Euler-Cromer algorithm that yield accurate solutions for oscillatory problems. Appendix 1 contains a description and discussion of the advantages and disadvantages of two other methods.

Exercises

1. Investigate the stability of the Euler-Cromer method. Modify our program so that it also calculates the total energy, kinetic plus potential, of the pendulum as a function of time. Show that the energy is conserved over each complete cycle of the motion.

2. Repeat the previous problem using the Runge-Kutta method described in Appendix 1. Compare the accuracy of the Runge-Kutta method with that of the Euler-Cromer algorithm using the same time step.

3. Use the Euler method to simulate the motion of a pendulum as in Figure 3.2. Study the behavior as a function of the step size, Δt, and show that the total energy always increases with time.

*4. For simple harmonic motion, the general form for the equation of motion is

$$\frac{d^2x}{dt^2} = -k\,x^{\alpha}\,, \tag{3.6}$$

with $\alpha = 1$. This has the same form as (3.2), although the variables have different names. Begin by writing a program that uses the Euler-Cromer method to solve for x as a function of time according to (3.6), with $\alpha = 1$ (for convenience you can take $k = 1$). The program we developed in this section can be modified to accomplish this. Show that the period of the oscillations is independent of the amplitude of the motion. This is a key feature of simple harmonic motion. Then extend your program to treat the case $\alpha = 3$. This is an example of an *anharmonic* oscillator. Calculate the period of the oscillations for several different amplitudes (amplitudes in the range 0.2 to 1 are good choices), and show that the period of the motion now depends on the amplitude. Give an intuitive argument to explain why the period becomes longer as the amplitude is decreased.

3.2 Chaos in the Driven Nonlinear Pendulum

Now that we have a numerical method that is suitable for the simple pendulum problem, we are ready to take on a slightly more complicated and also more interesting situation. In deriving (3.2) we made several simplifications. First, we assumed that the amplitude of the oscillation would always be small, which allowed us to expand the $\sin\theta$ term in (3.1). In this section we will not make this assumption; this will enable us to treat situations in which the mass swings to large angles or even all the way around the pivot point of the pendulum.[3] Second, we will include the effect of friction. The manner in which friction enters the equations of motion depends on the origin of the friction. Possible sources of friction include the effective bearing, where the string of the pendulum connects to the support, air resistance, etc. In many cases the frictional force, which we will also refer to as damping, is proportional to the velocity, and that is the assumption we will make here.[4] The frictional force we will employ thus has

[3] To allow such motion in a real pendulum we would have to replace the string in Figure 3.1 with a rigid rod.

[4] Note, however, that other functional forms are possible. For example, in the case of air resistance, we noted in Chapter 2 that while the drag from air resistance is proportional to v for very small velocities, it varies as v^2 in many cases of practical interest. We will leave the investigation of the behavior with other forms for F_{friction} to the inquisitive reader.

the form $-q(d\theta/dt)$, since the velocity of the pendulum is $\ell(d\theta/dt)$. Here q is a parameter that is a measure of the strength of the damping, and the minus sign guarantees that this force will always oppose the motion of the pendulum. A third ingredient we wish to add to our model is a driving force; that is, an external force acting on the pendulum. The form of this force will depend on how the force is applied. A convenient choice is to assume that the driving force is sinusoidal with time with amplitude F_D and angular frequency Ω_D (which is not to be confused with the natural frequency of the simple pendulum, Ω). This might arise, for example, if the pendulum mass has an electric charge and we apply an oscillating electric field. Putting all of these ingredients together, we have the equation of motion[5]

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin(\theta) - q\frac{d\theta}{dt} + F_D\sin(\Omega_D t)\,. \tag{3.7}$$

