

Forbidden graph minors, Arkhipov's theorem, and linear system games

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Non-local games

Given a simple graph $G = (V, E)$ with a non-proper vertex 2-colouring $b : V \rightarrow \{0, 1\}$ let $E(v)$ denote the set of edges incident to $v \in V$.

To the pair (G, b) there is an associated two-player non-local game \mathcal{G} .

Each player receives a vertex u, v , and responds with $f, g : E \rightarrow \{0, 1\}$ such that $\sum_{e \in E(u)} f(e) = b(u)$ and $\sum_{e \in E(v)} g(e) = b(v)$.

They win if $f(e) = g(e)$ for every $e \in E(u) \cap E(v)$.

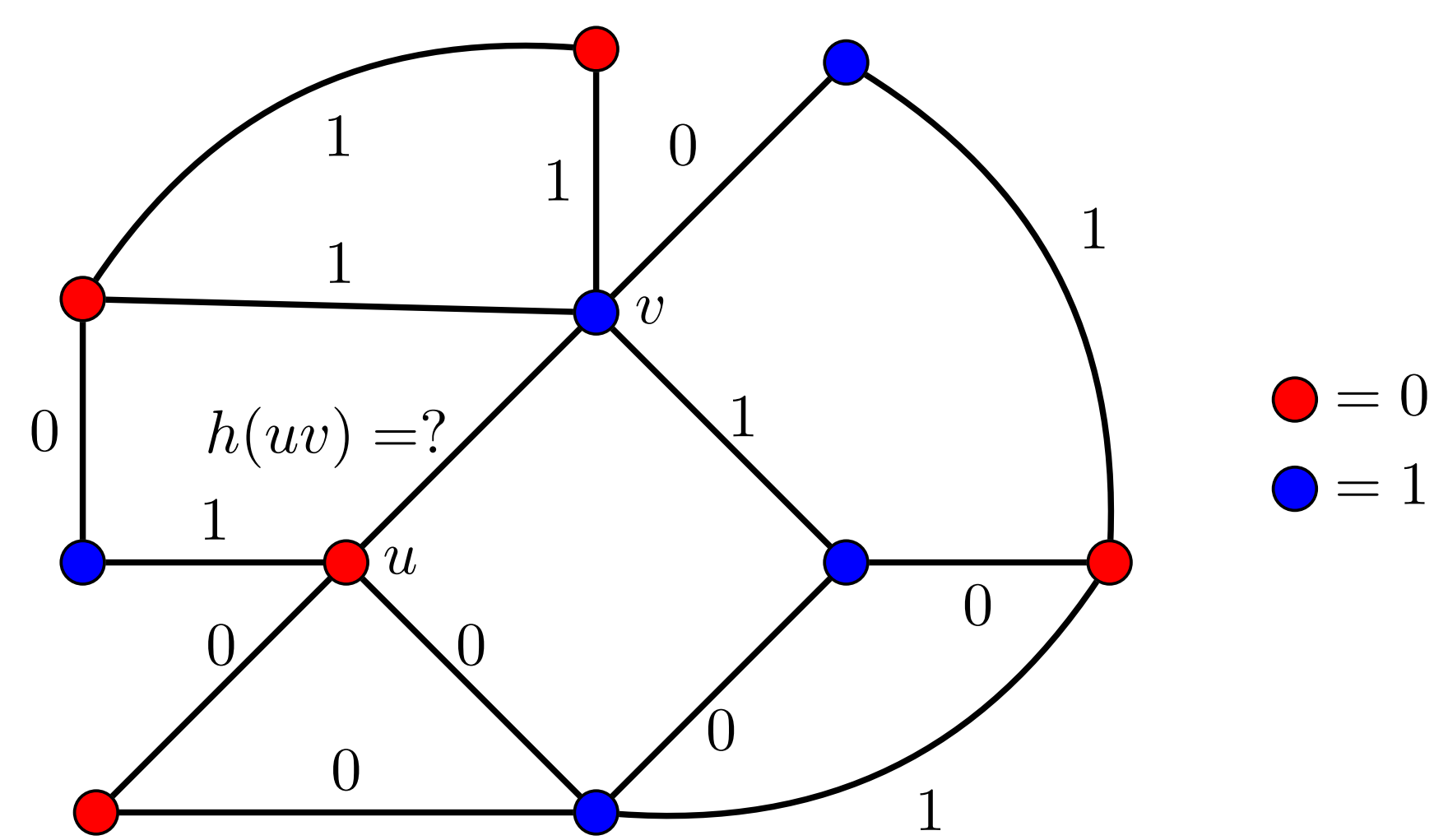


Figure 1: A game with no perfect strategy.

A *deterministic strategy* for \mathcal{G} is a function $h : E \rightarrow \{0, 1\}$, such that $\sum_{e \in E(v)} h(e) = b(v)$ for all $v \in V$.

This non-local game \mathcal{G} is a linear system game for the linear system $Ax = b$ over \mathbb{Z}_2 , where A is the *incidence matrix* of G .

