COMPUTATIONAL ASTROPHYSICS

Observatorio
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Computational Astrophysics

03A. Bayes' Rule. Monty Hall Problem

Eduard Larrañaga
Observatorio Astronómico Nacional
Universidad Nacional de Colombia

Bayes’ Rule

Random Variables and Probability Density Functions

A **random**, or **stochastic**, variable is that whose value results from the measurement of a quantity that is subject to random variations. It can take on a set of possible different values, each with an associated probability.

There are **discrete** and **continuos** random variables.

The function which ascribes a probability value to each outcome of the random variable is called **probability density function** or **pdf**.

Random Variables and Probability Density Functions

Independent Identically Distributed or **iid** random variables are drawn from the same distribution and are independent. Two random variables, x and y are *independent* if

$$p(x, y) = p(x)p(y)$$

i.e., the knowledge of the value of x tells nothing about the value of y.

When the variables are *not independent, we have*

$$p(x, y) = p(x \mid y)p(y) = p(y \mid x)p(x).$$

Marginal Probability Functions

The **Marginal Probability Function** is defined as

$$p(x) = \int p(x, y)dy$$

Or equivalently

$$p(x) = \int p(x \mid y)p(y)dy$$

Bayes' Rule

From the above results, we obtain the **Bayes' rule**,

$$p(y|x) = \frac{p(x|y)p(y)}{p(x)} = \frac{p(x|y)p(y)}{\int p(x|y)p(y)dy}.$$

For discrete random variables, y_i, this expression is writen as

$$p(y_j|x) = \frac{p(x|y_j)p(y_j)}{p(x)} = \frac{p(x|y_j)p(y_j)}{\sum_i p(x|y_i)p(y_i)}.$$

Monty Hall Problem

Monty Hall Problem

Doors : A, B and C

Assume that:

- We pick door A
- Monty opens door B
- Monty proposes to change door A for door C

We need to calculate two *posteriors:*

$P(A \mid B)$: *Probability that A contains the prize if Monty opened B*

$P(C \mid B)$: *Probability that C contains the prize if Monty opened B*

Bayes' Theorem Calculation

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Probability that the prize is at door A given that Monty opens door B

$$P(C \mid B) = \frac{P(B \mid C)P(C)}{P(B)}$$

Probability that the prize is at door C given that Monty opens door B

Bayes' Theorem Calculation

$$P(A\,|\,B) = \frac{P(B\,|\,A)P(A)}{P(B)}$$

$P(A)$: **Prior**. Probability that A contain the prize before any other event.

$P(B\,|\,A)$: **Likelihood**. Probability that Monty opens B if A contains the prize.

$P(B)$: **Normalizing Constant.** Probability that Monty opens B. This is usually calculated in terms of the other properties as

$$P(B) = P(B\,|\,A)P(A) + P(B\,|\,C)P(C)$$

Bayes' Theorem Calculation

$$P(C \mid B) = \frac{P(B \mid C)P(C)}{P(B)}$$

$P(C)$: **Prior**. Probability that C contain the prize before any other event.

$P(B \mid C)$: **Likelihood**. Probability that Monty opens B if C contains the prize.

$P(B)$: **Normalizing Constant.** Probability that Monty opens B. This is usualy calculated in terms of the other properties as

$$P(B) = P(B \mid A)P(A) + P(B \mid C)P(C)$$

Bayes' Theorem Calculation

Priors: P(A) and P(C)

The probability of any door to contain the prize before we pick a door is $1/3$. Therefore, the **priors** $P(A)$ and $P(C)$ are

$P(A) = 1/3$: Prior probability that A contains the prize

$P(C) = 1/3$: Prior probability that C contains the prize

Bayes' Theorem Calculation

Likelihood: P(B|A)

If the prize is behind door A, Monty can open B or C. Therefore the probability of opening either (B or C) is 50%. This means that

$P(B \mid A) = 1/2$: Likelihood that Monty opens B if A contains the prize

Bayes' Theorem Calculation

Likelihood: P(B|C)

If the prize is behind door C, Monty can only open B because he cannot open A since that door is picked. This means that

$P(B \mid C) = 1$: Likelihood that Monty opens B if C contains the prize

Bayes' Theorem Calculation

Numerators

$$P(B \mid A)P(A) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$P(B \mid C)P(C) = 1 \times \frac{1}{3} = \frac{1}{3}$$

Bayes' Theorem Calculation

Normalization Constant

Since the three events considered cover all possible options and they do not overlap, we can take the sum of the numerators:

$$P(B) = P(B\,|\,A)P(A) + P(B\,|\,C)P(C) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

Bayes' Theorem Calculation

Posterior P(A|B)

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} = \frac{1/6}{1/2}$$

$$P(A \mid B) = \frac{1}{3}$$

Bayes' Theorem Calculation

Posterior P(C|B)

$$P(C \mid B) = \frac{P(B \mid C)P(C)}{P(B)} = \frac{1/3}{1/2}$$

$$P(C \mid B) = \frac{2}{3}$$

Results

$$P(A \mid B) = \frac{1}{3}$$: Probability that the prize is at door A given that Monty opens door B

$$P(C \mid B) = \frac{2}{3}$$: Probability that the prize is at door C given that Monty opens door B

THE BEST OPTION IS TO CHANGE THE DOOR!

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