Our model for a nonlinear, damped, driven pendulum, (3.7), contains some very rich and interesting behavior. We will only be able to touch on a few of its intriguing properties here, although you can explore others through the exercises. Let us begin by examining the behavior of θ as a function of time for several typical cases. First we must construct a program to calculate a numerical solution, since there is no known exact solution to (3.7). Our program is similar in form to the one we used to study the simple pendulum. The only major difference is that we must use a slightly more complicated equation for ω. We again rewrite (3.7) as two first-order differential equations and obtain

$$\frac{d\omega}{dt} = -\frac{g}{\ell}\sin(\theta) - q\frac{d\theta}{dt} + F_D\sin(\Omega_D t)\,, \tag{3.8}$$

$$\frac{d\theta}{dt} = \omega\,.$$

These can be converted into difference equations for θ_i and ω_i at time step i, as we did in (3.5). These difference equations can then be translated into a program. The complete listing is given in Appendix 4; below we give only the subroutine that does the work of calculating θ and ω.

```
! use the Euler-Cromer method for a damped, nonlinear, driven pendulum
sub calculate(theta(),omega(),t(),length,dt,q,drive_force,drive_frequency)
   i = 0
   g = 9.8
   period = 2 * pi / sqr(g/length)  ! period of the corresponding simple
   do                               ! pendulum
      i = i + 1
      t(i+1) = t(i) + dt            ! use the Euler-Cromer method
      omega(i+1) = omega(i) - (g/length) * sin(theta(i)) * dt
         - q * omega(i) * dt + drive_force * sin(drive_frequency * t(i)) * dt
      theta(i+1) = theta(i) + omega(i+1) * dt
      if theta(i+1) > pi then theta(i+1) = theta(i+1) - 2 * pi ! keep theta in
      if theta(i+1) < -pi then theta(i+1) = theta(i+1) + 2 * pi! range -pi to pi
   loop until t(i+1) >= 10 * period
   mat redim omega(i+1),theta(i+1),t(i+1)
end sub
```

[5]Note that, strictly speaking, the parameter F_D in (3.7) is not the actual force since other factors of m and l also enter. However, F_D is proportional to the driving force, so when we speak of larger drive forces this will mean larger values of F_D, etc.

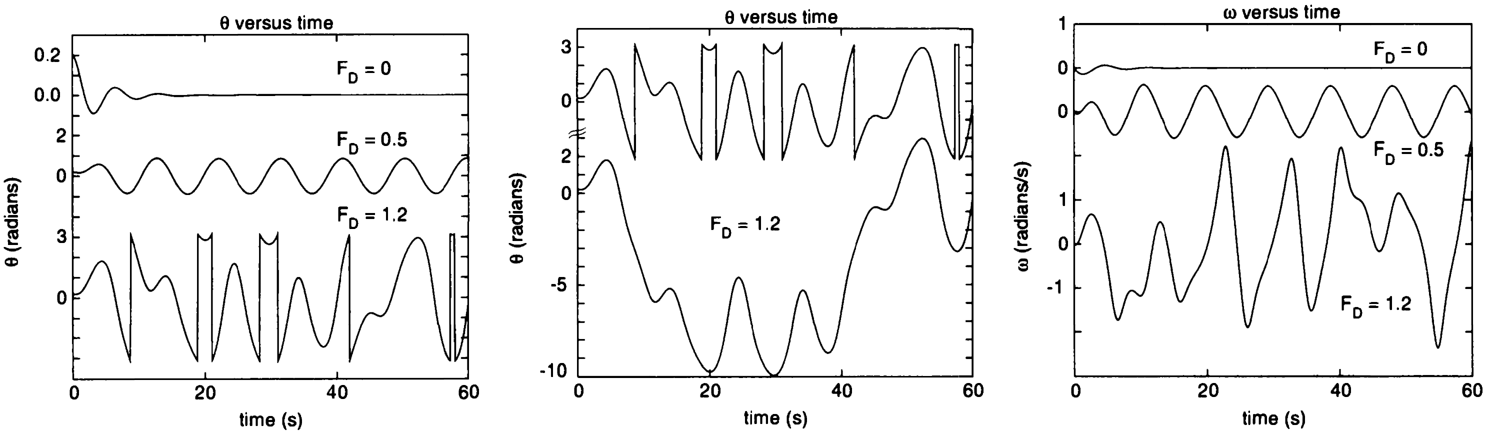

Figure 3.4: Left: behavior of θ as a function of time for our driven, damped, nonlinear pendulum, for several different values of the driving force. The vertical "jumps" in θ occur when the angle is reset so as to keep it in the range $-\pi$ to $+\pi$; they do not correspond to discontinuities in $\theta(t)$. Center: behavior of $\theta(t)$ for $F_D = 1.2$ with and without these "resets." Right: corresponding behavior of the angular velocity of the pendulum, ω. The parameters for the calculation were $q = 1/2$, $\ell = g = 9.8$, $\Omega_D = 2/3$, and $dt = 0.04$, all in SI. The initial conditions were $\theta(0) = 0.2$ and $\omega(0) = 0$.

