Is Absolute Separability Determined by the Partial Transpose?

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Abstract

The absolute separability problem asks for a characterization of the quantum states $\rho \in M_m \otimes M_n$ with the property that $U\rho U^\dagger$ is separable for all unitary matrices U. We provide evidence that ρ is absolutely separable if and only if $U\rho U^\dagger$ has positive partial transpose for all unitary matrices U. In particular, we show that many well-known separability criteria are unable to detect entanglement in any such state, including the range criterion, the realignment criterion, the Choi map and its generalizations, and the Breuer–Hall map. We also show that these two properties coincide for the families of isotropic and Werner states, and several eigenvalue results for entanglement witnesses are proved along the way that are of independent interest.

1 Introduction

In quantum information theory, a quantum state $\rho \in M_m \otimes M_n$ (where M_n denotes the space of $n \times n$ complex matrices) is called *separable* [1] if there exist constants $p_i \geq 0$ and states $\rho_i^{(1)} \in M_m$ and $\rho_i^{(2)} \in M_n$ such that $\sum_i p_i = 1$ and

$$\rho = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)}.$$

Finding methods for determining whether a given quantum state is separable or *entangled* (i.e., not separable) is one of the most active areas of quantum information theory research [2, 3]. Although this problem is believed to be difficult in general [4, 5], many partial results are known. For example, the *positive-partial-transpose (PPT) criterion* states that if ρ is separable then $(id_m \otimes T)(\rho) \geq 0$, where ≥ 0 indicates positive semidefiniteness, $id_m : M_m \to M_m$ is the identity map, and $T : M_n \to M_n$ is the transpose map [6]. However, the converse of the PPT criterion only holds when $mn \leq 6$ [7, 8], so additional tests for separability are required in higher dimensions.

The most natural generalization of the PPT criterion says that a state $\rho \in M_m \otimes M_n$ is separable if and only if $(id_m \otimes \Phi)(\rho)$ is positive semidefinite for all positive maps $\Phi : M_n \to M_m$ [9]. Thus each fixed positive $\Phi : M_n \to M_m$ gives a necessary condition for separability.

The *absolute separability* problem [10] (sometimes called the *separability from spectrum* problem [11]) asks for a characterization of the states $\rho \in M_m \otimes M_n$ with the property that $U\rho U^\dagger$ is separable for all unitary matrices $U \in M_m \otimes M_n$, which is equivalent to asking which sets of real numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_{mn}\}$ are such that every state $\rho \in M_m \otimes M_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{mn} \geq 0$

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is separable. This question was first answered in the $m = n = 2$ case in [12], where it was shown that $\rho \in M_2 \otimes M_2$ is absolutely separable if and only if its eigenvalues satisfy $\lambda_1 \leq \lambda_3 + 2\sqrt{\lambda_2\lambda_4}$, however the problem remains open in general.

One motivation for this problem comes from the fact that it is sometimes easier to determine the eigenvalues of a quantum state than it is to determine the entire structure of that state [13, 14]. Thus, the absolute separability problem asks for the strongest separability test that can be devised given this restricted information. In another direction, the exact largest size of a ball of separable states centered at the maximally-mixed state $\frac{1}{mn}(I \otimes I) \in M_m \otimes M_n$ is known [15], and it is not difficult to show that every state within this ball is absolutely separable. However, there are also absolutely separable states outside of this ball, and it would be nice to have a characterization of where they are. Alternatively, we can think of states that are *not* absolutely separable as those that can be used to generate entanglement when the operations at our disposal are global unitary channels [16].

One approach to characterizing the states that are absolutely separable would be to instead fix some necessary test for separability and determine the set of states $\rho \in M_m \otimes M_n$ with the property that $U\rho U^\dagger$ satisfies that separability test for all unitary matrices $U \in M_m \otimes M_n$. This approach was initiated in [17], where the set of states $\rho \in M_m \otimes M_n$ that are *absolutely PPT* (i.e., states such that $(id_m \otimes T)(U\rho U^\dagger)$ is positive semidefinite for all unitary U) were characterized. Since the PPT criterion implies separability when $mn \leq 6$, this result immediately showed that a state $\rho \in M_2 \otimes M_3$ is absolutely separable if and only if $\lambda_1 \leq \lambda_5 + 2\sqrt{\lambda_4\lambda_6}$, as well as recovering the already-known characterization in the $m = n = 2$ case.

It was then shown in [18] that the set of absolutely PPT states coincides with the set of absolutely separable states when $m = 2$ and n is arbitrary, despite the fact that the set of PPT states is strictly larger than the set of separable states when $m = 2$ and $n \geq 4$. The question was then asked whether or not the set of absolutely PPT states and absolutely separable states coincide when $m, n \geq 3$. The goal of this note is to provide evidence that these two sets do indeed coincide.

Some of the evidence of this claim that we provide is several results of the form "if ρ is absolutely PPT, then it is also absolutely <other separability criterion>". For example, we show that every absolutely PPT $\rho \in M_m \otimes M_n$ is also "absolutely realignable"—i.e., $U\rho U^\dagger$ always satisfies the *realignment criterion* introduced in [19, 20], even though there are PPT states ρ that violate the realignment criterion. This difference between the usual separability problem and the absolute separability problem is illustrated in Figure 1. We also prove that the absolute separability and absolute PPT properties coincide when restricted to the well-known families of Werner and isotropic states.

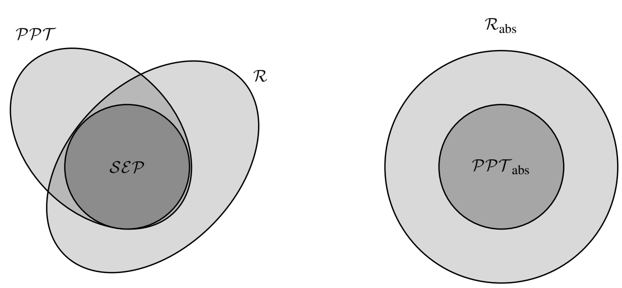

Figure 1: The figure on the left represents the relationship between the set of separable states $\mathcal{SEP}$, the set of PPT states $\mathcal{PPT}$, and the set of states that satisfy the realignment criterion $\mathcal{R}$. The figure on the right represents the relationship between the set of absolutely PPT states $\mathcal{PPT}_{\text{abs}}$ and the set $\mathcal{R}_{\text{abs}}$ of states that are "absolutely realignable" (see Theorem 1).

2 Preliminaries

The proofs of our results rely heavily on *semidefinite programming*. Given Hermitian matrices $A \in M_n$ and $B \in M_m$ and a Hermiticity-preserving linear map $\Phi : M_n \to M_m$ (i.e., a map such that $\Phi(X^\dagger) = \Phi(X)^\dagger$ for all $X \in M_n$), the semidefinite program associated with the triple (A, B, Φ) is the following pair of optimization problems:

$$\begin{array}{rlcrl} & \text{Primal problem} & \qquad & & \text{Dual problem} \\ \text{maximize:} & \mathrm{Tr}(AX) & & \text{minimize:} & \mathrm{Tr}(BY) \\ \text{subject to:} & \Phi(X) \leq B & & \text{subject to:} & \Phi^\dagger(Y) \geq A \\ & X \geq 0 & & & Y \geq 0, \end{array}$$

where $\Phi^\dagger : M_m \to M_n$ is the *dual map* of Φ defined by $\mathrm{Tr}(\Phi(X)Y) = \mathrm{Tr}(X\Phi^\dagger(Y))$ for all $X \in M_n$ and $Y \in M_m$. Semidefinite programs can be efficiently solved [21], and furthermore *weak duality* always holds, which tells us that $\mathrm{Tr}(AX) \leq \mathrm{Tr}(BY)$ for all feasible points $X \in M_n$ and $Y \in M_m$. In particular, this means that we can get upper bounds on the optimal value of the primal problem by simply finding a single feasible point for the dual problem (and similarly, feasible points of the primal problem give lower bounds on the optimal value of the dual problem). For a more thorough introduction to semidefinite programming, see [22, 23].

Given a linear map $\Phi : M_n \to M_m$, we recall that its *Choi matrix* is the operator

$$J(\Phi) \stackrel{\text{def}}{=} n(id_n \otimes \Phi)(|\psi^+\rangle\langle\psi^+|) \in M_n \otimes M_m,$$

where $|\psi^+\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |i\rangle \otimes |i\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ is the standard maximally-entangled pure state. It is well-known that Φ is *completely positive* (i.e., satisfies $(id_n \otimes \Phi)(X) \geq 0$ whenever $0 \leq X \in M_n \otimes M_n$) if and only if $J(\Phi)$ is positive semidefinite [24].

Our proofs will also be heavily reliant on the notion of *entanglement witnesses*, which are Hermitian operators $W \in M_m \otimes M_n$ with the property that $\mathrm{Tr}(W\sigma) \geq 0$ for all separable $\sigma \in M_m \otimes M_n$, but $\mathrm{Tr}(W\rho) < 0$ for some (necessarily entangled) $\rho \in M_m \otimes M_n$. Here we say that W detects the entanglement in ρ, and we note that every entangled ρ is detected by some entanglement witness W. Finally, we will also make frequent use of the family of Schatten p-norms, defined for $p \in [1, \infty]$ by

$$\|X\|_p := \left[\mathrm{Tr}\left((X^\dagger X)^{p/2}\right)\right]^{1/p},$$

where we define $\|X\|_{\mathrm{tr}} := \|X\|_1$, $\|X\|_F := \|X\|_2$, and $\|X\| := \|X\|_\infty$ (and in these special cases, these norms are often called the *trace norm*, *Frobenius norm*, and *operator norm*, respectively).

The remainder of this article is organized as follows. In Section 3, we briefly review the characterization of states that are absolutely PPT that was originally derived in [17]. We then present our primary conjecture on the equivalence of absolute PPT and absolute separability, and state the implications of this conjecture in Section 4. The next sections are dedicated to presenting various forms of evidence for the truth of our conjecture. For instance, in Section 5, we show many separability criteria are unable to detect entanglement in any absolutely PPT state. In Section 6, we show that for specific classes of states that absolute separability and absolute PPT coincide. Finally, in Section 7, we conclude and list a number of open problems and directions for future research.

