PHYS 633: Introduction to Stellar Astrophysics

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Solving the Equations

As we have discussed, the structure of a star during nuclear burning changes on timescales much longer than the dynamical and Kelvin-Helmholtz timescales, and we can neglect the time derivatives in the momentum and energy equations. Then, the governing differential equations for the structure are

$$\frac{\mathrm{d}r}{\mathrm{d}m} = \frac{1}{4\pi r^2 \rho} \tag{1}$$

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} \tag{2}$$

$$\frac{\mathrm{d}l}{\mathrm{d}m} = \epsilon - \epsilon_\nu \tag{3}$$

$$\frac{\mathrm{d}T}{\mathrm{d}m} = -\frac{GmT}{4\pi r^4 P}\nabla. \tag{4}$$

where we have written ordinary rather than partial derivatives. Likewise the governing differential equations for the evolution are

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = \sum_j \frac{\epsilon_{ji}}{q_{ji}} - \sum_k \frac{\epsilon_{ik}}{q_{ik}} \tag{5}$$

where there are $I - 1$ such equations (one less than the number of elements I that evolve, due to the closure relation associated with mass fractions).

To solve these equations, we first need to specify boundary conditions. For the evolution equations, the theory of star formation tells us that a star goes through a period where it is fully convective, before it arrives at the zero-age main sequence. Therefore, we can assume that the initial chemical composition is uniform throughout the star. For the Sun, this initial composition is given by $X = 0.7$, $Y = 0.28$ and $Z = 0.02$, and this is often assumed as the default' composition of other stars.

For the structure equations, at the center of the star ($m = 0$) we have, by definition, $r = 0$. Also, since the luminosity is related to the flux F by

$$l = 4\pi r^2 F, \tag{6}$$

and the flux must be finite at the center, we have $l = 0$ at the center too. The two remaining boundary conditions required to solve the equations must be specified at the surface. In the simplest case, we could assume the surface of the star is defined by zero pressure and density, corresponding to 'empty' space.

This 'zero boundary condition', with $P = 0$ and $T = 0$ at $m = M$, allows us to construct crude stellar models, but it isn't really realistic or useful. Interstellar space isn't empty, it's filled with the interstellar medium (ISM). In effect, there is no real physical boundary to the star — that is, there is no radius inside of which there is matter, and outside of which there is no matter.

A better way to think about the boundary of the star is the point at which photons have their last interaction with the stellar matter. This point, the 'photosphere', determines the properties of the photons escaping from the star, and therefore determines how the star will appear to us. To fix the position of the photosphere, we introduce the optical depth,

$$\tau(r) = \int_r^\infty \kappa\rho \, \mathrm{d}r, \tag{7}$$

being the number of mean free paths a photon must go through to escape to infinity from radius r within the star. The theory of stellar atmospheres tells us that, on average, photons escaping from the star come from an optical depth $2/3$. Thus, we can define the photospheric radius of the star, R, to be such that

$$\tau(R) = \int_R^\infty \kappa\rho \, \mathrm{d}r = 2/3. \tag{8}$$

Assuming that the opacity κ is approximately constant, and equal to $\bar{\kappa}$, throughout the photospheric layers, we can write this boundary condition as

$$\bar{\kappa} \int_R^\infty \rho \, \mathrm{d}r = 2/3. \tag{9}$$

The integral in this expression is often referred to as the photospheric column density; it is the total amount of mass per unit area sitting above the photosphere.

Usually, we wish to express the boundary condition above in terms of the photospheric pressure. By integrating the equation of hydrostatic equilibrium upward, we have

$$P(R) = -\int_R^\infty \frac{\mathrm{d}P}{\mathrm{d}r} \, \mathrm{d}r, \tag{10}$$

$$= \int_R^\infty \frac{Gm}{r^2}\rho \, \mathrm{d}r, \tag{11}$$

where we assume that the pressure goes to zero at infinity. Since typical pressure and density scale heights are very small in the photosphere ($\sim 10^{-3}R$), we approximate $m = M$ and $r = R$ in the integrand, so that

$$P(R) = \frac{GM}{R^2} \int_R^\infty \rho \, \mathrm{d}r. \tag{12}$$

Eliminating the column density using eqn. (9), we obtain the pressure boundary condition

$$P(R) = \frac{2GM}{3R^2\bar{\kappa}}. \tag{13}$$

The corresponding boundary condition for the temperature, to replace the zero condition $T = 0$, is also obtained from the theory of stellar atmospheres. At the $\tau = 2/3$ photosphere, the temperature is equal to the effective temperature defined by

$$L = 4\pi R^2 F = 4\pi R^2 \sigma T_{\rm eff}^4. \tag{14}$$

Hence, we have

$$T(R) = \left(\frac{L}{4\pi R^2 \sigma} \right)^2 \tag{15}$$

The boundary conditions (13) and (15) are a considerable improvement over the zero boundary conditions, but still have their flaws. During our derivation of the radiative diffusion equation, we assumed that the mean free path of photons is very much less than the typical scales on which the temperature changes throughout a star, and therefore that the radiation field is very close to isotropic. While extremely good deep within a star, this assumption breaks down near the photospheric layers, since the mean free path of photons becomes large as they escape from the star.

To deal with this breakdown, we chose a fitting point within the star, with mass coordinate $m_{\rm fit}$, that is deep enough that the radiative diffusion equation still holds; but close enough to the stellar surface for us to assume that both the flux F and the gravity g are constant, and equal to $L/4\pi R^2$ and and GM/R^2, in the overlying regions. Then, we apply the usual structure equations for masses $m < m_{\rm fit}$, but connect this interior solution smoothly onto a stellar atmosphere extending from $m_{\rm fit} < m < M$.