This subroutine is organized much like the `calculate` routine for our simple pendulum program, but there are several differences of note. First, the equation for `omega(i+1)` is more complicated since we have a different equation of motion. Second, we have two `if` statements that keep an eye on the value of `theta(i+1)`. Recall that our pendulum can now swing all the way around its pivot point, which corresponds to $|\theta| > \pi$. Since θ is an angular variable, values of θ that differ by 2π correspond to the *same* position of the pendulum. For plotting purposes it is convenient to keep θ in the range $-\pi$ to π, and that is accomplished with the two `if` statements. If θ becomes less than $-\pi$, then its value is increased by 2π; likewise, if it becomes greater than $+\pi$, its value is decreased by 2π. This procedure keeps $-\pi \leq \theta \leq +\pi$, which will be handy although not absolutely necessary in some of our analyses. Finally, note that we again use the Euler-Cromer method.

Some typical results for θ and ω as functions of time, as calculated with our program, are shown in Figure 3.4 where we plot the behavior for several different values of the driving force, with all of the other parameters held fixed. With a driving force of zero the motion is damped and the pendulum comes to rest after only a few oscillations. These damped oscillations have a frequency close to the natural frequency of the undamped pendulum, Ω, and are a vestige of simple harmonic motion. With a small driving force, $F_D = 0.5$, we find two regimes. The first few oscillations are affected by the decay of an initial transient as in the case of no driving force. That is, the initial displacement of the pendulum leads to a component of the motion that decays with time and has an angular frequency of $\sim \Omega$. After this transient is damped away, the pendulum settles into a steady oscillation in response to the driving force. The pendulum then moves at the driving frequency, Ω_D, *not* at its natural frequency, with an amplitude determined by a balance between the energy added by the driving force and the energy dissipated by the damping. In a sense, the motion of the pendulum can be viewed as an interplay of the two frequencies Ω and Ω_D, the natural frequency of the pendulum and the frequency of the driving force.

The behavior changes radically when the driving force is increased to $F_D = 1.2$. Now the motion is no longer simple, even at long times. The vertical jumps in θ are due to our resetting of the angle to keep it in the range $-\pi$ to π and thus correspond to the pendulum swinging "over the top." To make this clear, the center plot in Figure 3.4 shows $\theta(t)$ for $F_D = 1.2$, with and without this resetting of the angle. We see that the pendulum does not settle into any sort of repeating steady-state behavior, at least in the range shown here. We might suspect that we have not waited long enough for the transients to decay and that a steady oscillation might be found if we simply waited a little longer. This is not the case; for this value of the driving force the behavior *never* repeats. This is an example of *chaotic* behavior, which will be our main concern for the rest of this chapter.

It is important to appreciate the behavior illustrated in Figure 3.4. At low drive the motion is a simple oscillation (after the transients have decayed), which would, if we were sufficiently patient, repeat forever. On the other hand, at high drive the motion is chaotic; it is a very complicated nonrepeating function of time. But what does it really mean to be chaotic? Your intuition probably tells you that chaotic behavior is random and unpredictable, and the behavior of our pendulum at high drive certainly has that appearance. However, if the behavior is truly unpredictable, then how was our program able to calculate it? This conundrum can be put another way when we realize that the behavior of our pendulum is described by the differential equation (3.7). From the theory of such equations we know that once the initial conditions (at $t = 0$, for example) are specified, the solution for θ is then *completely determined* for all future times. Indeed, we took this for granted in constructing our program. But how can the behavior be both deterministic and unpredictable at the same time?

We are thus faced with an apparent contradiction between analytic theory (the theory of differential equations) and numerical calculations (our program). Since our only evidence, so far, that the pendulum can be chaotic is from our numerical results, we might be tempted to suspect that we have made some sort of programming error, or that we have somehow misinterpreted the meaning of the numerical results. For example, we could imagine that if we waited long enough, a (predictable) pattern might emerge even at high drive, including at $F_D = 1.2$ in Figure 3.4. Indeed, how could such behavior ever be ruled out? Rather than pursue the question in this way, let us raise another possibility; namely, that the behavior is deterministic *and* unpredictable at the same time! This may seem to be impossible, but we will now show how to reconcile these two, apparently contradictory notions.