3 Absolute Positive Partial Transpose

We now briefly recall some of the key points of the characterization of absolutely PPT states given in [17]. Indeed, the main result of that paper shows that, for each $m, n \in \mathbb{N}$, there exists a finite family of linear matrix inequalities (LMIs) with the property that $\rho \in M_m \otimes M_n$ is absolutely PPT if and only if its eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{mn}$ satisfy each of the LMIs.

In the $m = 2$ case, the LMI that determines absolute PPT is

$$L_1 := \begin{bmatrix} 2\lambda_{2n} & \lambda_{2n-1} - \lambda_1 \\ \lambda_{2n-1} - \lambda_1 & 2\lambda_{2n-2} \end{bmatrix} \geq 0,$$

which is easily seen to be equivalent to the previously-discussed inequalities $\lambda_1 \leq \lambda_3 + 2\sqrt{\lambda_2\lambda_4}$ when $n = 2$ and $\lambda_1 \leq \lambda_5 + 2\sqrt{\lambda_4\lambda_6}$ when $n = 3$.

In the $m = 3$ case, there are two LMIs that determine absolute PPT:

$$\begin{aligned} L_1 &:= \begin{bmatrix} 2\lambda_{3n} & \lambda_{3n-1} - \lambda_1 & \lambda_{3n-3} - \lambda_2 \\ \lambda_{3n-1} - \lambda_1 & 2\lambda_{3n-2} & \lambda_{3n-4} - \lambda_3 \\ \lambda_{3n-3} - \lambda_2 & \lambda_{3n-4} - \lambda_3 & 2\lambda_{3n-5} \end{bmatrix} \geq 0, \\ L_2 &:= \begin{bmatrix} 2\lambda_{3n} & \lambda_{3n-1} - \lambda_1 & \lambda_{3n-2} - \lambda_2 \\ \lambda_{3n-1} - \lambda_1 & 2\lambda_{3n-3} & \lambda_{3n-4} - \lambda_3 \\ \lambda_{3n-2} - \lambda_2 & \lambda_{3n-4} - \lambda_3 & 2\lambda_{3n-5} \end{bmatrix} \geq 0. \end{aligned} \tag{1}$$

That is, $\rho \in M_3 \otimes M_n$ is absolutely PPT if and only if its eigenvalues satisfy both of the positive semidefiniteness conditions (1).

In general, once we have fixed m, n we use $L_1, L_2, L_3, \ldots$ to denote the matrices of eigenvalues whose positive semidefiniteness determine absolute PPT, and these matrices always look quite similar to the matrices (1) in the $m = 3$ case. For example, each L_i is of size $\min\{m, n\} \times \min\{m, n\}$, the diagonal entry of each L_i is 2 times one of the λ_j's, and each off-diagonal entry is the difference of two of the λ_j's. Furthermore, the top-left 2×2 sub-matrix of L_1 is always of the form

$$\begin{bmatrix} 2\lambda_{mn} & \lambda_{mn-1} - \lambda_1 \\ \lambda_{mn-1} - \lambda_1 & 2\lambda_{mn-2} \end{bmatrix}, \tag{2}$$

so positive semidefiniteness of (2) is a necessary (but not sufficient when $m, n \geq 3$) condition for ρ to be absolutely PPT.

We note that the number of L_i's that must be checked to be positive semidefinite grows exponentially in $\min\{m, n\}$ (for example, when $\min\{m, n\} = 7$ the number of L_i's is $107,498$ [25]), and their exact construction is slightly complicated. However, it is not important for our purposes to be familiar with their exact construction—the properties of these matrices that we presented above are all we need.

We now present, without proof, a lemma that is well-known in matrix analysis (see, for example, [26, Problem III.6.14]).

Lemma 1. *Let $A, B \in M_n$ be Hermitian matrices with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$, respectively. Then*

$$\min\left\{ \mathrm{Tr}(AUBU^\dagger) : U \in M_n \text{ is unitary} \right\} = \sum_{j=1}^{n} \lambda_j \mu_{n-j+1}.$$

We can make use of Lemma 1 to see that semidefinite programming can be used to determine whether or not a given entanglement witness is capable of detecting entanglement in an absolutely PPT state. In particular, if we have an entanglement witness $W \in M_m \otimes M_n$ with eigenvalues $\mu_1 \geq \cdots \geq \mu_{mn}$ then W can detect the entanglement in some absolutely PPT state if and only if the optimal value of the following semidefinite program is strictly less than zero:

$$\begin{aligned} \text{minimize:} \quad & \sum_{j=1}^{mn} \lambda_j \mu_{mn-j+1} \\ \text{subject to:} \quad & L_i \geq 0 \quad \forall\, i \\ & \lambda_j \geq \lambda_{j+1} \geq 0 \quad \forall\, 1 \leq j \leq mn-1 \\ & \sum_{j=1}^{mn} \lambda_j = 1. \end{aligned} \tag{3}$$

Indeed, the constraints in the SDP (3) are simply enforcing the fact that $\lambda_1 \geq \cdots \geq \lambda_{mn} \geq 0$ are the eigenvalues of some absolutely PPT state. It then follows from Lemma 1 that the SDP (3) computes

$$\min\big\{\operatorname{Tr}(W\rho) : \rho \in M_m \otimes M_n \text{ is absolutely PPT}\big\}.$$

4 The Conjecture

We now present our main conjecture, which states that the absolute separability and absolute PPT properties coincide. Recall that this conjecture was already proved in the $m = 2$ case in [18].

Conjecture 1. *A quantum state $\rho \in M_m \otimes M_n$ is absolutely separable if and only if it is absolutely PPT.*

The rest of the paper is devoted to presenting evidence that Conjecture 1 is true. Or at the very least, we show that many of the "standard" techniques from entanglement theory cannot be used to find a counterexample to the conjecture. We first need the following proposition.

Proposition 1. *Suppose that there exists a state $\rho \in M_m \otimes M_n$ that is absolutely PPT but not absolutely separable. Then ρ has full rank.*

Proof. Suppose that ρ is absolutely PPT with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{mn} = 0$ (notice that we set the smallest eigenvalue equal to 0, so that ρ does not have full rank). Our goal is to show that ρ is absolutely separable.

We recall from Section 3 that the matrix (2) must be positive semidefinite. However, by using the fact that $\lambda_{mn} = 0$, we then see that $\lambda_1 = \lambda_{mn-1}$, which implies that (up to a positive scalar multiple), $\rho = I - |v\rangle\langle v|$ for some pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$. We now use [15, Theorem 1], which says that every operator of the form $I - X$ with $\|X\|_F \leq 1$ is separable (and even absolutely separable). Since $\||v\rangle\langle v|\|_F = 1$, it follows that ρ is absolutely separable, as desired. □

Proposition 1 immediately implies that the range criterion [27] for detecting entanglement, which states that the range of a separable state is spanned by product pure states, cannot possibly detect entanglement in any absolutely PPT state. To see this, simply note that the range of a full-rank state is the entire Hilbert space, which is always spanned by product states (such as the standard basis). Furthermore, Proposition 1 also shows that most of the "usual" ways of creating PPT entangled states cannot possibly create absolutely PPT entangled states, since almost all such methods result in states that are *not* of full rank (e.g., chessboard states [28], states constructed by unextendible product bases [29], the 1-parameter family of states constructed by the Horodeckis [30], and so on). Only a couple of families of bound entangled states with full rank are known [31], and we have not been able to find any that are absolutely PPT (see Section 6.3, for example).

5 The Absolute Separablity "Collapse"

In this section, we present the main results of the paper, which show that the set of absolutely PPT states is "closer" to the set of absolutely separable states than the set of PPT states is to the set of separable states in the following sense: there are (many) separability criteria that are capable of detecting entanglement in PPT states, but become weaker than the PPT criterion in the "absolute" regime (see Figure 1, for example). We already saw this for the range criterion in the previous section. We now prove that the same result holds for the realignment criterion [19, 20], the Choi map [24] and its generalizations [32], and the Breuer–Hall map [33, 34]. That is, each of these separability criteria are incapable of detecting any entanglement in absolutely PPT states.

Before dealing with any specific separability criteria, we first need the following very important lemma, which we will make repeated use of. This lemma lets us determine that an entanglement witness cannot detect entanglement in absolutely PPT states, based only on very limited information about the eigenvalues of the witness (specifically, its largest eigenvalue and the sum of its negative eigenvalues).

Lemma 2. *Let $W \in M_m \otimes M_n$ be a Hermitian operator scaled so that $\mathrm{Tr}(W) = 1$. Let μ_1 be the maximum eigenvalue of W and define ℓ to be the sum of its negative eigenvalues:*

$$\ell \stackrel{\text{def}}{=} (1 - \|W\|_{\text{tr}})/2.$$

Furthermore, define a function $f : [-\frac{1}{2}, 0] \to [\frac{1}{2}, 1]$ by:

$$f(x) \stackrel{\text{def}}{=} \frac{1}{4}\begin{cases} \sqrt{1-4x^2} - 2x + 1 & \text{if } -\frac{1}{2} \leq x \leq -\frac{1}{2\sqrt{2}} \\ 1 + \sqrt{2} & \text{if } -\frac{1}{2\sqrt{2}} < x < \frac{1-\sqrt{2}}{2} \\ \sqrt{1+4x-4x^2} - 2x + 3 & \text{if } \frac{1-\sqrt{2}}{2} \leq x \leq 0. \end{cases}$$

If $\ell \geq -\frac{1}{2}$ and $\mu_1 \leq f(\ell)$ then $\mathrm{Tr}(W\rho) \geq 0$ for all absolutely PPT states $\rho \in M_m \otimes M_n$.

Before proving the lemma, we note that we have shown numerically that the bound $\mu_1 \leq f(\ell)$ is optimal at least in the $m = n = 3$ case. That is, given any choice of ℓ and μ_1 such that $\mu_1 > f(\ell)$, we can find a Hermitian operator $W \in M_3 \otimes M_3$ and an absolutely PPT state $\rho \in M_3 \otimes M_3$ such that $\mathrm{Tr}(W) = 1$, W has a single negative eigenvalue equal to ℓ, the maximum eigenvalue of W is μ_1, and $\mathrm{Tr}(W\rho) < 0$.

The function $f(x)$ is plotted in Figure 2, where we have highlighted some important special cases. For example, $f(-\frac{1}{2}) = \frac{1}{2}$, $f(-\frac{2}{5}) = \frac{3}{5}$, $f(-\frac{1}{5}) = \frac{9}{10}$, and $f((1-\sqrt{2})/2) = (2+\sqrt{2})/4$.

