BEATTY SEQUENCES, FIBONACCI NUMBERS, AND THE GOLDEN RATIO

VINCENT RUSSO AND LOREN SCHWIEBERT

Abstract. $(\lfloor n\phi \rfloor)_{n\geq 1}$ and $\left(\lfloor n\phi^2 \rfloor\right)_{n\geq 1}$ are well-known complementary Beatty sequences. An infinite set of complementary Beatty sequences, based on a generalization of ratios of Fibonacci numbers and higher powers of ϕ, is proved. An open problem posed by Clark Kimberling, the *Swappage Problem*, is resolved in the affirmative as a special case of this set of complementary Beatty sequences.

1. Introduction

1.1. **Beatty Sequences.** A Beatty sequence ([9], [1], [10]) is generated by an irrational $\alpha > 1$ as follows:

$$(\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots) = (\lfloor n\alpha \rfloor)_{n\geq 1}.$$

Letting β be the number satisfying $1/\alpha + 1/\beta = 1$, the sequences $(\lfloor n\alpha \rfloor)_{n\geq 1}$ and $(\lfloor n\beta \rfloor)_{n\geq 1}$ are *complementary Beatty sequences*. See for example [6].

It is well known that if $\alpha = \phi$, where ϕ denotes the golden ratio, then $\beta = \alpha + 1$. The corresponding sequence, $(\lfloor n\alpha \rfloor)_{n\geq 1} = (1, 3, 4, 6, \ldots)$, is the *lower Wythoff sequence* [7]. The complementary Beatty sequence, $(\lfloor n\beta \rfloor)_{n\geq 1} = (2, 5, 7, 10, \ldots)$, is the *upper Wythoff sequence* [7].

2. Beatty Sequences

As noted above, when $\alpha = \phi$, we obtain $\beta = \alpha + 1$. From the identity $\phi^2 = \phi + 1$, we may rewrite β as $\beta = \phi^2$. These complementary Beatty sequences form the basis for theorem 2.2, which uses the following well-known result:

Lemma 2.1.

$$\phi^{k+1} = F_{k+1}\phi + F_k.$$

Proof. This is easily proved by induction. For the base case, observe that $F_1 = F_2 = 1$ and $\phi^2 = \phi + 1$. For the inductive case, assume equality holds $\forall k < n$. Then,

$$\begin{aligned}
\phi^{n+1} &= \phi(F_n\phi + F_{n-1}) \\
&= F_n(\phi + 1) + F_{n-1}\phi \\
&= (F_n + F_{n-1})\phi + F_n \\
&= F_{n+1}\phi + F_n
\end{aligned}$$

□

Theorem 2.2.

$$\forall i \geq 1 \qquad \left(\left\lfloor \frac{n\phi^i}{F_{i+1}} \right\rfloor \right)_{n\geq 1} \; \textit{and} \; \left(\left\lfloor \frac{n\phi^{i+1}}{F_i} \right\rfloor \right)_{n\geq 1}$$

are complementary Beatty sequences.

Proof.

$$\begin{aligned} \frac{F_{k+1}}{\phi^k} + \frac{1}{\beta} &= 1 \\ F_{k+1}\beta + \phi^k &= \phi^k\beta \\ \beta &= \frac{\phi^k}{\phi^k - F_{k+1}} \\ &= \frac{\phi^{k+1}}{\phi^{k+1} - F_{k+1}\phi} \\ &= \frac{\phi^{k+1}}{\phi^{k+1} - \phi^{k+1} + F_k} \qquad \text{by Lemma 2.1} \\ \beta &= \frac{\phi^{k+1}}{F_k} \end{aligned}$$

□

This theorem can be applied as a special case to an open problem posed by Clark Kimberling, the *Swappage Problem* [8], which is resolved in §3. Through some mathematical manipulation, an alternative derivation of the swappage sequence is provided, which is used to resolve the *Swappage Problem* in the affirmative.

3. The Swappage Problem

3.1. **Problem Statement.** Let $L = (1, 3, 4, 6, 8, \ldots)$ be the *Lower Wythoff Sequence* [2]. Similarly, let U be the complement L' of L; i.e., $U = (2, 5, 7, 10, \ldots)$ is the *Upper Wythoff Sequence* [3]. For each odd $U(n)$, let $L(m)$ be the least number in L such that after swapping $U(n)$ and $L(m)$, the resulting new sequences are both increasing. The resulting sequence derived by swapping these elements, called the *swappage* of L, is $V = (2, 4, 6, 10, 12, 14, 18, 20, 22, 26, \ldots)$ [5].