Let us consider the stability of the solutions to our pendulum equation of motion. We imagine that we have two *identical* pendulums, with exactly the same lengths and damping factors. We set them in motion at the same time with the same driving forces. The only difference is that we start them with *slightly* different initial angles. We thus must calculate the angular positions of two pendulums, θ_1 and θ_2, and we can do this using a program very similar to the one described above (an exercise we will leave to the reader). Some results for $\Delta\theta \equiv |\theta_1 - \theta_2|$ are shown in Figure 3.5 for two different values of the drive amplitude. The smaller value of F_D is the one for which we found simple oscillatory motion in Figure 3.4. To understand these results we first call your attention to the very sharp dips that occur approximately every 3 s. These dips in $\Delta\theta$ occur when one of the pendulums reaches a turning point. $\Delta\theta$ will vanish near each turning point since the trajectories $\theta_1(t)$ and $\theta_2(t)$ must then cross. It is more useful to focus on the plateau regions away from these dips in Figure 3.5. These plateau values of $\Delta\theta$ exhibit a steady and fairly rapid decrease with t. This means that the motion of the two pendulums becomes more and more similar, since the difference in the two angles approaches zero as the motion proceeds. (Indeed, this angular difference decreased by *six orders of magnitude* after about a dozen oscillations.) This in turn means that the motion is *predictable*. If, for some reason, we did not know the initial conditions of one of the pendulums, we could still predict its future motion since our results show that $\theta(t)$ converges to a particular solution (that of the other pendulum). This is what our intuition tells us to expect for predictable, nonchaotic motion.

In contrast, for the larger value of F_D we find that $\Delta\theta$ increases rapidly and irregularly with t; this trend is indicated by the dashed line on the right in Figure 3.5. This is usually described by saying that

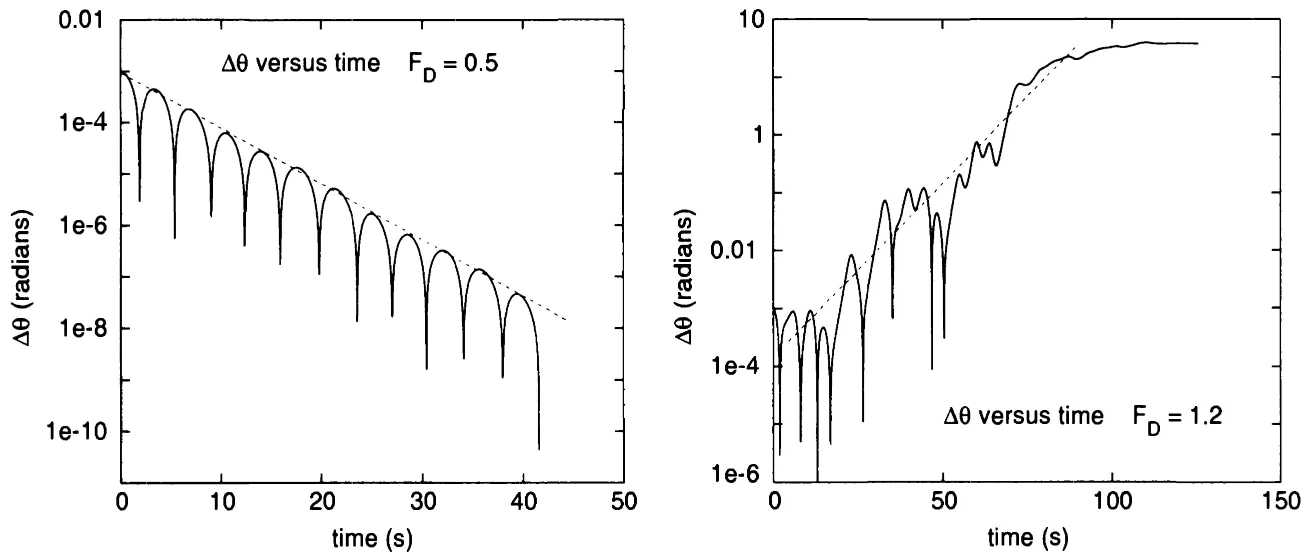

Figure 3.5: Results for $\Delta\theta$ from our comparision of two identical pendulums. The parameters were the same as in Figure 3.4. The initial values of θ for the two pendulums differed by 0.001 rad. On the left are results for low drive, while the results on the right were obtained from the chaotic regime. The dashed lines indicate the overall trends, that is, that $\Delta\theta$ decreases approximately exponentially for low drive, and increases roughly exponentially at high drive.