Proof of Lemma 2. We prove the result by showing that the semidefinite program (3) has optimal value ≥ 0 whenever $\ell \geq -\frac{1}{2}$ and $\mu_1 \leq f(\ell)$. First, we replace the complicated set of LMI constraints $L_i \geq 0$ for all i in this SDP with the single constraint that the 2×2 matrix (2) is positive semidefinite. Since this new SDP is a minimization problem subject to a weaker set of constraints, its optimal value is no larger than the optimal value of the SDP (3).

Second, notice that it suffices to consider the case where $\mu_{mn} = \ell$, where μ_{mn} is the minimal eigenvalue of W. The reason for this is that the objective function of the semidefinite program (3) can only decrease when we decrease μ_{mn} (while fixing $\sum_i \mu_i = 1$). Similarly, increasing μ_1 while fixing $\sum_i \mu_i = 1$, or increasing μ_2 while fixing μ_1 and $\sum_i \mu_i = 1$ will also decrease the value of the objective function, and similarly for μ_3. For example, when $\ell = -2/5$ and $\mu \leq f(-2/5) = 3/5$, it suffices to consider the case when $\mu_{mn} = -2/5, \mu_1 = 3/5, \mu_2 = 3/5, \mu_3 = 1/5$ (μ_1, μ_2, μ_3 are determined by making μ_1 as large as possible subject to $\mu_1 \leq f(\ell)$, then μ_2 as large as possible while subject to $\mu_2 \leq \mu_1$, and so on until

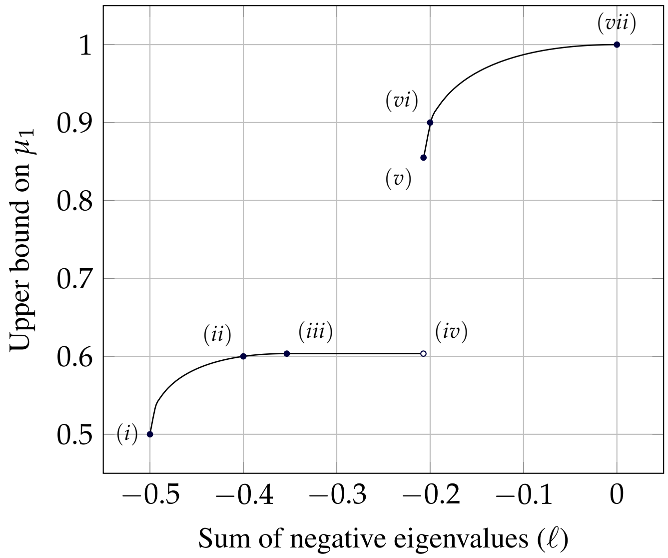

Figure 2: A plot of the upper bound $f(\ell)$ given by Lemma 2. For example, point (i) is $(-\frac{1}{2}, \frac{1}{2})$, which tells us that if the sum of the negative eigenvalues (ℓ) of W equals $-\frac{1}{2}$ then W cannot detect entanglement in absolutely PPT states if the largest eigenvalue (μ_1) of W is $\leq \frac{1}{2}$. Point (ii) corresponds to $\ell = -\frac{2}{5}$ and $\mu_1 \leq \frac{3}{5}$, (iii) corresponds to $\ell = -\frac{1}{2\sqrt{2}}$ and $\mu_1 \leq \frac{1}{4}(1+\sqrt{2})$, (v) corresponds to $\ell = \frac{1}{2}(1-\sqrt{2})$ and $\mu_1 \leq \frac{1}{4}(2+\sqrt{2})$, (vi) corresponds to $\ell = -\frac{1}{5}$ and $\mu_1 \leq \frac{9}{10}$, and (vii) corresponds to $\ell = 0$ and $\mu_1 \leq 1$ (in which case the result is trivial).

$\sum_i \mu_i = 1$). In general, we set $\mu_2 = \min\{\mu_1, 1-\mu_1-\ell\}$, and $\mu_3 = \max\{0, 1-2\mu_1-\ell\}$ (and $\mu_i = 0$ for all $4 \leq i \leq mn-1$).

Thus it suffices to show that the optimal value of the following SDP is ≥ 0, where we recall that μ_1 and ℓ are fixed constants in this SDP, and we optimize over $\lambda_1, \lambda_2, \ldots, \lambda_{mn}$:

$$\begin{array}{rl} & \text{Primal problem} \\ \text{minimize:} & \mu_1\lambda_{mn} + \min\{\mu_1, 1-\mu_1-\ell\}\lambda_{mn-1} + \max\{0, 1-2\mu_1-\ell\}\lambda_{mn-2} - \ell\lambda_1 \\ \text{subject to:} & \begin{bmatrix} 2\lambda_{mn} & \lambda_{mn-1}-\lambda_1 \\ \lambda_{mn-1}-\lambda_1 & 2\lambda_{mn-2} \end{bmatrix} \geq 0 \\ & \lambda_i \geq \lambda_{i+1} \geq 0 \quad \forall 1 \leq i \leq mn-1 \\ & \displaystyle\sum_{i=1}^{mn} \lambda_i = 1. \end{array} \tag{4}$$

The dual problem can be constructed using standard techniques of semidefinite programming as found in [23].

$$
\begin{array}{ll}
\multicolumn{2}{c}{\text{Dual problem}} \\
\text{maximize:} & t \\
\text{subject to:} & t - 2b + y_1 \le \ell \\
& t + 2b + y_{mn-1} - y_{mn-2} \le \min\{\mu_1, 1-\mu_1-\ell\} \\
& t + 2c + y_{mn-2} - y_{mn-3} \le \max\{0, 1-2\mu_1-\ell\} \\
& t + 2a - y_{mn-1} \le \mu_1 \\
& t + y_{i+1} - y_i \le 0 \quad \forall 1 \le i \le mn-4 \\
& y_i \ge 0 \quad \forall 1 \le i \le mn-1 \\
& \begin{bmatrix} a & b \\ b & c \end{bmatrix} \ge 0.
\end{array} \tag{5}
$$

It thus suffices to find a feasible point of the above dual problem with $t = 0$. We note that code that implements the above SDP in MATLAB via the CVX package [35] can be downloaded from [36]. We now split into three cases, depending on which branch of f we are working with.

Case a): $-\frac{1}{2} \le \ell \le -\frac{1}{2\sqrt{2}}$. In this case, we have $\mu_1 = (\sqrt{1-4\ell^2} - 2\ell + 1)/4$, $\min\{\mu_1, 1-\mu_1-\ell\} = \mu_1$, and $\max\{0, 1-2\mu_1-\ell\} = 1-2\mu_1-\ell$. It is then straightforward to verify the following defines a feasible point of the dual problem of the semidefinite program (4):

$$
\begin{aligned}
t = 0, \qquad a = \frac{\ell + 2\mu_1}{2}, \qquad b = -\frac{\ell}{2}, \qquad c = \frac{1-\ell-2\mu_1}{2} \\
y_i = 0 \quad \forall 1 \le i \le mn-2, \qquad y_{mn-1} = \mu_1 + \ell.
\end{aligned}
$$

The only condition in the dual problem that is not obviously satisfied is the fact that $\begin{bmatrix} a & b \\ b & c \end{bmatrix} \ge 0$. However, this follows from the fact that $b^2 = ac$ for this particular choice of a, b, c, and μ_1. Since this dual feasible point has $t = 0$, it follows that the semidefinite program (4) has optimal value ≥ 0, as desired.

Case b): $-\frac{1}{2\sqrt{2}} < \ell < \frac{1-\sqrt{2}}{2}$. This case follows immediately from choosing $\ell = -\frac{1}{2\sqrt{2}}$ in case a) and noting that increasing ℓ without increasing μ_1 does not decrease the optimal value of the SDP (4).

Case c): $\frac{1-\sqrt{2}}{2} \le \ell \le 0$. In this case, we have $\mu_1 = (\sqrt{1+4\ell-4\ell^2} - 2\ell + 3)/4$, $\min\{\mu_1, 1-\mu_1-\ell\} = 1-\mu_1-\ell$, and $\max\{0, 1-2\mu_1-\ell\} = 0$. It is then straightforward to verify the following defines a feasible point of the dual problem of the semidefinite program (4):

$$
\begin{aligned}
t = 0, \qquad a = \frac{\mu_1}{2}, \qquad b = \frac{1-\ell-\mu_1}{2}, \qquad c = \frac{1-\mu_1}{2} \\
y_i = 1-\mu_1 \quad \forall 1 \le i \le mn-3, \qquad y_{mn-2} = y_{mn-1} = 0.
\end{aligned}
$$

Similar to case a), we have $b^2 = ac$ for this particular choice of a, b, c, and μ_1, so the above point indeed satisfies all of the constraints of the dual problem. Since $t = 0$, it follows that the semidefinite program (4) has optimal value ≥ 0, which completes the proof. □

5.1 The Realignment Criterion

The *realignment criterion* [19, 20] for entanglement states that all separable states $\rho \in M_m \otimes M_n$ satisfy $\|R(\rho)\|_{\mathrm{tr}} \le 1$, where $R : M_m \otimes M_n \to M_{m,n} \otimes M_{m,n}$ is the linear "realignment" map defined on elementary tensors by $R(|i\rangle\langle j| \otimes |k\rangle\langle \ell|) = |i\rangle\langle k| \otimes |j\rangle\langle \ell|$. Thus if $\|R(\rho)\|_{\mathrm{tr}} > 1$ then we know that ρ is entangled, and we say that the realignment criterion detected the entanglement in ρ. This criterion is particularly useful, as it is one of the simplest tests that can detect entanglement in PPT states. The main result of this section shows that the realignment criterion *cannot* detect entanglement in any absolutely PPT states.