Let $S(n) = \frac{V(n)}{2}$ for every n. Is the complement of S (in the set of nonnegative integers) the same set of numbers that comprise the sequence:

$$\left(\lfloor n\phi^3 \rfloor \right)_{n\geq 0} = (0, 4, 8, 12, 16, 21, 25, 29, \ldots)?$$

3.2. **Solution.** The sequence V is generated by the swapping algorithm described in the problem statement. We first prove that V can be derived by an entirely different method, one that requires no swapping. We then prove that $S' = \left(\lfloor n\phi^3 \rfloor \right)_{n\geq 0}$. The following sequence, labeled W, is also derived from U without any swapping.

$$W(n) = \begin{cases} U(n) & \text{if } U(n) \text{ is even.} \\ U(n) - 1 & \text{if } U(n) \text{ is odd.} \end{cases}$$

The first few terms of W are given below:

$$\begin{aligned} U &= (2, 5, 7, 10, 13, 15, 18, 20, \dots) \\ W &= (2, 4, 6, 10, 12, 14, 18, 20, \dots) \end{aligned}$$

We shall now prove that $W = V$.

Proof. First, note that if $U(n)$ is even, then $U(n) = V(n) = W(n)$. So, assume that $U(n)$ is odd. Observe that $\forall n > 0, U(n) - 1 \in L$, because $U(n) = \lfloor n\phi^2 \rfloor$ and $\phi^2 > 2.6$. Hence, $U(n) \in U \implies U(n) - 1 \notin U$. $U(n) - 1 \in L$ is trivial and follows directly from the problem statement where U is defined as the complement of L.

For the odd $U(n)$, we prove equality by induction. For the base case, observe that $U(2) = 5$ and $V(2) = W(2) = 4$ as shown in the above sequences. Assume that $V(k) = W(k)$ for $k < n$ and consider the case where $U(n)$ is odd. According to the definition of W, $W(n) = L(m)$ for some element m such that $L(m) = U(n) - 1$. Since $L(m) = U(n) - 1$, if $U(n)$ and $L(m)$ are swapped the resulting sequences, U_n and L_n, are both increasing sequences. For instance:

$$\begin{aligned} L_n &= (1, 3, 5, 7, \dots, U(n), \dots) \\ U_n &= (2, 4, 6, \dots, U(n) - 1, \dots) \end{aligned}$$

Now consider which element is swapped to generate $V(n)$. In §3.1, we generate V by swapping the *least* element of L such that L and U both remain increasing sequences. Note that $L(m) \in L$ when we are choosing an element to swap for $V(n)$, because $\forall k < n, W(k) = V(k)$ and $W(k) \neq L(m)$. Since $L(m)$ can be swapped with $U(n)$ while maintaining increasing sequences, we need to consider swapping $L(k)$ with $U(n)$ only for $k < m$. However, any such $L(k)$ would result in the sequence $L = (\dots, U(n), \dots, L(m), \dots)$ and since $L(m) = U(n) - 1$, then L is no longer an increasing sequence. Therefore, we must swap $L(m)$ and $U(n)$. Thus, the sequence V has the same elements as sequence W. □

Let us now obtain an alternative expression for S. Observe that because of the relationship between U and W, $W = \left(2\left\lfloor \frac{U(n)}{2} \right\rfloor\right)_{n\geq 1} = \left(2\left\lfloor \frac{n\phi^2}{2} \right\rfloor\right)_{n\geq 1}$. Because $W = V$, $S(n) = \frac{W(n)}{2}$, so $S = \left(\left\lfloor \frac{n\phi^2}{2} \right\rfloor\right)_{n\geq 1}$. Hence, S and S' are complementary Beatty sequences as a special case of theorem 2.2, which proves the problem statement. Specifically,

$$\left(\left\lfloor \frac{n\phi^2}{F_3} \right\rfloor\right)_{n\geq 1} \text{ and } \left(\left\lfloor \frac{n\phi^3}{F_2} \right\rfloor\right)_{n\geq 1}$$

are complementary sequences. Note that since $0 \notin S$, $S' = \left(\lfloor n\phi^3 \rfloor\right)_{n\geq 0}$ if S' is defined for the set of nonnegative integers, as in the problem statement and integer sequence A004976 [4].

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Department of Computer Science, Wayne State University, Detroit, MI 48202
E-mail address: `vincentrusso1@gmail.com`

Department of Computer Science, Wayne State University, Detroit, MI 48202
E-mail address: `loren@wayne.edu`