the two trajectories, $\theta_1(t)$ and $\theta_2(t)$, *diverge* from one another. Note that this divergence is extremely rapid at short times ($t < 75$ in Figure 3.5). $\Delta\theta$ saturates (i.e., stops changing) at long time periods, but this is only because it has reached a value of order 2π and simply can't get any larger! We have used a logarithmic scale in Figure 3.5, so the increase of $\Delta\theta$ with time is *very* rapid. The irregular variation of $\Delta\theta$ cannot be described by any simple function. However, if we were to repeat this calculation for a range of different initial values of θ_1 (keeping $\Delta\theta(0)$ fixed) and average the results, we would find a much smoother behavior, such as the dashed line in Figure 3.5. This line corresponds to the relation $\log(\Delta\theta) \sim \lambda t$, which implies

$$\Delta\theta \approx e^{\lambda t} . \tag{3.9}$$

It turns out that this functional form for $\Delta\theta$ is very common and the parameter λ is known as a Lyapunov exponent.[6]

For our pendulum the numerical results show that λ is positive at high drive, which means that two pendulums that start with nearly, but not exactly, the same initial conditions will follow trajectories that diverge exponentially fast.[7] Since we can never hope to know the initial conditions or any of the other pendulum parameters exactly, this means that the behavior at $F_D = 1.2$ is for all practical purposes unpredictable. Our system is thus both deterministic and unpredictable. Put another way, a system can obey certain deterministic laws of physics, but still exhibit behavior that is unpredictable due to an extreme sensitivity to initial conditions. *This* is what it means to be chaotic. One more point should

[6]In general, systems such as a pendulum possess several different Lyapunov exponents. In order to be chaotic, at least one of these exponents must be positive. The accurate extraction of the values of the Lyapunov exponents from results such as those computed in this section is a somewhat complicated procedure. For one approach to this extraction see Wolf et al. (1985).

[7]Similar results would be found if the two pendulums had slightly different lengths or driving forces, or if any other of the parameters were different.

be noted from Figure 3.5. The behavior of $\Delta\theta$ can be described by a Lyapunov exponent in both the chaotic and nonchaotic regimes. In the former case $\lambda > 0$, while in the latter, $\lambda < 0$. The transition to chaos thus occurs when $\lambda = 0$.

Now that we have seen how $\theta(t)$ for our pendulum can be unpredictable, you might give up all hope of developing a useful theoretical description of the chaotic regime. However, it turns out that this view is too pessimistic. It is possible to make certain accurate predictions concerning θ, even in the chaotic regime! To demonstrate this we need to consider the trajectory in a different way. Instead of plotting θ as a function of t, let us plot the angular velocity ω as a function of θ. This is sometimes referred to as a *phase-space* plot. Since we have already constructed a program to calculate θ and ω, it is straightforward to modify it to make the desired plot; results for two values of the drive amplitude are shown in Figure 3.6.

With a small driving force the trajectory in phase space (ω-θ space) is easy to understand in terms of the behavior we found earlier for $\theta(t)$. For short times there is a transient that depends on the initial conditions (in this case we started at $\theta = 0.2$ with $\omega = 0$), but the pendulum quickly settles into a regular orbit in phase space corresponding to the oscillatory motion of both θ and ω. It can be shown that this final orbit is independent of the initial conditions; this is also what our results for the Lyapunov exponent imply. The behavior in the chaotic regime is a bit more surprising. The phase-space trajectory exhibits many orbits that are nearly closed and that persist for only one or two cycles. While this pattern is certainly not a simple one, it is not completely random, as might have been expected for