To phrase our result in another way, we can consider the sets of absolutely PPT states and "absolutely realignable" states:

$$\mathcal{PPT}_{\text{abs}} \stackrel{\text{def}}{=} \{\rho : (id_m \otimes T)(U\rho U^\dagger) \geq 0 \quad \forall \text{ unitary } U\},$$
$$\mathcal{R}_{\text{abs}} \stackrel{\text{def}}{=} \{\rho : \|R(U\rho U^\dagger)\|_{\text{tr}} \leq 1 \quad \forall \text{ unitary } U\}.$$

Our result states that $\mathcal{PPT}_{\text{abs}} \subseteq \mathcal{R}_{\text{abs}}$, so the realignment criterion becomes a weaker entanglement test than the PPT criterion in the "absolute" setting:

Theorem 1. *Let $m, n \geq 3$. If $\rho \in M_m \otimes M_n$ is absolutely PPT then $\|R(\rho)\|_{\text{tr}} \leq \frac{3}{\sqrt{mn}}$. In particular, the realignment criterion cannot detect entanglement in any absolutely PPT state.*

We note that the result of Theorem 1 provides the best possible upper bound on $\|R(\rho)\|_{\text{tr}}$, up to a global multiplicative constant, since even the maximally-mixed state $\rho = \frac{1}{mn} I \in M_m \otimes M_n$ (which is trivially absolutely PPT) has $\|R(\rho)\|_{\text{tr}} = \frac{1}{\sqrt{mn}}$.

It will be helpful to note that for any state $\rho \in M_m \otimes M_n$, one can write ρ in terms of its *operator-Schmidt decomposition*, which is defined as

$$\rho = \sum_i \lambda_i A_i \otimes B_i,$$

where $\lambda_i \geq 0$ for all i and the sets of operators $\{A_i\}$ and $\{B_i\}$ form orthonormal bases of M_m and M_n in the Hilbert–Schmidt inner product. There is a well-known correspondence between the realignment criterion and the operator-Schmidt decomposition of any state ρ. Specifically, it is the case that $\|R(\rho)\|_{\text{tr}} = \sum_i \lambda_i$ (a proof of this fact can be found in [37]), so any state with $\sum_i \lambda_i > 1$ is entangled. In the case when $\sum_i \lambda_i > c$ for some real constant c, it is straightforward to see that the operator W_c, defined by

$$W_c := cI - \sum_i A_i \otimes B_i, \tag{6}$$

satisfies $\text{Tr}(W_c\rho) < 0$, since orthonormality of $\{A_i\}$ and $\{B_i\}$ implies that $\text{Tr}(W_c\rho) = c - \sum_i \lambda_i < 0$. In particular, in the $c = 1$ case, W_c is an entanglement witness that detects the entanglement in ρ. Thus, to show that $\|R(\rho)\|_{\text{tr}} \leq c$ for all absolutely PPT states, it suffices to show that operators W_c of the form (6) have $\text{Tr}(W_c\rho) \geq 0$ whenever ρ is absolutely PPT. In order to prove this result for the value of $c = 3/\sqrt{mn}$ claimed by Theorem 1, we first need the following two auxiliary lemmas.

Lemma 3. *Let $\{A_i\}_{i=1}^{m^2} \subset M_m$ and $\{B_i\}_{i=1}^{n^2} \subset M_n$ be orthonormal sets in the Hilbert–Schmidt inner product. Then $\big|\text{Tr}\big(\sum_{i=1}^k A_i \otimes B_i\big)\big| \leq \sqrt{mn}$ for all $k \leq \min\{m^2, n^2\}$.*

Proof. Define vectors $u, v \in \mathbb{C}^k$ by $u_i := \text{Tr}(A_i)$ and $v_i := \text{Tr}(B_i)$. By the Cauchy–Schwarz inequality, it follows that

$$\left|\text{Tr}\left(\sum_{i=1}^k A_i \otimes B_i\right)\right| = \big|\langle \overline{u}, v\rangle\big| \leq \|u\| \cdot \|v\|.$$

To prove our result, we show that $\|u\| \leq \sqrt{m}$, with a similar argument holding for $\|v\| \leq \sqrt{n}$. We first rewrite u_i as $u_i = \langle I, A_i\rangle$. Note that the identity operator can be rewritten as

$$I = \sum_{i=1}^{m^2} \langle A_i, I\rangle A_i. \tag{7}$$

Computing the norms of the quantities in Equation (7), we obtain

$$\|I\|_F^2 = \sum_{i=1}^{m^2} |\langle A_i, I\rangle|^2 \geq \sum_{i=1}^{k} |u_i|^2 = \|u\|^2,$$

where the first equality follows from the definition of the Frobenius norm, and by noting that the set of operators $\{A_i\}_{i=1}^{m^2}$ are orthonormal. Since $\|I\|_F = m$, it directly follows that $\|u\| \leq \sqrt{m}$, as desired. $\square$

Lemma 4. *Suppose that $c \in \mathbb{R}$ is such that $\frac{1}{\sqrt{mn}} < c \leq 1$, and let $W_c \in M_m \otimes M_n$ be an entanglement witness of the form (6), scaled so that $\mathrm{Tr}(W_c) = 1$. Then*

$$\|W_c\|_F \leq \sqrt{\frac{2c}{cmn - \sqrt{mn}} + \frac{1-c^2}{(cmn - \sqrt{mn})^2}}.$$

Proof. Consider the Hermitian operator $\widetilde{W_c} := cI - \sum_{i=1}^k A_i \otimes B_i$ and let W_c be its normalization: $W_c = \widetilde{W_c}/\mathrm{Tr}(\widetilde{W_c})$. For brevity, define $t := \mathrm{Tr}\big(\sum_{i=1}^k A_i \otimes B_i\big)$. Then

$$\begin{aligned}
\|W_c\|_F^2 &= \mathrm{Tr}(W_c^\dagger W_c) \\
&= (c^2mn - 2ct + k)/\mathrm{Tr}(\widetilde{W_c})^2 \\
&\leq ((c^2+1)mn - 2ct)/(cmn - t)^2 \\
&= 2c/(cmn - t) + (1-c^2)/(cmn-t)^2 \\
&\leq 2c/(cmn - \sqrt{mn}) + (1-c^2)/(cmn - \sqrt{mn})^2
\end{aligned}$$

where the first inequality above comes from simply noting that $k \leq \min\{m^2, n^2\} \leq mn$ and the second inequality comes from Lemma 3 and using standard calculus techniques to notice that $((c^2+1)mn - 2ct)/(cmn-t)^2$ is an increasing function of t whenever $\frac{1}{\sqrt{mn}} < c \leq 1$. $\square$

We are now in a position to prove Theorem 1.

Proof of Theorem 1. We first fix $c = \frac{3}{\sqrt{mn}}$, as this is the value of c used in the statement of Theorem 1. Our goal is to make use of Lemma 2, so we want to find bounds on $\ell := (1 - \|W_c\|_{\mathrm{tr}})/2$ and the maximum eigenvalue μ_1 of W_c, where W_c is an entanglement witness of the form (6), scaled so that $\mathrm{Tr}(W_c) = 1$. Bounds on both of these quantities follow straightforwardly from Lemma 4, which tells us that when $c = \frac{3}{\sqrt{mn}}$ we have

$$\|W_c\|_F \leq \sqrt{\frac{13mn - 9}{4m^2n^2}}. \tag{8}$$

It is always the case that $\|\cdot\|_{\mathrm{tr}} \leq \sqrt{d}\|\cdot\|_F$ for $d \times d$ matrices, so Inequality (8) immediately implies that

$$\|W_c\|_{\mathrm{tr}} \leq \sqrt{mn}\|W_c\|_F \leq \frac{1}{2}\sqrt{13 - \frac{9}{mn}}.$$

Similarly, $\|\cdot\| \leq \|\cdot\|_F$ always, so we get the following bounds on ℓ and μ_1:

$$\ell \geq \frac{1}{4}\Big(2 - \sqrt{13 - \frac{9}{mn}}\Big) \geq \frac{1}{4}\big(2 - \sqrt{13}\big)$$
$$\mu_1 \leq \|W_c\| \leq \sqrt{\frac{13mn - 9}{4m^2n^2}}.$$

Furthermore, when $m, n \geq 3$ we then have the looser bound $\mu_1 \leq \frac{1}{\sqrt{3}}$. Since $f\big(\frac{1}{4}(2 - \sqrt{13})\big) \approx 0.59976\ldots \geq \frac{1}{\sqrt{3}} \geq \mu_1$, it follows from Lemma 2 that if ρ is absolutely PPT then $\mathrm{Tr}(W_c\rho) \geq 0$, so $\|R(\rho)\|_{\mathrm{tr}} \leq c = \frac{3}{\sqrt{mn}}$, as desired. □

5.2 The Choi Map

The *Choi map* [38] is a positive map on M_3 that is defined as follows:

$$\Phi_C(X) \stackrel{\text{def}}{=} \frac{1}{2}\begin{bmatrix} x_{11} + x_{22} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} + x_{33} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} + x_{11} \end{bmatrix}.$$

This map is of note because it was the first map found with the property that there are states $\rho \in M_3 \otimes M_3$ such that $(id_3 \otimes T)(\rho) \geq 0$, but $(id_3 \otimes \Phi_C)(\rho) \not\geq 0$. In other words, the Choi map was the first known example of a positive map that can detect entanglement in PPT states.

Our main result of this section is that the Choi map *cannot* detect entanglement in absolutely PPT states. Equivalently, we show that the set of "absolutely Choi map" states:

$$\mathcal{C}_{\text{abs}} \stackrel{\text{def}}{=} \big\{\rho : (id_3 \otimes \Phi_C)(U\rho U^\dagger) \geq 0 \quad \forall\,\text{unitary } U\big\}$$

is a superset of the absolutely PPT states: $\mathcal{PPT}_{\text{abs}} \subseteq \mathcal{C}_{\text{abs}}$. Thus the Choi map is a weaker entanglement test than the PPT criterion in the "absolute" setting:

Theorem 2. *If $\rho \in M_3 \otimes M_3$ is absolutely PPT then $(id_3 \otimes \Phi_C)(\rho) \geq 0$.*

As with the previous section, our first goal here is to rephrase the condition $(id_3 \otimes \Phi_C)(\rho) \not\geq 0$ in terms of entanglement witnesses, so that we can make use of Lemma 2. To this end, simply note that if $(id_3 \otimes \Phi_C)(\rho) \not\geq 0$ then there exists a pure state $|v\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$ such that $\langle v|(id_3 \otimes \Phi_C)(\rho)|v\rangle < 0$. Thus $\mathrm{Tr}((id_3 \otimes \Phi_C^\dagger)(|v\rangle\langle v|)\rho) < 0$, so

$$W := (id_3 \otimes \Phi_C^\dagger)(|v\rangle\langle v|) \tag{9}$$

is a witness that detects entanglement in ρ. It thus suffices to show that witnesses of the form (9) cannot detect entanglement in absolutely PPT states ρ. In order to prove this claim, we present the following lemma, which bounds the eigenvalues of $(id_3 \otimes \Phi_C^\dagger)(|v\rangle\langle v|)$.