Solution groups

The *solution group* $\Gamma(G, b)$ is the finitely-presented group with generators $\{x_e : e \in E\} \cup \{J\}$ and relations:

- (a) $x_e^2 = 1$ for all $e \in E$,
- (b) $\prod_{e \in E(v)} x_e = J^{b(v)}$ for all $v \in V$,
- (c) $[x_e, x_{e'}] = 1$ for all $v \in V$ and pairs $e, e' \in E(v)$,
- (d) $J^2 = 1$ and $[x_e, J] = 1$ for all $e \in E$.

A linear system game over \mathbb{Z}_2 has a perfect quantum strategy iff there is a representation $\pi : \Gamma(G, b) \rightarrow \mathcal{U}(\mathcal{H})$ such that $\pi(J) \neq \mathbb{1}_{\mathcal{H}}$.

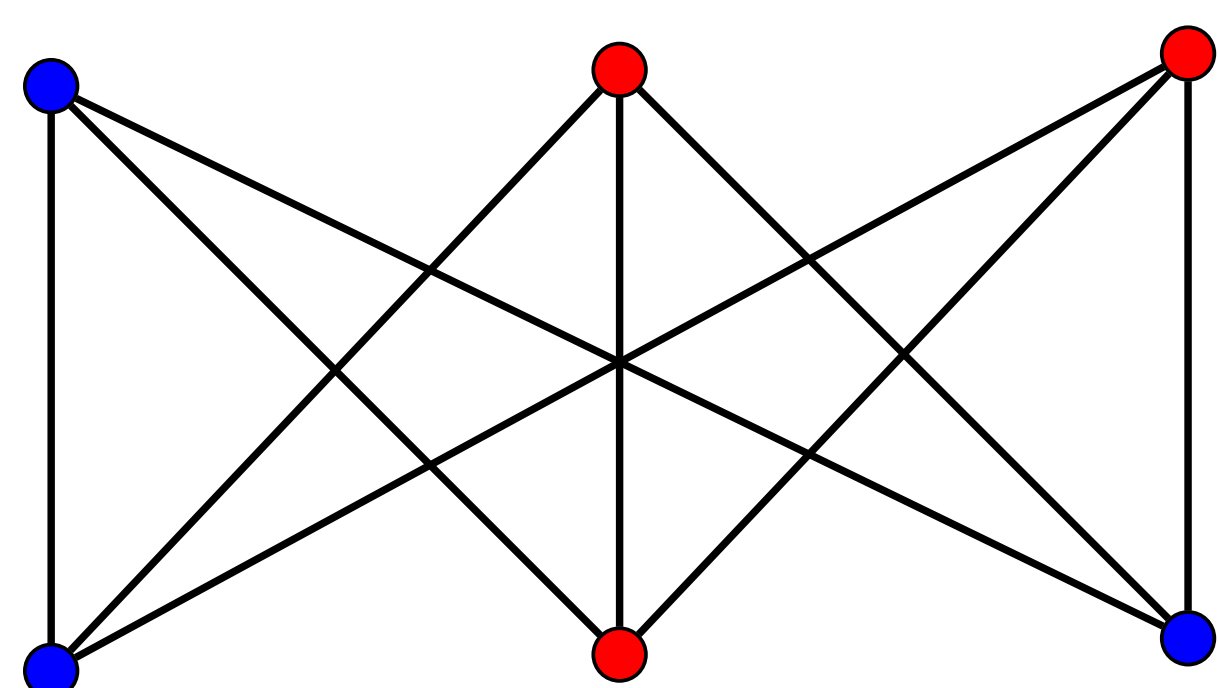


Figure 2: $\mathcal{G}(K_{3,3}, b)$ with b odd-parity is the Mermin-Peres magic square game, which has a perfect quantum strategy but no perfect classical strategy.

Graph minor operations

The standard *graph minor* operations are edge deletion, vertex deletion, and edge contraction.

On 2-coloured graphs (G, b) , the edge contraction minor combines the parity of two endpoints.