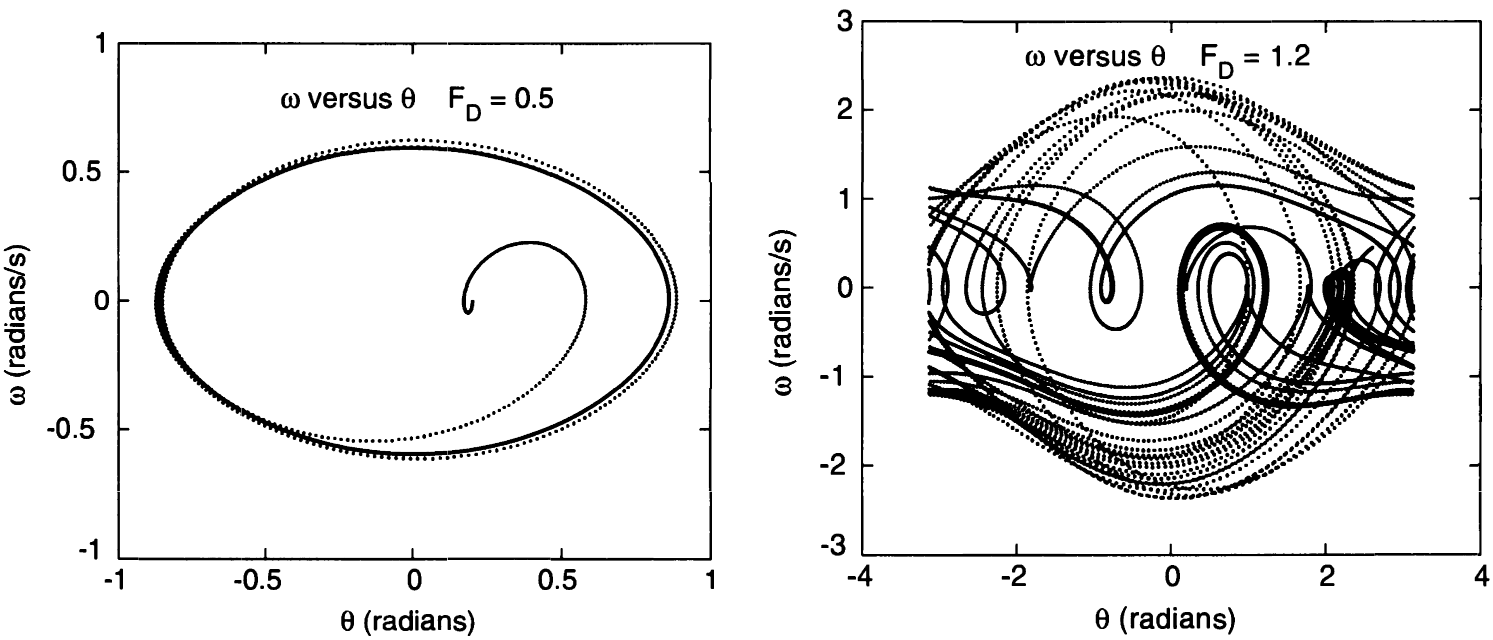

Figure 3.6: Results for ω as a function of θ for a pendulum. The parameters were the same as in Figure 3.4. For high drive (on the right), many trajectories go beyond $|\theta| = \pi$ and thus "jump" from $\theta = \pi$ to $\theta = -\pi$, or vice-versa in this plot.