Lemma 5. *Let $|v\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$ be a unit vector. Then the eigenvalues of $(id_3 \otimes \Phi_C^\dagger)(|v\rangle\langle v|)$ are contained within the interval $[-1/6, 2/3]$.*

Before proving this result, we note that both of its bounds on the eigenvalues are tight. A minimal eigenvalue of $-1/6$ is obtained when $|v\rangle = |\psi^+\rangle$ (the standard maximally-entangled state), and a maximal eigenvalue of $2/3$ is obtained when $|v\rangle = \frac{1}{\sqrt{3}}|0\rangle \otimes (|0\rangle + \sqrt{2}|1\rangle)$.

Proof. Let $\mu_1 \geq \cdots \geq \mu_9$ be the eigenvalues of $(id_3 \otimes \Phi_C^\dagger)(|v\rangle\langle v|)$. Our first goal is to show that $\mu_9 \geq -1/6$. To this end, first notice that $\Phi_C^\dagger$ is trace-preserving, so we have

$$\sum_{i=1}^{9} \mu_i = 1. \tag{10}$$

Also notice that

$$\left(\sum_{i=1}^{8} \mu_i\right) - \mu_9 \leq \sum_{i=1}^{9} |\mu_i| \leq \left\|\Phi_C^\dagger\right\|_\diamond, \tag{11}$$

where $\|\cdot\|_\diamond$ is the *diamond norm* [39] defined by

$$\left\|\Phi\right\|_\diamond \stackrel{\text{def}}{=} \sup\left\{\left\|(id_3 \otimes \Phi)(X)\right\|_{\text{tr}} : \left\|X\right\|_{\text{tr}} \leq 1\right\}.$$

By subtracting Inequality (11) from Equation (10), we see that

$$\mu_9 \geq \frac{1}{2}\left(1 - \left\|\Phi_C^\dagger\right\|_\diamond\right). \tag{12}$$

The diamond norm $\left\|\Phi_C^\dagger\right\|_\diamond$ can be computed via semidefinite programming [40], and in particular is equal to the optimal value of the following problem [41]:

$$\begin{aligned} \text{minimize:} \quad & \frac{1}{2}\left\|\text{Tr}_2(Y_0)\right\| + \frac{1}{2}\left\|\text{Tr}_2(Y_1)\right\| \\ \text{subject to:} \quad & \begin{bmatrix} Y_0 & -J(\Phi_C^\dagger) \\ -J(\Phi_C^\dagger)^\dagger & Y_1 \end{bmatrix} \geq 0 \\ & Y_0, Y_1 \geq 0, \end{aligned} \tag{13}$$

where we optimize over $Y_0, Y_1 \in M_3 \otimes M_3$ and denote $\text{Tr}_2(\cdot)$ as the partial trace with respect to the second subsystem. It is straightforward to verify that the following are feasible values of Y_0 and Y_1, written with respect to the standard basis of $\mathbb{C}^3 \otimes \mathbb{C}^3$:

$$Y_0 = Y_1 = \frac{1}{6}\begin{bmatrix} 5 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & 5 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 5 \end{bmatrix}.$$

Since $\frac{1}{2}\left\|\text{Tr}_2(Y_0)\right\| + \frac{1}{2}\left\|\text{Tr}_2(Y_1)\right\| = 4/3$, it follows that $\left\|\Phi_C\right\|_\diamond \leq 4/3$ (it is not difficult to show that it actually equals $4/3$, but we only need the upper bound). By plugging this upper bound into Inequality (12), we see that $\mu_9 \geq -1/6$, as desired.

Our next goal is to show that $\mu_1 \leq 2/3$. To show this, we first note that

$$\mu_1 \leq \sup_{|v\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3} \left\{\left\|(id_3 \otimes \Phi_C^\dagger)(|v\rangle\langle v|)\right\|\right\}.$$

By making use of [42, Theorem 4], we see that the id_3 in this quantity can be omitted, giving

$$\mu_1 \le \sup_{|v\rangle \in \mathbb{C}^3} \Big\{ \big\| \Phi_C^\dagger(|v\rangle\langle v|) \big\| \Big\}. \tag{14}$$

It was shown in [43] that the quantity on the right in Inequality (14) equals

$$\sup_{|v\rangle, |w\rangle \in \mathbb{C}^3} \big\{ (\langle v| \otimes \langle w|) J(\Phi_C^\dagger)(|v\rangle \otimes |w\rangle) \big\}. \tag{15}$$

Optimizing over separable states in general is expected to be difficult, but we can compute upper bounds of the quantity (15) via the semidefinite programming methods of [43, 44]. In particular, the following SDP takes the supremum over the set of PPT states, rather than the set of separable states, and thus computes an upper bound of the quantity (15) (and hence of μ_1):

$$\begin{array}{rlcrl} & \text{Primal problem} & & & \text{Dual problem} \\ \text{maximize:} & \mathrm{Tr}(J(\Phi_C^\dagger)\rho) & & \text{minimize:} & \lambda_{\max}\big((id_3 \otimes T)(Y) + J(\Phi_C^\dagger)\big) \\ \text{subject to:} & (id_3 \otimes T)(\rho) \ge 0 & & \text{subject to:} & Y \ge 0, \\ & \mathrm{Tr}(\rho) \le 1 & & & \\ & \rho \ge 0 & & & \end{array} \tag{16}$$

where we optimize over density matrices $\rho \in M_3 \otimes M_3$ in the primal problem and over Hermitian $Y \in M_3 \otimes M_3$ in the dual problem. It is straightforward to use semidefinite programming solvers to numerically verify that the optimal value of this semidefinite program is $2/3$, from which it follows that $\mu_1 \le 2/3$. To obtain a completely analytic proof of this fact, it suffices to find a single positive semidefinite $Y \in M_3 \otimes M_3$ such that $\lambda_{\max}\big((id \otimes T)(Y) + J(\Phi_C^\dagger)\big) = 2/3$. One such matrix is as follows:

$$Y = \frac{1}{6}\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 4 & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & 2 & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 2 & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & 4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

□

With the above lemma in hand, we are now in a position to prove the main result of this section.

Proof of Theorem 2. By Lemma 5, we know that every entanglement witness W of the form $W = (id_3 \otimes \Phi_C^\dagger)(|v\rangle\langle v|)$ has eigenvalues in the interval $[-1/6, 2/3]$. From the definition of Φ_C it is straightforward to see that every such W has at most 1 negative eigenvalue, so we know that $\ell := (1 - \|W\|_{\mathrm{tr}})/2 \ge -1/6$ as well. Since $f(-1/6) = \frac{1}{12}(10 + \sqrt{2}) \approx 0.9512\ldots \ge \frac{2}{3}$, it follows from Lemma 2 that W cannot detect entanglement in any absolutely PPT state, so neither can the Choi map Φ_C. □

5.3 Generalized Choi Maps

In the proof of Theorem 2, there was a rather large gap between the largest eigenvalue of $(id_3 \otimes \Phi_C^\dagger)(|v\rangle\langle v|)$ (which was $2/3$) and the quantity that we needed to bound this eigenvalue by (which was $\frac{1}{12}(10+\sqrt{2}) \approx 0.9512\ldots$). This suggests that any positive map that is sufficiently close to the Choi map also cannot detect entanglement in absolutely PPT states, since perturbing the Choi map will only slightly change both of these values. This section is devoted to making this statement more rigorous and precise, by investigating a well-known family of positive maps that generalize the Choi map.

We now introduce an infinite family of positive maps based on two real parameters $b, c \geq 0$ that were first studied in [32] (see also [45, 46]). If we let $a := 2 - b - c$, then these maps are defined as follows:

$$\Phi_{b,c}(X) \stackrel{\text{def}}{=} \frac{1}{2} \begin{bmatrix} ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{22} + bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11} + cx_{22} + ax_{33} \end{bmatrix}.$$

Notice that the Choi map Φ_C is recovered in the $(b, c) = (1, 0)$ case. Additionally, in the $(b, c) = (1, 1)$ case the map has the form $\Phi_{1,1}(X) = \frac{1}{2}\big(\mathrm{Tr}(X)I - X\big)$, which is the well-known *reduction map* [47].

It is known that $\Phi_{b,c}$ is positive but not completely positive (i.e., capable of detecting entanglement in some state) if and only if $(b, c) \neq (0, 0)$, and either $b + c \leq 1$ or $bc \geq (b + c - 1)^2$ (or both). Furthermore, it is *indecomposable* (i.e., it detects some PPT entanglement) if and only if $b \neq c$, and it is an exposed point in the convex set of positive maps (and hence extreme and optimal) [48] if $b \neq c$, $b + c > 1$, and $bc = (b + c - 1)^2$ (see Figure 3).

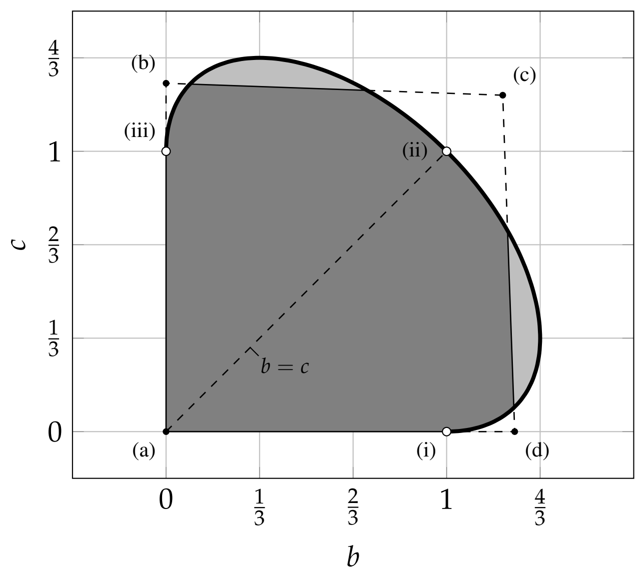

Figure 3: A plot of the set of positive but not completely positive maps $\Phi_{b,c}$. The gray region indicates the values of (b, c) for which $\Phi_{b,c}$ is positive. For every point (b, c) in the gray region, with the exception of the dashed line $b = c$, the map $\Phi_{b,c}$ is indecomposible, and hence is able to detect entanglement in some PPT state. The curved part of the boundary (i.e., the thick black line) consists of exposed positive maps. The four points (a)–(d) are the points described by Theorem 3, and hence every map in the dark gray region is incapable of detecting entanglement in absolutely PPT states. The point (i) is the Choi map Φ_C, (iii) is its dual $\Phi_C^\dagger$, and (ii) is the reduction map. This figure is also reproduced and considered in the following works [46, 45]

As with the previous sections, our goal is to show that the maps $\Phi_{b,c}$ cannot detect entanglement in any absolutely PPT state. While we are not able to prove this for *all* positive $\Phi_{b,c}$, we are able to prove it for most of them, including many of which are exposed in the set of positive maps.