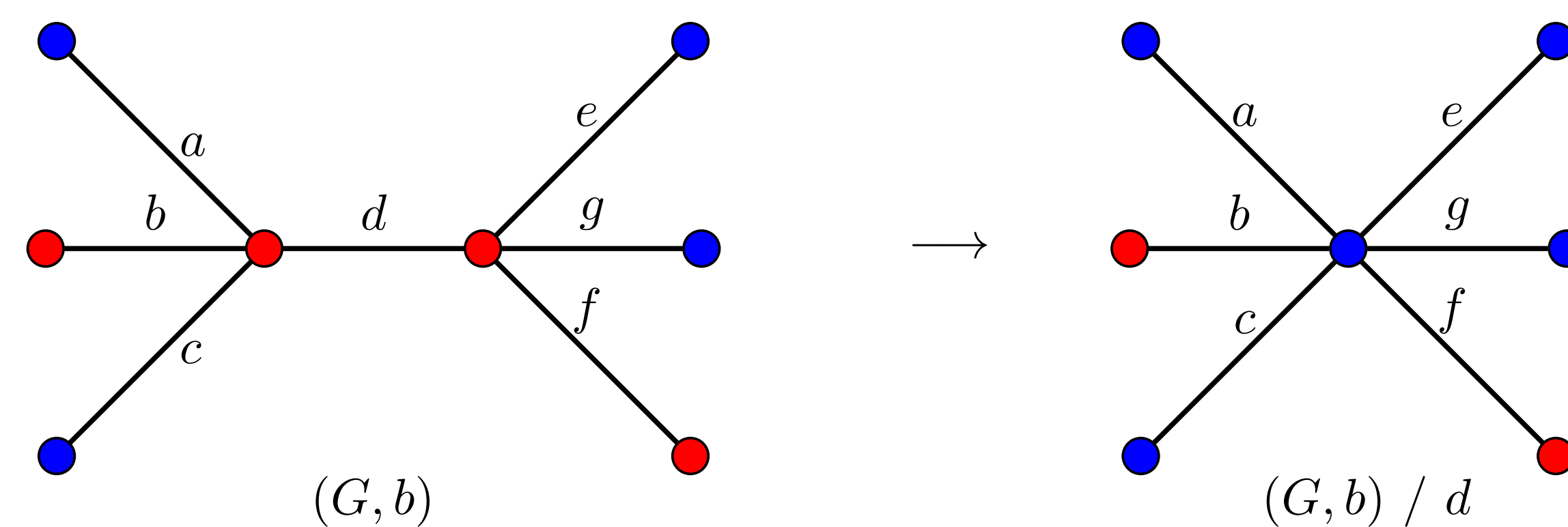


Figure 3: Two red vertices become one blue vertex under the edge contraction of d .

We add an additional minor operation to graphs (G, b) , swapping the colours at the ends of an edge, and disallow deletion of blue vertices.

A graph (G, b) contains (H, b') as a minor if there exists a sequence of graph minor operations that take $(G, b) \rightarrow (H, b')$.

Wagner's theorem. A graph is *planar* iff it does not contain $K_{3,3}$ or K_5 as minors.

Robertson-Seymour theorem. For every graph minor closed property P , there is a finite list of graphs \mathcal{L} such that a graph satisfies P iff it does not contain any minors in \mathcal{L} .

Deciding whether G contains any minors in a finite list \mathcal{L} can be done in time $O(V^3)$.

Main question and results

Arkhipov's theorem. If b has odd-parity, then $\mathcal{G}(G, b)$ has a perfect quantum strategy if and only if G is non-planar.

Question: Can we explain the connection in Arkhipov's theorem between solution group properties and graph minors?

Answer: Yes! We can even prove analogous results:

Theorem 1. If b has even-parity, then $\Gamma(G, b)$ is an *abelian* group if and only if (G, b) avoids $K_{3,4}$ and two vertex disjoint cycles $C \sqcup C$.

Theorem 2. $\Gamma(G, b)$ is a *finite* group if and only if it avoids $K_{3,6}$ and two vertex disjoint cycles $C \sqcup C$.

Solution groups and graph minors

Main lemma. If (H, b') is a minor of (G, b) then there is a *surjective group homomorphism* $\phi : \Gamma(G, b) \twoheadrightarrow \Gamma(H, b')$.

Example. The minor $G \rightarrow C \sqcup C$ corresponds to $\Gamma(G, b) \twoheadrightarrow \mathbb{Z}_2 * \mathbb{Z}_2$.

Corollary. Every *quotient-closed* property of solution groups $\Gamma(G, b)$ is characterized by a finite set of forbidden minors.

Proofs of Theorems 1 and 2: use classification of graphs which do not contain two vertex disjoint cycles (Lovász, 1965).

Solution groups and quantum strategies

Abelianness (see Theorem 1)

$\Gamma(G, b)$ abelian \Leftrightarrow all irreducible perfect strategies are 1-dimensional (1-dimensional = deterministic \subset classical).

If b has even-parity then the game \mathcal{G} has a perfect deterministic strategy.

If $\Gamma(G, b)$ is non-abelian and b has even-parity, then there are both deterministic and higher-dimensional irreducible perfect strategies.

Finiteness (see Theorem 2)

$\Gamma(G, b)$ finite \Rightarrow all irreducible perfect strategies are finite dimensional.

Open problems

The property that all perfect strategies of $\mathcal{G}(G, b)$ have classical correlation matrices is a quotient closed property of $\Gamma(G, b)$.

Can we find the minors for this property?

Conjecture: All perfect strategies of $\mathcal{G}(G, b)$ have classical correlation matrices if and only if b has even-parity.

What are the forbidden graph minors for other quotient closed properties of the solution group (e.g. *amenability*)?

What other group theory features of $\Gamma(G, b)$ can be related to graph theory features of (G, b) ?

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