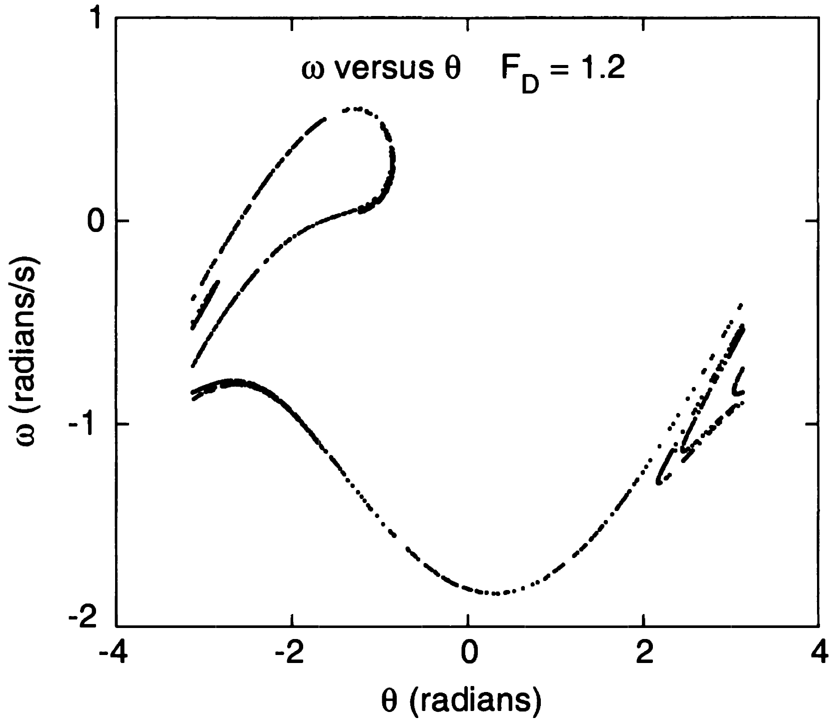

Figure 3.7: More results for ω as a function of θ for a pendulum; here we only plot points at times that are in phase with the driving force. The parameters were the same as in Figure 3.4. This surface of points is known as a strange attractor.

a chaotic system. This is a common property of chaotic systems; they generally exhibit phase-space trajectories with significant structure.

If we examine these trajectories in a slightly different manner we find a very striking result. In Figure 3.7 we show the same type of phase-space graph, but here we plot ω versus θ only at times that are *in phase* with the driving force. That is, we only display the point when $\Omega_D t = n\pi$, where n is an integer.[8] This is an example of what is known as a *Poincaré section* and is a very useful way to plot and analyze the behavior of a dynamical system. The motivation for plotting the results in this way can be appreciated from an analogy with the function of a stroboscope. It sometimes happens that we want to examine an object that is rotating at a high rate. A good example is an old-fashioned (vinyl) record as it is rotated on a record player. When in operation, the record rotates too fast for a human eye to read the label. However, if the record is illuminated with a light source (a stroboscope), which is turned on and off at the frequency of the record player, our eye will receive input only when the record has a particular orientation; as a result we will be able to read the label as if the record were not moving at all. The key point is that things will look simpler when we observe them at a rate (i.e., frequency) that matches the problem. This lesson can be applied to the pendulum by observing the behavior, that is, recording the values of θ and ω, at a rate that matches the drive frequency, and this is effectively what we have done in the Poincaré section shown in Figure 3.7. If we had constructed this plot in the nonchaotic regime,

[8]When constructing this plot numerically you must be careful to account for the fact that time increases in steps of size Δt. Thus, the points in Figure 3.7 were actually plotted when $|t - n\pi/\Omega_D| < \Delta t/2$.

with $F_D = 0.5$ for example, it would yield a single point (after allowing the initial transient to decay), since at any particular point of the drive cycle we would always find the same values of θ and ω.

The result of such a stroboscopic plot is very different in the chaotic regime, Figure 3.7. It turns out that except for the initial transient this phase-space trajectory is the same for a wide range of initial conditions. In other words, even though we cannot predict the behavior of $\theta(t)$, we do know that the system will possess values of ω and θ, which put it on this surface of points. The trajectory of our pendulum is drawn to this surface, which is known as an attractor. Actually, there are attractors in both the nonchaotic and chaotic regimes; the single point that would be found with $F_D = 0.5$ would also be an attractor. While the attractors have simple forms in the nonchaotic case, they have a very complicated structure in the chaotic regime. The "fuzziness" of the chaotic attractor in Figure 3.7 is not due to numerical uncertainties or plotting errors. It is a property of the attractor. Chaotic attractors have a *fractal* structure and are usually referred to as *strange attractors*.[9] We will discuss the nature of fractals at some length in Chapter 7.

There is much more that the damped, nonlinear, driven pendulum can tell us about chaos, and we will explore a few of these lessons in the exercises. Our key results are: (1) it is possible for a system to be both deterministic and unpredictable—in fact, this is what we mean by the term chaos; and (2) the behavior in the chaotic regime is not completely random, but can be described by a strange attractor in phase space. We will amplify and expand on these themes in the following sections.

Exercises

1. Study the effects of damping by starting the pendulum with some initial angular displacement, say $\theta = 0.5$ radians, and study how the motion decays with time. Use $q = 0.1$ and estimate the time constant for the decay. Compare your result with approximate analytic estimates for the decay time. Note: Any of the exercises in the section can be conveniently done with either the Euler-Cromer algorithm or the Runge-Kutta method described in Appendix 1.

2. Calculate $\theta(t)$ for $F_D = 0.1, 0.5$, and 0.99, with the other parameters as in Figure 3.4. Compare the waveforms, with special attention to the deviations from a purely sinusoidal form at high drive.