Theorem 3. *Suppose that the point (b,c) is contained within the convex hull of the following 4 points:*

(a) $(0,0)$,

(b) $(0,3(\sqrt{2}-1))$,

(c) $(6/5,6/5)$,

(d) $(3(\sqrt{2}-1),0)$.

If $\rho \in M_3 \otimes M_3$ is absolutely PPT then $(id_3 \otimes \Phi_{b,c})(\rho) \geq 0$.

As with Theorem 2, our method of proof is to come up with bounds on the eigenvalues of $(id_3 \otimes \Phi_{b,c}^\dagger)(|v\rangle\langle v|)$. We now present a lemma that gives a lower bound.

Lemma 6. *Let $|v\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$ and suppose $b,c \geq 0$ satisfy $b+c \leq 3$. Then the eigenvalues of $(id_3 \otimes \Phi_{b,c}^\dagger)(|v\rangle\langle v|)$ are no smaller than $-\frac{1}{6}(b+c)$.*

Proof. We note that the nuts and bolts of the proof are almost identical to the proof of the lower bound given in Lemma 5, so it suffices to just present a solution to the semidefinite program (13) that works for all $\Phi_{b,c}$ rather than just Φ_C. Indeed, it is straightforward to check that the following is a feasible point of the semidefinite program:

$$Y_0 = Y_1 = \frac{1}{6}\begin{bmatrix} 6-b-c & \cdot & \cdot & \cdot & 2b+2c-3 & \cdot & \cdot & \cdot & 2b+2c-3 \\ \cdot & 3b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 3c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 3c & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2b+2c-3 & \cdot & \cdot & \cdot & 6-b-c & \cdot & \cdot & \cdot & 2b+2c-3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 3b & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3c & \cdot \\ 2b+2c-3 & \cdot & \cdot & \cdot & 2b+2c-3 & \cdot & \cdot & \cdot & 6-b-c \end{bmatrix},$$

where we have used the constraints on b and c given in the statement of the lemma to ensure that the positive semidefiniteness requirements of the SDP (13) are satisfied. The corresponding value of the objective function is

$$\begin{aligned}\big\|\mathrm{Tr}_2(Y_0)\big\| &= \frac{1}{3}\left\|\begin{bmatrix} 3+b+c & 0 & 0 \\ 0 & 3+b+c & 0 \\ 0 & 0 & 3+b+c \end{bmatrix}\right\| \\ &= \frac{1}{3}(3+b+c).\end{aligned}$$

It follows that $\|\Phi_{b,c}^\dagger\|_\diamond \leq \frac{1}{3}(3+b+c)$, so from Equation (12) it follows that if μ_9 is the minimal eigenvalue of $(id_3 \otimes \Phi_{b,c}^\dagger)(|v\rangle\langle v|)$ then $\mu_9 \geq \frac{1}{2}(1 - \|\Phi_{b,c}^\dagger\|_\diamond) \geq -\frac{1}{6}(b+c)$, as desired. □

Upper bounds on the eigenvalues of $(id_3 \otimes \Phi_{b,c}^\dagger)(|v\rangle\langle v|)$ seem to be much messier than the lower bound given by Lemma 6, so we defer their discussion to Appendix A, where we complete the proof of Theorem 3.

Theorem 3 shows analytically that all of the maps in the dark shaded region of Figure 3 are unable to detect entanglement in absolutely PPT states. It is natural to ask whether or not the same is true of the positive maps in the light shaded region. We do not have an analytic proof that this is the case, but numerical evidence suggests that it is. In particular, we randomly generated 10^8 pure states $|v\rangle$ and values of (b,c) in the light shaded region of Figure 3, and found that every time the eigenvalues of $(id_3 \otimes \Phi_{b,c}^\dagger)(|v\rangle\langle v|)$ satisfied the hypotheses of Lemma 2.

5.4 Breuer–Hall Map

The *Breuer–Hall map* [33, 34] is a positive map on M_n, where $n \geq 4$ is even, that is defined as follows:

$$\Phi_{BH}(X) \stackrel{\text{def}}{=} \frac{1}{n-2}\left(\mathrm{Tr}(X)I - X - VX^TV^\dagger\right),$$

where $V \in M_n$ is a given unitary matrix that is skew symmetric (anti-symmetric), i.e. $V^T = -V$. We note that the reason for restricting to even n is because such unitary matrices exist if and only if n is even. One such unitary matrix is as follows:

$$V = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & -1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and it follows straightforwardly from [49, Theorem 2.3] that for our purposes it suffices to restrict attention to this particular V (i.e., there exists a Breuer–Hall map detecting entanglement in some absolutely PPT state if and only if the Breuer–Hall map that arises from this particular V detects entanglement in some absolutely PPT state).

Much like the Choi map and generalized Choi maps, the Breuer–Hall map is capable of detecting entanglement in PPT states. We now show, in a manner similar to the proof of Theorem 2, that these maps *cannot* detect entanglement in absolutely PPT states. Phrased differently, we show that the set of "absolutely Breuer–Hall" states:

$$\mathcal{BH}_{\text{abs}} \stackrel{\text{def}}{=} \{\rho : (id_n \otimes \Phi_{BH})(U\rho U^\dagger) \geq 0 \quad \forall \,\text{unitary } U\}$$

is a superset of the absolutely PPT states: $\mathcal{PPT}_{\text{abs}} \subseteq \mathcal{BH}_{\text{abs}}$.

Theorem 4. *Let $n \geq 4$ be even. If $\rho \in M_n \otimes M_n$ is absolutely PPT then $(id_n \otimes \Phi_{BH})(\rho) \geq 0$.*

Just like in the proof of Theorem 2, our goal is to bound the eigenvalues of operators of the form $(id_n \otimes \Phi_{BH}^\dagger)(|v\rangle\langle v|)$. The following lemma provides such a bound.

Lemma 7. *Let $|v\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$, where $n \geq 4$ is even. Then the eigenvalues of $(id_n \otimes \Phi_{BH}^\dagger)(|v\rangle\langle v|)$ are contained within the interval $[-1/n, 1/(n-2)]$.*

Proof. The proof follows the same construction as the proof of Theorem 2 and uses the same semidefinite programs. Let $\mu_1 \geq \cdots \geq \mu_{n^2}$ be the eigenvalues of $(id_n \otimes \Phi_{BH}^\dagger)(|v\rangle\langle v|)$. We first show that $\mu_1 \leq$

$1/(n-2)$. To achieve this, we must find a feasible point Y of the SDP (16) (replacing Φ_C by Φ_{BH}) such that the objective function has the value $1/(n-2)$ at that point. One choice of Y that works is $Y = \frac{n}{n-2}(I \otimes V)|\psi^+\rangle\langle\psi^+|(I \otimes V^\dagger)$, which is clearly positive semidefinite and thus a feasible point. The corresponding value of the objective function is

$$\lambda_{\max}\big(J(\Phi_{BH}^\dagger) + (id_n \otimes T)(Y)\big) = \frac{1}{n-2}\lambda_{\max}\big(I - n|\psi^+\rangle\langle\psi^+|\big) = \frac{1}{n-2},$$

as desired.

We next show that $\mu_{n^2} \geq -1/n$. To proceed, we calculate the diamond norm $\|\Phi_{BH}^\dagger\|_\diamond$ using the SDP (13), where we optimize over $Y_0, Y_1 \in M_n \otimes M_n$. We now show that the following choice of Y_0 and Y_1 is a dual feasible point that achieves this bound:

$$Y_0 = Y_1 = J(\Phi_{BH}^\dagger) + 2|\psi^+\rangle\langle\psi^+|.$$

It is straightforward to show that $P := Y_0 - |\psi^+\rangle\langle\psi^+|$ is (up to scaling) an orthogonal projection, so Y_0 is positive semidefinite. Next observe that

$$\begin{bmatrix} Y_0 & -J(\Phi_{BH}^\dagger) \\ -J(\Phi_{BH}^\dagger)^\dagger & Y_1 \end{bmatrix} = \begin{bmatrix} P & -P \\ -P & P \end{bmatrix} + \begin{bmatrix} |\psi^+\rangle\langle\psi^+| & |\psi^+\rangle\langle\psi^+| \\ |\psi^+\rangle\langle\psi^+| & |\psi^+\rangle\langle\psi^+| \end{bmatrix} \geq 0,$$

from which it follows that the given choice of Y_0, Y_1 is a feasible point of the SDP (13).

To compute the corresponding value of the objective function, we note that $\mathrm{Tr}_2(Y_0) = \frac{n}{n-2}I - \frac{1}{n-2}I - \frac{1}{n-2}I + \frac{2}{n}I = \frac{n+2}{n}I$, so $\frac{1}{2}\big\|\mathrm{Tr}_2(Y_0)\big\| + \frac{1}{2}\big\|\mathrm{Tr}_2(Y_1)\big\| = \frac{n+2}{n}$. It follows that $\big\|\Phi_{BH}^\dagger\big\|_\diamond \leq \frac{n+2}{n}$. By plugging this upper bound into Inequality (12) we see that $\mu_{n^2} \geq -1/n$, as desired, which concludes the proof. $\square$

Now that we have bounds on the eigenvalues of the witnesses of the form $(id_n \otimes \Phi_{BH}^\dagger)(|v\rangle\langle v|)$, we can prove the main result of this section.

Proof of Theorem 4. By Lemma 7, we know that every entanglement witness W of the form $W = (id_n \otimes \Phi_{BH}^\dagger)(|v\rangle\langle v|)$ has eigenvalues in the interval $[-1/n, 1/(n-2)]$. From the definition of Φ_{BH} it is straightforward to see that every such W has at most 1 negative eigenvalue, so we know that $\ell := (1 - \|W\|_{\mathrm{tr}})/2 \geq -1/n$ as well.