In the theory of stellar atmospheres, an atmosphere may be parameterized in terms of its effective temperature $T_{\rm eff}$ and their gravity g. So, for a given luminosity and radius at $m = m_{\rm fit}$, we calculate a stellar atmosphere from the corresponding $T_{\rm eff}$ and g. Setting the total mass in the atmosphere to $M - m_{\rm fit}$, we find the pressure $P_{\rm fit}$ and temperature $T_{\rm fit}$at its base, and use these values to apply boundary conditions to the interior solution at the fitting point.

$$P = P_{\rm fit}, \qquad T = T_{\rm fit}. \tag{16}$$

With the boundary conditions discussed, we now look at how we go about actually solving the equations. Clearly, they are far too complex to admit analytical solutions (except in special cases, such as stars obeying the polytropic pressure-density-relation), and so we must resort to numerical solution techniques. Typically, we adopt an operator splitting approach, alternately solving the equations in the space (structure) and time (evolution) directions. For the evolution steps, if we know the chemical composition at time t, then we can approximate the composition at some later time $t + \Delta t$ as

$$X_i(t + \Delta t) \approx \frac{{\rm d}X_i}{{\rm d}t} \Delta t. \tag{17}$$

This procedure is known as 'Euler's first-order explicit method'; there are other more-sophisticated methods of updating the composition, but all are based on

similar reasoning. The size of the time step, Δt, must be small compared with the typical time on which the stellar structure is changing, otherwise inaccuracy and/or numerical instability will develop.

After we have taken one time step, we use the updated composition to resolve the stellar structure equations. For this, we cannot use Euler's method, because unlike the evolution equations, the structure equations are not an initial-value problem. Rather, they are what is known as a boundary value problem (BVP) – a set of differential equations defined over a region, with boundary conditions defined at *both* ends of the interval. A familiar examples of a BVP is the quantum mechanical description of a particle trapped in a 1-dimensional potential well; boundary conditions must be applied to the solutions of the Schrödinger equation at the two walls of the well.

So, how do we solve a BVP numerically? There are two principal techniques. The first is known as 'shooting'; basically, we shoot a solution from one boundary to the other, adjusting our aim until we satisfy both sets of boundary conditions. In the case of the stellar structure equations, we can start at the center $m = 0$, and guess values for the central pressure P_c and temperature T_c. Based on these values, and the values $l = 0$ and $r = 0$ set by the central boundary conditions, we use Euler's method or some equivalent to integrate the pressure, temperature, luminosity and radius up to the outer boundary. In general, the values of the pressure and temperature at this boundary will not fulfill the boundary conditions we describe above. Thus, we go back and adjust P_c and T_c and repeat the procedure. By using an appropriate algorithm for calculating the adjustments, we can make sure that we end up closer to fulfilling the outer boundary conditions. Then, we just repeatedly adjust P_c and T_c until we eventually converge to the solution.

The second method for numerical solution of BVPs is known as finite differences, relaxation, or – in the specific case of stellar structures – the Henyey method. We divide the star up into some number n of discrete mass shells, which we label with an index k. We then describe the structure of the star using the values of the mass coordinate m_k, pressure P_k, temperature T_k, radius r_k and luminosity l_k in the center of each mass shell. To find these values, we replace the spatial derivatives appearing in the structure equations with numerical approximations based on finite differences. For instance, the mass equation

$$\frac{\mathrm{d}P}{\mathrm{d}m} = \frac{1}{4\pi r^2 \rho} \tag{18}$$

is represented between the k and $k+1$ shells by the difference equation

$$\frac{P_{k+1} - P_k}{m_{k+1} - m_k} = \frac{1}{4\pi r_{k+1/2}^2 \rho_{k+1/2}} \tag{19}$$

Here, $r_{k+1/2}$ represents the radius at the boundary between the mass shells, which can be approximated by

$$r_{k+1/2} = \frac{1}{2}(r_k + r_{k+1}). \tag{20}$$

Likewise, $\rho_{k+1/2}$ represents the density at the boundary; given the equation of state $\rho = \rho(P, T)$, it can be approximated by

$$\rho_{k+1/2} = \rho\left(\frac{1}{2}[P_k + P_{k+1}], \frac{1}{2}[T_k + T_{k+1}]\right). \tag{21}$$

We can write down four difference equations (corresponding to the four structure equations) for each pair of adjacent mass shells, to give a total of $4(n-1)$ difference equations. Adding in the four boundary conditions (two at the center, and two at the surface), we end up with a total of $4n$ algebraic equations, governing the $4n$ unknowns (P_k, T_k, r_k and l_k). To solve these (non-linear) equations, we use what is essentially a Newton-Raphson method. Consider the equation

$$f(x) = 0 \tag{22}$$

for which we wish to find the solutions x. If we already have an approximate solution, x_j, then an improved value will be given by

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}, \tag{23}$$

where $f'(x)$ is the first derivative of the function $f(x)$. By applying this procedure iteratively, we can rapidly converge to a solution.

In the case of the stellar structure equations, if we simultaneously apply the Newton-Raphson to the $4n$ difference equations plus boundary conditions, then we end up with a set of $4n$ simultaneous, linear equations giving the improved values of the $4n$ unknowns in terms of their current values. These simultaneous equations can be written in a matrix form, and solved by inverting the matrix; because of the form of the equations, the inversion can be done very efficiently. By repeatedly inverting this matrix and updating the unknowns, we can converge toward an approximate solution of the structure equations.

Typically, the Henyey method is the favored approach to solving the structure equations. After each evolution time step, the previous solution – with updated chemical composition values – is used as a the approximate solution to the equations, and convergence to the true solution is usually very rapid.