3. In constructing the Poincaré section in Figure 3.7 we plotted points only at times that were in phase with the drive force; that is, at times $t \approx 2\pi n/\Omega_D$, where n is an integer. At these values of t the driving force passed through zero [see (3.7)]. However, we could just as easily have chosen to make the plot at times corresponding to a maximum of the drive force, or at times $\pi/4$ out-of-phase with this force, etc. Construct the Poincaré sections for these cases and compare them with Figure 3.7.

4. Write a program to calculate and compare the behavior of two, nearly identical pendulums. Use it to calculate the divergence of two nearby trajectories in the chaotic regime, as in Figure 3.5, and make a qualitative estimate of the corresponding Lyapunov exponent from the slope of a plot of $\log(\Delta\theta)$ as a function of t.

5. Repeat the previous problem, but give the two pendulums slightly different damping factors. How does the value of the Lyapunov exponent compare with that found in Figure 3.5?

[9]Physicists like to use provocative names.

6. Study the shape of the chaotic attractor for different initial conditions. Keep the drive force fixed at $F_D = 1.2$ and calculate the attractors found for several different initial values of θ. Show that you obtain the same attractor even for different initial conditions, provided that these conditions are not changed by too much. Repeat your calculations for different values of the time step to be sure that it is sufficiently small that it does not cause any structure in the attractor.

*7. Investigate how a strange attractor is altered by small changes in one of the pendulum parameters. Begin by calculating the strange attractor in Figure 3.7. Then change either the drive amplitude or drive frequency by a small amount and observe the changes in the attractor.

*8. Construct a very high-resolution plot of the chaotic attractor in Figure 3.7, concentrating on the region $\theta > 2$ rad. You should find that there is more structure in the attractor than is obvious on the scale plotted in Figure 3.7. In fact, an important feature of chaotic attractors is that the closer you look, the more structure you find. We will see later that this property is related to fractals. It turns out that a strange attractor is a fractal object. Hint: In order to get accurate results for a high resolution plot of the attractor, it is advisable, in terms of the necessary computer time, to use the Runge-Kutta method.

3.3 Routes to Chaos: Period Doubling

We have seen that at low driving forces the damped, nonlinear pendulum exhibits simple oscillatory motion, while at high drive it can be chaotic. This raises an obvious question: Exactly how does the transition from simple to chaotic behavior take place? It turns out that the pendulum exhibits transitions to chaotic behavior at several different values of the driving force. We have already observed in Figure 3.4 that one of these transitions must take place between $F_D = 0.5$ and 1.2. However, this transition is not the clearest one to study numerically, so we will instead consider the behavior at somewhat higher driving forces.[10]

Figure 3.8 shows results for θ as a function of time for several values of the driving force calculated using the Euler-Cromer program described earlier. At these high values of the drive the pendulum often swings all the way around its support; this can be seen from the vertical steps in θ as our program resets[11] this angle to keep it in the range $-\pi$ to π. These steps notwithstanding, the behavior in Figure 3.8 is a periodic, repeating function of t in all three cases (after the initial transients have damped away). The drive frequency used here was $\Omega_D = 2/3$ so the period of the driving force was $2\pi/\Omega_D = 3\pi$, and this is precisely the period of the motion found at $F_D = 1.35$. Hence, in this case, the pendulum moves at the same frequency as the driving force.

The behavior at $F_D = 1.44$ is a bit more subtle. While we again have periodic motion, the period is now *twice* the drive period. This can be seen most clearly by comparing the θ-t waveforms at $F_D = 1.35$ and 1.44, and noticing that in the latter case the bumps alternate in amplitude. Our pendulum has already surprised us on several occasions, and we might be tempted to add this behavior to our list of pendulum puzzles. However, this surprise is a very special and important one. When a nonlinear

[10]The pendulum exhibits an extraordinarily rich behavior, some of which we discuss later. Unfortunately, we only have time to explore a small portion of the chaotic regime here. We will, therefore, limit ourselves to one value of both the drive frequency and damping, and only certain ranges of the drive amplitude. The interested reader is encouraged to investigate other parameter values. Additional results are also described by Baker and Gollub (1990).

[11]This distraction can be avoided by plotting ω instead of θ. We will leave this to the exercises.