For all $n \geq 4$, we have $f(\ell) = f(-1/n) \geq f(-1/4) = (1+\sqrt{2})/4 \approx 0.6035\ldots \geq 1/2 \geq 1/(n-2) = \mu_1$. It follows from Lemma 2 that W cannot detect entanglement in any absolutely PPT state, so neither can Φ_{BH}. $\square$

6 Special Classes of States

Up until now we have focused on proving that certain separability criteria are incapable of detecting entanglement in absolutely PPT states. Now we shift focus a bit and prove that the absolute separability and absolute PPT properties coincide on certain important families of quantum states.

6.1 Werner States

The first family of states that we investigate are the *Werner states* [1], which are defined via the single real parameter $\alpha \in [-1, 1]$ by

$$\rho = (I - \alpha S)/(n^2 - n\alpha) \in M_n \otimes M_n,$$

where $S \in M_n \otimes M_n$ is the *swap* operator defined by $S(|a\rangle \otimes |b\rangle) = |b\rangle \otimes |a\rangle$. It is well-known that the Werner state ρ is separable if and only if it is PPT if and only if $\alpha \leq 1/n$. We now show that the Werner states that are absolutely separable and those that are absolutely PPT also coincide.

Theorem 5. *Let $\rho = (I - \alpha S)/(n^2 - n\alpha) \in M_n \otimes M_n$ be a Werner state. Then ρ is absolutely separable if and only if ρ is absolutely PPT if and only if $\alpha \leq 1/n$.*

Proof. Throughout this proof, we work with the operator $X := I - \alpha S$, which is equal to ρ up to normalization (and thus has the same separability and PPT properties as ρ). We first prove that $\alpha \leq 1/n$ implies that X is absolutely separable (and hence absolutely PPT as well). To this end, notice that $UXU^\dagger = I - \alpha USU^\dagger$. Since $\|\alpha USU^\dagger\|_2 = \|\alpha S\|_2 = \alpha\|S\|_2 = \alpha n$, it follows from [15, Theorem 1] that $UXU^\dagger$ is separable whenever $\alpha n \leq 1$ (i.e., $\alpha \leq 1/n$).

We now prove that if X is absolutely PPT then $\alpha \leq 1/n$. Notice that the eigenvalues of X are $1 - \alpha$ with multiplicity $n(n+1)/2$ and $1 + \alpha$ with multiplicity $n(n-1)/2$. It follows from a straightforward calculation that each of the LMIs described by [17] corresponding to the set of absolutely PPT states is of the form

$$\begin{bmatrix} 2-2\alpha & -2\alpha & -2\alpha & \cdots & -2\alpha \\ -2\alpha & 2-2\alpha & -2\alpha & \cdots & -2\alpha \\ -2\alpha & -2\alpha & 2-2\alpha & \cdots & -2\alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2\alpha & -2\alpha & -2\alpha & \cdots & 2-2\alpha \end{bmatrix} \geq 0.$$

It is straightforward to see that $[1, 1, \ldots, 1]^T$ is an eigenvector of the above matrix with corresponding eigenvalue $2 - 2n\alpha$. It follows that $2 - 2n\alpha \geq 0$, so $\alpha \leq 1/n$, as desired. □

6.2 Isotropic States

The family of isotropic states is defined by the single real parameter $\alpha \in \left[-\frac{1}{n^2-1}, 1\right]$ via

$$\rho = \frac{1-\alpha}{n^2} I + \alpha|\psi^+\rangle\langle\psi^+| \in M_n \otimes M_n, \tag{17}$$

where we recall that $|\psi^+\rangle = \frac{1}{\sqrt{n}}\sum_{i=1}^n |i\rangle \otimes |i\rangle$ is the standard maximally-entangled state. It is well-known that the isotropic state ρ is separable if and only if it is PPT if and only if $\alpha \leq \frac{1}{n+1}$. We now show that the isotropic states that are absolutely separable and those that are absolutely PPT also coincide.

Theorem 6. *Let $\rho = \frac{1-\alpha}{n^2} I + \alpha|\psi^+\rangle\langle\psi^+| \in M_n \otimes M_n$ be an isotropic state. Then ρ is absolutely separable if and only if ρ is absolutely PPT if and only if $\alpha \leq \frac{2}{2+n^2}$.*

Proof. We first show that if ρ is absolutely PPT then $\alpha \leq \frac{2}{2+n^2}$. Notice that the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n^2}$ of ρ are

$$\lambda_1 = \alpha + \frac{1-\alpha}{n^2}, \quad \lambda_i = \frac{1-\alpha}{n^2} \quad \forall\, 2 \leq i \leq n^2.$$

Since we are assuming that ρ is absolutely PPT, we can plug these eigenvalues into the LMIs described in [17] to obtain

$$\frac{1}{n^2}\begin{bmatrix} 2-2\alpha & -n^2\alpha & 0 & \cdots & 0 \\ -n^2\alpha & 2-2\alpha & 0 & \cdots & 0 \\ 0 & 0 & 2-2\alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2-2\alpha \end{bmatrix} \geq 0.$$

The positivity of the above matrix implies that $n^2\alpha \leq 2 - 2\alpha$, so $\alpha \leq \frac{2}{2+n^2}$, as desired.

We now prove that if $\alpha \leq \frac{2}{2+n^2}$ then ρ is absolutely separable. First notice that $U\rho U^\dagger = \frac{1-\alpha}{n^2}I + \alpha|v\rangle\langle v|$, where $|v\rangle := U|\psi^+\rangle$. If we absorb constants in a different way, we obtain

$$U\rho U^\dagger = \alpha\left(\frac{1-\alpha}{n^2\alpha}I + |v\rangle\langle v|\right), \tag{18}$$

which we want to show is separable for all $|v\rangle$.

To this end, we recall that it was shown in [50] that the operator (18) is separable if and only if $\frac{1-\alpha}{n^2\alpha} \geq \gamma_1\gamma_2$, where $\gamma_1 \geq \gamma_2$ are the two largest Schmidt coefficients of $|v\rangle$. We can see that this is true for all $|v\rangle$ simply by noting that $\frac{1-\alpha}{n^2\alpha} \geq 1/2$ (which follows from the fact that $\alpha \leq \frac{2}{2+n^2}$), and $1/2 \geq \gamma_1\gamma_2$ (which can be seen by maximizing $\gamma_1\gamma_2$ subject to the constraint that $\gamma_1^2 + \gamma_2^2 \leq 1$). It follows that ρ is absolutely separable, as desired. □

6.3 UPB States

One of the more well-known methods of constructing PPT entangled states comes from the notion of an *unextendible product basis (UPB)* [29], which is a set of mutually orthogonal product states with no other product state orthogonal to all of them. Given a UPB $\mathcal{S} \subset \mathbb{C}^m \otimes \mathbb{C}^n$, it is straightforward to check that the following state is PPT and entangled:

$$\rho = \frac{1}{mn - |\mathcal{S}|}\left(I - \sum_{|v_i\rangle\in\mathcal{S}} |v_i\rangle\langle v_i|\right). \tag{19}$$

States of the form (19) cannot possibly be absolutely PPT since they are entangled yet have rank $mn - |\mathcal{S}| < mn$, which contradicts Proposition 1. However, we can follow the approach of [31] by letting $0 < p < 1$ be a real number and considering the full-rank state $\rho_p := pI/(mn) + (1-p)\rho$. As p increases from 0 to 1, the state ρ_p becomes arbitrarily close to the maximally-mixed state $I/(mn)$ and thus becomes absolutely PPT (and even absolutely separable) when p is large enough. We now investigate the question of whether or not there exists ρ_p that is absolutely PPT but not absolutely separable.

Theorem 7. *Let $\rho \in M_3 \otimes M_3$ be a state constructed via a UPB, as in Equation (19). Then ρ_p is absolutely PPT if and only if $p \geq 9(10-\sqrt{17})/83 \approx 0.6373\ldots$ Furthermore, ρ_p is absolutely separable if $p \geq 1 - 1/\sqrt{10} \approx 0.6838\ldots$*

Proof. We first show that ρ_p is absolutely PPT if and only if $p \geq 9(10-\sqrt{17})/83$. We first recall that all UPBs in $\mathbb{C}^3 \otimes \mathbb{C}^3$ have five states [51], so ρ_p has eigenvalues $p/9$ with multiplicity 5 and $p/9 + (1-p)/4 = (9-5p)/36$ with multiplicity 4. It follows that the two LMIs determining whether or not ρ_p is absolutely PPT are both as follows:

$$\frac{1}{36}\begin{bmatrix} 8p & 9p-9 & 9p-9 \\ 9p-9 & 8p & 9p-9 \\ 9p-9 & 9p-9 & 18-10p \end{bmatrix} \geq 0. \tag{20}$$

It is straightforward to check that the LMI (20) holds if and only if $p \geq 9(10-\sqrt{17})/83$, as claimed.

To see that ρ_p is absolutely separable when $p \geq 1 - 1/\sqrt{10}$, note that it suffices to consider the $p = 1 - 1/\sqrt{10}$ case. Define the operator $X := 8\rho_p = 8(1-1/\sqrt{10})I/9 + 8\rho/\sqrt{10}$. Then

$$\begin{aligned}\|X - I\|_F^2 &= \|8\rho/\sqrt{10} - (1 + 8/\sqrt{10})I/9\|_F \\ &= 4\big(2/\sqrt{10} - (1+8/\sqrt{10})/9\big)^2 + 5\big((1+8/\sqrt{10})/9\big)^2 \\ &= 1.\end{aligned}$$

It follows from [15, Theorem 1] that X is absolutely separable, so ρ_p is too. $\square$

From Theorem 7 we see that there is an interval approximately equal to $[0.6373, 0.6838)$ for which ρ_p is absolutely PPT, but we do not know whether or not it is absolutely separable. In the $p = 0.6373$ case, we have tried to detect entanglement in the state $U\rho_p U^\dagger$ for 10^5 randomly-generated (according to Haar measure) unitary matrices U. We used every entanglement criterion that is known to us, including the tests based on covariance matrices [52] and the extremely strong tests based on the 3-copy PPT symmetric extensions [53], and no entanglement was ever detected in any of these states. These numerical results seem to suggest that the state ρ_p is absolutely separable when $p = 0.6373$ (and thus for all $p \in [0.6373, 0.6838)$).

