \left\langle v^a, v^{a \oplus z} \right\rangle^2$$

- Using the property that $\left\langle v^a, v^b \right\rangle = 1 - 2d(a,b)/n$:

$$\frac{1}{n} \sum_{a \in x} \left(1 - \frac{2d(a, a \oplus z)}{n} \right)^2$$

Khot-Vishnoi Game: Obtaining Quantum Value (2)

- Note that, $d(a, a \oplus z)$ is simply the Hamming weight of z. So $d(a, a \oplus z) = |z|$. Hence:

$$\frac{1}{n} \sum_{a \in x} \left(1 - \frac{2|z|}{n}\right)^2$$

- Winning probability is the expectation over z:

$$\mathbb{E}_z \left[\left(1 - \frac{2|z|}{n}\right)^2 \right]$$

- Using convexity, we can take the square out:

$$\mathbb{E}_z \left[\left(1 - \frac{2|z|}{n}\right)^2 \right] \leq \left[\mathbb{E}_z \left(1 - \frac{2|z|}{n}\right) \right]^2$$

Khot-Vishnoi Game: Obtaining Quantum Value (3)

- Since z is an η-random string, its Hamming weight is $\eta \cdot n$. So $|z| = \eta \cdot n$:

$$\left[\mathbb{E}_z \left(1 - \frac{2\eta \cdot n}{n}\right)\right]^2$$

- Thus we achieve our bound:

$$\omega^*(KV) = (1 - 2\eta)^2 \approx \frac{1}{(\log n)^2}$$

Khot-Vishnoi Game: Achieved Violation

- Classical Strategy:

$$1/n^{\eta/(1-\eta)} \approx \frac{1}{n}$$

- Quantum Strategy:

$$(1-2\eta)^2 \approx \frac{1}{(\log n)^2}$$

- Violation:

$$\frac{\omega^*(KV)}{\omega(KV)} = \Omega\left(\frac{n}{(\log n)^2}\right)$$

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