7 Outlook

We have investigated the absolute separability problem and provided various forms of evidence to support Conjecture 1 that the sets of absolutely separable and absolutely PPT states coincide.

Certainly the most notable open problem resulting from this work is to prove or disprove Conjecture 1, but other interesting questions also arise from our work. For instance, are there other separability criteria that can be shown to be incapable of detecting entanglement in absolutely PPT states? We have shown that this is the case for the range criterion, realignment criterion, Choi map and its generalizations, and the Breuer–Hall map. However, there are other separability criteria such as those based on covariance matrices [52] and symmetric extensions [53] for which we still do not know the answer.

Other open problems in this work include proving that *all* of the generalized Choi maps $\Phi_{b,c}$ in Section 5.3 are incapable of detecting absolutely PPT entanglement (including those in the light gray region of Figure 3), and proving that the UPB states of Section 6.3 are absolutely separable when $p \in [0.6373, 0.6838)$. We have provided numerical evidence that both of these claims are true, but we have been unable to find an analytic proof of either fact.

It would also be beneficial to develop other easily-checkable conditions that imply that a given entanglement witness cannot detect entanglement in absolutely PPT states. Almost all of our results follow from Lemma 2, which gives a test based on the witness's maximal eigenvalue and the sum of its negative eigenvalues. However, it suffers from the drawback that the function f (see Figure 2) is not continuous, which makes it difficult to use sometimes. Are there other functions of the eigenvalues of an entanglement witness that can be used in its place?

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8 Appendix A. Proof of Theorem 3

We now prove some upper bounds on the eigenvalues of $(id_3 \otimes \Phi_{b,c}^\dagger)(|v\rangle\langle v|)$, which allow us to prove Theorem 3. We note that these upper bounds are much more complicated than the lower bound given by Lemma 6, so the upper bounds are illustrated in Figure 4 for clarity.

Lemma 8. *Let $|v\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$ and suppose $b, c \geq 0$ are such that $b + c \geq \frac{2}{3}$. We split into two cases:*

- *If $2b + c \geq 3$ or $b + 2c \geq 3$, then the eigenvalues of $(id_3 \otimes \Phi_{b,c}^\dagger)(|v\rangle\langle v|)$ are no larger than $\max\{b, c\}/2$.*

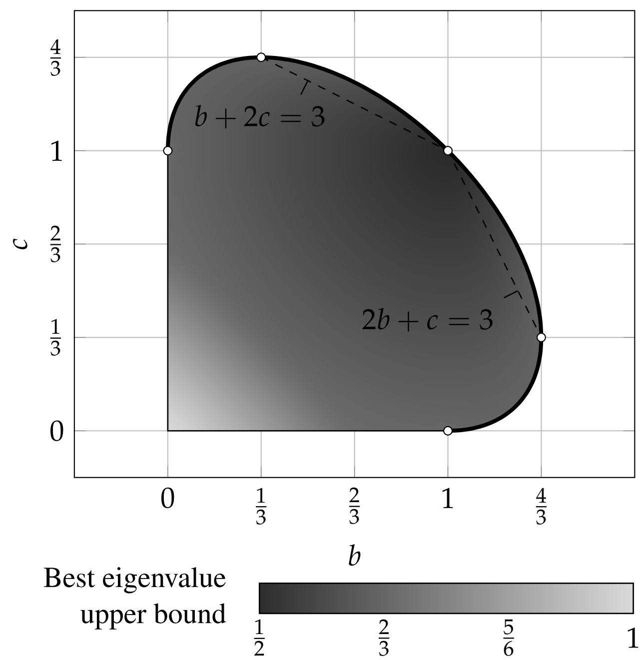

Figure 4: A plot of the best possible upper bound on the eigenvalues of $(id_3 \otimes \Phi_{b,c}^{\dagger})(|v\rangle\langle v|)$, which was computed numerically using the semidefinite program (16). This bound agrees with the bounds provided by Lemma 8 whenever $b + c \geq \frac{2}{3}$. The area above and to the right of the two dashed lines is covered by the first case of that lemma, whereas the area below and to the left of the dashed lines is covered by the second case of the lemma.

- *Otherwise, the eigenvalues of* $(id_3 \otimes \Phi_{b,c}^{\dagger})(|v\rangle\langle v|)$ *are no larger than*

$$\frac{b^2 + c^2 - 6(b+c) + bc + 9}{6(2-b-c)}.$$

Proof. We can get an upper bound of the eigenvalues of $(id_3 \otimes \Phi_{b,c}^{\dagger})(|v\rangle\langle v|)$ in a manner similar to that which was used in the proof of Lemma 5. In particular, it suffices to give a feasible point for the dual program of the SDP (16), with Φ_C replaced by $\Phi_{b,c}$.

To this end, we start by considering the first case (i.e., the case where $2b + c \geq 3$ or $b + 2c \geq 3$). It suffices to take $Y = 0$ in the dual program of the SDP (16). Then we can write $J(\Phi_{b,c}^{\dagger})$ in the standard basis of $\mathbb{C}^3 \otimes \mathbb{C}^3$ as follows:

$$\frac{1}{2}\begin{bmatrix} 2-b-c & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & 2-b-c & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 2-b-c \end{bmatrix},$$

from which it follows that $\lambda_{\max}\big(J(\Phi_C^\dagger)\big) = \max\{b, c, 3-b-c\}/2$. Since $2b + c \geq 3$ or $b + 2c \geq 3$ it follows that $\max\{b, c, 3-b-c\}/2 = \max\{b, c\}/2$, which shows that the SDP (16) has optimal value no larger than $\max\{b, c\}/2$ and competes the proof of this case.

We now consider the other case (i.e., we assume that $2b + c < 3$ and $b + 2c < 3$). Define the two quantities $x := (3-2b-c)^2/(6(2-b-c))$ and $y := (3-b-2c)^2/(6(2-b-c))$. Then consider the following operator, written with respect to the standard basis of $\mathbb{C}^3 \otimes \mathbb{C}^3$:

$$Y = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & x & \cdot & \sqrt{xy} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & y & \cdot & \cdot & \cdot & \sqrt{xy} & \cdot & \cdot \\ \cdot & \sqrt{xy} & \cdot & y & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot & \sqrt{xy} & \cdot \\ \cdot & \cdot & \sqrt{xy} & \cdot & \cdot & \cdot & x & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{xy} & \cdot & y & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

It is straightforward to see that $Y \geq 0$ and is thus a feasible point of the SDP (16). To see what the corresponding value of the objective function is, we compute

$$(id \otimes T)(Y) + J(\Phi_{b,c}^\dagger) =$$
$$\frac{1}{2}\begin{bmatrix} 2-b-c & \cdot & \cdot & \cdot & 2\sqrt{xy}-1 & \cdot & \cdot & \cdot & 2\sqrt{xy}-1 \\ \cdot & b+2x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c+2y & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c+2y & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2\sqrt{xy}-1 & \cdot & \cdot & \cdot & 2-b-c & \cdot & \cdot & \cdot & 2\sqrt{xy}-1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & b+2x & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b+2x & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c+2y & \cdot \\ 2\sqrt{xy}-1 & \cdot & \cdot & \cdot & 2\sqrt{xy}-1 & \cdot & \cdot & \cdot & 2-b-c \end{bmatrix}.$$

It is straightforward to verify that $b + 2x = c + 2y = 2 - b - c - (2\sqrt{xy} - 1)$ for the given choice of x and y, from which it follows that

$$\begin{aligned} &(id \otimes T)(Y) + J(\Phi_{b,c}^\dagger) \\ &= \frac{1}{2}\Big((b+2x)I + 3(2\sqrt{xy}-1)|\psi^+\rangle\langle\psi^+|\Big). \end{aligned}$$

Since $2\sqrt{xy} \leq 1$ whenever b and c satisfy the constraints of this case, we see that the maximum eigenvalue of $(id \otimes T)(Y) + J(\Phi_{b,c}^\dagger)$ equals $b/2 + x = \frac{b^2+c^2-6(b+c)+bc+9}{6(2-b-c)}$, so this quantity is an upper bound on the optimal value of the SDP (16), as desired. $\square$

Now that we have these upper bounds of Lemma 8 to work with, we are finally in a position to prove Theorem 3.

Proof of Theorem 3. We first note that it suffices to prove Theorem 3 for the four maps $\Phi_{b,c}$ given by the points (a)–(d) that it describes, since the result then immediately follows for any convex combination of those maps. We consider these four maps now, one at a time.

Case (a)**:** $(b,c) = (0,0)$. It is straightforward to check that $\Phi_{0,0}$ is completely positive, so the result is trivial.

For each of the remaining cases, we follow the notation of Lemma 2 and use ℓ to denote the sum of the negative eigenvalues of $(id_3 \otimes \Phi_{b,c}^{\dagger})(|v\rangle\langle v|)$, we use μ_1 to denote its maximal eigenvalue, and we use f to denote the function described by Lemma 2. Furthermore, we note that $(id_3 \otimes \Phi_{b,c}^{\dagger})(|v\rangle\langle v|)$ has at most one negative eigenvalue whenever $b + c \leq 3$, so any lower bound on the eigenvalues of $(id_3 \otimes \Phi_{b,c}^{\dagger})(|v\rangle\langle v|)$ immediately applies to ℓ as well.

Case (b)**:** $(b,c) = (0, 3(\sqrt{2}-1))$. We know from Lemma 6 that $\ell \geq -\frac{1}{6}(b+c) = \frac{1}{2}(1-\sqrt{2})$. Furthermore, plugging this choice of b and c into Lemma 8 shows that $\mu_1 \leq \frac{1}{7}(9-3\sqrt{2})$. Since $\frac{1}{7}(9-3\sqrt{2}) \approx 0.6796\ldots \leq f(\frac{1}{2}(1-\sqrt{2})) = \frac{1}{4}(2+\sqrt{2}) \approx 0.8535\ldots$, the result follows from Lemma 2.

Case (c)**:** $(b,c) = (6/5, 6/5)$. We know from Lemma 6 that $\ell \geq -\frac{1}{6}(b+c) = -\frac{2}{5}$. Furthermore, plugging this choice of b and c into Lemma 8 shows that $\mu_1 \leq \frac{3}{5}$. Since $\frac{3}{5} \leq f(-\frac{2}{5}) = \frac{3}{5}$, the result follows from Lemma 2.

Case (d)**:** $(b,c) = (3(\sqrt{2}-1), 0)$. Almost identical to case (b). □